TRANSLATING THE CLASS OF ABELIAN p-GROUPS INTO AN ELEMENTARY FIRST-ORDER THEORY

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ABSTRACT. The class of abelian p-groups are an example of some very interesting phenomena in computable structure theory and descriptive set theory. We will give an elementary first-order theory T_p whose models are each bi-interpretable with the disjoint union of an abelian p-group and a pure set (and so that every abelian p-group is bi-interpretable with a model of T_p) using computable infinitary formulas. This answers a question of Knight by giving an example of an elementary first-order theory with the following property: The computable infinitary theory of any model (whether hyperarithmetic or not) with computable Scott rank is $\aleph_0$-categorical. It also gives a new example of an elementary first-order theory whose isomorphism problem is $\mathbf{\Sigma}^1_1$-complete but not Borel complete.

1. INTRODUCTION

The class of abelian p-groups is a well-studied example in computable structure theory. A simple compactness argument shows that abelian p-groups are not axiomatizable by an elementary first-order theory, but they are definable by the conjunction of the axioms for abelian p-groups (which are first-order $\forall\exists$ sentences) and the infinitary Π^0_2 sentence which says that every element is torsion of order some power of p.

Abelian p-groups are classifiable by their Ulm sequences [Ulm33]. Due to this classification, abelian p-groups are examples of some very interesting phenomena in computable structure theory and descriptive set theory. We will define a theory T_p whose models behave like the class of abelian p-groups, giving a first-order example of these phenomena. In particular, Theorem 1.6 below answers a question of Knight.

1.1. **Infinitary Formulas.** The infinitary logic $\mathcal{L}_{\omega_1\omega}$ is the logic which allows countably infinite conjunctions and disjunctions but only finite quantification. If the conjunctions and disjunctions of a formula φ are all over computable sets of indices for formulas, then we say that φ is computable. We use $\Sigma^{\mathtt{in}}_\alpha$ and $\Pi^{\mathtt{in}}_\alpha$ to denote the classes of all infinitary Σ_α and Π_α formulas respectively. We will also use $\Sigma^{\mathtt{c}}_\alpha$ and $\Pi^{\mathtt{c}}_\alpha$ to denote the classes of computable Σ_α and Π_α formulas, where $\alpha < \omega_1^{CK}$ the least non-computable ordinal. See Chapter 6 of [AK00] for a more complete description of computable formulas.

1.2. **Bi-Interpretability.** One way in which we will see that the models of T_p are essentially the same as abelian p-group is using bi-interpretations using infinitary formulas [Mon, HTMMM, HTMM]. A structure $\mathcal{A}$ is infinitary interpretable in a structure $\mathcal{B}$ if there is an interpretation of $\mathcal{A}$ in $\mathcal{B}$ where the domain of the interpretation is allowed to be a subset of $\mathcal{B}^{<\omega}$ and where all of the sets in the interpretation are definable using infinitary formulas. This differs from the classical notion of interpretation, as in model theory [Mar02, Definition 1.3.9], where the domain is required to be a subset of $\mathcal{B}^n$ for some n, and the sets in the interpretation are first-order definable.

Definition 1.1. We say that a structure $\mathcal{A} = (A; P_0^{\mathcal{A}}, P_1^{\mathcal{A}}, \dots)$ (where $P_i^{\mathcal{A}} \subseteq A^{a(i)}$) is *infinitary interpretable* in $\mathcal{B}$ if there exists a sequence of relations $(\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}, \sim, R_0, R_1, \dots)$, definable using infinitary formulas (in the language of $\mathcal{B}$, without parameters), such that

1. $\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}} \subseteq \mathcal{B}^{<\omega}$,
2. $\sim$ is an equivalence relation on $\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}$,
3. $R_i \subseteq (B^{<\omega})^{a(i)}$ is closed under $\sim$ within $\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}$,

and there exists a function $f_{\mathcal{A}}^{\mathcal{B}}: \mathcal{D}om_{\mathcal{A}}^{\mathcal{B}} \to \mathcal{A}$ which induces an isomorphism:

$$(\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}/\sim; R_0/\sim, R_1/\sim, \dots) \cong (A; P_0^{\mathcal{A}}, P_1^{\mathcal{A}}, \dots),$$

where $R_i/\sim$ stands for the $\sim$-collapse of R_i.

Two structures $\mathcal{A}$ and $\mathcal{B}$ are infinitary bi-interpretable if they are each effectively interpretable in the other, and moreover, the composition of the interpretations—i.e., the isomorphisms which map $\mathcal{A}$ to the copy of $\mathcal{A}$ inside the copy of $\mathcal{B}$ inside $\mathcal{A}$, and $\mathcal{B}$ to the copy of $\mathcal{B}$ inside the copy of $\mathcal{A}$ inside $\mathcal{B}$—are definable.

Definition 1.2. Two structures $\mathcal{A}$ and $\mathcal{B}$ are *infinitary bi-interpretable* if there are infinitary interpretations of each structure in the other as in Definition 1.1 such that the compositions

$$f_{\mathcal{B}}^{\mathcal{A}} \circ \tilde{f}_{\mathcal{A}}^{\mathcal{B}}: \mathcal{D}om_{\mathcal{B}}^{(\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}})} \to \mathcal{B} \quad \text{and} \quad f_{\mathcal{A}}^{\mathcal{B}} \circ \tilde{f}_{\mathcal{B}}^{\mathcal{A}}: \mathcal{D}om_{\mathcal{A}}^{(\mathcal{D}om_{\mathcal{B}}^{\mathcal{A}})} \to \mathcal{A}$$

are definable in $\mathcal{B}$ and $\mathcal{A}$ respectively. (Here, we have $\mathcal{D}om_{\mathcal{B}}^{(\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}})} \subseteq (\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}})^{<\omega}$, and $\tilde{f}_{\mathcal{A}}^{\mathcal{B}}: (\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}})^{<\omega} \to \mathcal{A}^{<\omega}$ is the obvious extension of $f_{\mathcal{A}}^{\mathcal{B}}: \mathcal{D}om_{\mathcal{A}}^{\mathcal{B}} \to \mathcal{A}$ mapping $\mathcal{D}om_{\mathcal{B}}^{(\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}})}$ to $\mathcal{D}om_{\mathcal{B}}^{\mathcal{A}}$.)

If we ask that the sets and relations in the interpretation (or bi-interpretation) be (uniformly) relatively intrinsically computable, i.e., definable by both a $\Sigma_1^{\mathtt{c}}$ formula and a $\Pi_1^{\mathtt{c}}$ formula, then we say that the interpretation (or bi-interpretation) is effective. Any two structures which are effectively bi-interpretable have all of the same computability-theoretic properties; for example, they have the same degree spectra and the same Scott rank. See [Mon, Lemma 5.3].

Here, we will use interpretations which use (lightface) $\Delta_2^{\mathtt{c}}$ formulas. It is no longer true that any two structures which are $\Delta_2^{\mathtt{c}}$-bi-interpretable have all of the same computability-theoretic properties, but it is true, for example, that any two such structures either both have computable, or both have non-computable, Scott rank.

Theorem 1.3. *Each abelian p-group is effectively bi-interpretable with a model of T_p. Each model of T_p is $\Delta_2^{\mathtt{c}}$-bi-interpretable with the disjoint union of an abelian p-group and a pure set.*

This theorem will follow from the constructions in Sections 3 and 4. Given a model $\mathcal{M}$ of T_p, $\mathcal{M}$ is bi-interpretable with an abelian p-group G and a pure set. The domain of the copy of G inside of $\mathcal{M}$ is definable by a $\Sigma_1^{\mathtt{c}}$ formula but not by a $\Pi_1^{\mathtt{c}}$ formula. This is the only part of the bi-interpretation which is not effective.

1.3. **Classification via Ulm Sequences.** Let G be an abelian group. For any ordinal α, we can define $p^\alpha G$ by transfinite induction:

- $p^0 G = G$;
- $p^{\alpha+1} G = p(p^\alpha G)$;
- $p^\beta G = \bigcap_{\alpha<\beta} p^\alpha G$ if β is a limit ordinal.

These subgroups $p^\alpha G$ form a filtration of G. This filtration stabilizes, and we call the smallest ordinal α such that $p^\alpha G = p^{\alpha+1} G$ the length of G. We call the intersection $p^\infty G$ of these subgroups, which is a p-divisible group, the p-divisible part of G. Any countable p-divisible group is isomorphic to some direct product of the Prüfer group

$$\mathbb{Z}(p^\infty) = \mathbb{Z}[1/p, 1/p^2, 1/p^3, \ldots]/\mathbb{Z}.$$

Denote by $G[p]$ the subgroup of G consisting of the p-torsion elements. The αth Ulm invariant $u_\alpha(G)$ of G is the dimension of the quotient

$$\big(p^\alpha G\big)[p] \,/\, \big(p^{\alpha+1} G\big)[p]$$

as a vector space over $\mathbb{Z}/p\mathbb{Z}$.

Theorem 1.4 (Ulm's Theorem, see [Fuc70])**.** *Let G and H be countable abelian p-groups such that for every ordinal α their αth Ulm invariants are equal, and the p-divisible parts of G and H are isomorphic. Then G and H are isomorphic.*

1.4. **Scott Rank and Computable Infinitary Theories.** Scott [Sco65] showed that if $\mathcal{M}$ is a countable structure, then there is a sentence φ of $\mathcal{L}_{\omega_1\omega}$ such that $\mathcal{M}$ is, up to isomorphism, the only countable model of φ. We call such a sentence a Scott sentence for $\mathcal{M}$. There are many different definitions [AK00, Sections 6.6 and 6.7] of the Scott rank of $\mathcal{M}$, which differ only slightly in the ranks they assign. The one we will use, which comes from [Mon15], defines the Scott rank of $\mathcal{A}$ to be the least ordinal α such that $\mathcal{A}$ has a $\Pi^{\mathtt{in}}_{\alpha+1}$ Scott sentence. We denote the Scott rank of a structure $\mathcal{A}$ by $\mathrm{SR}(\mathcal{A})$. It is always the case that $\mathrm{SR}(\mathcal{A}) \le \omega_1^{\mathcal{A}} + 1$ [Nad74]. We could just as easily use any of the other definitions of Scott rank; for all of these definitions, given a computable structure $\mathcal{A}$:

1. $\mathcal{A}$ has computable Scott rank if and only if there is a computable ordinal α such that for all tuples $\bar{a}$ in $\mathcal{A}$, the orbit of $\bar{a}$ is defined by a computable Σ_α formula.
2. $\mathcal{A}$ has Scott rank ω_1^{CK} if and only if for each tuple $\bar{a}$, the orbit is defined by a computable infinitary formula, but for each computable ordinal α, there is a tuple $\bar{a}$ whose orbit is not defined by a computable Σ_α formula.
3. $\mathcal{A}$ has Scott rank $\omega_1^{CK} + 1$ if and only if there is a tuple $\bar{a}$ whose orbit is not defined by a computable infinitary formula.

Given a structure $\mathcal{M}$, define the computable infinitary theory of $\mathcal{M}$, $\mathrm{Th}_\infty(\mathcal{M})$, to be collection of computable $\mathcal{L}_{\omega_1\omega}$ sentences true of $\mathcal{M}$. We can ask, for a given structure $\mathcal{M}$, whether $\mathrm{Th}_\infty(\mathcal{M})$ is $\aleph_0$-categorical, or whether there are other countable models of $\mathrm{Th}_\infty(\mathcal{M})$. For $\mathcal{M}$ a hyperarithmetic structure:

1. If $\mathrm{SR}(\mathcal{M}) < \omega_1^{CK}$, then $\mathrm{Th}_\infty(\mathcal{M})$ is $\aleph_0$-categorical. Indeed, $\mathcal{M}$ has a computable Scott sentence [Nad74].
2. If $\mathrm{SR}(\mathcal{M}) = \omega_1^{CK}$, then $\mathrm{Th}_\infty(\mathcal{M})$ may or may not be $\aleph_0$-categorical [HTIK].
3. If $\mathrm{SR}(\mathcal{M}) = \omega_1^{CK} + 1$, then $\mathrm{Th}_\infty(\mathcal{M})$ is not $\aleph_0$-categorical as $\mathcal{M}$ has a non-principal type which may be omitted.

In the case of abelian p-groups, we can say something even when we replace the assumption that $\mathcal{M}$ is hyperarithmetic with the assumption that $\omega_1^G = \omega_1^{CK}$:

Theorem 1.5. *Let G be an abelian p-group with $\omega_1^{CK} = \omega_1^G$. Then:*

1. *G is the only countable model of $\mathrm{Th}_\infty(G)$ with $\omega_1^G = \omega_1^{CK}$, and*
2. *if $\mathrm{SR}(G) < \omega_1^{CK} = \omega_1^G$, then $\mathrm{Th}_\infty(G)$ is $\aleph_0$-categorical.*

This theorem is well-known but as far as we are aware does not appear in the literature. We will give a proof in Section 2. We also note that there are indeed non-hyperarithmetic abelian p-groups G with $\mathrm{SR}(G) < \omega_1^{CK}$.

Knight asked whether there was a (non-trivial) first-order theory with this same property. By a non-trivial example, we mean that the elementary first-order theory should have non-hyperarithmetic models. Our theory T_p is such an example.

Theorem 1.6. *Given $\mathcal{M} \models T_p$ with $\omega_1^{CK} = \omega_1^{\mathcal{M}}$:*

1. *$\mathcal{M}$ is the only countable model of $\mathrm{Th}_\infty(\mathcal{M})$ with $\omega_1^{\mathcal{M}} = \omega_1^{CK}$, and*
2. *if $\mathrm{SR}(\mathcal{M}) < \omega_1^{CK} = \omega_1^{\mathcal{M}}$, then $\mathrm{Th}_\infty(\mathcal{M})$ is $\aleph_0$-categorical.*

Proof. Let $\mathcal{M}$ be a model of T_p. Now $\mathcal{M}$ is bi-interpretable, using computable infinitary formulas, with the disjoint union of an abelian p-group G and a pure set. Thus $\mathcal{M}$ inherits these properties from G. □

Of course, there will be non-hyperarithmetic models of T_p with Scott rank below ω_1^{CK}.

1.5. **Borel Incompleteness.** In their influential paper [FS89], Friedman and Stanley introduced Borel reductions between invariant Borel classes of structures with universe ω in a countable language. Such classes are of the form $\mathrm{Mod}(\varphi)$, the set of models of φ with universe ω, for some $\varphi \in \mathcal{L}_{\omega_1\omega}$. A Borel reduction from $\mathrm{Mod}(\varphi)$ to $\mathrm{Mod}(\psi)$ is a Borel map $\Phi\colon \mathrm{Mod}(\varphi) \to \mathrm{Mod}(\psi)$ such that

$$\mathcal{M} \cong \mathcal{N} \iff \Phi(\mathcal{M}) \cong \Phi(\mathcal{N}).$$

If such a Borel reduction exists, we say that $\mathrm{Mod}(\varphi)$ is Borel reducible to $\mathrm{Mod}(\psi)$ and write $\varphi \leq_B \psi$. If $\varphi \leq_B \psi$ and $\psi \leq_B \varphi$, then we say that $\mathrm{Mod}(\varphi)$ and $\mathrm{Mod}(\psi)$ are Borel equivalent and write $\varphi \equiv_B \psi$. Friedman and Stanley showed that graphs, fields, linear orders, trees, and groups are all Borel equivalent, and form a maximal class under Borel reduction.

If $\mathrm{Mod}(\varphi)$ is Borel complete, then the isomorphism relation on $\mathrm{Mod}(\varphi)\times\mathrm{Mod}(\varphi)$ is $\mathbf{\Sigma}^1_1$-complete. The converse is not true, and the most well-known example is abelian p-groups, whose isomorphism relation is $\mathbf{\Sigma}^1_1$-complete but not Borel complete. Until very recently, they were one of the few such examples, and there were no known examples of elementary first-order theories with similar properties. Recently, Laskowski, Rast, and Ulrich [URL] gave an example of a first-order theory which is not Borel complete, but whose isomorphism relation is not Borel. (We do not know, but we expect, that the isomorphism problem is also $\mathbf{\Sigma^1_1}$-complete.) Our theory T_p is another such example.

Theorem 1.7. *The class of models of T_p is Borel equivalent to abelian p-groups.*

Because abelian p-groups are not Borel complete, but their isomorphism relation is $\mathbf{\Sigma^1_1}$-complete, we get:

Corollary 1.8. *The class of models of T_p is not Borel complete but the isomorphism relation is $\mathbf{\Sigma^1_1}$-complete.*

Theorem 1.7 is a specific instance of the following general question asked by Friedman:

Question 1.9. Is it true that for every $\mathcal{L}_{\omega_1\omega}$ sentence there is a Borel equivalent first-order theory?

2. Proof of Theorem 1.5

The proof of Theorem 1.5 consists essentially of expressing the Ulm invariants via computable infinitary formulas.

Definition 2.1. Let G be an abelian p-group. For each ordinal $\alpha < \omega_1^{CK}$, there is a computable infinitary sentence $\psi_\alpha(x)$ which defines $p^\alpha G$ inside of G:

- $\psi_0(x)$ is just $x = x$;
- $\psi_{\alpha+1}(x)$ is $(\exists y)[\psi_\alpha(y) \wedge py = x]$;
- $\psi_\beta(x)$ is $\bigwedge\!\!\!\bigwedge_{\alpha<\beta} \psi_\alpha(x)$ for limit ordinals β.

Definition 2.2. For each ordinal $\alpha < \omega_1^{CK}$ and $n \in \omega \cup \{\omega\}$, there is a computable infinitary sentence $\varphi_{\alpha,n}$ such that, for G an abelian p-group,

$$G \vDash \varphi_{\alpha,n} \Leftrightarrow u_\alpha(G) = n.$$

For $n \in \omega$, define $\varphi_{\alpha,\geq n}$ to say that there are $x_1, \ldots, x_n$ such that:

- $\psi_\alpha(x_1) \wedge \cdots \wedge \psi_\alpha(x_n)$,
- $px_1 = \cdots = px_n = 0$, and
- for all $c_1, \ldots, c_n \in \mathbb{Z}/p\mathbb{Z}$ not all zero, $\neg\psi_{\alpha+1}(c_1x_1 + \cdots + c_nx_n)$.

Then for $n \in \omega$, $\varphi_{\alpha,n}$ is $\varphi_{\alpha,\geq n} \wedge \neg\varphi_{\alpha,\geq n+1}$, and $\varphi_{\alpha,\omega}$ is $\bigwedge\!\!\!\bigwedge_{n\in\omega} \varphi_{\alpha,\geq n}$.

Lemma 2.3 (Theorem 8.17 of [AK00])**.** *Let G be an abelian p-group. Then:*

(1) the length of G is at most ω_1^G, and
(2) if G has length ω_1^G then G is not reduced (in fact, its p-divisible part has infinite rank) and $\mathrm{SR}(G) = \omega_1^G + 1$.

We are now ready to give the proof of Theorem 1.5.

Proof of Theorem 1.5. Since $\omega_1^{CK} = \omega_1^G$, G has length at most ω_1^{CK}. Note that $\mathrm{Th}_\infty(G)$ contains the sentences $\varphi_{\alpha,u_\alpha(G)}$ for $\alpha < \omega_1^{CK}$. Thus any model of $\mathrm{Th}_\infty(G)$ has the same Ulm invariants as G, for ordinals below ω_1^{CK}.

If $\mathrm{SR}(G) < \omega_1^{CK}$, let λ be the length of G. Then $\mathrm{Th}_\infty(G)$ includes the computable formula $(\forall x)[\psi_\lambda(x) \leftrightarrow \psi_{\lambda+1}(x)]$, so that any countable model of $\mathrm{Th}_\infty(G)$ has length at most λ. Note that in such a model, ψ_λ defines the p-divisible part. Let $n \in \omega \cup \{\omega\}$ be such that $p^\infty G$ is isomorphic to $\mathbb{Z}(p^\infty)^n$. Then, if $n \in \omega$, $\mathrm{Th}_\infty(G)$ contains the formula which says that there are $x_1, \ldots, x_n$ such that

- $\psi_\lambda(x_1) \wedge \cdots \wedge \psi_\lambda(x_n)$,
- for all $c_1, \ldots, c_n < p$ not all zero and $k_1, \ldots, k_n \in \omega$,
$$\frac{c_1}{p^{k_1}}x_1 + \cdots + \frac{c_n}{p^{k_n}}x_n \neq 0,$$
- for all y with $\psi_\lambda(y)$, there are $c_1, \ldots, c_n < p$ and $k_1, \ldots, k_n \in \omega$ such that
$$y = \frac{c_1}{p^{k_1}}x_1 + \cdots + \frac{c_n}{p^{k_n}}x_n.$$

If $n = \omega$, then $\mathrm{Th}_\infty(G)$ contains the formula which says that for each $m \in \omega$, there are $x_1, \ldots, x_m$ such that

- $\psi_\lambda(x_1) \wedge \cdots \wedge \psi_\lambda(x_m)$, and
- for all $c_1, \ldots, c_m < p$ not all zero and $k_1, \ldots, k_m \in \omega$,
$$\frac{c_1}{p^{k_1}} x_1 + \cdots + \frac{c_m}{p^{k_m}} x_m \neq 0.$$

Any countable model of $\mathrm{Th}_\infty(G)$ has p-divisible part isomorphic to $\mathbb{Z}(p^\infty)^n$. So any countable model of $\mathrm{Th}_\infty(G)$ has the same Ulm invariants and p-divisible part as G, and hence is isomorphic to $\mathrm{Th}_\infty(G)$. Hence $\mathrm{Th}_\infty(G)$ is $\aleph_0$-categorical. This gives (2), and (1) for the case where $\mathrm{SR}(G) < \omega_1^{CK}$.

If $\mathrm{SR}(G) = \omega_1^{CK} + 1$, let H be any other countable model of $\mathrm{Th}_\infty(G)$ with $\omega_1^H = \omega_1^G = \omega_1^{CK}$. Thus G and H both have length ω_1^{CK} and their p-divisible parts have infinite rank. As remarked before, they have the same Ulm invariants, and so they must be isomorphic. This completes the proof of (1). □

3. The Theory T_p

Fix a prime p. The language $\mathcal{L}_p$ of T_p will consist of a constant 0, unary relations R_n for $n \in \omega$, and ternary relations $P^n_{\ell,m}$ for $\ell, m \in \omega$ and $n \le \max(\ell, m)$. The following transformation of an abelian p-group into an $\mathcal{L}_p$-structure will illustrate the intended meaning of the symbols.

Definition 3.1. Let G be an abelian p-group. Define $\mathfrak{M}(G)$ to be $\mathcal{L}_p$-structure obtained as follows, with the same domain as G, and the symbols of $\mathcal{L}_p$ interpreted as follows:

- Set $0^{\mathfrak{M}(G)}$ to be the identity element of G.
- For each n, let $R_n^{\mathfrak{M}(G)}$ be the elements which are torsion of order p^n.
- For each $\ell, m \in \omega$ and $n \le \max(\ell, m)$, and $x, y, z \in G$, set $P^{n,\mathfrak{M}(G)}_{\ell,m}(x, y, z)$ if and only if $x + y = z$, $x \in R_\ell^{\mathfrak{M}(G)}$, $y \in R_m^{\mathfrak{M}(G)}$, and $z \in R_n^{\mathfrak{M}(G)}$.

One should think of such $\mathcal{L}_p$-structures as the canonical models of T_p. The theory T_p will consist of following axiom schemata:

(A1) For all $\ell, m, n \in \omega$:
$$(\forall x \forall y \forall z)\left[P^n_{\ell,m}(x, y, z) \to (R_\ell(x) \wedge R_m(x) \wedge R_n(z))\right].$$

(A2) *(R_n contains the elements which are torsion of order p^n.)*
$$(\forall x)[R_0(x) \leftrightarrow x = 0].$$
and, for all $n \ge 1$:
$$(\forall x)\left[x \in R_n \leftrightarrow (\exists x_2 \cdots \exists x_{p-1})\left[P^n_{n,n}(x, x, x_2) \wedge P^n_{n,n}(x, x_2, x_3) \wedge \cdots \wedge P^{n-1}_{n,n}(x, x_{p-1}, x_p)\right]\right].$$

(A3) *(P defines a partial function.)* For all $\ell, m, n, n' \in \omega$:
$$(\forall x \forall y \forall z \forall z')\left[\left(P^n_{\ell,m}(x, y, z) \wedge P^{n'}_{\ell,m}(x, y, z')\right) \to z = z'\right].$$

(A4) *(P is total.)* For all $\ell, m \in \omega$:
$$(\forall x \forall y)\left[\left(R_\ell(x) \wedge R_m(y)\right) \to \bigvee_{n \le \max(\ell,m)} (\exists z) P^n_{\ell,m}(x, y, z)\right].$$

(A5) *(Identity.)* For all $\ell \in \omega$:

$$(\forall x)[R_\ell(x) \to [P^\ell_{0,\ell}(0,x,x) \wedge P^\ell_{\ell,0}(x,0,x)]].$$

(A6) *(Inverses.)* For all $\ell \in \omega$:

$$(\forall x)(\exists y)[R_\ell(x) \to [P^0_{\ell,\ell}(x,y,0) \wedge P^0_{\ell,\ell}(y,x,0)]].$$

(A7) *(Associativity.)* For all $\ell, m, n \in \omega$:

$$(\forall x \forall y \forall z)\Bigg[\Big[R_\ell(x) \wedge R_m(y) \wedge R_n(z)\Big] \longrightarrow$$

$$\bigvee_{\substack{r \le \max(\ell,m) \\ s \le \max(m,n) \\ t \le \max(r,n), \max(\ell,s)}} (\exists u \exists v \exists w)\Big[P^r_{\ell,m}(x,y,u) \wedge P^t_{r,n}(u,z,w) \wedge P^s_{m,n}(y,z,v) \wedge P^t_{\ell,s}(x,v,w)\Big]\Bigg].$$

(A8) *(Abelian.)* For all $\ell, m \in \omega$ and $n \le \max(\ell, m)$:

$$(\forall x \forall y \forall z)[[R_\ell(x) \wedge R_m(y) \wedge R_n(z) \wedge P^n_{\ell,m}(x,y,z)] \to P^n_{m,\ell}(y,x,z)].$$

We must now check that the definition of T_p works as desired, that is, that if G is an abelian p-group, then $\mathfrak{M}(G)$ is a model of T_p.

Lemma 3.2. *If G is an abelian p-group, then $\mathfrak{M}(G) \vDash T_p$.*

Proof. We must check that each instance of the axiom schemata of T_p holds in $\mathfrak{M}(G)$.

(A1) Suppose that x, y, and z are elements of G with $P^{n,\mathfrak{M}(G)}_{m,\ell}(x,y,z)$. Then, by definition, $x + y = z$, $x \in R^{\mathfrak{M}}_\ell(G)$, $y \in R^{\mathfrak{M}(G)}_m$, and $z \in R^{\mathfrak{M}(G)}_n$.

(A2) $R^{\mathfrak{M}(G)}_0$ contains the elements of G which are torsion of order $p^0 = 1$, so R_0 contains just the identity. For each $n > 0$, $R^{\mathfrak{M}(G)}_n$ contains the elements of order p^n. An element x has order p^n if and only if px has order p^{n-1}. It remains only to note that if x has order p^n, then $x, 2x, 3x, \ldots, (p-1)x$ all have order p^n as well. The existential quantifier is witnessed by $x_2 = 2x$, $x_3 = 3x$, and so on.

(A3) If, for some x, y, z, and z', $P^{n,\mathfrak{M}(G)}_{\ell,m}(x,y,z)$ and $P^{n',\mathfrak{M}(G)}_{\ell,m}(x,y,z')$, then $x + y = z$ and $x + y = z'$, so that $z = z'$.

(A4) Given x and y in G which are of order p^m and p^ℓ respectively, $x + y$ is of order p^n for some $n \le \max(m, \ell)$, and so we have $P^{n,\mathfrak{M}(G)}_{m,\ell}(x, y, x + y)$.

(A5) If $x \in G$ is of order p^ℓ, then $x + 0 = 0 + x = x$ and so we have $P^{\ell,\mathfrak{M}(G)}_{\ell,0}(x, 0, x)$.

(A6) If $x \in G$ is of order p^ℓ, then $-x$ is also of order p^ℓ, and $x + (-x) = 0 = (-x) + x$. So we have $P^{0,\mathfrak{M}(G)}_{\ell,\ell}(x, -x, 0)$.

(A7) Given $x, y, z \in G$ of order p^ℓ, p^m, and p^n respectively, there are $r \le \max(\ell, m)$ and $s \le \max(m, n)$ such that $x + y$ and $y + z$ are of order p^r and p^s respectively. Then there is t such that $x + y + z$ is of order p^t; $t \le \max(r, n)$ and $t \le \max(\ell, s)$.

(A8) Given $x, y, z \in G$ of order p^ℓ, p^m, and p^n respectively, $n \le \max(\ell, m)$, and with $x + y = z$, we have $y + x = z$ as G is abelian.

Thus we have shown that $\mathfrak{M}(G)$ is a model of T_p. □

Note that G and $\mathfrak{M}(G)$ are effectively bi-interpretable, proving one half of Theorem 1.3.

4. From a model of T_p to an abelian p-group

Given an abelian p-group G, we have already described how to turn G into a model of T_p. In this section we will do the reverse by turning a model of T_p into an abelian p-group.

Definition 4.1. Let $\mathcal{M}$ be a model of T_p. Define $\mathfrak{G}(\mathcal{M})$ to be the group obtained as follows.

- The domain of $\mathfrak{G}(\mathcal{M})$ will be the subset of the domain of $\mathcal{M}$ given by $\bigcup_{n\in\omega} R_n^{\mathcal{M}}$.
- The identity element of $\mathfrak{G}(\mathcal{M})$ will be $0^{\mathcal{M}}$.
- We will have $x + y = z$ in $\mathfrak{G}(\mathcal{M})$ if and only if, for some ℓ, m, and n, $P_{\ell,m}^{n,\mathcal{M}}(x,y,z)$.

We will now check that $\mathfrak{G}(\mathcal{M})$ is always an abelian p-group.

Lemma 4.2. *If $\mathcal{M}$ is a model of T_p, then $\mathfrak{G}(\mathcal{M})$ is an abelian p-group.*

Proof. First we check that the operation $+$ on $\mathfrak{G}(\mathcal{M})$ defines a total function. Given $x, y \in \mathfrak{G}(\mathcal{M})$, choose ℓ and m such that $x \in R_\ell^{\mathcal{M}}$ and $y \in R_m^{\mathcal{M}}$. Then by (A3) and (A4), there is a unique $n \le \max(\ell, m)$ and a unique z such that $P_{\ell,m}^{n,\mathcal{M}}(x,y,z)$. Thus $x + y = z$, and z is unique.

Second, we check that $\mathfrak{G}(\mathcal{M})$ is in fact a group. To see that $0^{\mathcal{M}}$ is the identity, given $x \in \mathfrak{G}(\mathcal{M})$, there is ℓ such that $x \in R_\ell^{\mathcal{M}}$. By (A5), $P_{\ell,0}^{\ell,\mathcal{M}}(x, 0^{\mathcal{M}}, x)$ and $P_{0,\ell}^{\ell,\mathcal{M}}(0^{\mathcal{M}}, x, 0^{\mathcal{M}})$. Thus $x + 0^{\mathcal{M}} = 0^{\mathcal{M}} + x = x$, and $0^{\mathcal{M}}$ is the identity of $\mathfrak{G}(\mathcal{M})$. To see that $\mathfrak{G}(\mathcal{M})$ has inverses, given $x \in \mathfrak{G}(\mathcal{M})$, there is ℓ such that $x \in R_\ell^{\mathcal{M}}$, and by (A6) there is $y \in R_\ell^{\mathcal{M}}$ such that $P_{\ell,\ell}^{0,\mathcal{M}}(x,y,0^{\mathcal{M}})$ and $P_{\ell,\ell}^{0,\mathcal{M}}(y,x,0^{\mathcal{M}})$. Thus $x + y = y + x = 0^{\mathcal{M}}$, and so y is the inverse of x. Finally, to see that $\mathfrak{G}(M)$ is associative, given $x, y, z \in \mathfrak{G}(\mathcal{M})$, there are ℓ, m, and n such that $x \in R_\ell^{\mathcal{M}}$, $y \in R_m^{\mathcal{M}}$, and $z \in R_n^{\mathcal{M}}$. Then by (A7) there are r, s, and t, and u, v, and w, such that $P_{\ell,m}^{r,\mathcal{M}}(x,y,u)$, $P_{r,n}^{t,\mathcal{M}}(u,z,w)$, $P_{m,n}^{s,\mathcal{M}}(y,z,v)$, and $P_{\ell,s}^{t,\mathcal{M}}(x,v,w)$. Thus $x + y = u$, $u + z = w$, $y + z = v$, and $x + v = w$. So $(x + y) + z = x + (y + z)$. Thus $\mathfrak{G}(\mathcal{M})$ is associative.

Third, to see that $\mathfrak{G}(\mathcal{M})$ is abelian, let $x, y \in \mathfrak{G}(\mathcal{M})$. There are ℓ and m such that $x \in R_\ell^{\mathcal{M}}$ and $y \in R_m^{\mathcal{M}}$. Let $n \le \max(\ell, m)$ be such that $z = x + y \in R_n^{\mathcal{M}}$. (Such an n and z exist by the arguments above that $+$ is total, via (A3) and (A4).) Then $P_{\ell,m}^{n,\mathcal{M}}(x,y,z)$, and so by (A8), $P_{m,\ell}^{n,\mathcal{M}}(y,x,z)$. Thus $y + x = z$ and so $\mathfrak{G}(\mathcal{M})$ is abelian.

Finally, we need to see that $\mathfrak{G}(\mathcal{M})$ is a p-group. We claim, by induction on $n \ge 0$, that $R_n^{\mathcal{M}}$ consists of the elements of $\mathfrak{G}(\mathcal{M})$ which are of order p^n. From this claim, it follows that $\mathfrak{G}(\mathcal{M})$ is a p-group. For $n = 0$, the claim follows directly from (A2). Given $n > 0$, suppose that $x \in R_n^{\mathcal{M}}$. Then the witnesses $x_2, x_3, \ldots, x_p$ to (A2) must be $2x, 3x, \ldots, px$. Note that since $P_{n,n}^{n-1,\mathcal{M}}(x, (p-1)x, px)$, $px \in R_{n-1}^{\mathcal{M}}$. Thus px is of order p^{n-1}, and so x is of order p^n. On the other hand, if x is of order p^n, then px is of order p^{n-1} and so $px \in R_{n-1}^{\mathcal{M}}$. Moreover, $x_2 = 2x, x_3 = 3x, \ldots, x_{p-1} = (p-1)x$ are all of order p^n. So we have $P_{n,n}^{n,\mathcal{M}}(x,x,x_2), P_{n,n}^{n,\mathcal{M}}(x,x_2,x_3), \ldots, P_{n,n}^{n-1,\mathcal{M}}(x,x_{p-1},x_p)$. By (A2), $x \in R_n^{\mathcal{M}}$. This completes the inductive proof. $\square$

We now have two operations, one which turns an abelian p-group into a model of T_p, and another which turns a model of T_p into an abelian p-group. These two operations are almost inverses to each other. If we begin with an abelian p-group, turn it into a model of T_p, and then that model into an abelian p-group, we will obtain the original group. However, if we start with a $\mathcal{M}$ model of T_p, turn it into an abelian p-group, and then turn that abelian p-group into a model of T_p, we may obtain a different model of T_p. The problem is that the of elements of $\mathcal{M}$ which are not in any of the sets $R_n^{\mathcal{M}}$ are discarded when we transform $\mathcal{M}$ into an abelian p-group. However, these elements form a pure set, and so the only pertinent information is their size.

Definition 4.3. Given a model $\mathcal{M}$ of T_p, the size of $\mathcal{M}$, $\#\mathcal{M} \in \omega \cup \{\infty\}$, is the number of elements of M not in any relation R_n.

Lemma 4.4. *Given an abelian p-group G, $\mathfrak{G}(\mathfrak{M}(G)) = G$.*

Proof. Since $\#\mathfrak{M}(G) = 0$, we see that G, $\mathfrak{M}(G)$, and $\mathfrak{G}(\mathfrak{M}(G))$ all have the same domain. The identity of $\mathfrak{G}(\mathfrak{M}(G))$ is $0^{\mathfrak{M}(G)}$ which is the identity of G. If $x + y = z$ in G, then, for some $\ell, m, n \in \omega$, we have $P_{\ell,m}^{n,\mathfrak{M}(G)}(x, y, z)$. Thus, in $\mathfrak{G}(\mathfrak{M}(G))$, we have $x + y = z$. So $\mathfrak{G}(\mathfrak{M}(G)) = G$. □

We make a simple extension to $\mathfrak{M}$ as follows.

Definition 4.5. Let G be an abelian p-group and $m \in \omega \cup \{\infty\}$. Define $\mathfrak{M}(G, m)$ to be $\mathcal{L}_p$-structure with domain $G \cup \{a_1, \dots, a_m\}$ with the relations interpreted as in $\mathfrak{M}(G)$. Thus, no relations hold of any of the elements $a_1, \dots, a_m$.

Lemma 4.6. *Given a model $\mathcal{M}$ of T_p, $\mathfrak{M}(G(\mathcal{M}), \#\mathcal{M}) \cong \mathcal{M}$.*

Proof. We will show that if $\#\mathcal{M} = 0$, then $\mathfrak{M}(\mathfrak{G}(\mathcal{M})) = \mathcal{M}$. From this one can easily see that $\mathfrak{M}(G(\mathcal{M}), \#\mathcal{M}) \cong \mathcal{M}$ in general.

If $\#\mathcal{M} = 0$, then $\mathcal{M}$, $\mathfrak{G}(\mathcal{M})$, and $\mathfrak{M}(\mathfrak{G}(\mathcal{M}))$ all share the same domain. It is clear that $0^{\mathcal{M}} = 0^{\mathfrak{G}(\mathcal{M})} = 0^{\mathfrak{M}(\mathfrak{G}(\mathcal{M}))}$. From the proof of Lemma 4.2, we see that for each n, $R_n^{\mathcal{M}}$ defines the set of elements of $\mathfrak{G}(\mathcal{M})$ which are torsion of order p^n, and so $R_n^{\mathcal{M}} = R_n^{\mathfrak{M}(\mathfrak{G}(\mathcal{M}))}$. Given $\ell, m \in \omega$ and $n \le \max(\ell, m)$, and x, y, and z elements of the shared domain, we have $P_{\ell,m}^{n,\mathcal{M}}(x, y, z)$ if and only if

$$x + y = z \text{ in } \mathfrak{G}(\mathcal{M}) \text{ and } x \in R_\ell^{\mathcal{M}},\ y \in R_m^{\mathcal{M}}, \text{ and } z \in R_n^{\mathcal{M}}.$$

Since $R_i^{\mathcal{M}} = R_i^{\mathfrak{M}(\mathfrak{G}(\mathcal{M}))}$ for each i, this is the case if and only if $P_{\ell,m}^{n,\mathfrak{M}(\mathfrak{G}(\mathcal{M}))}(x, y, z)$. Thus we have shown that $\mathfrak{M}(\mathfrak{G}(\mathcal{M})) = \mathcal{M}$. □

Note that $\mathcal{M}$ and the disjoint union of $\mathfrak{G}(\mathcal{M})$ with a pure set of size $\#\mathcal{M}$ are bi-interpretable, using computable infinitary formulas, completing the proof of Theorem 1.3.

5. Borel Equivalence

In this section we will prove Theorem 1.7 by showing that the class of models of T_p and the class of abelian p-groups are Borel equivalent. $G \mapsto \mathfrak{G}(\mathfrak{M}(G)) = \mathfrak{G}(\mathfrak{M}(G, 0))$ is a Borel reduction from isomorphism on abelian p-groups to isomorphism on models of T_p. However, $\mathcal{M} \mapsto \mathfrak{G}(\mathcal{M})$ is not a Borel reduction in the other direction, because two non-isomorphic models of T_p might be mapped to isomorphic groups. We need to find a way to turn $\mathfrak{G}(\mathcal{M})$ and $\#\mathcal{M}$ into an abelian p-group $\mathfrak{H}(\mathfrak{G}(\mathcal{M}), \#\mathcal{M})$, so that $\mathcal{M}$ and $\#\mathcal{M}$ can be recovered from $\mathfrak{H}(\mathfrak{G}(\mathcal{M}), \#\mathcal{M})$.

We will define $\mathfrak{H}(G,m)$ for any abelian p-group H and $m \in \omega \cup \{\infty\}$. It is helpful to think about what this reduction will do to the Ulm invariants: The first Ulm invariant of $\mathfrak{H}(G,m)$ will be m, and for each α, then $1+\alpha$th Ulm invariant of $\mathfrak{H}(G,m)$ will be the same as the αth Ulm invariant of G.

Definition 5.1. Given an abelian p-group G, and $m \in \omega \cup \{\infty\}$, define an abelian p-group $\mathfrak{H}(G,m)$ as follows. Let $\hat{\mathcal{B}}$ be a basis for the $\mathbb{Z}_p$-vector space G/pG. Let $\mathcal{B} \subseteq G$ be a set of representatives for $\hat{\mathcal{B}}$. Let G^* be the abelian group $\langle G, a_b : b \in \mathcal{B} \mid pa_b = b\rangle$. Then define $\mathfrak{H}(G,m) = G^* \oplus (\mathbb{Z}_p)^m$.

To make this Borel, we can take $\mathcal{B}$ to be the lexicographically first set of representatives for a basis. It will follow from Lemma 5.4 that the isomorphism type of $\mathfrak{H}(G,m)$ does not depend on these choices. First, we require a couple of lemmas.

Lemma 5.2. *Each element of G can be written uniquely as a (finite) linear combination $h + \sum_{b\in\mathcal{B}} x_b b$ where $h \in pG$ and each $x_b < p$.*

Proof. Given $g \in G$, let $\hat{g}$ be the image of g in G/pG. Then, since $\hat{\mathcal{B}}$ is a basis for G/pG, we can write

$$\hat{g} = \sum_{b\in\mathcal{B}} x_b \hat{b}$$

with $x_b < p$, where $\hat{b}$ is the image of b in G/pG. Thus setting

$$h = g - \sum_{b\in\mathcal{B}} x_b b \in pG$$

we get a representation of g as in the statement of the theorem.

To see that this representation is unique, suppose that

$$h + \sum_{b\in\mathcal{B}} x_b b = h' + \sum_{b\in\mathcal{B}} y_b b.$$

Then, modulo pG,

$$\sum_{b\in\mathcal{B}} x_b \hat{b} = \sum_{b\in\mathcal{B}} y_b \hat{b}.$$

Since $\hat{\mathcal{B}}$ is a basis, $x_b = y_b$ for each $b \in \mathcal{B}$. Then we get that $h = h'$ and the two representations are the same. □

Lemma 5.3. *Each element of G^* can be written uniquely in the form $h + \sum_{b\in\mathcal{B}} x_b a_b$ where $h \in G$ and each $x_b < p$.*

Proof. It is clear that each element of G^* can be written in such a way. If

$$h + \sum_{b\in\mathcal{B}} x_b a_b = h' + \sum_{b\in\mathcal{B}} y_b a_b$$

then, in G,

$$ph + \sum_{b\in\mathcal{B}} x_b b = ph' + \sum_{b\in\mathcal{B}} y_b b.$$

This representation is unique, so $x_b = y_b$ for each $b \in \mathcal{B}$, and so $h = h'$. □

Lemma 5.4. *The isomorphism type of $\mathfrak{H}(G,m)$ depends only on the isomorphism type of G, and not on the choice of $\mathcal{B}$.*

Proof. It suffices to show that if $\mathcal{C}$ is another choice of representatives for a basis of G/pG, then $G^*_{\mathcal{B}} = G^*_{\mathcal{C}}$, where the former is constructed using $\mathcal{B}$, and the later is constructed using $\mathcal{C}$. Let $f : \mathcal{B} \to \mathcal{C}$ be an bijection.

Given $g \in G^*_{\mathcal{B}}$, write $g = g' + \sum_{b \in \mathcal{B}} x_b a_b$ with $g' \in G$ and $0 \leq x_b < p$. This representation of g is unique by Lemma 5.3. Define $\varphi(g) = g' + \sum_{b \in \mathcal{B}} x_b a_{f(b)}$. It is not hard to check that φ is a homomorphism. The inverse of φ is the map ψ which is defined by $\psi(h) = h' + \sum_{c \in \mathcal{C}} y_c a_{f^{-1}(c)}$ where $h = h' + \sum_{c \in \mathcal{C}} y_c a_c$. □

The next two lemmas will be used to show that if G is not isomorphic to G', or if m is not equal to m', then $\mathfrak{H}(G, m)$ will not be isomorphic to $\mathfrak{H}(G', m')$.

Lemma 5.5. $G = pG^*$.

Proof. Each element of G can be written as $g + \sum_{b \in \mathcal{B}} x_b b$ with $g \in pG$. Let $g' \in G$ be such that $pg' = g$. Then

$$p(g' + \sum_{b \in \mathcal{B}} x_b a_b) = g + \sum_{b \in \mathcal{B}} x_b b.$$

Hence $G \subseteq pG^*$. Given $h \in G^*$, write $h = g + \sum_{b \in \mathcal{B}} x_b a_b$. Then $ph = pg + \sum_{b \in \mathcal{B}} x_b b \in G$. So $pG^* \subseteq G$, and so $G = pG^*$. □

If G is a group, recall that we denote by $G[p]$ the elements of G which are torsion of order p.

Lemma 5.6. $\mathfrak{H}(G, m)[p] \,/\, (p\mathfrak{H}(G, m))[p] \cong (\mathbb{Z}_p)^m$.

Proof. Note that

$$\begin{aligned} \mathfrak{H}(G, m)[p]/(p\mathfrak{H}(G, m))[p] &\cong \big(G^*[p]/(pG^*)[p]\big) \oplus \big((\mathbb{Z}_p)^m[p]/(p(\mathbb{Z}_p)^m)[p]\big) \\ &\cong (G^*[p]/G[p]) \oplus (\mathbb{Z}_p)^m. \end{aligned}$$

We will show that $(G^*[p]/G[p])$ is the trivial group by showing that if $g \in G^*$, $pg = 0$, then $g \in G$. Indeed, write $g = g' + \sum_{b \in \mathcal{B}} y_b a_b$ with $g' \in G$. Then

$$0 = pg = pg' + \sum_{b \in \mathcal{B}} p y_b a_b = pg' + \sum_{b \in \mathcal{B}} y_b b.$$

Since $0 \in pG$ has a unique representation (by Lemma 5.2) $0 = 0 + \sum_{b \in \mathcal{B}} 0b$, we get that $y_b = 0$ for each $b \in \mathcal{B}$, and so $g = g' \in G$. □

By the previous lemma, we can recover m from $\mathfrak{H}(G, m)$. We have

$$p\mathfrak{H}(G, m) = pG^* \oplus p(\mathbb{Z}_p)^m \cong pG^* = G$$

so that we can also recover G.

Thus, using Lemma 4.6, $\mathcal{M} \mapsto \mathfrak{H}(\mathfrak{G}(\mathcal{M}), \#\mathcal{M})$ gives a Borel reduction from T_p to abelian p-groups. This completes the proof of Theorem 1.7.

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