WORLDIZATIONS OF POSSIBILITY MODELS

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Abstract. This paper is about the relation between two kinds of models for propositional modal logic: possibility models in the style of Humberstone and possible world models in the style of Kripke. We show that every countable possibility model $\mathcal{M}$ is completed by a Kripke model $\mathcal{K}$, its *worldization*; every total world of $\mathcal{K}$ is the limit of more and more refined possibilities in $\mathcal{M}$, and every possibility in $\mathcal{M}$ is realized by some total world of $\mathcal{K}$. In addition, we define a general notion of a *possibilization* of a Kripke model, which is a possibility model whose possibilities are sets of worlds from the Kripke model. We then characterize the class of possibility models that are isomorphic to the possibilization of some Kripke model. In particular, every possibility model in this class can be represented as a possibilization of one of its worldizations; and every possibility model can be naturally transformed into one in this class by, for example, deleting duplicated possibilities. This representation theorem clarifies the relationship between possibility models and Kripke models.

1. Introduction

Humberstone [Hum81] introduced *possibility models* as an alternative semantics for propositional modal logic. In a possibility model, the worlds of a Kripke model are replaced by partial possibilities. A possibility determines some parts or aspects of a world; as Edgington [Edg85] explains (see also Chapter 10 of [Hal13] and Chapter 6 of [Rum15]):

> Possibilities differ from possible worlds in leaving many details unspecified... I am counting the possibility that the die lands six-up as one possibility. There are indefinitely many possible worlds compatible with this one possibility which vary not only in the precise location and orientation of the landed die, but also as to whether it is raining in China at the time, or at any other time, and so on ad infinitum . . . . (564)

While a world in a Kripke model determines the truth or falsity of every sentence in the language under consideration, a possibility might determine the truth of some sentences, the falsity of others, and leave the truth of further sentences undetermined. One possibility might *refine* another. If a possibility Y refines a possibility X, then any atomic proposition true at X remains true at Y and any atomic proposition false at X remains false at Y; however, of the remaining atomic propositions, some may become true at Y, some may become false, and others may remain undecided. There is also a modal accessibility relation between possibilities.

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Holliday [Hol15] shows that at the level of *frames*, possibility semantics is more general than Kripke semantics: there are possibility frames as in Definition 2.1 below whose logic is a normal modal logic that is not the logic of any class of Kripke frames. This is explained in [Hol15] in terms of a duality theory relating possibility frames and certain modal algebras, which is contrasted with the well-known duality theory for Kripke frames. Our goal in this paper is to instead look at *models*: we will relate possibility models and Kripke models, in terms of structural transformations of one kind of model into the other. (As we will explain in Section 6, our results can also be viewed as relating general frames for possibility semantics and Kripke semantics.) For other recent work on possibility semantics, see [Gar13], [Hol14], [vBBH16], [Yam16], and [HT16] (and the related [BH16]).

One often thinks of possibilities as being sets of worlds; the possibility that the die lands six-up can be identified or at least associated with the set of total worlds in which the die lands six-up. There are objections to this on philosophical grounds—Edgington [Edg85, p. 564] asserts that when one thinks of a possibility, one is not thinking of a single possible world or even a set of possible worlds, but rather some other type of object—but the idea of possibilities as sets-of-total-worlds can be thought of as a simplified motivating example of possibility models. Even if a set of total worlds is not what one *thinks* of when thinking of a possibility, one might still *associate* possibilities with sets of total worlds.

On Edgington's view, no possibility is total, that is, no possibility decides every proposition, and thus every possibility should have a proper refinement. Of course, others would disagree and argue that a possibility might or might not be total. While this is an interesting discussion, in this paper we will not take either side. We will make no assumptions about whether or not a possibility might be total, and in our models we will neither prohibit total worlds (i.e., possibilities with no proper refinements) nor require that they exist.

In later sections, we will formally define what we mean by a possibility model, but for now we will informally outline the main results of this paper. Given a countable Kripke model, we can consider possibilities coming from that Kripke model, i.e., as sets of total worlds. These possibilities form a possibility model which is intimately connected to the Kripke model in the sense that a possibility makes some sentence true if and only if every world that it contains makes that sentence true. Moreover, any sufficiently rich[1] collection of possibilities generates a possibility model; we call a possibility model arising from such a collection a *possibilization* of the Kripke model. See Definition 2.9 for the full definition of a possibilization. A Kripke model can be viewed as a possibilization of itself, identifying each world with the possibility containing only itself. The possibilizations of Kripke models are among the prototypical examples of possibility models.

An abstract possibility model has certain conditions on the possibilities and the refinement relation—for example, that refinement should maintain the truth of sentences—but it need not have any relation with Kripke models, and the possibilities need not consist of sets of total worlds. Our first main result is that every countable possibility model $\mathcal{M}$ has a *worldization* $\mathcal{K}$. A worldization of $\mathcal{M}$ is a Kripke model $\mathcal{K}$ such that every possibility in $\mathcal{M}$ is refined by some total world of

[1]For example, for any two worlds, there should be a possibility containing one but not the other. The full list of requirements is given in Definition 2.9.

$\mathcal{K}$ and so that every total world of $\mathcal{K}$ is the limit of more and more refined possibilities in $\mathcal{M}$. The full definition of a worldization will be given later in Definition 3.1.

Theorem 1.1. *Let $\mathcal{M}$ be a countable possibility model in a countable language. Then there is a Kripke model $\mathcal{K}$ which is a worldization of $\mathcal{M}$.*

The difficulty in proving this theorem is that the accessibility relation of the worldization $\mathcal{K}$ should come from the accessibility relation on $\mathcal{M}$; it is obvious how to extend each individual possibility to a total world, but it is not obvious how to do this simultaneously for each possibility of $\mathcal{M}$ while respecting the accessibility relation. (See the definition of a worldization for the requirements we place on the accessibility relation.)

There are two important hypotheses in the statement of the theorem. First, we require that the language be countable, and second, that the number of possibilities be countable. We produce counteraxamples in both cases. Our counterexample when we allow the number of possibilities to be uncountable uses Aronszajn trees. (Aronszajn trees are trees with odd behaviour coming from set theory.) It is likely that most natural (i.e., non-pathological) possibility models for a countable language have a worldization.

Arguments have been given, from a philisophical perspective, for and against the idea of constructing worlds out of limits of possibilities. Rumfitt [Rum15], for example, describes how one might try to construct a total world:

> [T]he possibility that I have red hair leaves it undetermined whether Ed Miliband will win a General Election. But there is also the possibility that I have red hair while Miliband wins an election, and the distinct possibility that I have red hair while he does not. By iterating this process, it may be suggested, we shall eventually reach fully determinate possibilities which do settle the truth or falsity of all statements. These possibilities will be the points of modal space.... (159)

Our construction follows essentially the strategy described above, which has similarities to the construction of a generic in set theory (see [Coh66], [Jec03], or [Kun80]). However, our proof of Theorem 1.1 requires much more than the strategy just described; most of the work that we will do goes into picking appropriate sequences of refinements so that one can define the modal accessibility relation between the constructed points. It must also be noted that Rumfitt [Rum15] expresses doubts about the construction of the points themselves:

> [T]he business of making a possibility more determinate seems open-ended. There are possibilities that the child at home should be a boy, a six-year-old boy, a six-year-old boy with blue eyes, a six-year-old boy with blue eyes who weighs 3 stone, and so forth. So far from terminating in a fully determinate possibility, we seem to have an indefinitely long sequence of increasingly determinate possibilities, any one of which is open to further determination. But then, so far from conceiving of our rational activities as discriminating between regions of determinate points, we appear to have no clear conception of such a point at all. (159)

Here it is important that we restrict our attention to a countable set of propositions, so that we can define a countable sequence of possibilities such that each proposition is decided at some point in the sequence. Although every possibility in the sequence may be open to further determination, we can take the countable sequence itself as a "world" which decides each proposition in the given countable set. This is compatible with Rumfitt's assertion that we may never reach a possibility that "settles the truth or falsity of all statements" without restriction.

If a possibility model $\mathcal{M}$ is a possibilization of a Kripke model $\mathcal{K}$, then $\mathcal{K}$ is a worldization of $\mathcal{M}$. Of course, not all possibility models are possibilizations because, for example, the possibilities may not be sets of total worlds. Even up to isomorphism, a possibility model could include duplication of possibilities (two possibilities which have exactly the same refinements) or there might be possibilities X and Y such that every non-trivial refinement of Y is accessible from X, but Y is not. Neither of these situations can occur in a possibilization. However, every possibility model embeds in a very natural way into one which avoids these two issues (see Propositions 2.14 and 2.16). We call such a possibility model *separative* and *strong*. Then we are able to show that every countable, separative, and strong possibility model is (up to isomorphism) a possibilization. Thus, up to some equivalence such as allowing duplication of possibilities, every countable possibility model is a possibilization.

Theorem 1.2. *Let $\mathcal{M}$ be a countable, separative, and strong possibility model in a countable language. Then $\mathcal{M}$ is isomorphic to a possibilization of a countable Kripke model.*

We prove Theorem 1.2 by embedding $\mathcal{M}$ into a worldization $\mathcal{K}$ with a couple of additional properties. Then we interpret the possibilities in $\mathcal{M}$ as sets of total worlds from $\mathcal{K}$.

At the level of frames, we can define a notion of frame-worldization. A Kripke frame $\mathcal{F}$ is a frame-worldization of a possibility frame $\mathcal{G}$ if two conditions are met. First, the possibilities and the worlds of $\mathcal{F}$ and $\mathcal{G}$ are related as in a worldization of models (so that every possibility in $\mathcal{G}$ is refined by some total world of $\mathcal{F}$ and so that every total world of $\mathcal{F}$ is the limit of more and more refined possibilities in $\mathcal{G}$). Second, any Kripke model $\mathcal{K}$ based on $\mathcal{F}$ should induce a possibility model $\mathcal{M}$ based on $\mathcal{G}$, so that $\mathcal{K}$ is a worldization of $\mathcal{M}$, and vice versa.

There are countable possibility frames which have no frame-worldizations. This ties in to Holliday's [Hol15] result that there are possibility frames whose logic is not the logic of any class of Kripke frames. Such a possibility frame could not have a worldization.

On the other hand, it is natural to consider general possibility frames and general Kripke frames. If a general possibility frame has countably many admissible sets, then it has a frame-worldization.

Theorem 1.3. *Let $\mathcal{G}$ be a countable general possibility frame with countably many admissible sets. Then there is a general Kripke frame $\mathcal{F}$ which is a frame-worldization of $\mathcal{F}$.*

We will begin in Section 2 by defining possibility models, possibilizations, separative possibility models, and strong possibility models. In Section 3, we will define worldizations and prove Theorem 1.1. In Section 4, we will prove Theorem 1.2. In Section 5, we will give an example showing that *uncountable* possibility models need not have worldizations. In Section 6, we will give an example of a possibility frame which has no frame-worldizations, and finally we will prove Theorem 1.3.

2. Possibility Models

2.1. **Possibility Semantics.** Let P be a set of propositional variables, and let $\mathcal{L}(P)$ be the standard language of propositional modal logic with modal operators $\Box$ and $\Diamond$ and propositions coming from P.

The following frames may be viewed as a special case of the "full possibility frames" of [Hol15] and as a generalization of the frames of [Hum81].[2]

Definition 2.1. A (*basic*) *possibility frame* is a tuple $\mathcal{F} = (\mathcal{P}, \mathcal{R}, \leq)$ where:

(1) $\mathcal{P}$ is a non-empty set of *possibilities*,
(2) $\mathcal{R} \subseteq \mathcal{P} \times \mathcal{P}$ is a binary *accessibility relation*, and
(3) $\leq$ is a partial order on $\mathcal{P}$, the *refinement relation*,

satisfying the following three properties:

P1: For all X, X', and Y with $X' \geq X$, if $X'\mathcal{R}Y$ then $X\mathcal{R}Y$.

$$\begin{tikzcd}
X' \arrow[r, dashed] & Y \\
X \arrow[u] \arrow[ur, dashed, "\mathcal{R}"'] &
\end{tikzcd}$$

P2: For all X, Y, and Y' with $Y' \geq Y$, if $X\mathcal{R}Y$ then $X\mathcal{R}Y'$.

$$\begin{tikzcd}
 & Y' \\
X \arrow[r, dashed, "\mathcal{R}"'] \arrow[ur, dashed] & Y \arrow[u]
\end{tikzcd}$$

R: For all X and Y, if $X\mathcal{R}Y$ then there is $X' \geq X$ such that for all $X'' \geq X'$, there is $Y' \geq Y$ such that $X''\mathcal{R}Y'$.

$$\begin{tikzcd}
X'' \arrow[dr, dashed] & \\
X' \arrow[u, "\forall"] & Y' \\
X \arrow[u, "\exists"] \arrow[r, dashed, "\mathcal{R}"'] & Y \arrow[u, "\exists"]
\end{tikzcd}$$

We interpret $X\mathcal{R}Y$ as meaning that what is necessary at X is true at Y. $X \geq Y$ means that X determines each issue which Y does, in the same way, and possibly more. Our possibility frames are more general than those considered by Humberstone. Humberstone asked that a stronger version of the condition **R** be satisfied, namely:

R^{++}: For all X and Y, if $X\mathcal{R}Y$ then there is $X' \geq X$ such that for all $X'' \geq X'$, $X''\mathcal{R}Y$.[3]

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[2] Holliday [Hol15] writes '$X \sqsubseteq Y$' to mean that X is a refinement of Y, going "down" rather than "up" for refinements, while [Hum81] writes '$X \geqslant Y$' to mean that X is a refinement of Y. We will write '$X \geq Y$' to mean that X is a refinement of Y.

[3] There is also an intermediate condition $\mathbf{R}^+$ discussed in [HT16] and [Hol15].

See [Hol15, Section 2.3] for a discussion of why it is desirable to use the weaker condition on the refinability relation.

A partial function $f\colon D \to C$ is a function which is defined on some, possibly proper, subset of D. If $x \in D$ and f is defined at $x \in D$ and maps x to $y \in C$, we write $f(x) = y$; otherwise, if f is not defined at x, we write $f(x) = ?$.

Definition 2.2. A *possibility model* is a tuple $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ where $\mathcal{F} = (\mathcal{P}, \mathcal{R}, \leq)$ is a possibility frame and $V\colon \mathcal{P} \times P \to \{\mathtt{T}, \mathtt{F}\}$ is a partial function, the *valuation*, satisfying:

Persistence: For any $Y \geq X$ in $\mathcal{P}$ and any $p \in P$, if $V(X,p) = \mathtt{T}$ then $V(Y,p) = \mathtt{T}$, and similarly for $\mathtt{F}$.

Refinability: For any $X \in \mathcal{P}$, if $V(x,p) = ?$, then there exist $Y \geq X$ and $Z \geq X$ such that $V(Y,p) = \mathtt{F}$ and $V(Z,p) = \mathtt{T}$.

$\mathcal{M}$ is said to be *based on* $\mathcal{F}$.

If $X \in \mathcal{P}$, then we interpret $V(X,p) = \mathtt{T}$ as saying that p is true at X, $V(X,p) = \mathtt{F}$ as saying that p is false at X, and $V(X,p) = ?$ as saying that p is undetermined at X.

Definition 2.3. Given a possibility model $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$, the *satisfaction relation* is defined inductively as follows:

(1) $\mathcal{M}, X \models p$ if $V(X,p) = \mathtt{T}$.
(2) $\mathcal{M}, X \models \varphi \wedge \psi$ if $\mathcal{M}, X \models \varphi$ and $\mathcal{M}, X \models \psi$.
(3) $\mathcal{M}, X \models \neg\varphi$ if for all $Y \geq X$, $\mathcal{M}, Y \not\models \varphi$.
(4) $\mathcal{M}, X \models \Box\varphi$ if for all $Y \in \mathcal{P}$ such that $X\mathcal{R}Y$, $\mathcal{M}, Y \models \varphi$.

Humberstone [Hum81] proves all of the following lemmas and proposition (see [Hol15] for the proofs using the weaker refinability condition).

Lemma 2.4 (Persistence)**.** *Let $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ be a possibility model. If $Y \geq X$ and $\mathcal{M}, X \models \varphi$, then $\mathcal{M}, Y \models \varphi$.*

Lemma 2.5 (Refinability)**.** *Let $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ be a possibility model. If $\mathcal{M}, X \not\models \varphi$, then for some $Y \geq X$, $\mathcal{M}, Y \models \neg\varphi$.*

Lemma 2.6 (Double Negation Elimination)**.** *Let $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ be a possibility model. $\mathcal{M}, X \models \varphi$ if and only if $\mathcal{M}, X \models \neg\neg\varphi$*

As usual, we say that a sentence φ is *globally true* in a possibility model $\mathcal{M}$ if $\mathcal{M}, X \models \varphi$ for all X, and φ is *valid* if it is globally true in all possibility models. A sentence φ is *satisfiable* if there is a model $\mathcal{M}$ and possibility X with $\mathcal{M}, X \models \varphi$.

Proposition 2.7 (Soundness and Completeness)**.** *For any sentence φ, the following are equivalent:*

(1) φ is valid over all possibility models,
(2) φ is valid over all Kripke models,
(3) φ is provable in the minimal normal modal logic K*.*

2.2. **Possibilizations.** The simplest example of a possibilization is the powerset possibilization, where the set of possibilities is taken to be as large as possible.[4]

[4]Holliday [Hol15, Fact B.1] observes that a powerset possibilization might not satisfy Humberstone's stronger condition $\mathbf{R}^{++}$.

Definition 2.8. Let $\mathcal{K} = (\mathcal{W}, \mathcal{S}, U)$. A *powerset possibilization* of $\mathcal{K}$ is a possibility model $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ where:

(1) $\mathcal{P} = \wp(\mathcal{W}) \setminus \{\varnothing\}$;
(2) $X\mathcal{R}Y$ if and only if $Y \subseteq \mathcal{S}[X] = \{w' \colon (\exists w \in X) w\mathcal{S}w'\}$;
(3) for $X, Y \in \mathcal{P}$, $X \geq Y$ if and only if $X \subseteq Y$;
(4) $V(X, p) = \mathtt{T}$ if for all $w \in X$, $U(w, p) = \mathtt{T}$; $V(X, p) = \mathtt{F}$ if for all $w \in X$, $U(w, p) = \mathtt{F}$; and otherwise $V(X, p) = ?$.

More generally, a possibilization of a Kripke model will be a possibility model where the possibilities are sets of worlds from the Kripke model, with the requirement that the collection of possibilities is sufficiently rich to capture the structure of the Kripke model.

Given a Kripke model $\mathcal{K} = (\mathcal{W}, \mathcal{S}, U)$ and a set of worlds $Y \subseteq \mathcal{W}$, we define $\mathcal{S}[Y] = \{v \colon (\exists w \in Y) w\mathcal{S}v\}$ and $\Diamond Y = \{w \colon (\exists v \in Y) w\mathcal{S}v\}$.

Definition 2.9. Let $\mathcal{K} = (\mathcal{W}, \mathcal{S}, U)$ be a Kripke model. A *possibilization* of $\mathcal{K}$ is a possibility model $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ where:

(A1) $\mathcal{P}$ is a non-empty collection of non-empty subsets of $\mathcal{W}$ such that
  (i) if $v \neq w$, there is $X \in \mathcal{P}$ with $v \in X$ and $w \notin X$;
  (ii) if $X \in \mathcal{P}$ and $v \notin X$, there is $Y \in \mathcal{P}$ with $v \in Y$ and $X \cap Y = \varnothing$;
  (iii) if $\mathcal{K}, v \models \varphi$, there is $Y \in \mathcal{P}$ with $v \in Y$ and such that for all $w \in Y$, $\mathcal{K}, w \models \varphi$;
  (iv) if $v \in \mathcal{S}[X]$, there is $Y \in \mathcal{P}$ with $v \in Y \subseteq \mathcal{S}[X]$;
  (v) if $Y \not\subseteq \mathcal{S}[X]$, there is $Y' \in \mathcal{P}$ with $Y' \subseteq Y$ and $Y' \cap \mathcal{S}[X] = \varnothing$;
  (vi) if $\Diamond Y$ is non-empty, there is $X \in \mathcal{P}$ with $X \subseteq \Diamond Y$;
  (vii) if $X, Y \in \mathcal{P}$ and $v \in X \cap Y$, there is $Z \in \mathcal{P}$ with $v \in Z \subseteq X \cap Y$;
  (viii) If not vSw, there is $X \in \mathcal{P}$ with $v \in X$ and $w \notin \mathcal{S}[X]$.

The definitions of the accessibility relation, refinement relation, and valuation are exactly the same as for the powerset possibilization in Definition 2.8:

(A2) $X\mathcal{R}Y$ if and only if $Y \subseteq \mathcal{S}[X]$;
(A3) for $X, Y \in \mathcal{P}$, $X \geq Y$ if and only if $X \subseteq Y$;
(A4) $V(X, p) = \mathtt{T}$ if for all $w \in X$, $U(w, p) = \mathtt{T}$; $V(X, p) = \mathtt{F}$ if for all $w \in X$, $U(w, p) = \mathtt{F}$; and otherwise $V(X, p) = ?$.

The conditions (i)-(viii) are all natural conditions. For example, (iii) says that if φ is true at a world v, then v belongs to a possibility which makes v true, and (iv) says that if v is accessible from some world in X, then v belongs to a possibility Y which is accessible from X.

The powerset possibilization of a Kripke model is a possibilization, and a Kripke model can be viewed as a possibilization of itself. Possibilizations are, of course, possibility models.

Proposition 2.10. *If $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ is a possibilization of $\mathcal{K} = (\mathcal{W}, \mathcal{S}, U)$, then $\mathcal{M}$ is a possibility model.*

Proof. To see **P1**, suppose that $X' \geq X$ and $X'\mathcal{R}Y$. Then $Y \subseteq \mathcal{S}[X'] \subseteq \mathcal{S}[X]$ and so $X\mathcal{R}Y$. To see **P2**, suppose that $Y' \geq Y$ and $X\mathcal{R}Y$. Then $Y' \subseteq Y \subseteq \mathcal{S}[X]$ and so $X\mathcal{R}Y'$.

For **R**, suppose that $X\mathcal{R}Y$. By (vi), let $X' \subseteq X$ be such that $X' \subseteq \Diamond Y$. Then for all $X'' \subseteq X'$, by (iv) and (vii) there is $Y' \subseteq Y \cap \mathcal{S}[X'']$. Thus $X''\mathcal{R}Y'$.

For **Persistence**, suppose that $Y \geq X$ and $p \in P$ are such that $V(X,p) = \mathtt{T}$ or $V(X,p) = \mathtt{F}$. Without loss of generality, suppose that $V(X,p) = \mathtt{T}$. Then for all $v \in X$, $U(v,p) = \mathtt{T}$. Since $Y \subseteq X$, $V(Y,p) = \mathtt{T}$.

For **Refinability**, suppose that $V(X,p) = ?$. Then there are $v, w \in X$ with $U(v,p) = \mathtt{T}$ and $U(w,p) = \mathtt{F}$. So by (iii) and (vii) there are $Y, Z \in \mathcal{P}$ with $Y \geq X$ and $Z \geq X$ such that $V(Y,p) = \mathtt{T}$ and $V(Z,p) = \mathtt{F}$. □

Truth at a possibility in a possibilization is just truth at all of the total worlds in that possibility.

Proposition 2.11. *Let $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ be a possibilization of $\mathcal{K} = (\mathcal{W}, \mathcal{S}, U)$. For any sentence φ:*

$$\mathcal{M}, X \models \varphi \text{ if and only if } \mathcal{K}, v \models \varphi \text{ for all } v \in X.$$

Proof. The proof is by induction on the complexity of φ. We have

$$\mathcal{M}, X \models p \iff V(X,p) = \mathtt{T} \iff \forall v \in X, U(v,p) = \mathtt{T} \iff \forall v \in X, \mathcal{K}, v \models p.$$

The case of a conjunction is simple.

For a negation, if for all $v \in X$, $\mathcal{K}, v \models \neg\varphi$, then for all $Y \geq X$, $\mathcal{M}, Y \not\models \varphi$ by the induction hypothesis and hence $\mathcal{M}, X \models \neg\varphi$. On the other hand, if $\mathcal{M}, X \models \neg\varphi$, then given $v \in X$ we must show that $\mathcal{K}, v \models \neg\varphi$. Suppose towards a contradiction that $\mathcal{K}, v \models \varphi$ for some such v. Then by (iii) and (vii) there is $Y \geq X$ with $v \in Y$ such that for all $w \in Y$, $\mathcal{K}, w \models \varphi$. Since $\mathcal{M}, X \models \neg\varphi$, $\mathcal{M}, Y \not\models \varphi$. This is a contradiction.

Finally, suppose that for all $v \in X$, $\mathcal{K}, v \models \Box\varphi$. Suppose that $X\mathcal{R}Y$. Given $w \in Y$, there is $v \in X$ such that $v\mathcal{S}w$. Then since $\mathcal{K}, v \models \Box\varphi$, $\mathcal{K}, w \models \varphi$. So $\mathcal{M}, Y \models \varphi$. Since Y was arbitrary with $X\mathcal{R}Y$, $\mathcal{M}, X \models \Box\varphi$. On the other hand, suppose that $\mathcal{M}, X \models \Box\varphi$. Fix $v \in X$, and w such that $v\mathcal{S}w$. By (iii) there is Y such that $w \in Y \subseteq \mathcal{S}[X]$. Then $X\mathcal{R}Y$, and so $\mathcal{M}, Y \models \varphi$. Thus, by the induction hypothesis, $\mathcal{K}, w \models \varphi$ as desired. □

2.3. **Separative possibility models.** Suppose that $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ is a possibilization of a Kripke model $\mathcal{K}$. If $X, Y \in \mathcal{P}$ are two sets of total worlds, $X \nsubseteq Y$, then there is $v \in X \setminus Y$. Then (ii) and (vii) imply that there is an $X' \in \mathcal{P}$ with $v \in X' \subseteq X$ and $X' \cap Y = \varnothing$. Thus X' and Y have no common refinements, for which we write $X' \perp Y$.

The following natural class of possibility models is studied in Section 4.1 of [Hol15].[5]

Definition 2.12. Let $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ be a possibility model. $\mathcal{M}$ is *separative* if whenever $X \ngeq Y$, there is $X' \geq Y$ such that $X' \perp Y$.

Define

$$X \geq_s Y \iff (\forall X' \geq X)(\exists X'' \geq X')\ X'' \geq Y.$$

Then a possibility model $\mathcal{M}$ is separative if and only if the refinement relation $\geq$ is equal to $\geq_s$.

Not every possibility model is separative, though as remarked above, every possibilization is separative. However, every possibility model embeds in a natural way into a separative quotient by identifying equivalent possibilities, such as duplicated possibilities.

[5]The terminology comes from set-theoretic forcing; see for example p. 204 of [Jec03].

Definition 2.13. Let $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ be a possibility model. Let $X \simeq_s Y$ if and only if $X \geq_s Y$ and $Y \geq_s X$; this is an equivalence relation. Write $[X]$ for the equivalence class of X under $\simeq_s$. Let:

(1) $\mathcal{P}'$ be the equivalence classes under $\simeq_s$,
(2) $[X]\mathcal{R}'[Y]$ if there are $X' \simeq_s X$ and $Y' \simeq_s Y$ with $X'\mathcal{R}Y'$,
(3) $V'([X], p) = \mathtt{T}$ if $V(X, p) = \mathtt{T}$ and $V'([X], p) = \mathtt{F}$ if $V(X, p) = \mathtt{F}$; otherwise $V'([X], p) = ?$.

$\mathcal{M}_s = (\mathcal{P}', \mathcal{R}', \preceq_s, V')$ is the *separative quotient* of $\mathcal{M}$.

This is well-defined. There is a natural embedding of a possibility model into its separative quotient, and this embedding maintains truth.

Proposition 2.14 (Proposition 4.10 of [Hol15])**.** *Let $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ be a possibility model. $\mathcal{M}_s$ is a separative possibility model, and*

$$\mathcal{M}, X \models \varphi \Longleftrightarrow \mathcal{M}_s, [X] \models \varphi.$$

2.4. **Strong possibility models.** Suppose that $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ is a possibility model. If $X, Y \in \mathcal{P}$ are two possibilities, and for every $Y' > Y$, $X\mathcal{R}Y'$, then it is natural to expect that $X\mathcal{R}Y$.

Now suppose that $\mathcal{M}$ is in fact a possibilization of a Kripke model $\mathcal{K} = (\mathcal{W}, \mathcal{S}, U)$. If $X, Y \in \mathcal{P}$ are two sets of total worlds, suppose that for all $Y' \geq Y$, there is $Y'' \geq Y'$ with $X\mathcal{R}Y''$. We claim that $Y \subseteq \mathcal{S}[X]$. Suppose for a contradiction that $Y \nsubseteq \mathcal{S}[X]$. Then by (v), there is $Y' \geq Y$ with $Y' \cap \mathcal{S}[X] = \varnothing$. So for all $Y'' \geq Y'$, $Y'' \nsubseteq \mathcal{S}[X]$, i.e., for no $Y'' \geq Y'$ do we have $X\mathcal{R}Y''$. This is a contradiction. Hence $Y \subseteq \mathcal{S}[X]$ and $X\mathcal{R}Y$. Note that this is a refinability condition on $\mathcal{R}$. In fact, this condition has already been studied by Holliday.

Definition 2.15 (Section 2.3 of [Hol15])**.** A possibility model $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ is *strong* if, in addition to satisfying **P1**, **P2**, and **R**, it satisfies: whenever it is the case that for all $Y' \geq Y$ there is $Y'' \geq Y'$ such that $X\mathcal{R}Y''$, we already have $X\mathcal{R}Y$.

Any possibilization is strong. Once again, every possibility model embeds in a natural way into a strong model (Proposition 2.37 of [Hol15]).

Proposition 2.16. *Let $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ be a possibility model. Define a new accessibility relation $\mathcal{R}'$ by $X\mathcal{R}'Y$ if and only if for all $Y' \geq Y$ there is $Y'' \geq Y'$ with $X\mathcal{R}Y''$. Then $\mathcal{M}' = (\mathcal{P}, \mathcal{R}', \leq, V)$ is a strong possibility model and*

$$\mathcal{M}, X \models \varphi \Longleftrightarrow \mathcal{M}', X \models \varphi.$$

If $\mathcal{M}$ was separative, so is $\mathcal{M}'$, since we have not altered $\leq$.

3. Worldizations

We say that a Kripke model $\mathcal{K}$ is a *worldization* of a possibility model $\mathcal{M}$ if, informally speaking, each possibility in $\mathcal{M}$ is part of a total world from $\mathcal{K}$, and each total world in $\mathcal{K}$ is a limit of more and more refined possibilities. If $\mathcal{M}$ is a possibilization of $\mathcal{K}$, then $\mathcal{K}$ will be a worldization of $\mathcal{M}$ (though the opposite is not always true). We will prove this later in Proposition 4.2.

Definition 3.1. Let $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ be a possibility model and let $\mathcal{K} = (\mathcal{W}, \mathcal{S}, U)$ be a Kripke model. $\mathcal{K}$ is a *worldization* of $\mathcal{M}$ via an embedding $\Phi : \mathcal{W} \to \wp(\mathcal{P})$ which assigns to each total world $w \in \mathcal{W}$ a set of possibilities $\Phi(w) \subseteq \mathcal{P}$ if:

(W1) for each world $w \in \mathcal{W}$, $\Phi(w)$ is a *maximal order ideal*, i.e.,
  (a) $\Phi(w)$ is downwards-closed under refinement,
  (b) any two elements of $\Phi(w)$ have a common refinement in $\Phi(w)$, and
  (c) $\Phi(w)$ is maximal with these two properties;

(W2) for each possibility $X \in \mathcal{P}$, there is a world $w \in \mathcal{W}$ such that $X \in \Phi(w)$;

(W3) any two distinct total worlds $v, w \in \mathcal{W}$ are separated by possibilities, that is, there is $X \in \Phi(v) \setminus \Phi(w)$;

(W4) for each world $w \in \mathcal{W}$, and for each sentence φ, $\mathcal{K}, w \models \varphi$ if and only if there is some $X \in \Phi(w)$ such that $\mathcal{M}, X \models \varphi$; and

(W5) for each pair of worlds $w, v \in \mathcal{W}$, $w\mathcal{S}v$ if and only if for each $X \in \Phi(w)$ there is $Y \in \Phi(v)$ such that $X\mathcal{R}Y$.

We say that $\mathcal{K}$ is a worldization of $\mathcal{M}$ if the Φ which makes $\mathcal{K}$ a worldization is understood from the context, and that $\mathcal{M}$ is *worldizable* if there is a $\mathcal{K}$ which is a worldization of $\mathcal{M}$.

When our models are countable, $\Phi(w)$ is determined by some increasing chain in $\mathcal{M}$. Note that if $X, Y \in \Phi(w)$, then they have a common refinement, so we cannot have $\mathcal{M}, X \models \varphi$ and $\mathcal{M}, Y \models \neg\varphi$.

We will prove Theorem 1.1, which says that every countable possibility model in a language with countably many propositional variables has a worldization. The proof is essentially to construct infinite ascending chains while managing the accessibility relation to get the appropriate properties. Doing this is surprisingly complicated. We will begin with a warmup in which we use the stronger condition $\mathbf{R}^{++}$ from Section 2.1.

Theorem 3.2. *Let $\mathcal{M}$ be a countable possibility model in a countable language, satisfying $\mathbf{R}^{++}$. Then there is a Kripke model $\mathcal{K}$ which is a worldization of $\mathcal{M}$.*

Proof. For each $X \in \mathcal{M}$, we will define an increasing chain of possibilities $A_X = (A_X(n))_{n\in\omega}$. These chains will form the total worlds of the model $\mathcal{K}$. Let $(X_n)_{n\in\omega}$ be an enumeration of the possibilities in $\mathcal{M}$ and $\varphi_0, \varphi_1, \dots$ an enumeration of the sentences in the language. For simplicity, we occasionally write A_i for A_{X_i}. We define the chains A_X using a recursive construction. Begin with $A_X(0) = X$ for each X.

The idea is that we need to extend the chains in such a way that every formula is decided at some point in each chain, and also so that if the chain does not satisfy $\Box\varphi$ at some point X, there is a witnessing possibility Y which has $\neg\varphi$ so that any refinement of X is still related via the accessibility relation to Y. As we extend the chains, we alternate between these two requirements, at each step either deciding some new formula using the refinability property, or using $\mathbf{R}^{++}$ to lock in a witness to $\Diamond\neg\varphi$.

Suppose that we have defined $A_X(0), \dots, A_X(n)$ for each X. We will now define $A_X(n+1)$ for each X. We have two cases, depending on whether n is odd or even.

n is even: Write $n = 2k$. For each $X \in \mathcal{M}$, choose $X' \geq A_X(n)$ such that $\mathcal{M}, X' \models \varphi_k$ or $\mathcal{M}, X' \models \neg\varphi_k$. Now, if $X_k \geq X'$, set $A_X(n+1) = X_k$, and otherwise set $A_X(n+1) = X'$.

n is odd: Write $n = 2\langle k, i\rangle + 1$ where $\langle \cdot, \cdot \rangle : \omega^2 \to \omega$ is bijective. If there is Y such that $A_i(n)\mathcal{R}Y$ and $\mathcal{M}, Y \models \varphi_k$, then we also have $A_i(n)\mathcal{R}A_Y(n)$ since $A_Y(n) \geq Y$. Using $\mathbf{R}^{++}$, choose $X \geq A_i(n)$ such that for all $X' \geq$

X, $X'\mathcal{R}A_Y(n)$. Set $A_i(n+1) = X$. For each other possibility Z, set $A_Z(n+1) = A_Z(n)$. If no such Y exists, set $A_Z(n+1) = A_Z(n)$ for all Z.

$$\begin{tikzcd}
X' \arrow[ddr, dashed] & \\
X \arrow[u, "\forall"] \arrow[dr, dashed] & \\
A_i(n) \arrow[u] \arrow[r, dashed] \arrow[dr, dashed] & A_Y(n) \\
& Y \arrow[u]
\end{tikzcd}$$

FIGURE 1. The extensions of possibilities in the construction. The dotted line shows the relation $\mathcal{R}$. For all X' as shown, $\mathcal{R}$ relates X' and $A_Y(n)$.

This completes the construction of the sequences A_X. Let $\hat{A}_X$ be the order ideal which is the downwards closure of A_X. At even stages, we ensure that for each Y, either Y is part of the chain A_X or there is some n such that Y is not a refinement of $A_X(n)$. So $\hat{A}_X$ is maximal. Now let $\mathcal{W}$ be the set of these order ideals and note that there may be possibilities X and Y such that $\hat{A}_X = \hat{A}_Y$. Such an order ideal is included in F only once. We will define a total world model $\mathcal{K}$ with domain F which is a worldization of $\mathcal{M}$ via the identity function. The accessibility relation will be $\mathcal{S}$. For $I, J \in F$, define $I\mathcal{S}J$ if and only if there is a $Y \in J$ such that for all $X \in I$, $X\mathcal{R}Y$. Have an atomic proposition p hold at $I \in F$ if and only if for some $X \in I$, $\mathcal{M}, X \models p$. We make p false at $I \in F$ if and only if for some $X \in I$, $\mathcal{M}, X \models \neg p$. By construction, for each formula φ and $I \in F$, there is $X \in I$ such that either $\mathcal{M}, X \models \varphi$ or $\mathcal{M}, X \models \neg\varphi$. Also, if for some $Y \in I = \hat{A}_X$, $\mathcal{M}, Y \models \varphi$, then there is some n such that $A_X(n) \geq Y$ and so $\mathcal{M}, A_X(n) \models \varphi$.

Properties (W1), (W2), (W3), and (W5) of a worldization are immediate. To complete the proof, we check (W4) from the definition of worldization. The proof is by induction on the complexity of formulas. For an atomic proposition p, let $I \in F$ and let $X \in \mathcal{M}$ be such that $\hat{A}_X = I$. Then

$$\mathcal{K}, I \models p \Leftrightarrow (\exists n)\mathcal{M}, A_X(n) \models p$$

since, for some n, either $\mathcal{M}, A_X(n) \models p$ or $\mathcal{M}, A_X(n) \models \neg p$. For $\varphi \wedge \psi$,

$$\begin{aligned}
\mathcal{K}, I \models \varphi \wedge \psi \quad &\Leftrightarrow \quad \mathcal{K}, I \models \varphi \text{ and } \mathcal{K}, I \models \psi \\
&\Leftrightarrow \quad (\exists X \in I)\mathcal{M}, X \models \varphi \text{ and } (\exists Y \in I)\mathcal{M}, Y \models \psi \\
&\Leftrightarrow \quad (\exists Z \in I)\mathcal{M}, Z \models \varphi \wedge \psi
\end{aligned}$$

where, given X and Y witnesses to the second line, the witness Z to the third line is a common refinement of X and Y. For $\neg\varphi$,

$$\begin{aligned} \mathcal{K}, I \models \neg\varphi &\Leftrightarrow \mathcal{K}, I \not\models \varphi \\ &\Leftrightarrow (\forall X \in I)\mathcal{M}, X \not\models \varphi \\ &\Leftrightarrow (\exists X \in I)\mathcal{M}, X \models \neg\varphi \end{aligned}$$

since for some $X \in I$, either $\mathcal{M}, X \models \varphi$ or $\mathcal{M}, X \models \neg\varphi$.

Finally, we have the case $\Box\varphi$. Suppose that for all $X \in I$, $\mathcal{M}, X \not\models \Box\varphi$. Let X be such that $I = \hat{A}_X$ where $X = X_i$ and let k be such that $\neg\varphi = \varphi_k$. Then at stage $n = 2\langle k, i\rangle + 1$ of the construction, we have $A_X(n) \not\models \Box\varphi$, so there is some $Y \in \mathcal{M}$ with $X\mathcal{R}Y$ such that $\mathcal{M}, Y \not\models \varphi$; refining Y if necessary, we may assume that $\mathcal{M}, Y \models \neg\varphi$ while still maintaining $X\mathcal{R}Y$ by **P2**. Then (possibly for some different Y such that $Y \models \neg\varphi$ and $X\mathcal{R}Y$) we have $A_Y(n+1) \geq Y$ and for all $Z \geq A_X(n+1)$, $Z\mathcal{R}A_Y(n+1)$. Hence, for each $\ell \geq n+1$ and $m \geq n+1$, $A_X(\ell)\mathcal{R}A_Y(m)$. Thus $\hat{A}_X\mathcal{S}\hat{A}_Y$. Since $\mathcal{K}, \hat{A}_Y \models \neg\varphi$, $\mathcal{K}, \hat{A}_X \not\models \Box\varphi$. Thus we have shown that if $\mathcal{K}, I \models \Box\varphi$, then for some $X \in I$, $\mathcal{M}, X \models \Box\varphi$.

Now suppose that for some $Y \in \hat{A}_X$, $\mathcal{M}, Y \models \Box\varphi$. Then by persistence and the fact that $\hat{A}_X$ is the downwards closure of the chain A_X, $\mathcal{M}, A_X(n) \models \Box\varphi$ for some n. Let Z be such that $\hat{A}_X\mathcal{S}\hat{A}_Z$. Then there is some m such that $A_X(n)\mathcal{R}A_Z(m)$, and so $\mathcal{M}, A_Z(m) \models \varphi$. Hence $\mathcal{K}, \hat{A}_Z \models \varphi$. Since Z was arbitrary, $\mathcal{K}, \hat{A}_X \models \Box\varphi$. $\Box$

Now for Theorem 1.1, we must use $\mathbf{R}$ which is weaker than $\mathbf{R}^{++}$. While using $\mathbf{R}^{++}$ we were able to lock in the witness to $\Diamond\neg\varphi$ in a single step, this is no longer possible with $\mathbf{R}$. Instead, we have to constantly make sure that we maintain the same witness for each chain. We will keep track of the witnesses in a tree, so that there are no circular witness requirements. (By a circular witness requirement, we mean for example that Y is a witness for X, Z is a witness for Y, and X is a witness for Z.) This makes the proof somewhat complicated.

Proof of Theorem 1.1. For each $X \in \mathcal{P}$, we will define infinitely many increasing chains of possibilities $A_X^s = (A_X^s(n))_{n\in\omega}$ with $A_X^s(0) = X$. Let $(X_n)_{n\in\omega}$ be an enumeration of the worlds in $\mathcal{P}$ and $\varphi_0, \varphi_1, \ldots$ an enumeration of the sentences in the language $\mathcal{L}$. The chains A_X^s will be defined using a recursive construction. First, we must define an auxiliary object that we will build during the construction.

A *tree* is a graph such that between any two edges there is a unique path. A *rooted tree* is a tree with a distinguished node. Each edge in a rooted tree has a natural direction, towards or away from the root. Thus a rooted tree can be viewed as a *directed tree*, a tree in which each edge has a specified direction pointing away from the root. A (directed) *forest* is the disjoint union of directed trees. Let T be a tree. We denote the edge relation of T by T as well. We say that b is a *child* of a if $T(a, b)$. We say that a node a is a *leaf* if it has no children. A *connected component* of a forest is a maximal set of nodes which are pairwise connected by a path; each connected component of a forest is a tree.

At each stage n of the construction, we will have a forest T_n with domain $\omega \times \mathcal{P}$, representing the pair $\langle s, X\rangle$ corresponding to some chain A_X^s via some bijection. Each T_n will have only finitely many edges and the T_n will be nested; that is, if $m < n$, and $\langle s, X\rangle$ is a child of $\langle t, Y\rangle$ in T_m, then the same is true in T_n (but not necessarily vice versa). If there is an edge in T_n involving $\langle s, X\rangle$, then after stage n, we will only add edges outward from $\langle s, X\rangle$, and never inward. Thus the roots of any non-trivial connected components in T_n will remain the roots of their connected components. We will satisfy the requirement:

$(*)$: If $\langle s, X\rangle$ is a child of $\langle t, Y\rangle$ in T_n, then for all $Y' \geq A^t_Y(n)$, there is $X' \geq A^s_X(n)$ such that $Y'\mathcal{R}X'$.

Let $\langle\cdot,\cdot\rangle$ be a one-to-one function from $\omega \times \omega$ to ω. Begin the construction with $A^s_X(0) = X$ for each $X \in \mathcal{P}$ and $s \in \omega$. Suppose that we have defined $A^s_X(0), \dots, A^s_X(n)$ for each $\langle s, X\rangle$. Write $n+1 = 2\langle s, i, k\rangle + \epsilon$ where ϵ is 0 or 1. Let $X = X_i$ and $\varphi = \varphi_k$. Let $\langle t_0, Y_0\rangle, \langle t_1, Y_1\rangle, \dots, \langle t_\ell, Y_\ell\rangle, \langle s, X\rangle$ be a path from the root $\langle t_0, Y_0\rangle$ of the connected component of $\langle s, X\rangle$ in T_n. Essentially what we want to do is to extend $A^s_X(n)$ as we did in the warm-up proof. But to maintain $(*)$, we first need to "prepare" the path $\langle t_0, Y_0\rangle, \langle t_1, Y_1\rangle, \dots, \langle t_\ell, Y_\ell\rangle, \langle s, X\rangle$ by extending each of those chains using $(*)$ (and losing $(*)$ in the process), then extend $A^s_X(n)$, and then use $\mathbf{R}$ to recover the property $(*)$. See Figure 3 for a diagram showing how we do these extensions.

$$\begin{tikzcd}
 & & & & Y' \arrow[r, dashed] & X' \\
\tilde{Y}_0 & \tilde{Y}_1 & \tilde{Y}_2 & \cdots & \tilde{Y}_n \arrow[u, "\forall"] & \tilde{X} \arrow[u, "\exists"'] \\
 & \hat{Y}_1 \arrow[u] \arrow[r, dashed] & \hat{Y}_2 \arrow[u] \arrow[rr, dashed] & & \hat{Y}_\ell \arrow[u] \arrow[r, dashed] \arrow[ur, dashed] & \hat{X} \arrow[u] \\
A^{t_0}_{Y_0}(n) \arrow[uu] \arrow[ur, dashed] & A^{t_1}_{Y_1}(n) \arrow[u] & A^{t_2}_{Y_2}(n) \arrow[u] & \cdots & A^{t_\ell}_{Y_\ell}(n) \arrow[u] & A^s_X \arrow[u]
\end{tikzcd}$$

FIGURE 2. The extensions of possibilities in the construction. The dotted line shows the relation $\mathcal{R}$. For all Y' as shown, there is an X' filling in the diagram.

Let $\hat{Y}_0 = A^{t_0}_{Y_0}(n)$. By $(*)$, there is $\hat{Y}_1 \geq A^{t_1}_{Y_1}(n)$ such that $\hat{Y}_0\mathcal{R}\hat{Y}_1$. Then $\hat{Y}_1 \geq A^{t_1}_{Y_1}(n)$, so again by $(*)$ there is $\hat{Y}_2 \geq A^{t_2}_{Y_2}(n)$ such that $\hat{Y}_1\mathcal{R}\hat{Y}_2$. Continuing in this way, we get that $\mathcal{R}$ relates $\hat{Y}_0$ to $\hat{Y}_1$, $\hat{Y}_1$ to $\hat{Y}_2$, and so on until $\hat{Y}_\ell$ is related to $\hat{X} \geq A^s_X(n)$. This completes the "preparation."

Now in each case $\epsilon = 0$ or $\epsilon = 1$, we will define $\tilde{X} \geq \hat{X}$.

$\epsilon = 0$: Choose $\tilde{X} \geq \hat{X}$ such that either $\tilde{X} \models \varphi_k$ or $\tilde{X} \models \neg\varphi_k$, and so that either $\tilde{X} \geq X_k$ or $\tilde{X}$ is incomparable with X_k.

$\epsilon = 1$: If there is $Z \in \mathcal{P}$ such that $\hat{X}\mathcal{R}Z$ and $Z \models \varphi_k$, choose u such that $\langle u, Z\rangle$ has no edge in T_n, and is greater than any other pair connected to any edge in T_n. Let T_{n+1} be T_n with an additional edge from $\langle s, X\rangle$ to $\langle u, Z\rangle$. Using $\mathbf{R}$, choose $\tilde{X} \geq \hat{X}$ such that for all $X' \geq \tilde{X}$, there is $Z' \geq A^u_Z(n)$ with $X'\mathcal{R}Z'$ (this is to satisfy $(*)$).

Now we need to recover $(*)$. Note that $\mathcal{R}$ relates $\hat{Y}_\ell$ to $\tilde{X}$ by **P2**. Using $\mathbf{R}$, choose $\tilde{Y}_\ell \geq \hat{Y}_\ell$ such that for all $Y''_\ell \geq \tilde{Y}_n$, there is $X'' \geq \tilde{X}$ with $Y''\mathcal{R}X''$. Then using $\mathbf{R}$ again, choose $\tilde{Y}_{\ell-1} \geq \hat{Y}_{\ell-1}$ such that for all $Y''_{\ell-1} \geq \tilde{Y}_{\ell-1}$ there is $Y''_{\ell} \geq \tilde{Y}_{\ell}$ with $Y''_{\ell-1}\mathcal{R}Y''_{\ell}$. Continue in this way to define $\tilde{Y}_0, \ldots, \tilde{Y}_{\ell}$. Set $A^{t_i}_{Y_i}(n+1) = \tilde{Y}_i$. Set $A^s_X(n+1) = \tilde{X}$. For each other $\langle u, Z\rangle$, set $A^u_Z(n+1) = A^u_Z(n)$. It is easy to see that $(*)$ remains satisfied. Also, $A^{t_\ell}_{Y_\ell}(n)$ is related by $\mathcal{R}$ to $A^s_X(n+1)$.

This completes the construction. Let T be the union of the T_n (i.e., all of the edges which were in any of the T_n).

Claim 1. *For each $X \in \mathcal{P}$, $s \in \omega$, and formula φ, there is an n such that $A^s_X(n) \models \varphi$ or $A^s_X(n) \models \neg\varphi$. Similarly, for each $X \in \mathcal{P}$, $s \in \omega$, and possibility Y, there is an n such that $A^s_X(n) \geq Y$ or $A^s_X(n)$ is incomparable with Y.*

Proof. Let k be such that $\varphi = \varphi_k$ and i be such that $X = X_i$. Let $n+1 = 2\langle s, i, k\rangle$; then at stage $n+1$ of the construction, we set $A^s_X(n+1)$ to be a refinement of a possibility X' with $X' \models \varphi$ or $X' \models \neg\varphi$; by persistence, either $A^s_X(n+1) \models \varphi$ or $A^s_X(n+1) \models \neg\varphi$. The proof of the second claim is similar. ☐

Claim 2. *For each $X, Y \in \mathcal{P}$ and $s, t \in \omega$ with an edge from $\langle s, X\rangle$ to $\langle t, Y\rangle$ in T, and for every n, there is m such that $A^s_X(n)\mathcal{R}A^t_Y(m)$.*

Proof. Recall that if i is the index of Y, then at each stage $n+1 = 2\langle t, i, k\rangle + \epsilon$ for any k and ϵ, we ensured that $A^s_X(n)\mathcal{R}A^t_Y(n+1)$. Thus for infinitely many n, there is m such that $A^s_X(n)\mathcal{R}A^t_Y(m)$. So each n has some $n' \geq n$ and m such that $A^s_X(n')\mathcal{R}A^t_Y(m)$; by $\mathbf{P1}$, $A^s_X(n)\mathcal{R}A^t_Y(m)$. This suffices to prove the claim. ☐

Claim 3. *Let $X \in \mathcal{P}$, $s \in \omega$, and φ a formula. Suppose that for each m, there is Y_m with $Y_m \models \varphi$ and $A^s_X(m)\mathcal{R}Y_m$. Then there are Y and t, with $Y \models \varphi$, such that $T(\langle s, X\rangle, \langle t, Y\rangle)$.*

Proof. Let i be such that $X = X_i$ and k such that $\varphi = \varphi_k$. Let n be such that $n+1 = 2\langle s, i, k\rangle + 1$. Let Y be such that $Y \models \varphi$ and $A^s_X(n+1)\mathcal{R}Y$; then, for the $X' \leq A^s_X(n+1)$ defined at stage $n+1$, $X'\mathcal{R}Y$. So at stage $n+1$ of the construction, we find such a Y and t, and we put an edge between $\langle s, X\rangle$ and $\langle t, Y\rangle$ in T_{n+1}. ☐

We are now ready to define our Kripke model $\mathcal{K} = (\mathcal{W}, \mathcal{S}, U)$. For each $\langle s, X\rangle$, let $\hat{A}^s_X$ be the downwards closure of the chain A^s_X. By Claim 1, this is a maximal order ideal. Let $\mathcal{W} = \{\hat{A}^s_X : X \in \mathcal{P} \text{ and } s \in \omega\}$. Define $I\mathcal{S}J$ if for each $X \in I$, there is $Y \in J$ with $X\mathcal{R}Y$. Claim 2 implies that if in T there is an edge from $\langle s, X\rangle$ to $\langle t, Y\rangle$, then $\hat{A}^s_X\mathcal{S}\hat{A}^s_Y$. Define $U(I, p) = \mathtt{T}$ if, for some $X \in I$, $V(X, p) = \mathtt{T}$; similarly, define $U(I, p) = \mathtt{F}$ if, for some $X \in I$, $V(X, p) = \mathtt{F}$. By Claim 1, we are in exactly one of these two cases.

Claim 4. *For each sentence φ, $\mathcal{K}, I \models \varphi$ if and only if for some $X \in I$, $\mathcal{M}, X \models \varphi$.*

Proof. Let $\langle s, X\rangle$ be such that $I = \hat{A}^s_X$. Then for some $Y \in I$, $\mathcal{M}, Y \models \varphi$ if and only if for some n, $\mathcal{M}, A^s_X(n) \models \varphi$. The proof is by induction on the complexity of the formula φ. If φ is p, then this follows from the definition of U. For a sentence $\varphi \wedge \psi$,

$$\begin{aligned}
\mathcal{K}, \hat{A}^s_X \models \varphi \wedge \psi \quad &\Leftrightarrow \quad \mathcal{K}, \hat{A}^s_X \models \varphi \text{ and } \mathcal{K}, \hat{A}^s_X \models \psi \\
&\Leftrightarrow \quad (\exists m)\mathcal{M}, A^s_X(m) \models \varphi \text{ and } (\exists n)\mathcal{M}, A^s_X(n) \models \psi \\
&\Leftrightarrow \quad (\exists n)\mathcal{M}, A^s_X(n) \models \varphi \text{ and } \mathcal{M}, A^s_X(n) \models \psi \\
&\Leftrightarrow \quad (\exists n)\mathcal{M}, A^s_X(n) \models \varphi \wedge \psi
\end{aligned}$$

using persistence on the third line. For $\neg\varphi$,

$$\begin{aligned} \mathcal{K}, \hat{A}^s_X \models \neg\varphi &\Leftrightarrow \mathcal{K}, \hat{A}^s_X \not\models \varphi \\ &\Leftrightarrow (\forall n)\mathcal{M}, A^s_X(n) \not\models \varphi \\ &\Leftrightarrow (\exists n)\mathcal{M}, A^s_X(n) \models \neg\varphi \end{aligned}$$

since by Claim 1, for some n $\mathcal{M}, A^s_X(n) \models \varphi$ or $\mathcal{M}, A^s_X(n) \models \neg\varphi$.

For $\Box\varphi$, suppose that for all n, $\mathcal{M}, A^s_X(n) \not\models \Box\varphi$. Then, for each n, there is a Y such that $A^s_X(n)\mathcal{R}Y$ and $\mathcal{M}, Y \not\models \varphi$; refining Y if necessary, we may assume that $\mathcal{M}, Y \models \neg\varphi$. Then by Claim 3, there some such Y and $t \in \omega$ with an edge between $\langle s, X\rangle$ and $\langle t, Y\rangle$ in T. By Claim 2, $\hat{A}^s_X \mathcal{S} \hat{A}^t_Y$. Now $\mathcal{K}, \hat{A}^t_Y \models \neg\varphi$, so $\mathcal{K}, \hat{A}^s_X \not\models \Box\varphi$. Thus we have shown that if $\mathcal{K}, \hat{A}^s_X \models \Box\varphi$, then $\mathcal{M}, A^s_X(n) \models \Box\varphi$ for some n.

Now suppose that for some n, $\mathcal{M}, A^s_X(n) \models \Box\varphi$. Let $\langle t, Y\rangle$ be such that $\hat{A}^s_X \mathcal{S} \hat{A}^t_Y$. Then there is m such that $A^s_X(n)\mathcal{R}A^t_Y(m)$, and so $\mathcal{M}, A^t_Y(m) \models \varphi$. Hence $\mathcal{M}, \hat{A}^t_Y \models \varphi$. Since $\langle t, Y\rangle$ was arbitrary, $\mathcal{K}, \hat{A}^s_X \models \Box\varphi$. □

4. From possibility models to possibilizations

The goal in this section is to prove Theorem 1.2. Recall that Theorem 1.2 says that a countable, separative, and strong possibility model in a countable language is isomorphic to a possibilization of a countable Kripke model. We begin by proving a stronger form of the worldization theorem under the additional hypothesis that $\mathcal{M}$ is strong.

Theorem 4.1. *Let $\mathcal{M}$ be a strong countable possibility model in a countable language. Then there is a Kripke model $\mathcal{K}$ which is a worldization of $\mathcal{M}$ and such that:*

1. *$X\mathcal{R}Y$ if and only if for all w with $Y \in \Phi(w)$ there is v with $X \in \Phi(v)$ and $v\mathcal{S}w$.*
2. *if $X\mathcal{R}Y$, then there is $X' \geq X$ such that for all v' with $X' \in \Phi(v')$, there is w' with $Y \in \Phi(w')$ and $v'\mathcal{S}w'$.*

Proof. We modify the construction from Theorem 1.1. We will make a small modification to the trees from that Theorem. In T_n we will now have two types of edges, red and blue. The edges we added in Theorem 1.1 will be the red edges, and the blue edges will be added for the sake of (1) in the statement of this theorem. We call $\langle s, X\rangle$ a *red* child of $\langle t, Y\rangle$ if there is a red edge from $\langle t, Y\rangle$ to $\langle s, X\rangle$, and a *blue* child if there is a blue edge. $(*)$ from Theorem 1.1 will hold for the red edges:

$(*)$: If $\langle s, X\rangle$ is a red child of $\langle t, Y\rangle$ in T_n, then for all $Y' \geq A^t_Y(n)$, there is $X' \geq A^s_X(n)$ such that $Y'\mathcal{R}X'$.

We have a new property $(\dagger)$ for the blue edges:

$(\dagger)$: If $\langle s, X\rangle$ is a blue child of $\langle t, Y\rangle$ in T_n, then $A^s_X(n)\mathcal{R}A^t_Y(n)$.

Note that the direction of the accessibility relation here is the opposite of that in $(*)$.

The construction begins in the same way with $A^s_X(0) = X$ for each $X \in \mathcal{P}$ and $s \in \omega$. Suppose that we have defined $A^s_X(0), \ldots, A^s_X(n)$ for each $\langle s, X\rangle$. Write $n + 1 = 4\langle s, i, k\rangle + \epsilon$ where ϵ is 0, 1, 2, or 3. Let $X = X_i$. Let $\langle t_0, Y_0\rangle, \ldots, \langle t_\ell, Y_\ell\rangle, \langle s, X\rangle$ be a path from the root $\langle t_0, Y_0\rangle$ of the connected component of $\langle s, X\rangle$ in T_n; some of the edges in this path may be red, and others may be blue. Choose $\hat{Y}_0 = A^{t_0}_{Y_0}(n)$. Now, if $\langle t_1, Y_1\rangle$ is a red child of $\langle t_0, Y_0\rangle$, using $(*)$ choose $\hat{Y}_1 \geq A^{t_1}_{Y_1}(n)$ such that $\hat{Y}_0 \mathcal{R} \hat{Y}_1$. If $\langle t_1, Y_1\rangle$ is a blue child of $\langle t_0, Y_0\rangle$, using **R** choose $\hat{Y}_1 \geq A^{t_1}_{Y_1}(n)$ such that for all $\hat{Y}_1' \geq \hat{Y}_1$, there is $\hat{Y}_0' \geq \hat{Y}_0$ with $\hat{Y}_1' \mathcal{R} \hat{Y}_0'$. Continuing in this way, we get $\hat{Y}_0 \geq A^{t_0}_{Y_0}(n), \hat{Y}_1 \geq A^{t_1}_{Y_1}(n), \ldots, \hat{Y}_\ell \geq A^{t_\ell}_{Y_\ell}(n)$, and $\hat{X} \geq A^s_X(n)$ such that if $\langle t_{i+1}, Y_{i+1}\rangle$ is a red child of $\langle t_i, Y_i\rangle$, then $\hat{Y}_i \mathcal{R} \hat{Y}_{i+1}$, and if $\langle t_{i+1}, Y_{i+1}\rangle$ is a blue child of $\langle t_i, Y_i\rangle$, then for all $\hat{Y}_{i+1}' \geq \hat{Y}_{i+1}$, there is $\hat{Y}_i' \geq \hat{Y}_i$ with $\hat{Y}_{i+1}' \mathcal{R} \hat{Y}_i'$.

Recall that $n+1 = 4\langle s,i,k\rangle + \epsilon$. Now for each ϵ, we will define $\tilde{X} \geq \hat{X}$.

$\epsilon = 0$: Same as Theorem 1.1.

$\epsilon = 1$: Same as Theorem 1.1, adding a red edge.

$\epsilon = 2$: Let $Z = X_k$. If $Z \mathcal{R} \hat{X}$, then choose u such that $\langle u, Z\rangle$ has no edge in T_n, and is greater than any other pair connected to any edge in T_n. Let T_{n+1} be T_n with an additional blue edge from $\langle s, X\rangle$ to $\langle u, Z\rangle$.

$\epsilon = 3$: Let $k = \langle k_1, k_2, j\rangle$. Let $Z_1 = X_{k_1}$ and $Z_2 = X_{k_2}$. The j here just ensures that we visit this requirement infinitely many times. If Z_1 is such that for all $Z_1' \geq Z_1$ there is $Z_2' \geq Z_2$ with $Z_1' \mathcal{R} Z_2'$, and $Z_1 \leq \hat{X}$, then choose $Z_2' \geq Z_2$ such that $\hat{X} \mathcal{R} Z_2'$. Using **R**, choose $\tilde{X} \geq \hat{X}$ such that for all $\tilde{X}' \geq \tilde{X}$ there is $Z_2'' \geq Z_2'$ with $\tilde{X}' \mathcal{R} Z_2''$. Choose u such that $\langle u, Z_2'\rangle$ has no edge in T_n, and is greater than any other pair connected to any edge in T_n. Let T_{n+1} be T_n with an additional red edge from $\langle s, X\rangle$ to $\langle u, Z_2'\rangle$.

Now we need to recover $(*)$ and $(\dagger)$. If $\langle s, X\rangle$ is a red child of $\langle t_\ell, Y_\ell\rangle$, then note that $\mathcal{R}$ relates $\hat{Y}_\ell$ to $\hat{X}$ by **P2**. Using **R**, choose $\tilde{Y}_\ell \geq \hat{Y}_\ell$ such that for all $Y_\ell'' \geq \tilde{Y}_n$, there is $X'' \geq \tilde{X}$ with $Y'' \mathcal{R} X''$. Thus we have recovered $(*)$ between $\langle s, X\rangle$ and $\langle t_\ell, Y_\ell\rangle$. If $\langle s, X\rangle$ is a blue child of $\langle t_\ell, Y_\ell\rangle$, then by choice of $\hat{Y}_\ell$, there is $\tilde{Y}_\ell \geq \hat{Y}_\ell$ such that $\tilde{X} \mathcal{R} \tilde{Y}_\ell$. Thus we have recovered $(\dagger)$ between $\langle s, X\rangle$ and $\langle t_\ell, Y_\ell\rangle$. Continue in this way to define $\tilde{Y}_0, \ldots, \tilde{Y}_\ell$. Set $A^{t_i}_{Y_i}(n+1) = \tilde{Y}_i$. Set $A^s_X(n+1) = \tilde{X}$. For each other $\langle u, Z\rangle$, set $A^u_Z(n+1) = A^u_Z(n)$. Note that both $(*)$ and $(\dagger)$ have the property that if they held between $\langle u, Z\rangle$ and its child $\langle u', Z'\rangle$ at stage n, $A^u_Z(n+1) \geq A^u_Z(n)$, and $A^{u'}_{Z'}(n+1) = A^{u'}_{Z'}(n)$, then $(*)$ and $(\dagger)$ hold between $\langle u, Z\rangle$ and $\langle u', Z'\rangle$ at stage $n+1$. Thus $(*)$ and $(\dagger)$ both hold for T_{n+1}.

Define the model $\mathcal{K}$ in the same way as before. The proofs of the claims in Theorem 1.1 still hold for the red edges and so $\mathcal{K}$ is a worldization of $\mathcal{M}$ via Φ. Also, if there is a blue edge from $\langle s, X\rangle$ to $\langle t, Y\rangle$ in T, then $(\dagger)$ implies that $\hat{A}^t_Y \mathcal{S} \hat{A}^s_X$. We now have two new claims.

Claim 1. *$X \mathcal{R} Y$ if and only if for all w with $Y \in \Phi(w)$ there is v with $X \in \Phi(v)$ and $v \mathcal{S} w$.*

Proof. Suppose that $X \mathcal{R} Y$. Let $w = \hat{A}^s_Z$ be such that $Y \in \Phi(w)$. Then, for sufficiently large n, $X \mathcal{R} A^s_Z(n)$. So at some stage, we put a blue edge from $\langle s, Z\rangle$ to $\langle u, X\rangle$. Then $\hat{A}^u_X \mathcal{S} w$. Note that $X \in \Phi(\hat{A}^u_X \mathcal{S})$.

Suppose that $\neg X \mathcal{R} Y$. Then by $\mathcal{R}$-refinability, there is $Y' \geq Y$ such that for all $Y'' \geq Y'$, $\neg X \mathcal{R} Y''$. Fix w with $Y' \in \Phi(w)$. Then it follows from (W5) that for all v with $X \in \Phi(v)$, $\neg v \mathcal{S} w$. □

Claim 2. *If $X \mathcal{R} Y$, then there is $X' \geq X$ such that for all v' with $X' \in \Phi(v')$, there is w' with $Y \in \Phi(w')$ and $v' \mathcal{S} w'$.*

Proof. Suppose that $X\mathcal{R}Y$. By $\mathbf{R}$ there is $X' \geq X$ such that for all $X'' \geq X'$ there is $Y' \geq Y$ with $X''\mathcal{R}Y'$.

Suppose that v' is such that $X' \in \Phi(v')$. Choose s and Z such that $v' = \hat{A}^s_Z$. Then for sufficiently large n, $A^s_Z(n) \geq X'$. Let i, k_1, and k_2 be such that $Z = X_i$, $X' = X_{k_1}$, and $Y = X_{k_2}$. Then at stage $n = \langle s, i, \langle k_1, k_2, j\rangle\rangle + 3$ for some sufficiently large j (large enough that $A^s_Z(n) \geq X'$), we put into T_{n+1} a red edge from $\langle s, Z\rangle$ to $\langle u, Y'\rangle$ for some u and $Y' \geq Y$. Let $w' = \hat{A}^u_{Y'}$. Then $Y \in \Phi(w')$ and $v'\mathcal{S}w'$. $\square$

This completes the proof. $\square$

We will now prove Theorem 1.2: up to isomorphism of possibility models, every countable, separative, and strong possibility model in a countable language is the possibilization of a Kripke model.

Proof of Theorem 1.2. Let $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ be a countable, separative, and strong possibility model. Using Theorem 4.1, let $\mathcal{K} = (\mathcal{W}, \mathcal{S}, U)$ be a worldization of $\mathcal{M}$ via $\Phi\colon \mathcal{P} \to \wp(\mathcal{W})$. So we have (1) and (2) of Theorem 4.1.

Given $X \in \mathcal{P}$, let $S_X = \{w \in \mathcal{W}\colon X \in \Phi(w)\}$. We claim that $S_X = S_Y$ if and only if $X = Y$; this is where we use the fact that $\mathcal{M}$ is separative. Suppose that $S_X = S_Y$. We claim that $X \simeq_s Y$ where $\simeq_s$ is defined as in Definition 2.13. If $X' \geq X$, then there is v such that $X' \in \Phi(v)$. So $X \in \Phi(v)$ and since $S_X = S_Y$, $Y \in \Phi(v)$. So there is $X'' \in \Phi(v)$ with $X'' \geq X', Y$. Thus $X \geq_s Y$. By interchanging X and Y, we see that $X \simeq_s Y$. Since $\mathcal{M}$ is separative, $X = Y$ as desired.

Identify $X \in \mathcal{P}$ with S_X. We can interpret $\mathcal{R}$, $\leq$, and V as acting on the sets S_X. Let $\mathcal{M}'$ be the model with possibilities S_X. We claim that $\mathcal{M}'$ is a possibilization of $\mathcal{K}$. In verifying properties (A1)-(A4) of a possibilization, we will write X interchangeably with S_X (so that we write $v \in X$ for $v \in S_X$).

(A1): We must check each of (i)-(viii).

- (i): Given $v \neq w$, by (W3) there is X with $v \in X$ and $w \notin X$.
- (ii): Given X and $v \notin X$, suppose to the contrary that there is no Y with $v \in Y$ and $X \cap Y = \varnothing$. Then there is an order ideal containing $\Phi(v) = \{Y : v \in Y\}$ and X. This contradicts (W1).
- (iii): Given $v \in X$ and $\mathcal{K}, v \models \varphi$, by (W4) there is Y with $v \in Y$ and $\mathcal{M}, Y \models \varphi$.
- (iv): Suppose that $v \in \mathcal{S}[X]$. Let $w \in X$ be such that $w\mathcal{S}v$. Then by (W5) there is Y with $v \in Y$ and $X\mathcal{R}Y$. By (1) in the statement of Theorem 4.1, $Y \subseteq \mathcal{S}[X]$.
- (v): Suppose that $Y \not\subseteq \mathcal{S}[X]$. Then there is $w \in Y$ such that there is no $v \in X$ with $v\mathcal{S}w$. By (1) of Theorem 4.1, $\neg X\mathcal{R}Y$. Since $\mathcal{M}$ is strong, there is $Y' \geq Y$ such that for all $Y'' \geq Y'$, $\neg X\mathcal{R}Y''$. We claim that $Y' \cap \mathcal{S}[X] = \varnothing$. Suppose to the contrary that $v \in X$ and $w \in Y'$ are such that $v\mathcal{S}w$. By (W5), there is Y'' with $w \in Y''$ and $X\mathcal{R}Y''$. By (W1) and $\mathbf{P2}$, we may assume that $Y'' \geq Y'$. But this contradicts the choice of Y'.
- (vi): Given Y with $\Diamond Y$ non-empty, let v and $w \in Y$ be such that $v\mathcal{S}w$. Choose X such that $v \in X$. By (W5) and (W1), there is $Y' \geq Y$ with $w \in Y'$ and $X\mathcal{R}Y$. By (2) in the statement of Theorem 4.1, there is $X' \geq X$ such that for all $v' \in X$, there is $w' \in Y$ with $v'\mathcal{R}w'$. So $X \subseteq \Diamond Y$.

- (vii): Suppose that $v \in X \cap Y$. Then by (W1), there is Z with $v \in Z$ and $Z \geq X, Y$, i.e., $Z \subseteq X \cap Y$.
- (viii): Suppose that $\neg v \mathcal{S} w$. Then there is X with $v \in X$ such that for all Y with $w \in Y$, $\neg X \mathcal{R} Y$. We claim that $w \notin \mathcal{S}[X]$. If to the contrary we did have $w \in \mathcal{S}[X]$, then there would be $v' \in X$ with $v' \mathcal{S} w$. But then since $v' \in X$ we would have that for all Y with $w \in Y$, $X' \mathcal{R} Y$. This is a contradiction.

(A2): This follows from (1) in the statement of Theorem 4.1.
(A3): If $X \geq Y$, then by (W1), whenever $v \in X$, $v \in Y$. So $X \subseteq Y$.
(A4): If $V(X,p) = \mathtt{T}$, then it follows immediately from (W4) that for all $w \in X$, $U(w,p) = \mathtt{T}$. If, for all $w \in X$, $U(w,p) = \mathtt{T}$, then we argue that $V(X,p) = \mathtt{T}$ as follows. Suppose not; by **Refinability**, there is $X' \geq X$ with $V(X',p) = \mathtt{F}$. Then taking $v \in X'$, $U(v,p) = \mathtt{F}$. But $v \in X$, so this gives a contradiction. The same works for $\mathtt{F}$. $\square$

The following proposition should be thought of as a converse to Theorem 1.2.

Proposition 4.2. *Let $\mathcal{K} = (\mathcal{W}, \mathcal{S}, U)$ be a Kripke model and let $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ be a possibilization of $\mathcal{K}$. Then $\mathcal{K}$ is a worldization of $\mathcal{M}$.*

Proof. Define $\Phi : \mathcal{W} \to \wp(\mathcal{P})$ by letting $\Phi(w)$ be the set of possibilities X with $w \in X$. We must check (W1)-(W5):

(W1) For each world $w \in \mathcal{W}$, $\Phi(w)$ is a maximal order ideal as:
  (a) if $w \in X \in \mathcal{P}$ and $X \subseteq Y \in \mathcal{P}$, then $w \in Y$;
  (b) if $w \in X \in \mathcal{P}$ and $w \in Y \in \mathcal{P}$, by (vii) there is $Z \in \mathcal{P}$ with $w \in Z \subseteq X \cap Y$;
  (c) if $w \notin X \in \mathcal{P}$, then by (ii) there is $Y \in \mathcal{P}$ with $w \in Y$ and $X \cap Y = \varnothing$, so that X and Y have no common refinement and hence $\Phi(w)$ is maximal among order ideals.

(W2) This follows from the fact that each $X \in \mathcal{P}$ is non-empty.
(W3) This follows from (i).
(W4) This is just Proposition 2.11.
(W5) Fix $w, v \in \mathcal{W}$. First suppose that $v \mathcal{S} w$. Let $X \in \mathcal{P}$ be such that $v \in X$. Then since $w \in \mathcal{S}[X]$, there is $Y \in \mathcal{P}$ with $w \in Y \subseteq \mathcal{S}[X]$. Thus $X \mathcal{R} Y$.

On the other hand, suppose that $\neg v \mathcal{S} w$. By (viii) there is $X \in \mathcal{P}$ with $v \in X$ and $w \notin \mathcal{S}[X]$. Then for all Y with $w \in Y$, $Y \nsubseteq \mathcal{S}[X]$ and so $\neg X \mathcal{R} Y$.

$\square$

5. Uncountable models

Theorem 1.1 required that the possibility model and the language were countable. We will give two examples to show that this assumption was necessary. We will exhibit two possibility models with no worldizations, first with a countable set of possibilities and uncountably many propositional variables, and second with an uncountable set of possibilities and countably many propositional variables.

Proposition 5.1. *There is a possibility model $\mathcal{M}$ with countably many possibilities in a language with uncountably many propositional variables which does not have any worldizations.*

Proof. Let $2^{<\omega}$ be the infinite binary tree, that is, the elements of $2^{<\omega}$ are the finite string of 0's and 1's. Let 2^{ω} be the set of infinite binary strings, which we view as paths through $2^{<\omega}$. Let P be a set of continuum-many propositional variables. Let $f\colon P \to 2^\omega$ be a bijection between P and 2^ω. Let $\mathcal{P} = 2^{<\omega}$. The refinement relation $\leq$ is the natural extension relation on strings. The accessibility relation $\mathcal{R}$ is trivially empty. Define $V(\sigma, p)$ as follows. Let $\pi = f(p)$. Either σ is an initial segment of π, in which case we set $V(\sigma, p) = ?$, or some entry of σ differs from π. If, at the first such entry, σ has a 1, set $V(\sigma, p) = \mathtt{T}$, and otherwise if it is 0, set $V(\sigma, p) = \mathtt{F}$. Then $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ is a possibility model. Every ascending chain in $\mathcal{P}$ is corresponds to a path π through 2^ω, and this path does not decide whether $p = f^{-1}(\pi)$ is true or false. Thus by (W4) none of the ascending chains in $\mathcal{P}$ can be in a worldization. By (W2), there are no worldizations of this model. $\square$

Now we will give the second example, which is more complicated than the first.

Proposition 5.2. *There is a possibility model $\mathcal{M}$ with uncountably many possibilities in a language with countably many propositional variables which does not have any worldizations.*

Proof. By a tree, we now mean a poset $(T, \precsim)$ such that $\{b : b \prec a\}$ is well-ordered for each a. We call the order type of $\{b : b \prec a\}$ the height of a, height(a). The height of a tree is the supremum of the heights of its elements. A path through a tree is a linearly ordered set in the tree closed under predecessor. Let $(T, \precsim)$ be a well-pruned Aronszajn tree, that is, a tree with:

(1) height ω_1,
(2) every element of T has countable height,
(3) every path in T is countable,
(4) for each element a of height α, and each β with $\omega_1 > \beta > \alpha$, there is an element $b \succsim a$ of height β.

The first three properties are what it means to be an Aronszajn tree, and the last says that the tree is well-pruned (see [Kun80, pp. 69-72]). Let $\mathcal{P}$ be the disjoint union of ω_1 and T. Define the refinement relation $\leq$ on $\mathcal{P}$ by making it the natural ordering on ω_1, and the tree ordering on T, but having elements of ω_1 and of T be incomparable. Set $\alpha \mathcal{R} \sigma$ if $\alpha \in \omega_1$ and $\sigma \in T$ and height$(\sigma) \geq \alpha$. We will have one propositional variable p. Set $V(X, p) = \mathtt{T}$ for all $X \in \mathcal{P}$. Let $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$.

For **P1**, if $\alpha \mathcal{R} \sigma$ and $\beta \leq \alpha$, then height$(\sigma) \geq \alpha \geq \beta$ and so $\beta \mathcal{R} \sigma$. For **P2**, if $\alpha \mathcal{R} \sigma$ and $\tau \succsim \sigma$, then height$(\tau) \geq$ height$(\sigma) \geq \alpha$ and so $\alpha \mathcal{R} \tau$. For **R**, suppose that $\alpha \mathcal{R} \sigma$ so that height$(\sigma) \geq \alpha$. Then for all $\beta \geq \alpha$, since T is well-pruned there is $\tau \succsim \sigma$ of height at least β, and hence $\beta \mathcal{R} \tau$. **Refinement** and **Persistence** are clear. Thus $\mathcal{M}$ is a possibility model.

Now we claim that there is no worldization of $\mathcal{M}$. Suppose that there was, say $\mathcal{K} = (\mathcal{W}, \mathcal{S}, U)$. Then $U(w, p) = \mathtt{T}$ for all w by (W4). By (W2), let w be such that $0 \in \Phi(w)$ (where $0 \in \omega_1$); in fact, by (W1) we get $\Phi(w) = \omega_1$. Then, since $0 \mathcal{R} \varnothing$ (where $\varnothing \in T$ is the empty string) and $V(\varnothing, p) = \mathtt{T}$, $\mathcal{M}, 0 \models \Diamond p$. By (W4), $\mathcal{K}, w \models \Diamond p$. Let v be such that $w \mathcal{S} v$ and $\mathcal{K}, v \models p$. By (W1), $\Phi(v) \subseteq T$ is a path through T. Since T is an Aronszajn tree, there is a countable bound on the height of the elements of $\Phi(v)$. On the other hand, by (W5), for each $\alpha \in \omega_1$, there is $\sigma \in \Phi(v)$ with $\alpha \mathcal{R} \sigma$ and hence height$(\sigma) \geq \alpha$, so that the heights of elements of $\Phi(v)$ are unbounded below ω_1. This is a contradiction. So $\mathcal{M}$ has no worldization. $\square$

6. Frames

6.1. **No Worldizations of Basic Possibility Frames.** Recall from Definition 2.1 the definition of a basic possibility frame. In this section we will consider worldizations on the level of frames. By a frame-worldization of a possibility frame $\mathcal{F}$, we mean a Kripke frame $\mathcal{K}$ satisfying (W1)-(W3) and (W5) of the definition of a worldization, and such that any possibility model based on $\mathcal{F}$ gives rise to a model-worldization based on $\mathcal{K}$.

Definition 6.1. Let $\mathcal{G} = (\mathcal{P}, \mathcal{R}, \leq)$ be a (basic) possibility frame and let $\mathcal{F} = (\mathcal{W}, \mathcal{S})$ be a Kripke model. $\mathcal{F}$ is a *frame-worldization* of $\mathcal{G}$ via an embedding $\Phi : \mathcal{W} \to \wp(\mathcal{P})$ which assigns to each total world $w \in \mathcal{W}$ a set of possibilities $\Phi(w) \subseteq \mathcal{P}$ if:

(W1) for each world $w \in \mathcal{W}$, $\Phi(w)$ is a *maximal order ideal*, i.e.,
  (a) $\Phi(w)$ is downwards-closed under refinement,
  (b) any two elements of $\Phi(w)$ have a common refinement in $\Phi(w)$, and
  (c) $\Phi(w)$ is maximal with these two properties;

(W2) for each possibility $X \in \mathcal{P}$, there is a world $w \in \mathcal{W}$ such that $X \in \Phi(w)$;

(W3) any two distinct total worlds $v, w \in \mathcal{W}$ are separated by possibilities, that is, there is $X \in \Phi(v) \setminus \Phi(w)$; and

(W5) for each pair of worlds $w, v \in \mathcal{W}$, $w\mathcal{S}v$ if and only if for each $X \in \Phi(w)$ there is $Y \in \Phi(v)$ such that $X\mathcal{R}Y$;

and such that for any possibility model $\mathcal{M}$ based on $\mathcal{G}$, there is a Kripke model $\mathcal{K}$ based on $\mathcal{F}$ such that $\mathcal{K}$ is a worldization of $\mathcal{M}$ via Φ (and vice versa).

There are basic possibility frames without a worldization. The issue is that in the construction of Theorem 1.1, at some stages we extended a possibility X to a further refinement X' which decided some formula φ. This required us to have countably many definable sets of possibilities; but there may be uncountably many sets of possibilities which are definable in some model based on a countable frame.

Proposition 6.2. *There is a countable basic frame $\mathcal{F}$ with no frame-worldization.*

Proof. Consider the following example of a basic possibility frame $\mathcal{G}$ which is similar to Proposition 5.1. Let $\mathcal{P}$ be the infinite binary tree $2^{<\omega}$. The accessibility relation $\mathcal{R}$ is trivially empty, and $\leq$ is the natural relation on extension of strings. We claim that there cannot possibly be a frame-worldization of $\mathcal{G} = (\mathcal{P}, \mathcal{R}, \leq)$.

Any frame-worldization $\mathcal{F}$ would have to contain, among its total worlds, some world w corresponding to an ascending chain through $\mathcal{P}$. This ascending chain corresponds to some infinite path π through the binary tree. Now, using a single propositional variable, we can define a valuation V to get a possibility model $\mathcal{M}$ based on $\mathcal{G}$. Define $V(\sigma, p) = \mathtt{T}$ if the first place σ differs from π, it has an entry of 1, and $V(\sigma, p) = \mathtt{F}$ if σ first differs from π with an entry of 0. If σ is an initial segment of π, then set $V(\sigma, p) = ?$. Then V satisfies **Persistence** and **Refinability**. However, the ascending chain π never decides p, and so there is no model-worldization of $\mathcal{M}$ based on $\mathcal{F}$. Thus $\mathcal{F}$ is not a frame-worldization of $\mathcal{G}$. □

6.2. **Worldizations of General Possibility Frames.** If we are willing to work with general frames, then we can make a worldization construction. Holliday [Hol15, Definition 2.21] has a natural definition of a general possibility frame.

Definition 6.3. $\mathcal{F} = \langle \mathcal{P}, \mathcal{R}, \leq, \mathcal{A} \rangle$ is a (general) possibility frame if $\langle \mathcal{P}, \mathcal{R}, \leq \rangle$ is a basic possibility frame and $\mathcal{A} \subseteq \wp(\mathcal{P})$, the set of admissible propositions, satisfies:

(1) $\varnothing, \mathcal{P} \in \mathcal{A}$;
(2) Given $A, B \in \mathcal{A}$, $A \cap B \in \mathcal{A}$;
(3) Given $A \in \mathcal{A}$, $A^* = \{X \in \mathcal{P} : \forall Y \geq X, Y \notin A\} \in \mathcal{A}$;
(4) Given $A \in \mathcal{A}$, $\Box A = \{X \in \mathcal{P} : (\forall Y) X \mathcal{R} Y \Rightarrow Y \in A\} \in \mathcal{A}$;
(5) Each $A \in \mathcal{A}$ is regular open in the upset topology.

A set A is regular open set in the upset topology if and only if it satisfies the following conditions of persistence and refinability for sets:

(i) for each $X \in A$ and $X' \geq X$, $X' \in A$, and
(ii) for each $X \in \mathcal{P}$, if $X \notin A$, then there is $X' \geq X$ such that for all $X'' \geq X'$, $X'' \notin A$.

A possibility model $\mathcal{M} = \langle \mathcal{P}, \mathcal{R}, \leq, V \rangle$ is based on $\mathcal{F} = \langle \mathcal{P}, \mathcal{R}, \leq, \mathcal{A} \rangle$ if $\{X : V(X, p) = \mathtt{T}\} \in \mathcal{A}$ for each X and p.

Condition (3) corresponds to the usual condition (for general Kripke frames) of closure under complements. (For a review of general Kripke frames, see Section 5.5 of [BdRV01].) If $\mathcal{M}$ is a possibility model based on a general frame $\mathcal{F}$, then the sets of possibilities definable in $\mathcal{M}$ are all admissible in $\mathcal{F}$

If $\mathcal{F}$ is a general possibility frame, $(p_i)_{i \in I}$ are propositional variables, and $(A_i)_{i \in I}$ are admissible sets, then setting $V(X, p_i) = \mathtt{T}$ if $X \in A_i$, $V(X, p_i) = \mathtt{F}$ if $X \in A_i^*$, and $V(X, p_i) = ?$ otherwise determines a possibility model based on $\mathcal{F}$. The requirement that each admissible set be regular open ensures that **Persistence** and **Refinability** are satisfied.

We can define frame-worldizations of general possibility frames as follows.

Definition 6.4. Let $\mathcal{G} = (\mathcal{P}, \mathcal{R}, \leq, \mathcal{A})$ be a general possibility frame and let $\mathcal{F} = (\mathcal{W}, \mathcal{S}, \mathcal{B})$ be a general Kripke frame. $\mathcal{F}$ is a *frame-worldization* of $\mathcal{G}$ via an embedding $\Phi : \mathcal{W} \to \wp(\mathcal{P})$ which assigns to each total world $w \in \mathcal{W}$ a set of possibilities $\Phi(w) \subseteq \mathcal{P}$ if:

(W1) for each world $w \in \mathcal{W}$, $\Phi(w)$ is a *maximal order ideal*, i.e.,
  (a) $\Phi(w)$ is downwards-closed under refinement,
  (b) any two elements of $\Phi(w)$ have a common refinement in $\Phi(w)$, and
  (c) $\Phi(w)$ is maximal with these two properties;
(W2) for each possibility $X \in \mathcal{P}$, there is a world $w \in \mathcal{W}$ such that $X \in \Phi(w)$;
(W3) any two distinct total worlds $v, w \in \mathcal{W}$ are separated by possibilities, that is, there is $X \in \Phi(v) \setminus \Phi(w)$; and
(W5) for each pair of worlds $w, v \in \mathcal{W}$, $w \mathcal{S} v$ if and only if for each $X \in \Phi(w)$ there is $Y \in \Phi(v)$ such that $X \mathcal{R} Y$;

and such that for any possibility model $\mathcal{M}$ based on $\mathcal{G}$, there is a Kripke model $\mathcal{K}$ based on $\mathcal{F}$ such that $\mathcal{K}$ is a worldization of $\mathcal{M}$ via Φ (and vice versa).

(Note this is almost word-for-word the same definition as for basic frames, though some of the words, such as “based on”, now have a different meaning.)

We now prove Theorem 1.3, which says that a countable possibility frame with countably many admissible sets has a frame-worldization.

Proof of Theorem 1.3. Let $\mathcal{G} = (\mathcal{P}, \mathcal{R}, \leq, \mathcal{A})$ be a countable general possibility frame with countably many admissible sets. For each $A \in \mathcal{A}$, we will have a propositional variable p_A. Define $\mathcal{M}$ a possibility model based on $\mathcal{G}$ with valuation $V(X, p_A) = \mathtt{T}$ if $X \in A$, $V(X, p_A) = \mathtt{F}$ if $X \in A^*$, and $V(X, p_A) = ?$ otherwise.

$\mathcal{M}$ is a countable model in a countable language. By Theorem 1.1, $\mathcal{M}$ has a worldization $\mathcal{K} = (\mathcal{W}, \mathcal{S}, U)$, say via Φ. Let $\mathcal{B}$ be the collection of sets

$$B_A = \{w \in \mathcal{W} : \mathcal{K}, w \models p_A\} = \{w \in \mathcal{W} : (\exists X \in \Phi(w)) X \in A\}.$$

Claim 1. *$\mathcal{F} = (\mathcal{W}, \mathcal{S}, \mathcal{B})$ is a general Kripke frame.*

Proof. $U(w, p_\varnothing) = \mathtt{F}$ for all $w \in \mathcal{W}$, so $B_\varnothing = \varnothing \in \mathcal{B}$.

To see that $\mathcal{B}$ is closed under complements, we show that for $A \in \mathcal{A}$, the complement of B_A is B_A^*. We must show that for $w \in \mathcal{W}$, if $\mathcal{K}, w \not\models p_A$, then $\mathcal{K}, w \models p_{A^*}$. Since $\mathcal{K}, w \models \neg p_A$, there is some $X \in \Phi(w)$ such that $\mathcal{M}, X \models \neg p_A$. So for all $Y \geq X$, $Y \notin A$. Thus $X \in A^*$, and so $\mathcal{M}, X \models p_{A^*}$. But then $\mathcal{K}, w \models p_{A^*}$.

Now we will see that $\mathcal{B}$ is closed under intersections. Given $A, A' \in \mathcal{A}$, we will show that $B_A \cap B_{A'} = B_{A \cap A'}$. Suppose that $w \in B_A \cap B_{A'}$. Then $\mathcal{K}, w \models p_A \wedge p'_A$, and so there are $X \in \Phi(w)$ with $X \in A$ and $X' \in \Phi(w)$ with $X' \in A$. But then there is $X'' \in \Phi(w)$ with $X'' \geq X, X'$, and so $X'' \in A \cap A'$. Hence $\mathcal{M}, X \models p_{A \cap A'}$, and so $\mathcal{K}, w \models p_{A \cap A'}$. Thus $x \in B_{A \cap A'}$. The other direction is similar.

Finally, given $A \in \mathcal{A}$, we will show that $\Box B_A = \{w \in \mathcal{W} : (\forall v) w\mathcal{S}v \Rightarrow v \in B_A\}$ is equal to $B_{\Box A}$. First, suppose that for all v with $w\mathcal{S}v$, $v \in B_A$. Thus for all such v, $\mathcal{K}, v \models p_A$. So $\mathcal{K}, w \models \Box p_A$. There must be some $X \in \Phi(w)$ with $\mathcal{M}, X \models \Box p_A$. So for all Y with $X\mathcal{R}Y$, $\mathcal{M}, Y \models p_A$ and so $X \in \Box A$. Thus $\mathcal{M}, X \models p_{\Box A}$ and so $\mathcal{K}, w \models p_{\Box A}$. The other direction is similar. □

Finally, we want to check that $\mathcal{F}$ is a frame-worldization of $\mathcal{G}$ via Φ. Since $\mathcal{K}$ is a worldization of $\mathcal{M}$ via Φ, it suffices to check that for each possibility model $\mathcal{M}'$ based on $\mathcal{G}$, there is a Kripke model $\mathcal{K}'$ based on $\mathcal{F}$ such that $\mathcal{K}'$ is a worldization of $\mathcal{M}'$ via Φ (and that for each Kripke model $\mathcal{K}'$ based on $\mathcal{F}$, there is a possibility model $\mathcal{M}'$ based on $\mathcal{G}$ such that $\mathcal{K}'$ is a worldization of $\mathcal{M}'$ via Φ).

Claim 2. *For each possibility model $\mathcal{M}'$ based on $\mathcal{G}$, there a Kripke model $\mathcal{K}'$ based on $\mathcal{F}$ such that $\mathcal{K}'$ is a worldization of $\mathcal{M}'$ via Φ.*

Proof. Let $\mathcal{M}' = (\mathcal{P}, \mathcal{R}, \leq, V')$ be a possibility model based on $\mathcal{G}$. Define a valuation U' on $\mathcal{F}$ as follows. For each propositional variable q, let $A_q \in \mathcal{A}$ be such that $A_q = \{X : \mathcal{M}', X \models q\}$. Then define $U'(w, q) = \mathtt{T}$ if $w \in B_{A_q}$, and $U'(w, q) = \mathtt{F}$ if $w \notin B_{A_q}$. So $\mathcal{K}' = (\mathcal{W}, \mathcal{S}, U')$ a Kripke model based on $\mathcal{F}$.

Note that we have both that $\mathcal{M}', X \models q$ if and only if $\mathcal{M}, X \models p_{A_q}$, and that $\mathcal{K}', w \models q$ if and only if $\mathcal{K}, w \models p_{A_q}$. Given a formula φ in the language of $\mathcal{M}'$, we can translate φ to a formula φ^* in the language of $\mathcal{M}$ by replacing each variable q with p_{A_q}. Then $\mathcal{M}', X \models \varphi$ if and only if $\mathcal{M}, X \models \varphi^*$, and $\mathcal{K}', w \models \varphi$ if and only if $\mathcal{K}, w \models \varphi^*$. Since $\mathcal{K}$ is a worldization of $\mathcal{M}$, it follows that $\mathcal{K}'$ is a worldization of $\mathcal{M}'$. □

Claim 3. *For each Kripke model $\mathcal{K}'$ based on $\mathcal{F}$, there a possibility model $\mathcal{M}'$ based on $\mathcal{G}$ such that $\mathcal{K}'$ is a worldization of $\mathcal{M}'$ via Φ.*

Proof. Let $\mathcal{K}' = (\mathcal{W}, \mathcal{S}, U)$ be a Kripke model based on $\mathcal{F}$. Define a valuation V' on $\mathcal{G}$ as follows. For each propositional variable q, let $A_q \in \mathcal{A}$ be such that $B_{A_q} = \{w : \mathcal{K}', w \models q\}$. Then define $V'(X, q) = \mathtt{T}$ if $X \in A_q$, and $V'(X, q) = \mathtt{F}$ if $X \notin A_q$. The rest of the argument is similar to the previous claim. □

So we have shown that $\mathcal{F}$ is a frame-worldization of $\mathcal{G}$, completing the theorem. $\square$

A possibility model $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \leq, V)$ for a countable language induces a general possibility frame $\mathcal{G} = (\mathcal{P}, \mathcal{R}, \leq, \mathcal{A})$ with a countable set $\mathcal{A}$ of admissible sets, namely the sets definable by formulas in $\mathcal{M}$. Then, applying Theorem 1.3 to $\mathcal{G}$, we get a frame-worldization $\mathcal{F}$ of $\mathcal{G}$, and by the last sentence of Definition 6.4, there is a worldization of $\mathcal{M}$ based on $\mathcal{F}$. Thus Theorem 1.3 implies our earlier Theorem 1.1 on worldizations of countable possibility models in a countable language.

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