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The main theorem (stated roughly)

All structures are countable with domain $\omega$.

Throughout, $\mathcal{A}$ and $\mathcal{B}$ will be structures.

**Theorem**

There is a correspondence between “effective interpretations” and “computable functors”.

**Example**

Let $\mathcal{A}$ be the equivalence structure with one equivalence class of size $n$ for each $n$.

Let $\mathcal{B}$ be the graph which consists of a cycle of size $n$ for each $n$.

$\mathcal{A}$ is effectively interpretable in $\mathcal{B}$ (in fact, they are bi-interpretable).
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A relation on $\mathcal{A}$ is a subset of $\mathcal{A}^{<\omega}$ (not $\mathcal{A}^n$ for some $n$).

For example this allows us to code subsets of $\mathcal{A}^{<\omega} \times \omega$ as subsets of $\mathcal{A}^{<\omega}$ in an effective way using the length of tuples.

Many results which were originally proven for subsets of $\mathcal{A}^n$ still hold for subsets of $\mathcal{A}^{<\omega}$.
Let $R$ be a relation on $\mathcal{A}^{<\omega}$.

**Definition**

$R$ is *uniformly relatively intrinsically computably enumerable (u.r.i.c.e.*) if there is a c.e. operator $W$ such that for every copy $(\mathcal{B}, R^\mathcal{B})$ of $(\mathcal{A}, R)$, $R^\mathcal{B} = W^{D(\mathcal{B})}$.

$R$ is *uniformly relatively intrinsically computable (u.r.i. computable)* if there is a computable operator $\Psi$ such that for every copy $(\mathcal{B}, R^\mathcal{B})$ of $(\mathcal{A}, R)$, $R^\mathcal{B} = \Psi^{D(\mathcal{B})}$.

Recall:

**Theorem (Ash-Knight-Manasse-Slaman,Chisholm)**

$R$ is u.r.i.c.e. if and only if it is definable by a $\Sigma^c_1$ formula without parameters.
Let $\mathcal{A} = (A; P^A_0, P^A_1, ...)$ where $P^A_i \subseteq A^{a(i)}$.

**Definition**

$\mathcal{A}$ is *effectively interpretable* in $\mathcal{B}$ if there exist a u.r.i. computable sequence of relations $(\text{Dom}^B_\mathcal{A}, \sim, R_0, R_1, ...)$ such that

1. $\text{Dom}^B_\mathcal{A} \subseteq B^{<\omega}$,
2. $\sim$ is an equivalence relation on $\text{Dom}^B_\mathcal{A}$,
3. $R_i \subseteq (B^{<\omega})^{a(i)}$ is closed under $\sim$ within $\text{Dom}^B_\mathcal{A}$,

and a function $f^B_\mathcal{A}: \text{Dom}^B_\mathcal{A} \to A$ which induces an isomorphism:

$$(\text{Dom}^B_\mathcal{A}/\sim; R_0/\sim, R_1/\sim, ...) \cong (A; P^A_0, P^A_1, ...).$$

This is equivalent to $\Sigma$-reducibility without parameters.
Definition

Iso(\(A\)) is the category of copies of \(A\) with domain \(\omega\). The morphisms are isomorphisms between copies of \(A\).

Recall: a functor \(F\) from Iso(\(A\)) to Iso(\(B\))

1. assigns to each copy \(\widehat{A}\) in Iso(\(A\)) a structure \(F(\widehat{A})\) in Iso(\(B\)),
2. assigns to each isomorphism \(f: \widehat{A} \rightarrow \widehat{A}\) in Iso(\(A\)) an isomorphism \(F(f): F(\widehat{A}) \rightarrow F(\widehat{A})\) in Iso(\(B\)).

Definition

\(F\) is \emph{computable} if there are computable operators \(\Phi\) and \(\Phi_*\) such that

1. for every \(\widehat{A} \in \text{Iso}(A)\), \(\Phi_{D(\widehat{A})}\) is the atomic diagram of \(F(A)\),
2. for every isomorphism \(f: \widehat{A} \rightarrow \widehat{A}\), \(F(f) = \Phi_{D(\widehat{A})} \oplus f \oplus D(\widehat{A})\).
The main theorem

Theorem

\[ A \text{ is effectively interpretable in } B \]

\[ \iff \]

\[ \text{there is a computable functor } F \text{ from } B \text{ to } A. \]

Question

If \( A \) is a computable structure, is this vacuous?
Let $F, G : \text{Iso}(B) \rightarrow \text{Iso}(A)$ be computable functors.

**Definition**

$F$ is *effectively isomorphic* to $G$ if there is a computable Turing functional $\Lambda$ such that for any $\tilde{B} \in \text{Iso}(B)$, $\Lambda \tilde{B}$ is an isomorphism from $F(\tilde{B})$ to $G(\tilde{B})$, and the following diagram commutes:
Let $F : \text{Iso}(B) \to \text{Iso}(A)$ be a computable functor. Using the main theorem, we get an interpretation $\mathcal{I}$ of $A$ in $B$. Again using the main theorem, we get a functor $F_{\mathcal{I}}$ from this interpretation.

**Proposition**

*These two functors are effectively isomorphic.*

**Example**

Let $A = B = (\omega, 0, +)$. Consider the functors:

$$F \ := \ \text{identity functor}$$

$$G \ := \ \text{constant functor giving the standard presentation of } \omega$$

These are not effectively isomorphic, and the interpretations we get are faithful to the functor.
Definition

A and B are effectively bi-interpretable if there are effective interpretations of each in the other, and u.r.i. computable isomorphisms $\text{Dom}_A^{\text{Dom}_B^A} \rightarrow A$ and $\text{Dom}_B^{\text{Dom}_A^B} \rightarrow B$. 

$$
\text{Dom}_A^B \xrightarrow{g} \text{Dom}_B^{\text{Dom}_A^B}
$$
**Definition**

\( \mathcal{A} \) and \( \mathcal{B} \) are *computably bi-transformable* if there are computable functors \( F : \text{Iso}(\mathcal{A}) \to \text{Iso}(\mathcal{B}) \) and \( G : \text{Iso}(\mathcal{B}) \to \text{Iso}(\mathcal{A}) \) such that both \( F \circ G : \text{Iso}(\mathcal{B}) \to \text{Iso}(\mathcal{B}) \) and \( G \circ F : \text{Iso}(\mathcal{A}) \to \text{Iso}(\mathcal{A}) \) are effectively isomorphic to the identity functor.

So if \( \widehat{\mathcal{B}} \) is a copy of \( \mathcal{B} \), then \( F(G(\widehat{\mathcal{B}})) \cong \widehat{\mathcal{B}} \) and the isomorphism can be computed uniformly in \( \widehat{\mathcal{B}} \).

**Theorem**

\[ \mathcal{A} \text{ and } \mathcal{B} \text{ are effectively bi-interpretable} \]

\[ \iff \]

\[ \mathcal{A} \text{ and } \mathcal{B} \text{ are computably bi-transformable.} \]
Classes of structures

Let \( \mathcal{C} \) and \( \mathcal{D} \) be classes of structures.

**Definition**

\( \mathcal{C} \) is *uniformly transformally reducible* to \( \mathcal{D} \) if there is a subclass \( \mathcal{D}' \) of \( \mathcal{D} \) and computable functors \( F: \mathcal{C} \to \mathcal{D}' \), \( G: \mathcal{D}' \to \mathcal{C} \) such that \( F \circ G \) and \( G \circ F \) are effectively isomorphic to the identity functor.

**Definition**

\( \mathcal{C} \) is *reducible via effective bi-interpretability* to \( \mathcal{D} \) if for every \( \mathcal{C} \in \mathcal{C} \) there is a \( \mathcal{D} \in \mathcal{D} \) such that \( \mathcal{C} \) and \( \mathcal{D} \) are effectively bi-interpretable and the formulas involved do not depend on the choice of \( \mathcal{C} \) or \( \mathcal{D} \).

**Theorem**

\[ \mathcal{C} \text{ is reducible via effective bi-interpretability to } \mathcal{D} \]

\[ \iff \]

\[ \mathcal{C} \text{ is uniformly transformally reducible to } \mathcal{D}. \]
Examples

Theorem (Hirschfeldt, Khoussainov, Shore, Slinko)

Every class is reducible via effective bi-interpretability to each of the following classes:

1. undirected graphs,
2. partial orderings, and
3. lattices,

and, after naming finitely many constants,

1. integral domains,
2. commutative semigroups, and
3. 2-step nilpotent groups.

Theorem (Miller, Park, Poonen, Schoutens, Shlapentokh)

We can add fields of characteristic zero to the first list above.
Examples of interpretations above a jump

**Theorem (Marker, Miller)**

There is a computable functor from graphs to differentially closed fields (and an inverse functor, defined only on some differentially closed fields, which is $0'$-computable).

**Theorem (Ocasio)**

There is a computable functor from linear orders to real closed fields (and an inverse functor, defined only on some real closed fields, which is $0'$-computable).