RESEARCH STATEMENT

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My research is in computability theory ([6]), and particularly computable structure theory ([2, 4, 8, 12, 15, 17]) and its interactions with algebra ([3, 7, 14, 16]). I have also published work in differential algebra and model theory ([1, 13]), possibility semantics for modal logic ([5, 9]), and subjective probability ([10, 11]).

In computable structure theory, we use the tools of computability theory to study mathematical structures such as groups, graphs, or fields, and their computational properties. A structure consists of a domain together with relations and functions. Because we want to perform computations using these structures, we will only consider countable structures with domain $\mathbb{N}$. We will say that a structure is a computable structure if the relations and functions are computable as subsets of $\mathbb{N}^n$ and as functions $\mathbb{N}^n \to \mathbb{N}$ respectively.

Using the tools of computability theory, we can answer questions about which mathematical structures are complicated and which are simple, and about which constructions are complicated and which are simple. We can also find relations between different structures and constructions. To illustrate this, consider the well-known example of algebraically closed fields. We might be interested in how easy it is to find a transcendence basis for an infinite-dimensional computable algebraically closed field. It turns out that, in general, there is no computable algorithm. We need to be able to decide whether or not a finite tuple of elements is algebraically independent by searching for a polynomial which they satisfy. This is an infinite search, which cannot be completed in finite time. This is the only issue: It is known that if we had a computer which, along with performing mechanical calculations, also had the capability to ask an oracle whether an infinite search would ever be completed, then that computer could compute a transcendence basis for every computable algebraically closed field. (There are other, more difficult, problems for which an even stronger oracle would be required.)

An important observation is that there are computable structures which are isomorphic, but for which it is impossible to compute an isomorphism between the two. This can be seen in the case of infinite-dimensional computable algebraically closed fields as follows. On the one hand, there is a computable algebraically closed field which has a computable transcendence basis. On the other hand, since we cannot compute a transcendence basis for every computable algebraically closed field, there is a computable algebraically closed field with no computable transcendence basis. The two are not computably isomorphic (for otherwise we could take the image of the computable transcendence basis of the first field to get one for the second). And yet these two fields are isomorphic as they are both algebraically closed fields of countable dimensional. Now we ask: How hard is it to build an isomorphism between two algebraically closed fields? The answer, as one might expect, is that it is the same difficulty as finding a transcendence basis. (In [16] we studied phenomena of this type in many different algebraic structures with a notion of independence. See §1.4.)

Some types of structures, such as vectors spaces, algebraic fields, and algebraically closed fields, are very tame. Other types of structures, such as groups, fields, and graphs, are as complicated as structures could possibly be, in the sense that any structure of any type is computability-theoretically equivalent to a group, field, or graph. (More technically, for any structure $\mathcal{A}$, there is a group, field, or graph $\mathcal{G}$ and a *computable equivalence of categories* between the presentations of $\mathcal{A}$ and the presentations of $\mathcal{G}$. In [15, 17] we studied the relationship between computable functors and interpretations. See §1.3.) Finally, other types of structures, such as abelian groups, linear orders, and boolean algebras, fall somewhere in between. In general, we are trying to figure out which mathematical structures are complicated and which are simple, and we also want to understand complexity by drawing relations between structural and computational properties of structures.

Before describing my results in more detail, I will give a less technical description of two of my major projects:

(1) In practice, when we want to find structures with certain kinds of pathological behaviour, we must construct them explicitly. If $\mathcal{A}$ is a natural structure—by which I mean an informal notion of a

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structure that might show up in the normal course of mathematics, and which was not constructed explicitly as a computability-theoretic counterexample—then arguments about $\mathcal{A}$ will generally relativize. So we can study natural structures by studying arbitrary structures relativized "on a cone". This gives us a way of making precise statements about the imprecise notion of a natural structure. In [4], I studied what happens when you have an additional relation on a natural structure—for example, the algebraic independence relation of an algebraically closed field—which you want to compute, and with Csima [2], I studied the difficulty of finding isomorphisms between two copies of a natural structure (see §1.1).

(2) We can assign to a countable structure its Scott rank, which measures the complexity of describing that structure. If we have a class of structures that is described in a simple way, such as the class of groups or graphs, we might expect that the ranks of the structures in that class would have some regularity. For example, we might expect that any such class contains a structure of low rank, or that if there are structures in the class of high rank, then there are structures of many different high ranks. My work on Scott spectra [8] showed that there are simple classes of structures all of whose members are very complicated, as well as other results on similar problems (see §1.2).

1. Work in Computability Theory

1.1. **Degree spectra and computable categoricity on a cone.** Given a computable structure $\mathcal{A}$ with an additional relation R, R may have different Turing degrees in different computable copies of $\mathcal{A}$. For example, $\mathcal{A}$ might be the linear order $(\mathbb{N}, <)$, and R might be the successor relation. In some computable copies of $\mathcal{A}$, R is computable, while in others it is non-computable, but it is always co-c.e. The degree spectrum of R is the set of all Turing degrees of R in different computable copies of $\mathcal{A}$. This is a measure of the complexity of that relation.

Natural structures and relations—those which arise in the normal course of mathematics, such as vector spaces with the independence relation—have natural classes of degree spectra, such as the c.e. degrees, the d.c.e. degrees, the Δ^0_2 degrees, etc. On the other hand, many examples have been constructed with pathological degree spectra. Of course, there is no formal definition of a natural structure; instead, Montalbán has suggested that one can study natural behaviour as follows. Consider some property P of structures and relations. Usually in computability theory, results about natural structures relativize; so if a natural structure has property P, then it will have property P relative to any degree $\mathbf{d}$. On the other hand, with enough determinacy, for any structure, natural or not, that structure will either have property P relative to all sufficiently high degrees $\mathbf{d}$, or not have property P relative to all sufficiently high degrees $\mathbf{d}$. (More formally, there will be a degree $\mathbf{c}$ relative to which the structure will either have property P (or not have property P) relative to all degrees $\mathbf{d} \geq \mathbf{c}$.) If the structure has property P relative to all degrees $\mathbf{d} \geq \mathbf{c}$ for some $\mathbf{c}$, then we say that the structure has property P *on a cone*. So by studying the properties of structures on a cone, we can get at the properties of natural structures. Results about structures on a cone also shed light on the techniques required to construct various examples: if all structures have property P on a cone, then to construct a computable structure without property P, one must use a technique, such as diagonalization, which does not relativize.

Harizanov showed (though not in this language) that every degree spectrum on a cone is either just the computable degree, or contains all the c.e. degrees. In [4], my two main theorems were:

Theorem 1. *There are two incomparable degree spectra contained within the d.c.e. degrees.*

Theorem 2. *Every degree spectrum on a cone is either contained within the Δ^0_2 degrees or contains all of the 2-CEA degrees.*

The first theorem has consequences in pure computability theory: The degree spectra I build form natural classes of Turing degrees which are between the c.e. degrees and the d.c.e. degrees. By natural, I mean that they relativize. These classes of Turing degrees are the beginning of a finer refinement of Ershov's hierarchy.

The second theorem partially answers a question of Ash and Knight. They had asked for a generalization of Harizanov's result which was described above: Subject to some effectivity conditions, must a degree spectrum which is not contained in the Δ^0_α degrees contain all of the α-CEA degrees? My result answers this question for $\alpha = 2$.

My work forms the first steps of a program to try to understand the degree spectra of natural relations. My results give just the beginning, and there is a lot to know about degree spectra of relations. One hopes to eventually be able to completely classify the degree spectra of relations on a cone.

Later, with Csima [2], we applied similar techniques to degrees of categoricity. A computable structure $\mathcal{A}$ has degree of categoricity $\mathbf{d}$ if for every computable copy $\mathcal{B}$ of $\mathcal{A}$, $\mathbf{d}$ computes an isomorphism between $\mathcal{A}$ and $\mathcal{B}$, and $\mathbf{d}$ is the least degree with this property. A difficult open question is to classify the degrees which are the degree of categoricity of some structure. We showed that, on a cone, every structure has Δ^0_α-complete degree of categoricity for some ordinal α.

1.2. **Scott ranks of models of a theory.** To each countable structure we can assign a countable ordinal, the structure's *Scott rank*, which measures its internal complexity. Scott ranks arose from the proof of Scott's theorem that for each countable structure $\mathcal{A}$, there is a sentence φ of the infinitary logic $\mathcal{L}_{\omega_1\omega}$ that describes the isomorphism type of $\mathcal{A}$ among countable structures. This sentence φ is known as the Scott sentence of $\mathcal{A}$, and the Scott rank measures its quantifier complexity.

Given a theory T—by which we mean a sentence of infinitary logic—we define the Scott spectrum of T to be the set of Scott ranks of countable models of T. In [8], I considered the following general question: what sets of countable ordinals are the Scott spectrum of a theory? I characterized the Scott spectra in terms of a descriptive-set-theoretic class of ordinals:

Theorem 3 (ZFC + PD)**.** *The Scott spectra of sentences of $\mathcal{L}_{\omega_1\omega}$ are exactly the sets of countable ordinals of the form:*

(1) *the well-founded parts of orders in $\mathcal{C}$,*
(2) *the orders in $\mathcal{C}$ with the non-well-founded part collapsed to a single element, or*
(3) *the union of the two previous options.*

where $\mathcal{C}$ is a $\mathbf{\Sigma}^1_1$ class of linear orders of ω.

The majority of the work went into showing that each of these sets of ordinals is the Scott spectrum of a theory. The proof is a construction which takes a Σ^1_1 class $\mathcal{C}$ of linear orders and produces a class of two-sorted structures where the first sort is a linear order in $\mathcal{C}$ and the second sort is a tree with non-standard back-and-forth relations whose height is indexed by the first sort. In this way, the well-founded part of the first sort controls the Scott rank of the whole structure.

Adapting the same construction, I was also able to answer three open questions. First, Montalbán asked whether every $\Pi^{\mathtt{in}}_2$ theory T has a model of Scott rank two or less. I showed that for every countable ordinal α, there is a $\Pi^{\mathtt{in}}_2$ theory T all of whose countable models have Scott rank α, and in fact, assuming projective determinacy, every Scott spectrum is the Scott spectrum of a $\Pi^{\mathtt{in}}_2$ theory.

Second, it follows from a general counting argument that there is a least ordinal $\alpha < \omega_1$ such that if T is a computable $\mathcal{L}_{\omega_1\omega}$-sentence whose Scott spectrum is bounded below ω_1, then the Scott spectrum of T is bounded below α. We call this ordinal the Scott height of the computable $\mathcal{L}_{\omega_1\omega}$ sentences. Sacks and Marker asked which ordinal is the Scott height of the computable $\mathcal{L}_{\omega_1\omega}$ sentences. Sacks had previously shown that it was at most δ^1_2, the least ordinal which has no Δ^1_2 presentation, and I showed that it was exactly δ^1_2.

Finally, a computable structure is known to have Scott rank either a countable ordinal, ω_1^{CK}, or $\omega_1^{CK}+1$. A structure with Scott rank ω_1^{CK} or $\omega_1^{CK}+1$ is said to be a structure of high Scott rank. These structures are interesting because they are computable, but they do not have a computable Scott sentence. Very few examples of structures of high Scott rank are known. Because the known examples were all approximable by structures of low Scott rank, Calvert and Knight asked whether every computable model of high Scott rank is approximable in this way. I produced models of Scott rank ω_1^{CK} and $\omega_1^{CK}+1$ which are not computably approximable. Thus these are new examples of computable structures of high Scott rank.

In joint work with Igusa and Knight [12], we produced other new examples of computable structures of high Scott rank. Millar and Sacks asked whether there is a computable structure of Scott rank ω_1^{CK} whose computable infinitary theory is not $\aleph_0$-categorical. They produced a structure $\mathcal{A}$ which is not computable, but which has $\omega_1^{\mathcal{A}} = \omega_1^{CK}$, such that the computable theory of $\mathcal{A}$ is not $\aleph_0$-categorical. $\mathcal{A}$ does not live in $L_{\omega_1^{CK}}$, but in a fattening of it. We produced a computable structure of Scott rank ω_1^{CK} for which the conjunction of the computable infinitary theory is not a Scott sentence. This answers positively the question of Millar and Sacks mentioned above. We also produced structures of Scott rank $\omega_1^{CK}+1$ with no order indiscernible triple.

1.3. **Functors and interpretations.** Sometimes, a structure $\mathcal{A}$ may be represented inside a structure $\mathcal{B}$ by an *interpretation*; the domain of $\mathcal{A}$ is identified with with definable equivalence classes of tuples in $\mathcal{B}$ so that the function, relations, and constants of $\mathcal{A}$ are all definable in $\mathcal{B}$. If $\mathcal{A}$ is interpreted inside of $\mathcal{B}$, then we get a functor which takes copies of $\mathcal{B}$ and produces copies of $\mathcal{A}$. If the formulas of the interpretation are effective (Σ_1), then the functor is a computable functor. With Melnikov, Miller, and Montalbán [15], we showed that the converse is also true: each computable functor from copies of $\mathcal{B}$ to copies of $\mathcal{A}$ gives rise to an interpretation of $\mathcal{A}$ in $\mathcal{B}$ using Σ_1 formulas. Later, in [17], with Miller and Montalbán we generalized this to arbitary infinitary interpretations and Borel functors. We also considered bi-interpretations, which are in correspondence with adjoint equivalences of categories.

Computable functors have been of much interest recently. A number of classes of structures, such as graphs, have long been known to be universal in the sense that every countable structure is effectively bi-interpretable with one in that class. Recently, fields were added to this list. There have also recently been construction of an interpretation of graphs into differentially closed fields and into real closed fields.

1.4. **Computable algebra.** Downey and Kurtz showed that a computable group, even a computable abelian group, which is orderable need not have a computable order. Downey and Kurtz asked whether each computable group is classically isomorphic to a computable group which has a computable ordering. Using results of Dobritsa, Solomon showed that this is the case for abelian groups. The case of non-abelian groups was left open. In [7], I showed that there is a computable left-orderable group which is not classically isomorphic to a computable group with a computable left-order. This result uses a new way of coding into ordered groups using cycles in the orbits under conjugation. The case of bi-orderable groups is still open.

Together with Melnikov and Montalbán, in [16] we noticed that there were many instances in the literature of classes of structures with an independence relation (formally, a pre-geometry) such as linear independence in vector spaces, algebraic independence in algebraically closed fields, or $\mathbb{Z}$-linear independence in torsion-free abelian groups, with the property that every computable structure of infinite dimension in the class has a computable copy with a computable basis and a computable copy with no computable basis. We said that such a structure has the *Mal'cev property.* We gave a metatheorem, encompassing the existing examples, for showing that a class of structures has the Mal'cev property, and used this metatheorem to produce new examples such as differentially closed fields.

Given an algebraic field K, every automorphism of K can be extended to an automorphism of its algebraic closure. Together with Melnikov and Miller, in [14] we studied the effectivity of this and other related facts. We showed that every computable automorphism of K extends to a computable automorphism of its algebraic closure if and only if K has a splitting algorithm.

In [3], I studied the effectivity of a number of constructions for extending valuations from a valued field to an algebraic field extension.

1.5. **The Gamma problem.** A set is coarsely computable with density r if there is an algorithm for deciding membership in that set which always gives an answer, and the answer is correct with lower density r. To a (Turing or many-one) degree $\mathbf{a}$, we can assign a value $\Gamma(\mathbf{a})$ which measures the minimal degree of coarse computability of sets in $\mathbf{a}$. A simple argument shows that Γ cannot take on values strictly between 1/2 and 1. It is known that, for both Turing and many-one degrees, Γ can take on the values 0, 1/2, and 1. The Gamma questions asks: What are the other possible values of Γ? In [6], I showed that for many-one degrees, Γ can take on all values between 0 and 1/2. (The answer for Turing degrees has been shown to be the opposite: The only possible values are 0, 1/2, and 1.)

2. Work in Other Areas

2.1. **Differential algebra and model theory.** The Kolchin problem asks for a decision procedure to decide whether a radical differential ideal is prime. In joint work with Klys and Moosa [13], we used nonstandard methods to show that an existence of bounds version of the Kolchin problem is equivalent to five other problems. We also proved the existence of bounds for characteristic sets of prime differential ideals, and we gave a new proof of the effective differential Nullstellensatz.

With Chatzidakis and Moosa [1], we showed that the generic type of differential jet spaces have a property called preserving internality to the constants—a model-theoretic abstraction of the behaviour of jet spaces in complex-analytic geometry.

2.2. **Modal logic.** While Kripke models have been the standard models for modal logic, there has been much interest in alternative models. Possibility models, in the style of Humberstone, replace the total worlds of Kripke models with partial possibilities and a relation of refinement. Possibility models have been of recent interest for a number of reasons, among which is their ability to capture, via frame completeness, more normal modal logics than Kripke models. In [9], I showed that each possibility model can be transformed into a Kripke model via a process of *worldization*. The worlds of the Kripke model are formed by increasing limits in the possibility model, and every possibility is contained in some world. In [5], I presented possibility semantics for first-order modal logic. I also showed that it is possible to give a certain type of finitary completeness proof where each possibility has only finitely much information.

2.3. **Subjective probability.** We are interested in the following general question: given a number of judgments of subjective probability of the form "Event A is at least as likely as B", what further judgments can be inferred? Insua gave a complete set of principles for extracting all of the inferred judgments, the chief of which, Generalized Finite Cancellation, was a generalization of the principal of Finite Cancellation which had previously appeared in the literature. In joint work with Holliday and Icard [11], we showed that Generalized Finite Cancellation is in fact stronger than Finite Cancellation. In later work [10], we considered the case where the events in the initial comparisons are all disjoint, and we gave a simpler way to produce all of the inferred comparisons in this case.

References

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