FIRST-ORDER POSSIBILITY MODELS AND FINITARY COMPLETENESS PROOFS

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Abstract. This paper builds on Humberstone's idea of defining models of propositional modal logic where total possible worlds are replaced by partial possibilities. We follow a suggestion of Humberstone by introducing possibility models for quantified modal logic. We show that a simple quantified modal logic is sound and complete for our semantics. Although Holliday showed that for many propositional modal logics, it is possible to give a completeness proof using a canonical model construction where every possibility consists of finitely many formulas, we show that this is impossible to do in the first-order case. However, one can still construct a canonical model where every possibility consists of a computable set of formulas and thus still of finitely much information.

1. Introduction.

The standard Kripke models for modal logic involve possible worlds that are fully determinate, relative to the language in question. These worlds determine, for each of infinitely many propositions, whether that proposition is true or false. Humberstone [Hum81] outlines a reason that one might be opposed to the standard semantics. Our intuitive idea of the possible comes from activities like imagining what might happen in some hypothetical situation. Having only a bounded capacity for imagination, we cannot imagine a total situation that decides each of possibly infinitely many different facts. We seem to instead imagine some finite list of facts which does not contain every single aspect of an entire world, but rather only the relevant details. An agent who imagines that *Moriarty is the murderer* does not imagine a world or class of worlds in which Moriarty is the murderer, together with all the meals he eats for breakfast, lunch, and dinner, the books he wrote, what he did on his fifth birthday, where various other people went on vacation, and so on. The agent simply imagines Moriarty committing the murder, leaving other parts of his life indeterminate. Such a hypothetical situation could be further refined by specifying more details, and then even more details, yielding situations which become more and more detailed. A total world is a limit of these refinements in which every single detail is specified.

One might object to total worlds on the grounds that such limiting operations are inadmissible. Like Humberstone, we will not argue for the view that total worlds are inadmissible. Rather, we will follow a weaker suggestion of Humberstone: that we allow partial situations in our semantics, while not completely disallowing total worlds.

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Humberstone gives a semantics for classical propositional modal logic in which total worlds are replaced by partial objects which he calls *possibilities*. Each possibility decides that some of the atomic propositions are true, that some are false, and does not decide the rest. There is a relation of *refinement*: if a possibility Y refines a possibility X, then any atomic proposition true at X remains true at Y and any atomic proposition false at X remains false at Y; however, of the remaining atomic propositions, some may become true at Y, some may become false, and others may remain undecided. There is also a modal accessibility relation between possibilities, and the key is in how the refinement and accessibility relations interact. (Our models will be different from Humberstone's in that we place weaker conditions on this interaction.)

Humberstone shows that possibility models have two properties which he calls *persistence* and *refinability*. The former says that any sentence true at a possibility is true at any refinement, while the latter says that any sentence which is not decided at a possibility is decided as true in some refinement and false in some other refinement. He also shows that the modal logic **K** is sound and complete with respect to these models and that certain extensions of **K** are sound and complete with respect to restricted classes of models.

Holliday [Hol14, Hol15] has recently revisited Humberstone's possibility semantics. Holliday has shown that propositional possibility frames are more general than Kripke frames, in the sense that there are normal modal logics which can be characterized by possibility frames but not by Kripke frames. On the other hand, possibility semantics still maintains the geometric intuition that makes Kripke frames powerful, whereas other more general semantics, such as arbitrary modal algebras, lose the geometric intuition. Possibility semantics offers some of the best of both worlds. For other recent work on possibility semantics, see [vBBH16], [Yam16], and [HT16].

This paper has two main parts. First, we will begin by introducing first-order possibility models. This follows a suggestion of Humberstone at the end of his exposition of his possibility models. We will begin by introducing constant-domain models with only relation symbols in the language, and we will describe how to modify these models to obtain variable-domain models and languages with function symbols. We will prove soundness and completeness for a simple quantified modal logic based on **K** which we call *QML*—see, for example, [LZ94].

Second, we consider the possibility of using our first-order possibility models to give a finitary completeness proof. Hale [Hal13] writes:

> According to the modest conception I shall adopt here, ways for things to be (and so possibilities in my sense) are always *finitely specifiable*—that is, they can each be given a finite description. In the context of formal semantics, we can think of this as a partial assignment of truth-values to the sentences of some fixed language (or an assignment to sub-sentential expressions inducing such a partial assignment of truth-values, or a combination of the two). (229)

On such a view, one should restrict attention to models all of whose possibilities are finitely specifiable. It is then natural to ask whether a logic is complete with respect to such models. In the propositional case, by a finitely specified possibility, we mean that the possibility decides only finitely many of the atomic propositions.

A propositional modal logic has a finitary completeness proof if it has a canonical model all of whose possibilities are finitely specified in this sense. This was one of Humberstone's original motivations for considering possibility models. For many normal modal logics extending **K** with standard axioms (such as D, T, 4, B, and 5), Holliday [Hol14] is able to give such a completeness proof. On the other hand, there are normal modal logics which do not admit such a finitary completeness proof.[1]

In the case of first-order logics, it is not quite so clear what one might mean by a finitary description. We turn again to [Hal13]:

> The specification of a way for things to be may fix the truth-values of infinitely many sentences, perhaps all of the sentences of the language, even though its description is finite—but this is very much a special case....
>
> Being finite, specifications of possibilities cannot be closed under logical consequence—a finite description may have infinitely many logical consequences, most of which cannot form part of that description. But we might expect possibilities themselves to be so, in the sense that every proposition entailed by propositions *true at a possibility* is also *true at that possibility.* (229)

One interpretation is that a possibility is finitely specified if for each possibility, there is a finite set Γ of sentences whose consequences are exactly the sentences true at that possibility. Note that Γ will not only consist of atomic sentences: a sentence like $(\forall x)P(x)$ entails infinitely many different atomic facts. A possibility model all of whose possibilities are finitely specified in this sense will be called finitary. We will show that there is a consistent sentence of *QML* which is not satisfied in any finitary model. Thus *QML* is not complete with respect to finitary models.

However, there are also other interpretations of what it means to be finitely specified. One can ask that at each possibility, there is a *computable* set Γ of sentences whose consequences are exactly the sentences true at that possibility. (The consequences of Γ might not themselves be decidable.) A computable set of sentences is finitely specified by a finite algorithm which generates them. In the same way that the formula $(\forall x)P(x)$ determines the truth of infinitely many atomic formulas, a finite algorithm can determine an infinite set of formulas. A possibility model all of whose possibilities are described by computable sets of sentences in this way will be called a model with computable possibilities. We will show that *QML* is complete with respect to models with computable possibilities.

2. First-Order Possibility Models.

2.1. **Basic Relational Semantics.** In this section we will begin by describing our language and semantics. There are many varieties of quantified modal logic for which we could give a possibility semantics. The particular choice we make is of minor importance, since our main results will hold for a variety of choices. Nevertheless, we must make some choice, and we will make what is possibly the simplest choice: we will give possibility semantics for what has been called the "simplest quantified modal logic" [LZ94]. For an exposition of a variety of other possible choices, see [Gar84, FM98].

[1] See §7.3 of [Hol15] for a way around this by taking a detour through an extended language.

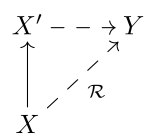
To begin, our signature σ will contain only relation symbols. We will later expand our language to include function and constant symbols in Section 2.6. For each symbol $P \in \sigma$ we have an arity $a(P)$. We will also have infinitely many variable symbols $x, y, z, \ldots$. Our language $\mathcal{L}$ is the standard language of quantified modal logic:

$$\varphi ::= s = t \mid P(t_1, \ldots, t_{a(P)}) \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \forall x \varphi \mid \Box\varphi$$

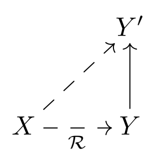
where s and t are terms (for now, just variables). We take $\wedge$, $\Box$, and $\forall$ as primitive; $\vee$, $\exists$, and $\Diamond$ can be defined in the usual way.

A *first-order constant-domain possibility model* is a tuple $\mathcal{M} = (\mathcal{W}, \mathcal{R}, \geq, \mathcal{D}, \mathcal{I})$. In standard models of *QML*, $(\mathcal{W}, \mathcal{R})$ is a Kripke frame. For us, the set $\mathcal{W}$ is a non-empty set of *possibilities*. $\mathcal{R}$ and $\geq$ are binary relations on $\mathcal{W}$, representing the accessibility relation and the refinement relation respectively. $\mathcal{R}(X, Y)$ means that what is necessary at X is true at Y. $X \geq Y$ means that X determines each issue which Y does, in the same way, and possibly more. We require that $\geq$ be a partial order. (One could instead require that $\geq$ be a pre-order, but we can always transform a pre-order into a partial order by taking a quotient; the two will be modally equivalent.) Following Humberstone, we impose three conditions on $\mathcal{R}$ and $\geq$:

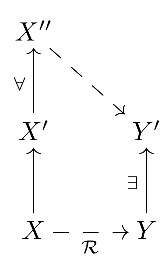
P1: For all X, X', and Y with $X' \geq X$, if $\mathcal{R}(X', Y)$ then $\mathcal{R}(X, Y)$.



P2: For all X, Y, and Y' with $Y' \geq Y$, if $\mathcal{R}(X, Y)$ then $\mathcal{R}(X, Y')$.



R: For all X and Y, if $\mathcal{R}(X, Y)$ then there is $X' \geq X$ such that for all $X'' \geq X'$, there is $Y' \geq Y$ such that $\mathcal{R}(X'', Y')$.



Humberstone justifies **P1** and **P2** as follows. For **P1**, we need to show that everything necessary at X is true at Y. Because of $\mathcal{R}(X', Y)$, everything necessary at X' is true at Y, and since $X' \geq X$, everything necessary at X is necessary at X'. For **P2**, we need to show that everything necessary at X is true at Y'. Since $\mathcal{R}(X, Y)$, everything necessary at X is true at Y and hence at Y' since $Y' \geq Y$. For the third condition, we deviate from Humberstone. For him, the condition was

$\mathbf{R}^{++}$: For all X and Y, if $\mathcal{R}(X,Y)$ then there is $X' \geq X$ such that for all $X'' \geq X'$, $\mathcal{R}(X'',Y)$.

We will discuss different refinability conditions, and our justification for the condition **R**, in Section 2.2.

$\mathcal{D}$ is a non-empty set called the object domain. The function $\mathcal{I}$ interprets the relation symbols in each possibility. For each possibility X and relation symbol P of arity $a(P)$, $\mathcal{I}(X,P)$ is a partial subset of $\mathcal{D}^{a(P)}$, that is, it determines for some tuples of objects that they are in the set, for others that they are not in the set, and leaves the rest undetermined. We write $\bar{c} \in \mathcal{I}(X,P)$ if $\bar{c}$ is in this set, $\bar{c} \notin \mathcal{I}(X,P)$ if $\bar{c}$ is not in this set, and $\bar{c}\uparrow\mathcal{I}(X,P)$ if it is undefined. We have two conditions on $\mathcal{I}$:

Persistence: For any X and Y in $\mathcal{W}$ and any relation symbol P, if $Y \geq X$ then $\mathcal{I}(Y,P)$ extends $\mathcal{I}(X,P)$; that is, if $\bar{a} \in \mathcal{I}(X,P)$, then $\bar{a} \in \mathcal{I}(Y,P)$, and if $\bar{a} \notin \mathcal{I}(X,P)$, then $\bar{a} \notin \mathcal{I}(Y,P)$.

Refinability: For any X in $\mathcal{W}$, if $\bar{a}\uparrow\mathcal{I}(X,P)$, then there exist $Y \geq X$ and $Z \geq X$ such that $\bar{a} \in \mathcal{I}(Y,P)$ and $\bar{a} \notin \mathcal{I}(Z,P)$.

A variable assignment v is a map which assigns to each variable an element of $\mathcal{D}$. We have the following definition of truth at a possibility X in a model $\mathcal{M}$, following Humberstone for the boolean and modal clauses:

(1) $\mathcal{M}, X \vDash_v P(x_1,\ldots,x_n)$ if $v(x_1),\ldots,v(x_n) \in \mathcal{I}(X,P)$.
(2) $\mathcal{M}, X \vDash_v x = y$ if $v(x) = v(y)$.
(3) $\mathcal{M}, X \vDash_v \varphi \wedge \psi$ if $\mathcal{M}, X \vDash_v \varphi$ and $\mathcal{M}, X \vDash_v \psi$.
(4) $\mathcal{M}, X \vDash_v \neg\varphi$ if for all $Y \geq X$, $\mathcal{M}, Y \nvDash_v \varphi$.
(5) $\mathcal{M}, X \vDash_v \Box\varphi$ if for all $Y \in W$ such that $\mathcal{R}(X,Y)$, $\mathcal{M}, Y \vDash_v \varphi$.
(6) $\mathcal{M}, X \vDash_v (\forall x)\varphi$ if $\mathcal{M}, X \vDash_w \varphi$ for every variable assignment w which agrees with v except possibly at x.

If the model $\mathcal{M}$ is understood, we omit it. We may also sometimes omit the variable assignment v.

We also get truth definitions for the defined connectives. For example:

(1) $X \vDash_v \varphi \vee \psi$ if for all $Y \geq X$, there is a $Z \geq Y$ such that $Z \vDash_v \varphi$ or $Z \vDash_v \psi$.
(2) $X \vDash_v \varphi \Rightarrow \psi$ if for all $Y \geq X$, if $Y \vDash_v \varphi$, then there is a $Z \geq Y$ such that $Z \vDash_v \psi$.
(3) $X \vDash_v \Diamond\varphi$ if for all $X' \geq X$ there is a Y such that $\mathcal{R}(X',Y)$ and $Y \vDash_v \varphi$.
(4) $X \vDash_v (\exists x)\varphi$ if for all $Y \geq X$, there is a variable assignment w which agrees with v except possibly at x and some $Z \geq Y$ such that $Z \vDash_w \varphi$.

(3) uses **P1**. As it was for Humberstone, it will be a fact, following from results in §2.4, that we have:

(5) $X \vDash_v \varphi \Rightarrow \psi$ if for all $Y \geq X$, if $Y \vDash_v \varphi$, then $Y \vDash_v \psi$.

We can give the usual definition for validity: a formula φ is valid if for every model $\mathcal{M}$, possibility X, and variable assignment v, $\mathcal{M}, X \vDash_v \varphi$.

Each total world model (by which we mean a standard constant-domain model, as in [LZ94]) can be viewed as a possibility model by viewing the total worlds as possibilities and taking the refinement relation to be equality.

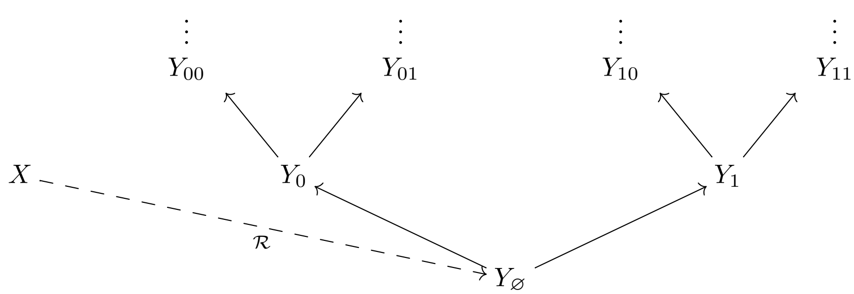
2.2. **The Condition R.** Holliday [Hol15, Appendix B.1] gives an example which shows that the condition $\mathbf{R}^{++}$ of Humberstone is too strong: it should be possible to have a possibility X which satisfies $\Box p \vee \Box q$ without satisfying either disjunct, and a possibility Y which satisfies $p \vee q$ without satisfying either disjunct, and to have $\mathcal{R}(X, Y)$. But $\mathbf{R}^{++}$ says that for some refinement X' of X, each $X'' \geq X'$ has $\mathcal{R}(X'', Y)$. Since X satisfies $\Box p \vee \Box q$, some such X'' can be chosen to satisfy either $\Box p$ or $\Box q$, contradicting the fact that Y satisfies neither p nor q.

Holliday briefly discusses a third condition $\mathbf{R}^{+}$ before concluding that $\mathbf{R}$ is the correct condition to impose:

> **$\mathbf{R}^{+}$:** For all X and Y, if $\mathcal{R}(X, Y)$ then there is $X' \geq X$ and $Y' \geq Y$ such that for all $X'' \geq X'$, $\mathcal{R}(X'', Y')$.

This is a weaker condition than $\mathbf{R}^{++}$, but stronger than $\mathbf{R}$. One can reject $\mathbf{R}^{+}$ for similar reasons.

Suppose that $p_1, p_2, \ldots$ and $q_1, q_2, \ldots$ are various independent atomic propositions. X is a possibility which satisfies $\Box p_i \vee \Box q_i$ for each i, and Y is a possibility which satisfies $p_i \vee q_i$ for each i, but neither X nor Y satisfies any of the disjuncts. Now Y has many refinements, but suppose that each refinement of Y decides only finitely many of the issues p_i and q_i. For example, the refinements above $Y = Y_\varnothing$ may form a binary tree $\langle Y_\sigma : \sigma \in 2^{<\omega} \rangle$, with Y_σ satisfying $p_i \wedge \neg q_i$ if $\sigma(i) = 0$ and $q_i \wedge \neg p_i$ if $\sigma(i) = 1$. We also have $\mathcal{R}(X, Y)$ (and hence $\mathcal{R}(X, Y_\sigma)$ for any $\sigma \in 2^{<\omega}$).



All of this is plausible and should be possible in a possibility model. However, $\mathbf{R}^{+}$ says that there are refinements X' of X and Y' of Y such that any further refinement of X' bears $\mathcal{R}$ to Y'. Now Y' may satisfy some of p_i or q_i, but for some sufficiently large i, Y' satisfies $p_i \vee q_i$ but neither p_i nor q_i. Since X satisfies $\Box p_i \vee \Box q_i$, there is a refinement X'' of X' which satisfies either $\Box p_i$ or $\Box q_i$, and $\mathcal{R}(X'', Y')$ by $\mathbf{R}^{+}$. This is a contradiction. If one agrees that this should be an acceptable model, one has to reject the condition $\mathbf{R}^{+}$.

2.3. **Truth Conditions for Quantifiers.** We chose to take $(\forall x)\varphi$ as primitive, defining its truth conditions without looking at refinements, while letting $(\exists x)\varphi$ have the truth conditions derived from those of universal quantification and negation. We could have tried to do the opposite, taking the existential quantifier as primitive. A universal quantifier is similar to an infinite conjunction over each element of the domain; thus it is natural to define the truth conditions for the universal quantifier in a similar way to those for a conjunction. An existential quantifier, on the other hand, may have an undetermined witness. For example, it could be determined at some possibility that there is an object with the property P without determining which object has property P, in a similar way to the way a possibility can determine that a disjunction is true without determining either disjunct. So to make an existential fact true, it should not be sufficient that there is some refinement with a witness; rather, we want it to be that for any refinement, there is a further refinement with a witness.

2.4. **Persistence and Refinability.** We say that a formula is *persistent* if whenever $X \vDash \varphi$ and $Y \geq X$, then $Y \vDash \varphi$. We say that a formula is *refinable* if whenever $X \nvDash \varphi$, there is some $Y \geq X$ such that $Y \vDash \neg\varphi$. Like Humberstone, we can prove persistence and refinability for all formulas, not just the atomic ones. Humberstone omitted many of these proofs, but for completeness, we will include the entire proof. The only essentially new steps are the quantifier cases.

Lemma 1 (Persistence). *If $Y \geq X$ and $X \vDash_v \varphi$, then $Y \vDash_v \varphi$.*

Proof. The proof is by induction on the complexity of the formula φ. The atomic cases are $x = y$, whose truth value depends only on the variable assignment v and is independent of the possibility, and $P(x_1, \dots, x_n)$. Suppose that $X \vDash_v P(x_1, \dots, x_n)$. By definition, $v(x_1), \dots, v(x_n) \in \mathcal{I}(X, P)$. By **Persistence**, we have $v(x_1), \dots, v(x_n) \in \mathcal{I}(Y, P)$ and hence $Y \vDash_v P(x_1, \dots, x_n)$.

For conjunctions, suppose that $X \vDash_v \varphi \wedge \psi$. Then $X \vDash_v \varphi$ and $X \vDash_v \psi$, so $Y \vDash_v \varphi$ and $Y \vDash_v \psi$. Hence $Y \vDash_v \varphi \wedge \psi$.

For negation, suppose that $X \vDash_v \neg\varphi$. Then for all $Z \geq X$, $Z \nvDash_v \varphi$. In particular, since $Y \geq X$, for all $Z \geq Y$, $Z \nvDash_v \varphi$. Hence $Y \vDash_v \neg\varphi$.

Now we do the modal operator. Suppose that $X \vDash_v \Box\varphi$. Let Z be such that $\mathcal{R}(Y, Z)$. Since $Y \geq X$, by **P1** we have $\mathcal{R}(X, Z)$. Then $Z \vDash_v \varphi$.

Finally, for quantification, suppose that $X \vDash_v (\forall x)\varphi$. Let w be a variable assignment that agrees with v except possibly at x. Then $X \vDash_w \varphi$; hence $Y \vDash_w \varphi$ and so $Y \vDash_v (\forall x)\varphi$. □

Lemma 2 (Refinability). *If $X \nvDash_v \varphi$, then for some $Y \geq X$, $Y \vDash_v \neg\varphi$.*

Proof. The proof is by induction on the complexity of the formula φ. For the atomic case, $x = y$ if and only if $v(x) = v(y)$; if $X \nvDash_v x = y$, then $v(x) \neq v(y)$, and hence for all $Y \geq X$, $Y \nvDash_v x = y$. So $X \vDash_v \neg(x = y)$. If $X \nvDash_v P(\bar{x})$, then either $v(\bar{x}) \notin \mathcal{I}(X, P)$ or $v(\bar{x}) \uparrow \mathcal{I}(X, P)$. In the first case, by **Persistence**, for all $Y \geq X$, $v(\bar{x}) \notin \mathcal{I}(X, P)$, and hence $X \vDash_v \neg P(\bar{x})$. In the second case, by **Refinability**, there is $Y \geq X$ such that $v(\bar{x}) \notin \mathcal{I}(Y, P)$. Hence $Y \vDash_v \neg P(\bar{x})$.

If $X \nvDash_v \varphi \wedge \psi$, then $X \nvDash_v \varphi$ or $X \nvDash_v \psi$, and hence there is some $Y \geq X$ with $Y \vDash_v \neg\varphi$ or $Y \vDash_v \neg\psi$. Then for all $Z \geq Y$, $Z \nvDash_v \varphi \wedge \psi$, and hence $Y \vDash \neg(\varphi \wedge \psi)$.

Suppose that $X \nvDash_v \neg\varphi$. Then there is $Y \geq X$ such that $Y \vDash_v \varphi$. By persistence, for all $Z \geq Y$, $Z \vDash_v \varphi$. Hence $Z \nvDash_v \neg\varphi$. So $Y \vDash_v \neg\neg\varphi$.

If $X \nvDash_v \Box\varphi$, then there is Z such that $\mathcal{R}(X, Z)$ and $Z \nvDash_v \varphi$. Then for some $Z' \geq Z$, $Z' \vDash_v \neg\varphi$; we also have, by **P2**, that $\mathcal{R}(X, Z')$. By **R** there is $Y \geq X$ such that for all $Y' \geq Y$, there is $Z'' \geq Z'$ with $\mathcal{R}(Y', Z'')$. Then $Z'' \vDash_v \neg\varphi$ and so $Y' \nvDash_v \Box\varphi$ since $\mathcal{R}(Y', Z'')$. As Y' was an arbitrary refinement of Y, $Y \vDash_v \neg\Box\varphi$.

Finally, suppose that $X \nvDash_v (\forall x)\varphi$. Then there is a variable assignment w which agrees with v except possibly at x for which $X \nvDash_w \varphi$. Hence, for some $Y \geq X$, $Y \vDash_w \neg\varphi$. For all $Z \geq Y$, $Z \nvDash_w \varphi$; hence $Z \nvDash_v (\forall x)\varphi$ as witnessed by w. Thus $Y \vDash_v \neg(\forall x)\varphi$. □

Corollary 3 (Double Negation Elimination). *$X \vDash \varphi$ if and only if $X \vDash \neg\neg\varphi$*

Proof. Suppose $X \vDash \varphi$. Let $Y \geq X$. Then $Y \nvDash \neg\varphi$, since by persistence $Y \vDash \varphi$; hence $X \vDash \neg\neg\varphi$.

On the other hand, suppose $X \not\models \varphi$. Then, for some $Y \geq X$, $Y \models \neg\varphi$ by refinability. Hence $X \not\models \neg\neg\varphi$. □

2.5. **Soundness and Completeness.** Humberstone showed that validity according to his propositional models is the same as provability in the modal logic **K**. For us, the same is true for validity according to our constant-domain models and provability in the constant-domain quantified modal logic based on **K**. The logic we use is the simple quantified modal logic from [LZ94]. We follow Linsky and Zalta by calling this logic *QML*. Unless otherwise stated, by $\vdash$ we mean provable in *QML*. The axioms of *QML* are as follows:

(1) All propositional tautologies.
(2) The K axiom:
$$\Box(\varphi \Rightarrow \psi) \Rightarrow [\Box\varphi \Rightarrow \Box\psi].$$
(3) Axioms of classical quantification:
$$(\forall x)[\psi \Rightarrow \varphi(x)] \Rightarrow [\psi \Rightarrow (\forall x)\varphi(x)]$$
where x does not appear freely in ψ, and
$$(\forall x)\varphi \Rightarrow \varphi$$
where x does not appear freely in φ.
(4) Axioms for equality:
$$x = x.$$
$$x = y \Rightarrow [P(\cdots, x, \cdots) \iff P(\cdots, y, \cdots)].$$
$$(x = y) \Rightarrow \Box(x = y).$$
$$x \neq y \Rightarrow \Box(x \neq y).$$
(5) The Barcan formula:
$$(\forall x)\Box\varphi \Rightarrow \Box(\forall x)\varphi.$$

We also have three rules:

Modus Ponens:
$$\frac{\varphi \quad \varphi \Rightarrow \psi}{\psi}.$$
Universal Generalization:
$$\frac{\varphi(x)}{(\forall x)\varphi(x)}.$$
Necessitation Rule:
$$\frac{\varphi}{\Box\varphi}.$$

Lemma 4 (Soundness and Completeness)**.** *For any formula φ in $\mathcal{L}$, φ is valid over first-order possibility models if and only if it is provable in QML.*

Proof. As noted at the end of §2.1, every first-order total world model is a first-order possibility model. So any formula which is valid over first-order possibility models is valid over first-order total world models. Since *QML* is complete with respect to first-order total world models, it is also complete with respect to first-order possibility models.

Now we need to check that the axioms above are valid with respect to our semantics and that the three rules preserve validity. Note that if in some model, every possibility X with $X \models \varphi$ has $X \models \psi$, then every possibility satisfies $\varphi \to \psi$.

Checking that each axiom is sound for our models is tedious but straightforward; the previous remark simplifies the verifications. □

2.6. **Functions and Constants.** So far, we have not included functions or constants in the language. We will now show how to add them. A constant is just a $\varnothing$-ary function, so it suffices to just talk about functions. The difficulty is in how to treat partiality of a function. At a particular possibility, we would want to be able to satisfy a sentence like $\eta(a) \neq b$ without having to select a particular value for η. Also, the sentence saying that η is a function should be valid. So we already want to treat functions and constants like relations (the relation being the function's graph) because we want to be able to decide the different possible equalities and inequalities of the function's value at some possibility separately. However, we would not want to say that functions are *just* relations, because we want the sentence which says that a particular relation is the graph of a function to be valid; moreover, we want to be able to build up terms using the functions.

If η is a function symbol, let $a(\eta)$ be its arity. An interpretation $\mathcal{I}$ now assigns to each possibility X and function symbol η a partial function $\mathcal{I}(X,\eta)\colon \mathcal{D}^{a(\eta)} \to \mathcal{D}$. We write $\mathcal{I}(X,\eta)(\bar{c}) = d$ if the partial function is defined at $\bar{c}$ and equals d, $\mathcal{I}(X,\eta)(\bar{c}) \neq d$ if the partial function has been determined not to take $\bar{c}$ to d, and $\mathcal{I}(X,\eta)(\bar{c})\uparrow d$ if it has not been determined whether or not the partial function takes $\bar{c}$ to d. We extend **Refinement** and **Persistence** in the natural way. We also add two conditions to our models to ensure that the function symbols are actually interpreted as functions:

- **Function Totality:** For any $X \in \mathcal{W}$ and $\bar{a} \in \mathcal{D}^{a(\eta)}$, there is $Y \geq X$ and $b \in \mathcal{D}$ such that $\mathcal{I}(Y,\eta)(\bar{a}) = b$.
- **Function Uniqueness:** For any $X \in \mathcal{W}$, $\bar{a} \in \mathcal{D}^{a(\eta)}$, and $b_1, b_2 \in \mathcal{D}$, if $\mathcal{I}(Y,\eta)(\bar{a}) = b_1$ and $\mathcal{I}(Y,\eta)(\bar{a}) = b_2$ then $b_1 = b_2$.

For a constant symbol c, we write $\mathcal{I}(X,c) = d$, $\mathcal{I}(X,c) \neq d$, and $\mathcal{I}(X,c)\uparrow d$ and have similar conditions. One can now talk about functions and constants in a relational language, by talking about their graphs. For example, to say that $f(g(x)) = y$, we can instead write $(\exists z)[g(x) = z \wedge f(z) = y]$.

Another approach, which we will now take, is to define the interpretations of terms. The terms in our language are now built up using the standard recursive definition:

$$t ::= x \mid c \mid \eta(t_1, \ldots, t_{a(\eta)})$$

where x is a variable symbol, c is a constant symbol, and η is a function symbol.

Given a possibility X and a variable assignment v, the interpretation $\mathcal{I}(X,t)$ of a term t at a possibility X is a set of objects which t might denote. (In the case when t is a constant term, this is an abuse of notation, but it will not cause any issues.) If t is a variable symbol x, then we define $\mathcal{I}(X,t) = \{v(x)\}$. If t is a constant symbol c, then we define

$$\mathcal{I}(X,t) = \{a \in \mathcal{D} : \text{ there is } Y \geq X \text{ with } \mathcal{I}(Y,c) = a\}.$$

Finally, if t is $f(s_1, \ldots, s_n)$ where $s_1, \ldots, s_n$ are terms and f is a function symbol, then we define

$$\mathcal{I}(X,t) = \{a \in \mathcal{D} : \text{ there are } Y \geq X \text{ and } \bar{b} \text{ with } \mathcal{I}(Y,s_i) = b_i \text{ and } \mathcal{I}(Y,f)(\bar{b}) = a\}.$$

We can define $\mathcal{I}(X,t) = a$ if $\mathcal{I}(X,t) = \{a\}$, $\mathcal{I}(X,t) \neq a$ if $a \notin \mathcal{I}(X,t)$, and $\mathcal{I}(X,t)\uparrow a$ if $a \in \mathcal{I}(X,t)$ but $\mathcal{I}(X,t)$ has more than one element.

Then $\mathcal{M}, X \vDash_v t_1 = t_2$ if for all $Y \geq X$, $\mathcal{I}(X, t_1) = \mathcal{I}(X, t_2)$. This agrees with the truth conditions one gets by translating into a relational language. Note that it is not enough to have $\mathcal{I}(X, t_1) = \mathcal{I}(X, t_2)$, as it might be, for example, that $t_1 = c$ and $t_2 = d$ are both constants, and there are refinements $Y \geq X$ where $\mathcal{I}(Y, c) = a$ and $\mathcal{I}(Y, d) = b$, and $Z \geq X$ where $\mathcal{I}(Z, c) = b$ and $\mathcal{I}(Z, d) = a$. Then $\mathcal{I}(X, c) = \mathcal{I}(X, d) = \{a, b\}$, but there are refinements $X' \geq X$ where $\mathcal{I}(X', c) \neq \mathcal{I}(X', d)$.

2.7. **Varying-Domain Models and Actualist Quantification.** The models we defined in Section 2.1 were constant-domain models, but we want to be able to deal with varying domains. In varying-domain models with total worlds, each world is assigned a domain. But a possibility should not have to completely decide its domain. At a possibility, some objects should be determined to exist, some objects should be determined to not exist, and the existence of other objects should be indeterminate. We will adapt a standard way of interpreting varying-domain models inside constant-domain models using an existence predicate.

We assume that among the relation symbols is a distinguished unary relation symbol E of arity one; we interpret $\mathsf{E}(a)$ as saying that a exists, allowing that it is contingent which objects exist, so the truth of $\mathsf{E}(a)$ may vary across possibilities. We add a condition to our models to ensure that at least one object exists.

Existence: For any X in $\mathcal{W}$, there is an $a \in \mathcal{D}$ and a refinement $Y \geq X$ such that $a \in \mathcal{I}(Y, \mathsf{E})$.

The predicate E, using our standard treatment of relations, allows a possibility to have exactly the kind of indeterminateness about which objects exist that we want.

The quantification we have already defined is *possibilist*; that is, the quantifiers range over every thing that does exist and also every thing that could possibly exist. We can define *actualist* quantifiers over only those objects which actually exist by writing $(\forall^{\mathsf{E}} x)\varphi$ for $(\forall x)[\mathsf{E}(x) \Rightarrow \varphi]$ and $(\exists^{\mathsf{E}} x)\varphi$ for $(\exists x)[\mathsf{E}(x) \wedge \varphi]$. By soundness and completeness, we could also have defined $(\exists^{\mathsf{E}} x)\varphi$ by $\neg(\forall^{\mathsf{E}} x)\neg\varphi$. The *actualist fragment* of our logic, $\mathcal{L}_A$, is the fragment where we allow only actualist quantifiers.

There are other possible approaches we could have taken for the domains in our models. We could also have assigned to each possibility a domain of objects which exist at that possibility and replaced our quantification by actualist quantification. If $Y \geq X$ are possibilities, then the domain assigned to Y would have to include that assigned to X. An object would only be determined not to exist at a possibility if it does not show up in the domain of any refinement. There is no essential difference between this and our actualist quantification.

Humberstone [Hum81, p. 331] says that his

> preferred answer to [the question of how to define universal quantification] is that the truth-condition should involve quantification not only over such objects as belong to the domain of the possibility in question, but over those inhabiting any of its refinements.

This statement should probably be taken to refer to the actualist quantifier $\forall^{\mathsf{E}}$. By writing out the derived truth conditions for the actualist quantifiers, we will see that the truth conditions correspond exactly with Humberstone's preferred solution.

Lemma 5.

(1) *$\mathcal{M}, X \vDash_v (\forall^{\mathsf{E}} x)\varphi$ if and only if for all $X' \geq X$ and for all variable assignments w which agree with v except possibly at x, with $w(x) \in \mathcal{I}(X', \mathsf{E})$, $\mathcal{M}, X' \vDash_w \varphi$.*

(2) $\mathcal{M}, X \vDash_v (\exists^{\mathsf{E}} x)\varphi$ *if for all* $Y \geq X$*, there is a variable assignment* w *which agrees with* v *except possibly at* x *and some* $Z \geq Y$ *with* $w(x) \in \mathcal{I}(Z, \mathsf{E})$ *such that* $\mathcal{M}, Z \vDash_w \varphi$*.*

Note that the truth conditions for $\exists^{\mathsf{E}}$ are the same as for $\exists$, except that we require $w(x) \in \mathcal{I}(Z, \mathsf{E})$.

Proof. (1) ($\Rightarrow$). Suppose that $\mathcal{M}, X \vDash_v (\forall^{\mathsf{E}} x)\varphi$. Then by definition $\mathcal{M}, X \vDash_v (\forall x)[\neg \mathsf{E}(x) \vee \varphi]$. Fix $X' \geq X$ and w an x-variant of v with $w(x) \in \mathcal{I}(X', \mathsf{E})$. Suppose, towards a contradiction, that $\mathcal{M}, X' \nvDash_w \varphi$. Then there is $X'' \geq X'$ with $\mathcal{M}, X'' \vDash_w \neg\varphi$. Then for all $X''' \geq X''$, we have neither $w(x) \notin \mathcal{I}(X''', \mathsf{E})$ nor $\mathcal{M}, X''' \vDash_w \varphi$. This contradicts $\mathcal{M}, X \vDash_v (\forall x)[\neg \mathsf{E}(x) \vee \varphi]$. Hence $\mathcal{M}, X' \vDash_w \varphi$ as desired.

($\Leftarrow$). Now suppose that for all $X' \geq X$ and for all w which agree with v except possibly at x, with $w(x) \in \mathcal{I}(X', \mathsf{E})$, $\mathcal{M}, X' \vDash_w \varphi$. Fix w an x-variant of v. Given $X' \geq X$, we must find $X'' \geq X'$ such that either $\mathcal{M}, X'' \vDash_w \varphi$ or $w(x) \notin \mathcal{I}(X', \mathsf{E})$. If there is no $X'' \geq X'$ with $w(x) \notin \mathcal{I}(X'', \mathsf{E})$, then by **Refinement**, $w(x) \in \mathcal{I}(X', \mathsf{E})$. Hence $\mathcal{M}, X' \vDash_w \varphi$ as desired.

(2) ($\Rightarrow$). Suppose that $X \vDash_v (\exists^{\mathsf{E}} x)\varphi$. Then for all $X' \geq X$, $X \nvDash_v (\forall^{\mathsf{E}} x)\neg\varphi$. By (1), for all $X' \geq X$, there is $X'' \geq X'$ and an x-variant w of v with $w(x) \in \mathcal{I}(X'', \mathsf{E})$ and $\mathcal{M}, X'' \nvDash_w \neg\varphi$. By Refinability and Double Negation Elimination, there is $X''' \geq X''$ with $\mathcal{M}, X''' \vDash_w \varphi$.

($\Leftarrow$). Suppose that for all $Y \geq X$, there is a variable assignment w which agrees with v except possibly at x and some $Z \geq Y$ with $w(x) \in \mathcal{I}(Z, \mathsf{E})$ such that $\mathcal{M}, Z \vDash_w \varphi$. Fix $X' \geq X$. Let $X'' \geq X'$ and let w be an x-variant of v with $w(x) \in \mathcal{I}(X'', \mathsf{E})$ and $\mathcal{M}, X'' \vDash_w \varphi$. Then by (1), $\mathcal{M}, X' \nvDash_v (\forall^{\mathsf{E}} x)\neg\varphi$. By Refinability, $\mathcal{M}, X \vDash \neg(\forall^{\mathsf{E}} x)\neg\varphi$, that is, $\mathcal{M}, X \vDash (\exists^{\mathsf{E}} x)\varphi$. ☐

2.8. **Indeterminate Objects.** One of the features of Humberstone's models is that they allow a formalization of the idea of a "belief possibility," namely a possibility in which exactly what an agent believes to be true is true, exactly what the agents believes to be false is false, and anything on which the agent is undecided is undecided by the possibility. This is a natural sort of object to represent belief from the first-person perspective of the agent, who cannot imagine every single total world which is consistent with their beliefs. Holliday [Hol14] has used Humberstone's models to define *functional* possibility models where the accessibility relation is replaced by a function which assigns to each possibility an agent's belief possibility. One could make a similar construction in the first-order case, but there are also similar phenomena which only show up in the first-order case. For example, suppose that our belief agent is a detective who has been investigating a murder, but has not yet solved it. He already knows some things about the murderer: that the murderer wears a cloak, that the murderer used a gun, and that the murderer was over six feet tall. Yet there are other things that he does not know, such as the identity of the murderer, or the murderer's favourite place to have lunch. Of course, the detective does not hold in his mind every possible suspect, all simultaneously. Instead, the detective probably imagines the murderer as a tall, cloaked figure holding a gun, perhaps with his face in shadow, or simply with some generic sort of face. This can be represented in the detective's belief possibility with a constant, say c, representing "the murderer." The detective's belief possibility would satisfy the statements which say that the object denoted by c wears a cloak, used a gun, or is over six feel tall. The belief possibility might also satisfy statements which say that c does not denote the detective's mother, because the detective was having dinner with her at the time of the murder. But there might be a number of suspects, for each of whom the belief possibility leaves open the possibility that c denotes them. At some refinements of the belief possibility, c denotes one suspect, while at other refinements, c denotes another. So c is a single constant whose denotation across possibilities has the properties which the agent believes the murderer to have but leaves undetermined those about which the agent is unsure.

3. On the Possibility of a Finitary Completeness Proof.

3.1. **Finitary Models, Internal Adjointness, and the Finite Existence Property.** One of Humberstone's original motivations for introducing possibility models was to give completeness proofs for various propositional modal logics in a finitary way, namely to construct a canonical model where every possibility is characterized by a finite set of formulas instead of a complete set of formulas. However, Humberstone does not give a proof and it seems that it would be impossible to do so with his condition $\mathbf{R}^{++}$: Holliday [Hol15, Fact B.2] shows that under a formal definition of a finitary model there are no non-trivial finitary models satisfying $\mathbf{R}^{++}$. Using the condition $\mathbf{R}$, Holliday [Hol14] is able to give a finitary canonical model construction for his functional possibility models. In this section, we will consider the question of whether such a construction is possible for our first-order models.

To begin, we will define what it means for a model to be finitary. Fix a logic L. First, suppose L is a propositional modal logic.

Definition 6 (L-finitary models for propositional modal logic)**.** $\mathcal{M} = (\mathcal{W}, \mathcal{R}, \leq, \mathcal{I})$ is L-finitary if for each $X \in \mathcal{W}$ there is a finite set Φ of sentences (or equivalently a single sentence) such that

$$\mathcal{M}, X \vDash \varphi \Longleftrightarrow \Phi \vdash_L \varphi.$$

When we have quantifiers in our language, we want the set Φ to determine which atomic facts are true about the elements of the domain $\mathcal{D}$. So we also need to have a variable assignment. A finite set of sentences Φ may not decide whether $x = y$ or $x \neq y$ for some variables x and y. Thus the possibility corresponding to Φ may have two extensions, at one of which $x = y$ and at the other $x \neq y$. But any single variable assignment determines whether or not $x = y$. So we must deal with partial variable assignments.

Definition 7. A partial variable assignments v is a partial function from variables to $\mathcal{D}$. It is a finite variable assignments if its domain is finite.

We adopt the natural notion of extensions of partial variable assignments.

Definition 8 (L-finitary models for quantified modal logic)**.** $\mathcal{M} = (\mathcal{W}, \mathcal{R}, \leq, \mathcal{D}, \mathcal{I})$ is L-finitary if for each $X \in \mathcal{W}$ there is a finite set Φ of formulas and a finite variable assignment v such that $\Phi \vdash_L \varphi$ if and only if for each total variable assignment w extending v, $\mathcal{M}, X \vDash_w \varphi$.

Note that if $\mathcal{M}$ is an L-finitary model, X is a possibility, and Φ and v are as above, then for each pair of variables x, y in the domain of v, $\Phi \vdash_L x = y$ or

$\Phi \vdash_L x \neq y$. Also, the interpretation of the relation symbols at X is completely determined by Φ and v (see Theorem 19).

We say that L has a finitary completeness proof if for each L-consistent formula φ, there is an L-finitary model $\mathcal{M}$ (and a variable assignment v in the quantified case) with a possibility X such that $\mathcal{M}, X \vDash_v \varphi$.

To begin, we will touch briefly on Holliday's functional possibility models for propositional modal logic. A functional possibility model replaces the accessibility relation $\mathcal{R}$ with a function f. Intuitively, $f(X)$ is the possibility Y which is refined by exactly those possibilities accessible from X. Thus we can turn a functional possibility model into a relational possibility model by defining $\mathcal{R}$ by

$$\mathcal{R}(X, Y) \text{ if and only if } f(X) \leq Y.$$

We cannot necessarily turn a relational possibility model into a functional possibility model without adding any new possibilities, but the only barrier is that for some possibility X, there might not be a greatest lower bound to all of the possibilities accessible from X, i.e., no possibility Y with:

$$\{Z : Z \geq Y\} = \{Z : \mathcal{R}(X, Z)\}.$$

See [Hol14] for the semantics of functional possibility models. The definitions are the natural ones which make the correspondence described above work.

Holliday finds that the key property for a finitary completeness proof is the following:

Definition 9. We say that a modal logic L has *internal adjointness* if whenever Γ is a finite set of sentences, there is a finite set Ψ of sentences such that for all φ:

$$\Gamma \vdash_L \Box\varphi \iff \Psi \vdash_L \varphi.$$

L may be either a propositional or first-order modal logic.

Holliday [Hol14] shows that a number of standard modal propositional logics have internal adjointness, and hence admit a finitary completeness proof. The idea behind the completeness proof is that the possibilities correspond to finite, consistent sets of sentences (modulo being equivalent sets of sentences under provability). A possibility corresponding to the finite set Φ refines a possibility corresponding to Ψ if and only if Ψ is a set of consequences of Φ. The accessibility function is the map which takes a possibility Γ to the possibility Ψ given by internal adjointness. This corresponds (if we consider the model as a relational possibility model) to having the possibility corresponding to Φ related by the accessibility relation $\mathcal{R}$ to the possibility corresponding to Ψ if and only if for all formulas α,

$$\Phi \vdash \Box\alpha \Longrightarrow \Psi \vdash \alpha.$$

If we are interested in constructing a relational possibility model, rather than a functional possibility model, then what we need is something weaker than internal adjointness:

Definition 10. L has the finite existence property if for each consistent finite set Φ of sentences and ψ such that $\Phi \nvdash_L \Box\psi$, there is a finite set Ψ such that $\Psi \nvdash_L \psi$ and for all φ:

$$\Phi \vdash_L \Box\varphi \Longrightarrow \Psi \vdash_L \varphi.$$

We will show that if L does not have the finite existence property, then L is not complete with respect to finitary models.

Proposition 11. *Let L be a propositional or first-order modal logic. If L does not have the finite existence property, then there is an L-consistent sentence φ which is not true in any finitary model.*

Proof. Let Φ and ψ be a witness to the fact that $\mathcal{L}$ does not have the finite existence property, that is, Φ is a consistent finite set of sentences and ψ is a sentence, such that $\Phi \nvdash_L \Box\psi$, but for all finite sets Ψ, if $\Psi \nvdash_L \psi$ then there is φ such that $\Phi \vdash_L \Box\varphi$ and $\Psi \nvdash_L \varphi$. So $\Phi \wedge \neg\Box\psi$ is consistent.

Suppose to the contrary that there is an L-finitary model $\mathcal{M}$ and a possibility X such that $\mathcal{M}, X \vDash \Phi \wedge \neg\Box\psi$. So there is a possibility Y with $X\mathcal{R}Y$ such that $\mathcal{M}, Y \vDash \neg\psi$. Let Ψ be such that $\Psi \vdash_L \varphi$ if and only if $\mathcal{M}, Y \vDash \varphi$. Then by choice of Φ and ψ, there is φ such that $\Phi \vdash_L \Box\varphi$ and $\Psi \nvdash_L \varphi$. Thus $\mathcal{M}, X \vDash \Box\varphi$, and since $X\mathcal{R}Y$, $\mathcal{M}, Y \vDash \varphi$. This contradicts the choice of Φ. $\Box$

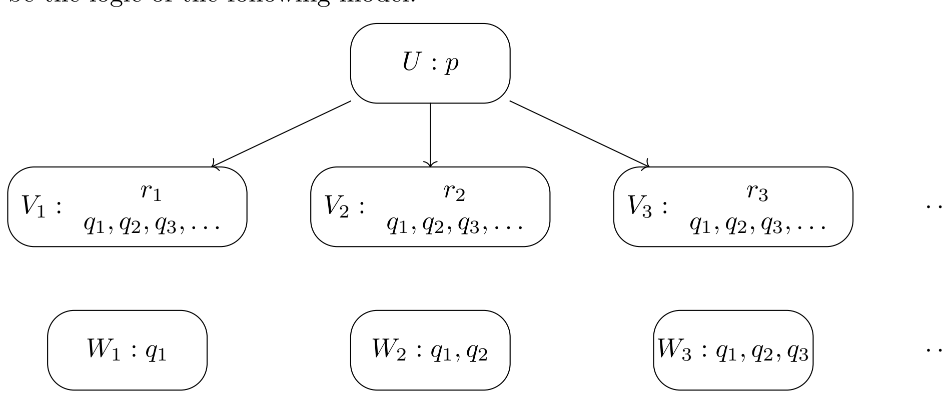
We will begin the rest of Section 3 by showing that the finitary existence property is in fact weaker than internal adjointness, by exhibiting a logic which has the finitary existence property but does not have internal adjointness. Then we will show that *QML* does not have the finitary existence property. Then by Proposition 11, *QML* is not complete with respect to finitary models.

There are examples from [Hol15] of logics that do not have a finitary completeness proof, but which do after expanding the language. Our counter-example to the finitary existence property fails when we add the quantifiers "there exist infinitely many" and "for all but finitely many" so we give a second counterexample—though only to internal adjointness—which works even for this expanded language. We leave open the question of whether *QML* has the finitary existence property after expanding the language in this way.

3.2. **Comparing Internal Adjointness and the Finitary Existence Property.** We will sketch a proof that it is possible to construct a normal modal logic which has the finitary existence property, but not internal adjointness. At first we will just give an example of a theory of a model which does not have the finitary existence property, and then we will adapt this example to get a logic of a frame which does not have the finitary existence property.

Theorem 12. *There is a Kripke model $\mathcal{M}$ such that the set of formulas globally true in $\mathcal{M}$ has the finitary existence property but not internal adjointness.*

Proof. Our language will have proposition symbols p, $(r_i)_{i\in\mathbb{N}}$, and $(q_i)_{i\in\mathbb{N}}$. Let L be the logic of the following model:



First, we claim that L does not have internal adjointness. We claim that there is no finite set Φ of sentences such that $p \vdash_L \Box\varphi$ if and only if $\Phi \vdash_L \varphi$. Suppose to the contrary that there was such a finite set Φ. Since Φ contains only finitely many sentences, there is some n such that Φ involves only p, $q_1, \ldots, q_n$, and $r_1, \ldots, r_n$. Note that for every $\varphi \in \Phi$, $p \vdash_L \Box\varphi$. So $\Box\varphi$ is true at U. Hence Φ is true at each of the worlds V_i. In particular, Φ is satisfied at the world V_{n+1}. Then it is also satisfied at the world W_n, since they satisfy exactly the same sentences involving p, $q_1, \ldots, q_n$, and $r_1, \ldots, r_n$. But q_{n+1} is not satisfied at W_n, and so $\Phi \nvdash_L q_{n+1}$. This is a contradiction, as $p \vdash_L \Box q_{n+1}$ (as U is the only world satisfying p) and hence $\Phi \vdash_L q_{n+1}$.

Now we claim that L has the finite existence property. Let Φ be a consistent set of sentences and ψ a sentence such that $\Phi \nvdash_L \Box\psi$. Note that Φ must be satisfied at U and $\neg\psi$ must be satisfied at one of the worlds V_m (since every world other than U has no successors under the accessibility relation). Fix this m. We claim that the sentence r_m witnesses the finite existence property for Φ. We have that $r_m \nvdash_L \psi$ since both r_m and $\neg\psi$ are satisfied at V_m. Now suppose that $\Phi \vdash_L \Box\varphi$. We must show that $r_m \vdash_L \varphi$. Since Φ is satisfied at U, φ is satisfied at each world W_i. It is not hard to see that if φ is satisfied at each world W_i, then $\{q_1, q_2, \ldots\} \vdash_L \varphi$. Since $r_m \vdash_L q_i$ for each i, we are done. □

This example just gives the theory of a model $\mathcal{M}$. Let $\mathcal{F}$ be the frame underlying $\mathcal{M}$. We will modify the example to get the logic of a frame $\mathcal{G}$ which has the finitary existence property but not internal adjointness.

Theorem 13. *There is a normal modal logic—the logic of a frame—which has the finitary existence property but not internal adjointness.*

Associate to each proposition symbol s a pair of even numbers $\langle m_s, n_s\rangle$ without any repetition. Then modify the frame $\mathcal{F}$ so that at a node w where the proposition s corresponding to $\langle m_s, n_s\rangle$ held, instead there are now chains of related nodes of length exactly m_s and n_s. Thus the frame $\mathcal{G}$ includes the frame $\mathcal{F}$, but has more worlds. Then we can replace the proposition s by the statement "there are chains of related nodes of length exactly m_s and n_s" which can be expressed as

$$\varphi_s := \Diamond^{m_s}(\top \wedge \Box\bot) \wedge \Diamond^{n_s}(\top \wedge \Box\bot).$$

Since m_s and n_s are even, for a world w from $\mathcal{G}$ and a proposition symbol s, $\mathcal{G}, w \vDash \varphi_s$ (noting that this is well-defined even though $\mathcal{G}$ is a frame, as φ_s has no proposition variables) if and only if w is a world in the original frame $\mathcal{F}$ and $\mathcal{M}, w \vDash s$. Thus the logic of this frame has the finitary existence property but not internal adjointness.

3.3. **No Finitary Completeness Proof.** We begin by showing that *QML* does not have the finitary existence property, and hence no finitary completeness proof.

Proposition 14. *QML does not have the finitary existence property.*

Proof. We prove the proposition by exhibiting a consistent finite set Γ of sentences such that there is no finite set Φ of sentences such that for all formulas φ:

$$\Gamma \vDash \Box\varphi \Longrightarrow \Phi \vDash \varphi.$$

The language for our example will be $\{0, \leq, S, P, R\}$ where 0 is a constant, $\leq$, S, and P are binary predicates, and R is a unary predicate. The set Γ will consist of the following sentences:

(1) $\leq$ is a total linear order,
(2) S is a total function which is a successor function relative to $\leq$,
(3) P is a total function which is a predecessor function relative to $\leq$,
(4) for all $x < 0$, $R(x)$ holds, and for all $x > 0$, $R(x)$ does not hold,
(5) at all accessible worlds, $\leq$, S, and P are interpreted as the empty set,
(6) for all x, $R(x)$ if and only if $\Box R(x)$,
(7) $\Box\Box\bot$, and
(8) $\Diamond\top$.

Note that Γ is clearly consistent. Suppose towards a contradiction that there is a finite set Φ such that for all φ, if $\Gamma \vDash \Box\varphi$ then $\Phi \vDash \varphi$.

First, we claim that for each formula φ, there is a modal-free formula ψ_φ which involves only the restricted language $\{0, R\}$ such that $\Gamma \vDash \Box(\varphi \iff \psi_\varphi)$. Let χ be the sentence which says that $\leq$, S, and P are the empty set, and $\Box\bot$. So $\Gamma \vDash \Box\chi$. We will have that $\chi \vDash \varphi \iff \psi_\varphi$. We define ψ_φ by induction on the complexity of φ:

(1) if φ is an atomic formula involving $\leq$, S, or P, then ψ_φ is $\bot$;
(2) if φ is an atomic formula involving R or $=$, then ψ_φ is φ;
(3) if φ is $\neg\phi$, then ψ_φ is $\neg\psi_\phi$;
(4) if φ is $\phi_1 \wedge \phi_2$, then ψ_φ is $\psi_{\phi_1} \wedge \psi_{\phi_2}$;
(5) if φ is $(\forall\alpha)\phi$ then ψ_φ is $(\forall\alpha)\psi_\phi$; and
(6) if φ is $\Box\phi$, then ψ_φ is $\top$.

It is easy to see that this definition satisfies the properties stated above. Thus, we can replace Φ, modulo χ, by a set Φ^* of modal-free formulas which involve only the restricted language $\{0, R\}$. Note that if φ is a sentence with no free variables, then so is ψ_φ.

Now note that $\Gamma \vDash (\exists_{\geq n} x)R(x)$ and $\Gamma \vDash (\exists_{\geq n} x)\neg R(x)$ for each n, because Γ entails that there are infinitely many elements greater than and infinitely many elements less than zero. But we know that Γ says that, for each x, $R(x)$ holds if and only if $\Box R(x)$ holds. So $\Gamma \vDash \Box(\exists_{\geq n} x)R(x)$ and $\Gamma \vDash \Box(\exists_{\geq n} x)\neg R(x)$ for each n. So the set $\{\varphi : \Gamma \vDash \Box\varphi\}$ contains, for each n, $(\exists_{\geq n} x)R(x)$ and $(\exists_{\geq n} x)\neg R(x)$.

Thus $\{\chi\} \cup \Phi^*$ entails each of these formulas $(\exists_{\geq n} x)R(x)$ and $(\exists_{\geq n} x)\neg R(x)$. Now we claim that Φ^* by itself entails each of these formulas (in first-order logic). Suppose that Φ^* does not entail one of these formulas for some m, and call that formula ϕ. Then there is a first-order model of $\Phi^* \cup \{\neg\phi\}$ in the language $\{0, R\}$. Now extend this first-order model to the language $\{0, R, \leq, S, P\}$ by setting the new symbols to be interpreted as the empty set, and make it into a total world model with a single world and empty accessibility relation. Then this total world model is still a model of Φ^* and $\neg\phi$, but it is also a model of χ. This is a contradiction since $\{\chi\} \cup \Phi^* \vDash \phi$. So Φ^* entails, for each n, $(\exists_{\geq n} x)R(x)$ and $(\exists_{\geq n} x)\neg R(x)$.

So we are left with the question of what sort of things we can say in the language $\{0, R\}$ in first-order logic using finitely many formulas (or equivalently, a single formula). Since R is unary, there is not much that we can say; indeed, it is well-known that this language has the finite model property: if there is a model of a formula φ, then there is a finite model of that formula. No finite model can satisfy, for all n, both $(\exists_{\geq n} x)R(x)$ and $(\exists_{\geq n} x)\neg R(x)$. Thus Φ^* does not entail all of these formulas. This is a contradiction. $\square$

Putting together Propositions 11 and 14, we have the following.

Corollary 15. *QML is not complete with respect to finitary models.*

3.4. **The quantifier "there exist infinitely many".** If we allow the quantifier $\exists^\infty x$ (there exist infinitely many x) then the counterexample to the finitary existence property from §3.3 is no longer a counterexample, as the two sentences $(\exists^\infty x)R(x)$ and $(\exists^\infty x)\neg R(x)$ are a possible choice for Φ. However, there is a more complicated counterexample even for formulas involving $\exists^\infty$, but we can only show that it is a counterexample for internal adjointness.

Proposition 16. *QML lacks internal adjointness in the expanded language including $\exists^\infty$.*[2]

Proof. Once again the proof is to construct a counterexample. The language is $\{0, 1, +, \cdot, <, F\}$ where 0 and 1 are constants, $+$ and $\cdot$ are binary functions, $<$ is a binary predicate, and F is a unary predicate. The set Γ consists of the following sentences:

(1) $(F, 0, 1, +, \cdot, <)$ is an ordered field on the domain F,
(2) at all accessible worlds, $<$ is empty,
(3) at all accessible worlds, $(F, 0, 1, +, \cdot)$ is a field,
(4) the field at any accessible world is a field extension of the field at this world,
(5) $\Box\Box\bot$, and
(6) $\Diamond\top$.

Now suppose towards a contradiction that there is a formula ψ, possibly involving the quantifier $\exists^\infty$, such that for all formulas φ:

$$\Gamma \vDash \Box\varphi \iff \psi \vDash \varphi.$$

As before, due to the fact that Γ includes $\Box\Box\bot$, $\Diamond\top$, etc., there is a modal-free formula ψ^* in the language of fields (with the quantifier $\exists^\infty$) which exactly axiomatizes those fields which are extensions of an ordered field. Note that an ordered field has characteristic zero, and that $\mathbb{Q}$ is an ordered field which is also the prime field in characteristic zero. Hence the fields which extend an ordered field are exactly the fields of characteristic zero.

So the models of ψ^* are exactly the fields of characteristic zero. We know that this is not possible if ψ^* is a standard formula of first-order logic, but we must show that it is impossible even though ψ^* is allowed to include the quantifier $\exists^\infty$. The strategy will be to reduce ψ^* to a formula of standard first-order logic using the elimination of $\exists^\infty$ in a strongly minimal model.

Consider the theory ACF of algebraically closed fields of any characteristic (whereas ACF_p, for p a prime or zero, is the theory of algebraically closed fields of characteristic p). See, for example, [Mar02] for some facts about the theory ACF. Since there are algebraically closed fields of characteristic zero, ψ^* is consistent with ACF. Moreover, $\mathrm{ACF} \cup \{\psi^*\} \vDash \mathrm{ACF}_0$. Although ACF is not a complete theory, it has quantifier elimination. We claim that, modulo ACF, ψ^* is equivalent to a formula in which $\exists^\infty$ does not appear. It suffices to show that a formula of the form $(\exists^\infty x)\varphi(x, \bar{a})$ (with φ containing only standard first-order quantifiers) is equivalent, modulo ACF, to one with only standard quantifiers (and even to a quantifier-free

[2]Technically we have not defined such a logic, but one can easily define the logic using the obvious semantics for $\exists^\infty$.

formula); by repeating this argument, we may eliminate[3] all of the quantifiers $\exists^\infty$ and $\forall^\infty$. Now modulo ACF, $\varphi(x,\bar{y})$ is equivalent to a quantifier-free formula of the form

$$(\bigwedge_{i=1}^{n} f_i(x,\bar{y}) = 0) \wedge g(x,\bar{y}) \neq 0.$$

There are infinitely many solutions to this formula if and only if each $f_i(x,\bar{y})$ is identically zero as a polynomial in x, and $g(x,\bar{y})$ is not identically zero. This is expressible in a quantifier-free way in terms of the variables $\bar{y}$. So $(\exists^\infty x)\varphi(x,\bar{y})$ is equivalent to a quantifier-free formula in the free variable $\bar{y}$. Hence we obtain a formula ψ^{**} which uses only standard quantifiers, or even one which is quantifier-free, such that $\mathrm{ACF} \cup \{\psi^{**}\} \vDash \mathrm{ACF}_0$. This is impossible by standard compactness arguments.

Thus we have a contradiction. So there is no formula ψ, even allowing the quantifier $\exists^\infty$, such that for all φ,

$$\Gamma \vDash \Box\varphi \Longleftrightarrow \psi \vDash \varphi.$$

This completes the proof. □

This is only a counterexample to internal adjointness, and not to the finite existence property, because we used the fact that the models of ψ are *exactly* the fields of characteristic zero to see that $\mathrm{ACF} \cup \{\psi^*\}$ is consistent. We leave open the question of whether this expanded language has the finitary existence property.

Question. Does Proposition 16 hold for the finitary existence property?

4. Canonical Models with Computable Possibilities.

The previous examples show that it is impossible to run Holliday's construction for the first-order case, and we cannot make his functional canonical model construction even when we allow the quantifier $\exists^\infty$. Holliday [Hol14] showed that even in the propositional case, there are logics which lack internal adjointness: **K5**, **K45**, **KD5**, and **KD45**. As observed in Section 7.3 of [Hol15], one way around these results is to pass to the minimal tense extension of a logic, which automatically has the property of internal adjointness. Then one can construct a canonical model for the original logic such that every possibility is generated by a finite set of sentences in the expanded tense language.

In this section, we will consider another way way to recover a finitary construction without changing the underlying language. We will show that *QML* has internal adjointness if we replace "finite" in the definition by "computable." (In fact, the proof is quite general and would work for many other logics as well.) See [Soa87] for an introduction to computability theory; we will only use the most basic concepts. For this section, we assume that the language is computable.

Proposition 17. *Let Γ be a computable set of sentences of QML. Then there is a computable set Φ of sentences such that for all sentences α,*

$$\Gamma \vdash \Box\alpha \Longleftrightarrow \Phi \vdash \alpha.$$

[3]This is a property which the theory ACF has by virtue of being strongly minimal. We say that ACF eliminates the quantifier $\exists^\infty$. We will not use this fact directly.

Proof. We can enumerate the proofs from Γ and collect in a computably enumerable set Ψ the sentences α such that $\Gamma \vdash \Box\alpha$. Then it follows that

$$\Gamma \vdash \Box\alpha \iff \Psi \vdash \alpha$$

where the right-to-left direction uses Necessitation and the K axiom for $\Box$.

It is a standard fact that for every computably enumerable set of sentences, there is a computable set of sentences with the same consequences. Let $\psi_0, \psi_1, \ldots$ be a computable enumeration of Ψ. Then let $\hat{\psi}_0$ be ψ_0. Let $\hat{\psi}_1$ be the conjunction of ψ_1 with itself sufficiently many times to ensure that $\hat{\psi}_1$ has Gödel number greater than that of $\hat{\psi}_0$. Continue in this way to choose $\hat{\psi}_2$ with Gödel number greater than that of $\hat{\psi}_1$ and so on. Let Φ be the set $\{\hat{\psi}_i : i \in \omega\}$. Since there is an increasing computable enumeration of Φ, it is a computable set. $\Box$

A computable set may not be finite, but it does contain only finitely much information. There is a procedure with a finite description which can list all of the sentences in a computable set. So, just as a formula such as $(\forall x)\varphi(x)$ should be regarded as finitary in the sense that it describes, in a finite way, the infinitely many facts $\varphi(a)$ for particular objects a, a computable set of formulas can also be regarded as finitary. There is a difference in that $(\forall x)\varphi(x)$ is equivalent to infinitely many facts in a schematic way—each of the infinitely many facts is just a substitution of some particular element for x. But a computable set of formulas is still finitely specified.

We generalize the finitary models of Definition 8 to allow the possibilities to be specified by a computable set of formulas.

Definition 18. Relative to a logic L, $\mathcal{M} = (\mathcal{W}, \mathcal{R}, \leq, \mathcal{D}, \mathcal{I})$ is a possibility model with computable possibilities if for each $X \in \mathcal{W}$ there is a computable set Φ of formulas and a finite variable assignment v such that $\Phi \vdash_L \varphi$ if and only if for each total variable assignment w extending v, $\mathcal{M}, X \vDash_w \varphi$.

Recall from the introduction that Hale [Hal13] suggested that possibilities should be finitely specifiable—that is, that they should have "a finite description." A finite algorithm describes each possibility of a model with computable possibilities, and so on Hale's view, a model with computable possibilities should be an acceptable possibility model. While *QML* is not complete with respect to finitary models, it is complete with respect to models with computable possibilities.

Theorem 19. *QML is complete with respect to possibility models with computable possibilities.*

Gangulia and Nerode [GN04] have shown that every decidable *QML* theory has a decidable Kripke model. Before proving Theorem 19, we will briefly describe the difference between our results and theirs. A decidable Kripke model is a Kripke model where the set of worlds, the object domain, and the accessibility relation are all computable, and truth at a world is decidable. A decidable theory is a computable set of sentences closed under logical consequence.

Asking that a model be decidable is a much stronger condition than asking that it have computable possibilities. On the other hand, there are sentences, such as the axioms for Robinson arithmetic, which are not contained within any decidable theory. So while every decidable theory has a decidable Kripke model, not every consistent sentence has a decidable Kripke model. Hence *QML* is not complete with respect to decidable models.

Proof of Theorem 19. We must show that for each consistent sentence γ, there is a model $\mathcal{M}$ with computable possibilities, a possibility X, and a variable assignment v such that $\mathcal{M}, X \vDash_v \gamma$.

Assume that γ is consistent with the existence of infinitely many elements. The case where γ is only consistent with the existence of finitely many elements is similar, though there are some small differences. Let $\mathcal{D}$ be a set of countably many elements. To keep the proof relatively short and focused only on the new elements, throughout the proof we will use many facts that are part of the standard completeness proofs.

Let $\mathcal{S}$ be the collection of pairs (Φ, v) where v is a finite variable assignment and Φ is a computable set of formulas such that:

(1) Φ is consistent,
(2) Φ mentions exactly the variables in the domain of v,
(3) $\Phi \vdash x = y$ or $\Phi \vdash x \neq y$ for each pair of variables x, y in the domain of v (depending on whether $v(x) = v(y)$ or $v(x) \neq v(y)$), and
(4) $\Phi \vdash \exists^{\geq n} x$ for each n.

We put an equivalence relation $\sim$ on $\mathcal{S}$ by setting $(\Phi, u) \sim (\Psi, v)$ if and only if $u = v$, $\Phi \vdash \Psi$, and $\Psi \vdash \Phi$. Let $\mathcal{W} = \mathcal{S}/\sim$. Define $(\Phi, u) \leq (\Psi, v)$ if $u \subseteq v$ and $\Psi \vdash \Phi$, and define $(\Phi, u)\mathcal{R}(\Psi, v)$ if $u \subseteq v$ and $\Psi \vdash \{\varphi : \Phi \vdash \Box\varphi\}$.

The following lemma allows us to extend any consistent computable set of formulas to form a possibility.

Lemma 20. *If Φ is a consistent computable (or even computably enumerate) set of formulas with finitely many free variables, and which is consistent with the existence of infinitely many elements, and v is a finite variable assignment such that if $\Phi \vdash x = y$ then $v(x) = v(y)$ if these are defined, and if $\Phi \vdash x \neq y$ then $v(x) \neq v(y)$ if these are defined. Then there are $\Phi' \vdash \Phi$ and $v' \supseteq v$ such that $(\Phi', v') \in \mathcal{S}$.*

Proof sketch. Suppose that Φ is computably enumerable, we can replace it by a computable set as in Theorem 17. Extend v to a finite variable assignment v' which contains in its domain all of the free variables of Φ, so that if $\Phi \vdash x = y$ then $v(x) = v(y)$, and if $\Phi \vdash x \neq y$ then $v(x) \neq v(y)$. Extend Φ to Φ' by adding, for each pair of variables x, y, $x = y$ if $v'(x) = v'(y)$, and $x \neq y$ if $v'(x) \neq v'(y)$. Also add to Φ the formulas $\exists^{\geq n} x$ for each n. This is consistent, and so $(\Phi', v') \in \mathcal{S}$. ☐

Note that the variables which appear in (Φ', v') are exactly the variables which appear in either Φ or v.

We must now define the interpretations of each of the relation symbols at each possibility. Let (Φ, v) be a possibility and P a relation symbol of arity n. Given $\bar{a} = (a_1, \ldots, a_n) \in \mathcal{D}$, set:

(1) $\bar{a} \in \mathcal{I}(X, P)$ if there is a variable assignment $w \supset v$ and a tuple of variables $\bar{y}$ with $w(\bar{y}) = \bar{a}$ and $\Phi \vdash P(\bar{y})$.
(2) $\bar{a} \notin \mathcal{I}(X, P)$ if there is a variable assignment $w \supset v$ and a tuple of variables $\bar{y}$ with $w(\bar{y}) = \bar{a}$ and $\Phi \vdash \neg P(\bar{y})$.
(3) $\bar{a} \uparrow \mathcal{I}(X, P)$ otherwise.

Lemma 21. *We cannot have both $\bar{a} \in \mathcal{I}(X, P)$ and $\bar{a} \notin \mathcal{I}(X, P)$ at a single possibility $X = (\Phi, v)$.*

Proof. Suppose that $w, w' \supset v$ and $\bar{y}, \bar{y}'$ are such that $w(\bar{y}) = w(\bar{y}') = \bar{a}$, $\Phi \vdash P(\bar{y})$, and $\Phi \vdash \neg P(\bar{y}')$. Write $\bar{y} = (y_1, \ldots, y_n)$, $\bar{y}' = (y_1', \ldots, y_n')$, and $\bar{a} = (a_1, \ldots, a_n)$.

Let $\bar{z}$ be the tuple of variables in $\bar{y}$ which do not appear freely in Φ, and similarly with $\bar{z}'$ and $\bar{y}'$. Then $\Phi \vdash (\forall \bar{z}) P(\bar{y})$ and $\Phi \vdash (\forall \bar{z}') P(\bar{y}')$. Note that if y_i and y_i' are both in the domain of v, then since w and w' extend v and $w(y_i) = a_i = w'(y_i')$, $\Phi \vdash y_i = y_i'$. Let $\bar{y}'' = (y_1'', \ldots, y_n'')$ be such that y_1'' is y_1 if y_1 is in the domain of v, and y_1' otherwise. Then $\Phi \vdash P(\bar{y}'')$ and $\Phi \vdash \neg P(\bar{y}'')$. This contradicts the consistency of Φ. ☐

We now give an alternate characterization of $\mathcal{I}$.

Lemma 22. *Let $X = (\Phi, v)$.*

(1) *$\bar{a} \in \mathcal{I}(X, P)$ if for any variable assignment $w \supset v$ and tuple of variables $\bar{y}$ with $w(\bar{y}) = \bar{a}$, $\Phi \vdash P(\bar{y})$.*
(2) *$\bar{a} \notin \mathcal{I}(X, P)$ if for any variable assignment $w \supset v$ and tuple of variables $\bar{y}$ with $w(\bar{y}) = \bar{a}$, $\Phi \vdash \neg P(\bar{y})$.*

Proof. For (1), suppose that $\bar{a} \in \mathcal{I}(X, P)$. Fix $w \supset v$ and a tuple of variables $\bar{y}$ with $w(\bar{y}) = \bar{a}$. Suppose to the contrary that $\Phi \nvdash P(\bar{y})$. Then $\Phi \cup \{\neg P(\bar{y})\}$ is consistent. A similar argument as for the previous lemma yields a contradiction. The proof of (2) is similar. ☐

Next we show that $\mathcal{M} = (\mathcal{W}, \mathcal{R}, \leq, \mathcal{D}, \mathcal{I})$ is a possibility model.

Lemma 23. *$\mathcal{M}$ satisfies **P1**, **P2**, and **R**.*

Proof. For **P1**, suppose that $X = (\Phi, u)$, $Y = (\Psi, v)$, and $Y' = (\Psi', v')$ are such that $Y' \geq Y$ and $X \mathcal{R} Y$. We claim that $X \mathcal{R} Y'$. Since $X \mathcal{R} Y$, $u \subseteq v$ and $\Psi \vdash \{\varphi : \Phi \vdash \Box\varphi\}$. Since $Y' \geq Y$, $v \subseteq v'$ and $\Psi' \vdash \Psi$. Thus $u \subseteq v'$ and $\Psi' \vdash \{\varphi : \Phi \vdash \Box\varphi\}$.

For **P2**, suppose that $X = (\Phi, u)$, $X' = (\Phi', u')$, and $Y = (\Psi, v)$ are such that $X' \geq X$ and $X' \mathcal{R} Y$. We claim that $X \mathcal{R} Y$. Since $X' \mathcal{R} Y$, $u' \subseteq v$ and $\Psi \vdash \{\varphi : \Phi' \vdash \Box\varphi\}$. Since $X' \geq X$, $u \subseteq u'$ and $\Phi' \vdash \Phi$. Thus $u \subseteq v$ and since $\{\varphi : \Phi \vdash \Box\varphi\} \subseteq \{\varphi : \Phi' \vdash \Box\varphi\}$, $\Psi \vdash \{\varphi : \Phi \vdash \Box\varphi\}$.

For **R**, suppose that $X = (\Phi, u)$ and $Y = (\Psi, v)$ are such that $X \mathcal{R} Y$. Consider the set $\Phi \cup \{\Diamond\varphi : \Psi \vdash \varphi\}$; this set is consistent since if not, $\Phi \vdash \neg\Diamond\alpha$ for some α with $\Psi \vdash \alpha$, that is, $\Phi \vdash \Box\neg\alpha$. (At first, we just get that for some $\alpha_1, \ldots, \alpha_n$ with $\Psi \vdash \alpha_1 \wedge \cdots \wedge \alpha_n$, $\Phi \vdash \neg\Diamond\alpha_1 \vee \cdots \vee \neg\Diamond\alpha_n$. But then $\Phi \vdash \Box\neg(\alpha_1 \wedge \cdots \wedge \alpha_n)$.) Since $X \mathcal{R} Y$, $\Psi \vdash \{\varphi : \Phi \vdash \Box\varphi\}$, and so $\Psi \vdash \neg\alpha$. Thus $\Psi \vdash \alpha \wedge \neg\alpha$, a contradiction. So $\Phi \cup \{\Diamond\varphi : \Psi \vdash \varphi\}$ is consistent, and by Lemma 20 there is a possibility $X' = (\Phi', v)$ with $\Phi' \supseteq \Phi \cup \{\Diamond\varphi : \Psi \vdash \varphi\}$. We have $X' \geq X$ since $u \subseteq v$ and $\Phi' \vdash \Phi$.

Given $X'' \geq X'$, we must find $Y' \geq Y$ such that $X'' \mathcal{R} Y'$. Write $X'' = (\Phi'', u'')$. Consider the computably enumerable set of formulas $\Psi \cup \{\varphi : \Phi'' \vdash \Box\varphi\}$. We claim that this set is consistent. If not, then $\Psi \vdash \neg\alpha$ for some α with $\Phi'' \vdash \Box\alpha$. (Again, at first, we get that there are $\alpha_1, \ldots, \alpha_n$ such that $\Psi \vdash \neg\alpha_1 \vee \cdots \vee \neg\alpha_n$ and $\Phi'' \vdash \Box\alpha_1 \wedge \cdots \wedge \Box\alpha_n$. But then $\Psi \vdash \neg(\alpha_1 \wedge \cdots \wedge \alpha_n)$ and $\Phi'' \vdash \Box(\alpha_1 \wedge \cdots \wedge \alpha_n)$.) Since $\Psi \vdash \neg\alpha$, $\Diamond\neg\alpha \in \Phi' \subseteq \Phi''$. But then Φ'' is inconsistent. Thus we conclude that $\Psi \cup \{\varphi : \Phi'' \vdash \Box\varphi\}$ is consistent. By Lemma 20 there is a possibility $Y' = (\Psi', u'')$ with $\Psi' \vdash \Psi \cup \{\varphi : \Phi'' \vdash \Box\varphi\}$. Then $Y' \geq Y$ since $v \subseteq u''$ and $\Psi' \vdash \Psi$, and $X'' \mathcal{R} Y'$ since $\Psi' \vdash \{\varphi : \Phi'' \vdash \Box\varphi\}$. ☐

Lemma 24. *$\mathcal{M}$ satisfies **Persistence** and **Refinability**.*

Proof. For **Persistence**, let $X = (\Phi, u)$ be a possibility and P a relation symbol of arity n. Let $Y = (\Psi, v)$ be such that $Y \geq X$. Given $\bar{a} = (a_1, \ldots, a_n) \in \mathcal{D}$, suppose that $\bar{a} \in \mathcal{I}(X,P)$. Then for every variable assignment $w \supset u$ and tuple of variables $\bar{y}$ with $w(\bar{y}) = \bar{a}$, $\Phi \vdash P(\bar{y})$. Since $\Psi \vdash \Phi$ and $u \subseteq v$, for every variable assignment $w \supset v$ and tuple of variables $\bar{y}$ with $w(\bar{y}) = \bar{a}$, $\Psi \vdash P(\bar{y})$. Hence $\bar{a} \in \mathcal{I}(X,P)$.

For **Refinability**, let $X = (\Phi, v)$ be a possibility and P a relation symbol of arity n. Suppose that $\bar{a} \uparrow \mathcal{I}(X,P)$. Fix a variable assignment $w \supset v$ and a tuple of variables $\bar{y}$ with $w(\bar{y}) = \bar{a}$. Then $\Phi \nvdash P(\bar{y})$ and $\Phi \nvdash \neg P(\bar{y})$. Thus $\Phi_1 = \Phi \cup \{P(\bar{y})\}$ and $\Phi_2 = \Phi \cup \{\neg P(\bar{y})\}$ are consistent. Let w' be a finite variable assignment, $v \subseteq w' \subset w$, which contains in its domain all of the variables $\bar{y}$. By Lemma 20, there are $\Phi_1' \supseteq \Phi_1$ and $\Phi_2' \supseteq \Phi_2$ such that $Y = (\Phi_1', w')$ and $Z = (\Phi_2', w')$ are possibilities with $X \leq Y, Z$. Moreover, $\bar{a} \in \mathcal{I}(Y,P)$ and $\bar{a} \notin \mathcal{I}(Z,P)$. □

Finally, we prove the Truth Lemma:

Lemma 25 (Truth Lemma)**.** *Let $X = (\Phi, u)$ be a possibility. For any formula φ:*

$$\Phi \vdash \varphi \iff \text{ for any } w \supset u \ X \vDash_w \varphi.$$

Proof. We argue by induction on φ. If φ is of the form $x = y$, then we have two cases. First, if x and y are both in the domain of u, then

$$\Phi \vdash x = y \iff u(x) = u(y) \iff X \vDash_w x = y$$

for each $w \supseteq u$. If one of x or y, say x, is not in the domain of u, then we cannot have $\Phi \vdash x = y$, since x does not appear in Φ and Φ proves that there are at least two distinct elements. Also, we can choose $w \supset u$ such that $w(x) \neq w(y)$, so that $X \vDash_w x \neq y$.

Suppose that $\Phi \vdash P(\bar{x})$. Fix $w \supset u$. Let $\bar{a} = w(\bar{x})$. Then we have $\bar{a} \in \mathcal{I}(X,P)$. Hence $X \vDash_w P(\bar{x})$.

Now suppose that for all $w \supseteq u$, $X \vDash_w P(\bar{x})$. Define $w \supset u$ so that if x is a variable not in the domain of u, then $w(x) \neq w(x')$ for any other variable x' (including x' in the domain of u). Let $\bar{a} = w(\bar{x})$. Then since $X \vDash_w P(\bar{x})$, $\bar{a} \in \mathcal{I}(X,P)$. Thus there is $w' \supset u$ and a tuple of variables $\bar{y}$ with $w'(\bar{y}) = \bar{a} = w(\bar{x})$ and such that $\Phi \vdash P(\bar{y})$. Let $\bar{z}$ be the variables in $\bar{y}$ that do not appear freely in Φ. Then $\Phi \vdash (\forall \bar{z}) P(\bar{y})$. Note that if y_i is in the domain of u, then by choice of w and using the fact that $w(\bar{x}) = w'(\bar{y})$, x_i is also in the domain of u. Moreover, since w and w' both extend u, $u(y_i) = u(x_i)$ and so $\Phi \vdash x_i = y_i$. Thus $\Phi \vdash P(\bar{x})$.

Suppose that for all $w \supseteq u$, $X \vDash_w \neg\varphi$. If $\Phi \nvdash \neg\varphi$, then $\Phi \cup \{\varphi\}$ is consistent. By Lemma 20 there is a possibility $Y = (\Psi, v)$ such that $\Psi \vdash \Phi \cup \{\varphi\}$. Then for all $w \supseteq v$, $Y \vDash_w \varphi$. This is a contradiction.

Suppose that $\Phi \vdash \neg\varphi$ and fix a variable assignment w extending u. Let $\bar{x}$ be the variables which appear in φ but not in Φ. Then $\Phi \vdash (\forall \bar{x}) \neg\varphi$. Let $Y = (\Psi, v)$ be such that $Y \geq X$. Then $\Psi \vdash (\forall \bar{x}) \neg\varphi$ and so $Y \nvDash_{w'} \varphi$ for all $w' \supset v$. Since w and w' agree on the variables which are free in $(\forall \bar{x}) \neg\varphi$, $Y \nvDash_w \varphi$. Hence $X \vDash_w \neg\varphi$.

Suppose that $\Phi \vdash \varphi \wedge \psi$ and fix $w \supset u$. Then $\Phi \vdash \varphi$ and so $X \vDash_w \varphi$, and similarly $\Phi \vdash \psi$ and so $X \vDash_w \psi$. Thus $X \vDash_w \varphi \wedge \psi$.

Now suppose that for all $w \supset u$, $X \vDash_w \varphi \wedge \psi$. Then for all $w \supset u$, $X \vDash_w \varphi$ and $X \vDash_w \psi$. So $\Phi \vdash \varphi$ and $\Phi \vdash \psi$, and so $\Phi \vdash \varphi \wedge \psi$.

Suppose that $\Phi \vdash (\forall x)\varphi$. Fix $w \supset u$. We may assume by substitution that x does not appear in the domain of u. Since $\Phi \vdash (\forall x)\varphi$, $\Phi \vdash \varphi$. Let w' be an x-variant of w, so that w' still extends u. Then $X \vDash_{w'} \varphi$. So $X \vDash_w (\forall x)\varphi$.

If $X \vDash_w (\forall x)\varphi$ for all $w \supset u$, we may assume by substitution that x does not appear in the domain of u. Then $X \vDash_{w'} \varphi$ for all w' which are x-variants of some $w \supset u$, i.e., for all $w' \supset u$. So $\Phi \vdash \varphi$. Since x does not appear in Φ, $\Phi \vdash (\forall x)\varphi$.

For the case of $\Box\varphi$, let $\bar{x}$ be the free variables in φ which do not appear in Φ. Recall that $\vdash (\forall \bar{x})\Box\varphi \iff \Box(\forall \bar{x})\varphi$. It suffices to show that

$$\Phi \vdash \Box(\forall \bar{x})\varphi \iff \text{ for any } w \supset u \ X \vDash_w \Box(\forall \bar{x})\varphi.$$

So we may assume that every variable which is free in φ also appears in Φ and in the domain of u.

Suppose that $\Phi \vdash \Box\varphi$. Fix $w \supset u$. Let $Y = (\Psi, v)$ be another possibility such that $X\mathcal{R}Y$. Then, by definition, $u \subseteq v$ and $\Psi \vdash \{\psi : \Phi \vdash \Box\psi\}$. Since $\Phi \vdash \Box\varphi$, $\Psi \vdash \varphi$. Thus, by the induction hypothesis, for any total variable assignment $w' \supseteq v$, $Y \vDash_{w'} \varphi$. Now w and w' agree on the variables which appear in φ since they are both extensions of u, and so $Y \vDash_w \varphi$. Thus $X \vDash_w \varphi$.

Suppose that $X \vDash_w \Box\varphi$ for all $w \supset u$. Suppose for the sake of contradiction that $\Phi \nvdash \Box\varphi$. Then by Proposition 17 there is Ψ such that $\Psi \vdash \varphi \iff \Phi \vdash \Box\varphi$. Then $\Psi \nvdash \varphi$, and so $\Psi \cup \{\neg\varphi\}$ is consistent. By Lemma 20, there is a possibility $Y = (\Psi', v)$ with $\Psi' \vdash \Psi$ and $v \supseteq u$. Then $X\mathcal{R}Y$. Since $Y \vDash_w \varphi$ for all $w \supset u$, $\Psi \vdash \varphi$. This is a contradiction. $\Box$

It follows from the Truth Lemma that $\mathcal{M}$ has computable possibilities and that γ is true at some possibility in $\mathcal{M}$. $\Box$

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