First-order Possibility Models and Worldizations

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Overview

We will expand possibility semantics (in the style of Humberstone and as described in Wesley Holliday's talk) to the case of first-order modal logic.

1. We will describe first-order possibility models and some of their features.
2. Humberstone suggested that possibility semantics could be used to give a finitary completeness proof for **K**. We will consider such completeness proofs in first-order modal logic.
3. We will talk about how possibility models sit inside classical Kripke-like model with total worlds; we call these worldizations.

First-order possibility models

For simplicity we consider constant-domain first-order modal logic.

A *first-order possibility model* is a tuple $\mathcal{M} = (\mathcal{W}, \mathcal{R}, \sqsubseteq, \mathcal{D}, \mathcal{I})$ where:

1. $\mathcal{W}$ is the set of *possibilities*
2. $\mathcal{R}$ is the *accessibility relation*
3. $\sqsubseteq$ is the *refinement relation*
4. $\mathcal{D}$ is the *domain of objects*
5. $\mathcal{I}$ is an *interpretation* of the symbols

For objects $\bar{a} \in \mathcal{D}$, a relation symbol P, and a possibility $X \in \mathcal{W}$, the interpretation either:

1. determines that $\bar{a} \in P$ at X OR
2. determines that $\bar{a} \notin P$ at X OR
3. leaves this undetermined

Three conditions on the accessibility relation

P1:
$$\begin{tikzcd} X & Y \\ X' \arrow[u] \arrow[ur, dashed] & \end{tikzcd} \Longrightarrow \begin{tikzcd} X \arrow[r, dashed] & Y \\ X' \arrow[u] \arrow[ur, dashed] & \end{tikzcd}$$

P2:
$$\begin{tikzcd} X \arrow[r, dashed] & Y \\ & Y' \arrow[u] \end{tikzcd} \Longrightarrow \begin{tikzcd} X \arrow[r, dashed] \arrow[dr, dashed] & Y \\ & Y' \arrow[u] \end{tikzcd}$$

R:
$$X \dashrightarrow Y \Longrightarrow \begin{tikzcd} X \arrow[r, dashed] & Y \\ X' \arrow[u] & Y' \arrow[u, "\exists"] \\ X'' \arrow[u, "\forall"] \arrow[ur, dashed] & \end{tikzcd}$$

Conditions on the interpretation

We also have conditions on the interpretation.

Persistence: If the interpretation puts $\bar{a} \in P$ at X, and $Y \sqsubseteq X$, then it puts $\bar{a} \in P$ at Y.

Refinability: If the interpretation does not decide whether or not $\bar{a} \in P$ at X, then there is $Y \sqsubseteq X$ where it decides $\bar{a} \in P$, and $Z \sqsubseteq X$ where it decides $\bar{a} \notin P$.

A function symbol is treated like a relation symbol, but we ask that the interpretation have the following two properties. Treat a constant as a 0-ary function.

Totality: For each X and $\bar{a}$, there is $Y \sqsubseteq X$ and b such that the interpretation decides that $f(\bar{a}) = b$ at Y.

Uniqueness: If $f(\bar{a}) = b$ and $f(\bar{a}) = c$ at X, then $a = c$.

These conditions ensure that it is valid over the models that a function symbol actually represents a function.

Evaluation of formulas

We evaluate formulas at points in our models in a similar way as propositional possibility models. The important new clause is the clause for the universal quantifier.

$\mathcal{M}, X \models_v P(x_1, \ldots, x_n)$	iff	$(v(x_1), \ldots, v(x_n)) \in \mathcal{I}(X, P)$
$\mathcal{M}, X \models_v x = y$	iff	$v(x) = v(y)$
$\mathcal{M}, X \models_v \varphi \wedge \psi$	iff	$\mathcal{M}, X \models_v \varphi$ and $\mathcal{M}, X \models_v \psi$
$\mathcal{M}, X \models_v \neg\varphi$	iff	$\forall Y \sqsubseteq X$, $\mathcal{M}, Y \not\models_v \varphi$
$\mathcal{M}, X \models_v \Box\varphi$	iff	$\forall Y \in W$ with $X\mathcal{R}Y$, $\mathcal{M}, Y \models_v \varphi$
$\mathcal{M}, X \models_v (\forall x)\varphi$	iff	$\mathcal{M}, X \models_w \varphi$ for every variable assignment w which agrees with v except possibly at x

$X \models_v (\exists x)\varphi$ iff for all $Y \sqsubseteq X$, there is a variable assignment w which agrees with v except possibly at x and some $Z \sqsubseteq Y$ such that $Z \models_w \varphi$.

Remarks

For evaluating formulas, universal quantifiers are treated as an infinite conjunction over all of the objects in the domain.

Lemma (Persistence)

If $X \models \varphi$, and $Y \sqsubseteq X$, then $Y \models \varphi$.

Lemma (Refinability)

If $X \not\models \varphi$, there is $Y \sqsubseteq X$ with $Y \models \neg\varphi$.

Lemma (Completeness)

The logic of first-order possibility models is the standard logic of first-order constant-domain models with total worlds.

Indeterminate objects

One of the philosophical motivations for Humberstone was that possibility models allow "belief worlds", i.e., single worlds which validate exactly an agent's beliefs.

First-order possibility models have objects which play an analogous role. The identity of a constant symbol with other elements of the domain may be undetermined. For example, we may have

$$X \models (M = A) \vee (M = B)$$

$$X \models M \text{ owns a gun}$$

M = the murderer   A = Alice   B = Bob

X is the belief possibility of a detective who knows that either Alice or Bob committed the murder with a gun, but cannot decide which.

Finitary completeness proofs

One of Humberstone's original reasons to introduce possibility models was to give finitary completeness proofs.

For propositional modal logic, the finitary canonical model of a logic **L** is the possibility model $\mathcal{M}$ where:

1. the possibilities $\mathcal{W}$ are the (equivalence classes) of finite consistent sets of formulas
2. the accessibility relation is defined by $[\Gamma]\mathcal{R}[\Psi]$ iff for all $\Box\varphi \in [\Gamma]$, $\varphi \in [\Psi]$
3. the refinability relation is defined by $[\Gamma] \sqsubseteq [\Psi]$ iff $\Gamma \vdash \Psi$
4. the valuation is given by putting p at $[\Gamma]$ if $\Gamma \vdash p$, and $\neg p$ if $\Gamma \vdash \neg p$

If $\mathcal{M}, [\Gamma] \models \varphi$ iff $\Gamma \vdash \varphi$, then $\mathcal{M}$ gives a completeness proof for **L**.

This is not always the case.

Finite existence property

In order for this to work, **L** must have a key property:

Definition

A logic **L** has the finite existence property if for each consistent finite set Φ of sentences and ψ such that $\Phi \nvdash \Box\psi$, there is a finite set $\Psi = f(\Phi, \psi)$ of sentences such that $f(\Phi, \psi) \nvdash \psi$ and for all φ:

$$\Phi \vdash \Box\varphi \Rightarrow f(\Phi, \psi) \vdash \varphi.$$

Proposition

First-order modal logic does not have the finite existence property. There is a finite set Φ of sentences such that

$$\{\varphi : \Box\varphi \in \Phi\}$$

are not consequences of any finite set of sentences.

Computable completeness proofs

Everything works with computable sets of formulas. First-order modal logic has the computable existence property:

Proposition

Let Γ be a computable set of sentences. Then there is a computable set Φ of sentences such that for all sentences ψ,

$$\Gamma \models \Box\alpha \Longleftrightarrow \Phi \models \alpha.$$

A computable set of formulas still consists of finitely much information. So there is still a completeness proof where the possibilities have only finitely much information.

Worldizations

A worldization of a possibility model is a total world model such that every possibility is embedded in some total world and so that every total world is the limit of more and more refined possibilities.

$$\begin{tikzcd}
X_1 \arrow[r, dashed] \arrow[d] & Y_1 \arrow[d] & & Z_1 \arrow[d] \\
X_2 \arrow[r, dashed] \arrow[d] & Y_2 \arrow[d] & & Z_2 \arrow[d] \\
X_3 \arrow[r, dashed] \arrow[d] & Y_3 \arrow[d] & & Z_3 \arrow[d] \\
\vdots & \vdots & & \vdots
\end{tikzcd}$$

Worldizations

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$$
\begin{array}{ccccc}
X_1 & \dashrightarrow & Y_1 & \qquad\qquad & Z_1 \\
\downarrow & & \downarrow & & \downarrow \\
X_2 & \dashrightarrow & Y_2 & & Z_2 \\
\downarrow & & \downarrow & & \downarrow \\
X_3 & \dashrightarrow & Y_3 & & Z_3 \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots \\
 & & & & \\
U & \dashrightarrow & V & & W
\end{array}
$$

Existence of worldizations

If W is a total world that is the limit of the possibilities

$$X_0 \sqsupseteq X_1 \sqsupseteq X_2 \sqsupseteq \cdots$$

then we want $W \models \varphi$ if and only if $X_i \models \varphi$ for all sufficiently large i.

The accessibility relation in the worldization should be related to the accessibility relation in the original model.

Theorem

Every (countable) first-order or propositional possibility model has a worldization.

If possibility semantics is the logic of sets of total worlds after we forget about the total worlds, then this theorem says that we can always add back in the total worlds.