Borel Functors and Infinitary Interpretations

Matthew Harrison-Trainor

University of California, Berkeley

Computability in Europe, Paris, June 2016

An interesting question

Let $\mathcal{F}$ and $\mathcal{G}$ be two structures.

Suppose that $\mathcal{F}$ and $\mathcal{G}$ have the same automorphism group:

$$\mathrm{Aut}(\mathcal{F}) \cong \mathrm{Aut}(\mathcal{G}).$$

How are $\mathcal{F}$ and $\mathcal{G}$ related?

The answer lies in infinitary interpretations and Borel functors.

I will talk about work from two papers:

- With R. Miller and Montalbán:
  Borel functors and infinitary interpretations
- With Melnikov, R. Miller, and Montalbán:
  Computable functors and effective interpretations

Infinitary logic

All of our structures will be countable structures with domain ω.

We will use the infinitary logic $\mathcal{L}_{\omega_1\omega}$ which allows countable conjunctions and disjunctions.

Infinitary interpretations

Let $\mathcal{A} = (A; P_0^{\mathcal{A}}, P_1^{\mathcal{A}}, \ldots)$ where $P_i^{\mathcal{A}} \subseteq A^{a(i)}$.

Definition

$\mathcal{A}$ is *infinitary interpretable* in $\mathcal{B}$ if there exists a sequence of $\mathcal{L}_{\omega_1\omega}$-definable relations $(\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}, \sim, R_0, R_1, \ldots)$ such that

(1) $\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}} \subseteq \mathcal{B}^{<\omega}$,

(2) $\sim$ is an equivalence relation on $\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}$,

(3) $R_i \subseteq (B^{<\omega})^{a(i)}$ is closed under $\sim$ within $\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}$,

and a function $f_{\mathcal{A}}^{\mathcal{B}}: \mathcal{D}om_{\mathcal{A}}^{\mathcal{B}} \to \mathcal{A}$ which induces an isomorphism:

$$(\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}/\sim; R_0/\sim, R_1/\sim, \ldots) \cong (A; P_0^{\mathcal{A}}, P_1^{\mathcal{A}}, \ldots).$$

Some examples

Example

If $(R, 0, 1, +, \cdot)$ is an integral domain, the fraction field and polynomial ring of R are interpretable in R.

The domain of the fraction field F is $R \times R - \{0\}$ modulo the equivalence relation

$$(a, b) \sim (c, d) \Leftrightarrow ad = bc.$$

Addition on the fraction field is defined by

$$(a, b) + (c, d) = (ad + cb, bd).$$

Multiplication on the fraction field is defined by

$$(a, b) \cdot (c, d) = (ac, bd).$$

Borel functors

Let R be an integral domain with fraction field F.

If S is an isomorphic copy of R, we can use the same construction to build its fraction field G viewing the domain as $S \times S - \{0\}$ (modulo an equivalence relation).

Obviously G is an isomorphic copy of F.

So the fraction field construction yields a way of turning copies of R into copies of its fraction field.

We view this as a functor on the following category:

Definition

$\mathrm{Iso}(\mathcal{A})$ is the category of copies of $\mathcal{A}$ with domain ω. The morphisms are isomorphisms between copies of $\mathcal{A}$.

Borel functors

Recall: a functor F from $\mathrm{Iso}(\mathcal{A})$ to $\mathrm{Iso}(\mathcal{B})$

(1) assigns to each copy $\widehat{\mathcal{A}}$ in $\mathrm{Iso}(A)$ a structure $F(\widehat{\mathcal{A}})$ in $\mathrm{Iso}(\mathcal{B})$,

(2) assigns to each isomorphism $f\colon \widehat{\mathcal{A}} \to \widetilde{\mathcal{A}}$ in $\mathrm{Iso}(\mathcal{A})$ an isomorphism $F(f)\colon F(\widehat{\mathcal{A}}) \to F(\widetilde{\mathcal{A}})$ in $\mathrm{Iso}(\mathcal{B})$.

It satisfies $F(f \circ g) = F(f) \circ F(g)$.

Definition

F is *Borel* if there are Borel operators Φ and Φ_*
such that

(1) for every $\widehat{\mathcal{A}} \in \mathrm{Iso}(\mathcal{A})$, $\Phi^{D(\widehat{\mathcal{A}})}$ is the atomic diagram of $F(\mathcal{A})$,

(2) for every isomorphism $f : \widehat{\mathcal{A}} \to \widetilde{\mathcal{A}}$, $F(f) = \Phi_*^{D(\widehat{\mathcal{A}}) \oplus f \oplus D(\widetilde{\mathcal{A}})}$.

Automorphism groups

Back to the example:
Let R be an integral domain with fraction field F.

Let φ be an automorphism of R.

Then we get an automorphism φ_* on F:

$$\varphi_*(a, b) = (\varphi_*(a), \varphi_*(b)).$$

In fact, $\varphi \mapsto \varphi_*$ is a homomorphism $\mathrm{Aut}(R) \to \mathrm{Aut}(F)$.

Automorphism groups as Polish groups

Given a structure $\mathcal{A}$, we can view $\mathrm{Aut}(\mathcal{A})$ as subgroup of S_∞, the permutations of ω.

This is a topological group (in fact a Polish group).

Some facts:

1. Every Baire-measureable homomorphism of Polish groups is continuous.
2. An isomorphism of Polish groups is continuous if and only if it is an isomorphism of topological groups.
3. There is a model of $ZF + DC$ where all homomorphisms of Polish groups are continuous. *(Solovay, Shelah)*
4. In *ZFC* there are automorphism groups which are isomorphic but not isomorphic as topological groups. *(Evans, Hewitt)*

The first main theorem

Theorem (H-T., Miller, Montalbán)

$\mathcal{A}$ is infinitary interpretable in $\mathcal{B}$

$\Updownarrow$

there is a Borel functor F from $\mathcal{B}$ to $\mathcal{A}$.

$\Updownarrow$

there is a continuous homomorphism from $Aut(\mathcal{B})$ to $Aut(\mathcal{A})$.

The complexities of the formulas used in the interpretation correspond to the level in the Borel hierarchy.

The effective version of this theorem:

Theorem (H-T., Melnikov, Miller, Montalbán)

$\mathcal{A}$ is effectively (Σ^c_1) interpretable in $\mathcal{B}$

$\Updownarrow$

there is a computable functor F from $\mathcal{B}$ to $\mathcal{A}$.

Which interpretation?

Given a functor, we get an interpretation.

From that interpretation, we get back a functor.

Are these functors the same?

Yes:

Theorem (H-T., Miller, Montalbán)

Given a Borel functor F from $\mathcal{B}$ to $\mathcal{A}$, there is an infinitary interpretation $\mathcal{I}$ of $\mathcal{A}$ in $\mathcal{B}$ such that the functor $F_{\mathcal{I}}$ induced by $\mathcal{I}$ is isomorphic to F.

What does isomorphic mean?

Isomorphisms of functors

Let $F, G\colon \mathrm{Iso}(\mathcal{B}) \to \mathrm{Iso}(\mathcal{A})$ be computable functors.

Definition

F is *Borel isomorphic* to G if there is a Borel operator Λ such that for any $\widetilde{\mathcal{B}} \in \mathrm{Iso}(\mathcal{B})$, $\Lambda(\widetilde{\mathcal{B}})$ is an isomorphism from $F(\widetilde{\mathcal{B}})$ to $G(\widetilde{\mathcal{B}})$, and the following diagram commutes:

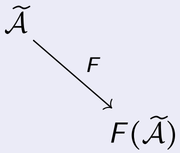
$$\widetilde{\mathcal{A}}$$

Isomorphisms of functors

Let $F, G \colon \mathrm{Iso}(\mathcal{B}) \to \mathrm{Iso}(\mathcal{A})$ be computable functors.

Definition

F is *Borel isomorphic* to G if there is a Borel operator Λ such that for any $\widetilde{\mathcal{B}} \in \mathrm{Iso}(\mathcal{B})$, $\Lambda(\widetilde{\mathcal{B}})$ is an isomorphism from $F(\widetilde{\mathcal{B}})$ to $G(\widetilde{\mathcal{B}})$, and the following diagram commutes:

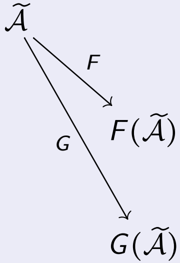


Isomorphisms of functors

Let $F, G\colon \mathrm{Iso}(\mathcal{B}) \to \mathrm{Iso}(\mathcal{A})$ be computable functors.

Definition

F is *Borel isomorphic* to G if there is a Borel operator Λ such that for any $\widetilde{\mathcal{B}} \in \mathrm{Iso}(\mathcal{B})$, $\Lambda(\widetilde{\mathcal{B}})$ is an isomorphism from $F(\widetilde{\mathcal{B}})$ to $G(\widetilde{\mathcal{B}})$, and the following diagram commutes:

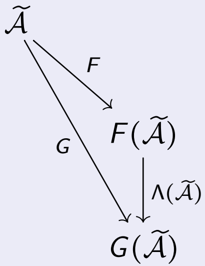


Isomorphisms of functors

Let $F, G\colon \mathrm{Iso}(\mathcal{B}) \to \mathrm{Iso}(\mathcal{A})$ be computable functors.

Definition

F is *Borel isomorphic* to G if there is a Borel operator Λ such that for any $\widetilde{\mathcal{B}} \in \mathrm{Iso}(\mathcal{B})$, $\Lambda(\widetilde{\mathcal{B}})$ is an isomorphism from $F(\widetilde{\mathcal{B}})$ to $G(\widetilde{\mathcal{B}})$, and the following diagram commutes:

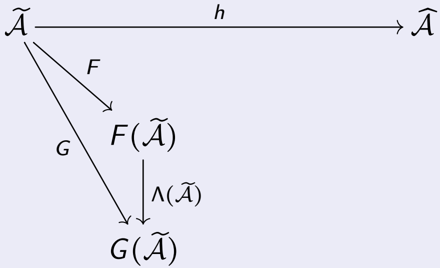


Isomorphisms of functors

Let $F, G\colon \mathrm{Iso}(\mathcal{B}) \to \mathrm{Iso}(\mathcal{A})$ be computable functors.

Definition

F is *Borel isomorphic* to G if there is a Borel operator Λ such that for any $\widetilde{\mathcal{B}} \in \mathrm{Iso}(\mathcal{B})$, $\Lambda(\widetilde{\mathcal{B}})$ is an isomorphism from $F(\widetilde{\mathcal{B}})$ to $G(\widetilde{\mathcal{B}})$, and the following diagram commutes:

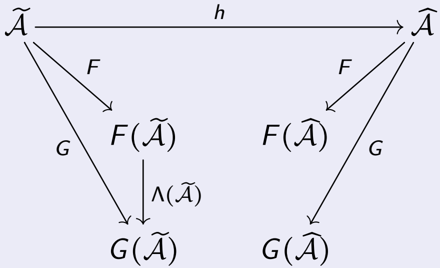


Isomorphisms of functors

Let $F, G\colon \mathrm{Iso}(\mathcal{B}) \to \mathrm{Iso}(\mathcal{A})$ be computable functors.

Definition

F is *Borel isomorphic* to G if there is a Borel operator Λ such that for any $\widetilde{\mathcal{B}} \in \mathrm{Iso}(\mathcal{B})$, $\Lambda(\widetilde{\mathcal{B}})$ is an isomorphism from $F(\widetilde{\mathcal{B}})$ to $G(\widetilde{\mathcal{B}})$, and the following diagram commutes:

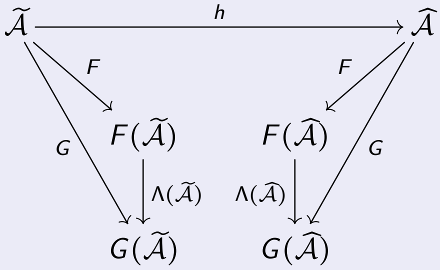


Isomorphisms of functors

Let $F, G \colon \mathrm{Iso}(\mathcal{B}) \to \mathrm{Iso}(\mathcal{A})$ be computable functors.

Definition

F is *Borel isomorphic* to G if there is a Borel operator Λ such that for any $\widetilde{\mathcal{B}} \in \mathrm{Iso}(\mathcal{B})$, $\Lambda(\widetilde{\mathcal{B}})$ is an isomorphism from $F(\widetilde{\mathcal{B}})$ to $G(\widetilde{\mathcal{B}})$, and the following diagram commutes:

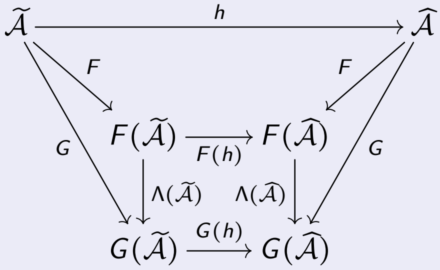


Isomorphisms of functors

Let $F, G\colon \mathrm{Iso}(\mathcal{B}) \to \mathrm{Iso}(\mathcal{A})$ be computable functors.

Definition

F is *Borel isomorphic* to G if there is a Borel operator Λ such that for any $\widetilde{\mathcal{B}} \in \mathrm{Iso}(\mathcal{B})$, $\Lambda(\widetilde{\mathcal{B}})$ is an isomorphism from $F(\widetilde{\mathcal{B}})$ to $G(\widetilde{\mathcal{B}})$, and the following diagram commutes:



Bi-interpretations

Definition

$\mathcal{A}$ and $\mathcal{B}$ are *infinitary bi-interpretable* if there are infinitary interpretations of each in the other, so that

$$f_{\mathcal{A}}^{\mathcal{B}} \circ f_{\mathcal{B}}^{\mathcal{A}} : \mathcal{D}om_{\mathcal{A}}^{(\mathcal{D}om_{\mathcal{B}}^{\mathcal{A}})} \to \mathcal{A} \text{ and } f_{\mathcal{B}}^{\mathcal{A}} \circ f_{\mathcal{A}}^{\mathcal{B}} : \mathcal{D}om_{\mathcal{B}}^{(\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}})} \to \mathcal{B}$$

are $\mathcal{L}_{\omega_1\omega}$-definable.

$$\begin{array}{ccc} & & \mathcal{B} \\ & & \cup\!\!\mid \\ \mathcal{A} & \xleftarrow{\;f_{\mathcal{A}}^{\mathcal{B}}\;} & \mathcal{D}om_{\mathcal{A}}^{\mathcal{B}} \end{array}$$

Bi-interpretations

Definition

$\mathcal{A}$ and $\mathcal{B}$ are *infinitary bi-interpretable* if there are infinitary interpretations of each in the other, so that

$$f_{\mathcal{A}}^{\mathcal{B}} \circ f_{\mathcal{B}}^{\mathcal{A}} : \mathcal{D}om_{\mathcal{A}}^{(\mathcal{D}om_{\mathcal{B}}^{\mathcal{A}})} \to \mathcal{A} \text{ and } f_{\mathcal{B}}^{\mathcal{A}} \circ f_{\mathcal{A}}^{\mathcal{B}} : \mathcal{D}om_{\mathcal{B}}^{(\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}})} \to \mathcal{B}$$

are $\mathcal{L}_{\omega_1\omega}$-definable.

$$\begin{array}{ccccc}
 & & & & \mathcal{B} \\
 & & & & \rotatebox{90}{$\subseteq$} \\
 & & \mathcal{A} & \xleftarrow{f_{\mathcal{A}}^{\mathcal{B}}} & \mathcal{D}om_{\mathcal{A}}^{\mathcal{B}} \\
 & & \rotatebox{90}{$\subseteq$} & & \\
\mathcal{B} & \xleftarrow{f_{\mathcal{B}}^{\mathcal{A}}} & \mathcal{D}om_{\mathcal{B}}^{\mathcal{A}} & &
\end{array}$$

Bi-interpretations

Definition

$\mathcal{A}$ and $\mathcal{B}$ are *infinitary bi-interpretable* if there are infinitary interpretations of each in the other, so that

$$f_{\mathcal{A}}^{\mathcal{B}} \circ f_{\mathcal{B}}^{\mathcal{A}} \colon \mathcal{D}om_{\mathcal{A}}^{(\mathcal{D}om_{\mathcal{B}}^{\mathcal{A}})} \to \mathcal{A} \text{ and } f_{\mathcal{B}}^{\mathcal{A}} \circ f_{\mathcal{A}}^{\mathcal{B}} \colon \mathcal{D}om_{\mathcal{B}}^{(\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}})} \to \mathcal{B}$$

are $\mathcal{L}_{\omega_1\omega}$-definable.

$$\begin{array}{ccccc}
 & & & & \mathcal{B} \\
 & & & & \cup\!| \\
 & & \mathcal{A} & \xleftarrow{f_{\mathcal{A}}^{\mathcal{B}}} & \mathcal{D}om_{\mathcal{A}}^{\mathcal{B}} \\
 & & \cup\!| & & \cup\!| \\
\mathcal{B} & \xleftarrow{f_{\mathcal{B}}^{\mathcal{A}}} & \mathcal{D}om_{\mathcal{B}}^{\mathcal{A}} & \xleftarrow{f_{\mathcal{A}}^{\mathcal{B}}} & \mathcal{D}om_{\mathcal{B}}^{(\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}})}
\end{array}$$

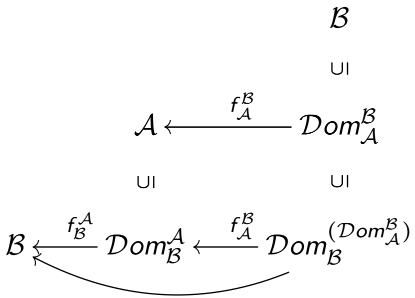
Bi-interpretations

Definition

$\mathcal{A}$ and $\mathcal{B}$ are *infinitary bi-interpretable* if there are infinitary interpretations of each in the other, so that

$$f_{\mathcal{A}}^{\mathcal{B}} \circ f_{\mathcal{B}}^{\mathcal{A}} : \mathcal{D}om_{\mathcal{A}}^{(\mathcal{D}om_{\mathcal{B}}^{\mathcal{A}})} \to \mathcal{A} \text{ and } f_{\mathcal{B}}^{\mathcal{A}} \circ f_{\mathcal{A}}^{\mathcal{B}} : \mathcal{D}om_{\mathcal{B}}^{(\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}})} \to \mathcal{B}$$

are $\mathcal{L}_{\omega_1\omega}$-definable.



Adjoint equivalences of categories

Definition

An adjoint equivalence of categories between $\mathsf{Iso}(\mathcal{A})$ and $\mathsf{Iso}(\mathcal{B})$ consists of functors

$$F\colon \mathsf{Iso}(\mathcal{A}) \to \mathsf{Iso}(\mathcal{B}) \text{ and } G\colon \mathsf{Iso}(\mathcal{B}) \to \mathsf{Iso}(\mathcal{A})$$

such that $F \circ G$ and $G \circ F$ are isomorphic to the identity (plus an extra condition on the isomorphisms).

The second main theorem

Theorem (H-T., Miller, Montalbán)

$\mathcal{A}$ and $\mathcal{B}$ are infinitary bi-interpretable

$$\Updownarrow$$

there is a Borel adjoint equivalence of categories between $\mathcal{A}$ and $\mathcal{B}$

$$\Updownarrow$$

there is a continuous isomorphism between $\mathrm{Aut}(\mathcal{A})$ and $\mathrm{Aut}(\mathcal{B})$.

An application

Let $\mathcal{A}$ be a countable structure.

Theorem (H-T., Miller, Montalbán)

The following are equivalent:

(1) *There is a continuous isomorphism between* $\mathrm{Aut}(\mathcal{A})$ *and* S_∞.

(2) *There is an* $\mathcal{L}_{\omega_1\omega}$*-definable* $D \subset A^n$ *and a* $\mathcal{L}_{\omega_1\omega}$*-definable equivalence relation* $E \subset D^2$ *with infinitely many equivalence classes, such that the* E*-equivalence classes are absolutely indiscernible and every other element is definable from this set.*

Thanks!