Scott Ranks of Models of a Theory

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$\mathcal{L}_{\omega_1\omega}$ theories

$\mathcal{L}_{\omega_1\omega}$ is the infinitary logic which allows countable conjunctions and disjunctions. By a "theory" we mean a sentence of $\mathcal{L}_{\omega_1\omega}$.

Many well-known classes of structures have Π_2 axiomatizations.

Example

There is a Π_2^c formula which describes the class of torsion groups. It consists of the group axioms together with:

$$(\forall x) \bigvee_{n\in\mathbb{N}} nx = 0.$$

Given a theory, what are the Scott ranks of its countable models?

Scott rank

Theorem (Scott)

Let $\mathcal{A}$ be a countable structure. There is an $\mathcal{L}_{\omega_1\omega}$-sentence φ such that:

$$\mathcal{B} \text{ countable}, \mathcal{B} \models \varphi \iff \mathcal{B} \cong \mathcal{A}.$$

φ is a *Scott sentence* of $\mathcal{A}$.

Definition (Scott rank)

$SR(\mathcal{A})$ is the least ordinal α such that $\mathcal{A}$ has a $\Pi^{\mathtt{in}}_{\alpha+1}$ Scott sentence.

Theorem (Montalbán)

Let $\mathcal{A}$ be a countable structure, and α a countable ordinal. TFAE:

- *$\mathcal{A}$ has a $\Pi^{\mathtt{in}}_{\alpha+1}$ Scott sentence.*
- *Every automorphism orbit in $\mathcal{A}$ is $\Sigma^{\mathtt{in}}_{\alpha}$-definable without parameters.*
- *$\mathcal{A}$ is uniformly (boldface) $\mathbf{\Delta}^0_{\alpha}$-categorical without parameters.*

Scott spectra

Let T be an $\mathcal{L}_{\omega_1\omega}$-sentence.

Definition

The *Scott spectrum* of T is the set

$$SS(T) = \{\alpha \in \omega_1 \mid \alpha \text{ is the Scott rank of a countable model of } T\}.$$

Main Question

What can we say about $SS(T)$?

Classifying the Scott spectra

Question

What are the possible Scott spectra of theories?

Definition

Let L be a linear order.

- $\mathrm{wfp}(L)$ is the well-founded part of L.
- $\mathrm{wfc}(L)$ is L with the non-well-founded part collapsed to a single element.

If $\mathcal{C}$ is a class of linear orders, we can apply to operations to each member of $\mathcal{C}$ to get $\mathrm{wfp}(\mathcal{C})$ and $\mathrm{wfc}(\mathcal{C})$.

Example

- $\mathrm{wfp}(\omega_1^{CK}(1+\mathbb{Q})) = \omega_1^{CK}$
- $\mathrm{wfc}(\omega_1^{CK}(1+\mathbb{Q})) = \omega_1^{CK} + 1$

Classifying the Scott spectra

Theorem (ZFC + PD)

The Scott spectra of $\mathcal{L}_{\omega_1\omega}$-sentences are exactly the sets of the form:

1. $\mathrm{wfp}(\mathcal{C})$,
2. $\mathrm{wfc}(\mathcal{C})$, *or*
3. $\mathrm{wfp}(\mathcal{C}) \cup \mathrm{wfc}(\mathcal{C})$

where $\mathcal{C}$ is a $\mathbf{\Sigma}^1_1$ class of linear orders.

Example

The admissible ordinals are a Scott spectrum.

Low-quantifier-rank theories with no simple models

Let T be a $\Pi_2^{\mathtt{in}}$ sentence.

Question (Montalbán)

Must T have a model of Scott rank two or less?

Theorem

Fix $\alpha < \omega_1$. There is a $\Pi_2^{\mathtt{in}}$ sentence T whose models all have Scott rank α.

In fact:

Theorem (ZFC + PD)

Every Scott spectrum is the Scott spectrum of a $\Pi_2^{\mathtt{in}}$ theory.

Scott height of $\mathcal{L}_{\omega_1\omega}$

Definition (Scott heights)

$\mathrm{sh}(\mathcal{L}_{\omega_1,\omega})$ is the least countable ordinal α such that, for all computable $\mathcal{L}_{\omega_1\omega}$-sentences T:

$$T \text{ has a model of Scott rank } \alpha$$
$$\Downarrow$$
$$T \text{ has models of arbitrarily high Scott ranks.}$$

Question (Sacks)

What is $\mathrm{sh}(\mathcal{L}_{\omega_1,\omega})$?

Theorem

$\mathrm{sh}(\mathcal{L}_{\omega_1,\omega}) = \delta^1_2$, *the least ordinal which has no* Δ^1_2 *presentation.*

Computable structures of high Scott rank

Theorem (Nadel)

A computable structure has Scott rank $\leq \omega_1^{CK} + 1$.

Theorem (Harrison)

There is a computable linear order of order type $\omega_1^{CK} \cdot (1 + \mathbb{Q})$ *with Scott rank* $\omega_1^{CK} + 1$.

Theorem (Makkai, Knight, Millar)

There is a computable structure of Scott rank ω_1^{CK}.

A computable structure has *high Scott rank* if it has Scott rank ω_1^{CK} or $\omega_1^{CK} + 1$.

High Scott rank and definability of orbits

Let $\mathcal{A}$ be a computable structure.

$SR(\mathcal{A}) < \omega_1^{CK}$ if for some computable ordinal α each automorphism orbit is definable by a Σ^c_α formula.

$SR(\mathcal{A}) = \omega_1^{CK}$ if each automorphism orbit is definable by a Σ^c_α formula for some α, but there is no computable bound on the ordinal α required.

$SR(\mathcal{A}) = \omega_1^{CK} + 1$ if there is an automorphism orbit which is not defined by a computable formula.

Approximations of structures

Let $\mathcal{A}$ be a computable structure of high Scott rank.

Definition

$\mathcal{A}$ is (strongly) computably approximable if every computable infinitary sentence φ true in $\mathcal{A}$ is also true in some computable $\mathcal{B} \not\cong \mathcal{A}$ with $SR(\mathcal{B}) < \omega_1^{CK}$.

Question (Calvert and Knight)

Is every computable model of high Scott rank computably approximable?

Theorem

No: There is a computable model $\mathcal{A}$ of Scott rank $\omega_1^{CK} + 1$ and a Π_2^c sentence ψ such that:

- $\mathcal{A} \vDash \psi$
- $\mathcal{B} \vDash \psi \Longrightarrow SR(\mathcal{B}) = \omega_1^{CK} + 1$.

Atomic models

Question (Millar, Sacks)

Is there a computable structure of Scott rank ω_1^{CK} whose computable infinitary theory is not $\aleph_0$-categorical?

Theorem (Millar, Sacks)

There is a structure $\mathcal{A}$ of Scott rank ω_1^{CK} whose computable infinitary theory is not $\aleph_0$-categorical.
$\mathcal{A}$ is not computable, but $\omega_1^{\mathcal{A}} = \omega_1^{CK}$. ($\mathcal{A}$ lives in a fattening of $L_{\omega_1^{CK}}$.)

Theorem (H., Igusa, Knight)

There is a computable structure of Scott rank ω_1^{CK} whose computable infinitary theory is not $\aleph_0$-categorical.

Open questions

Question

Classify the Scott spectra of $\mathcal{L}_{\omega_1\omega}$-sentences in ZFC.

Question

Classify the Scott spectra of computable $\mathcal{L}_{\omega_1\omega}$-sentences.

Question

Classify the Scott spectra of first-order theories.

Thanks!