Partially Magic Labelings & The Antimagic Graph Conjecture

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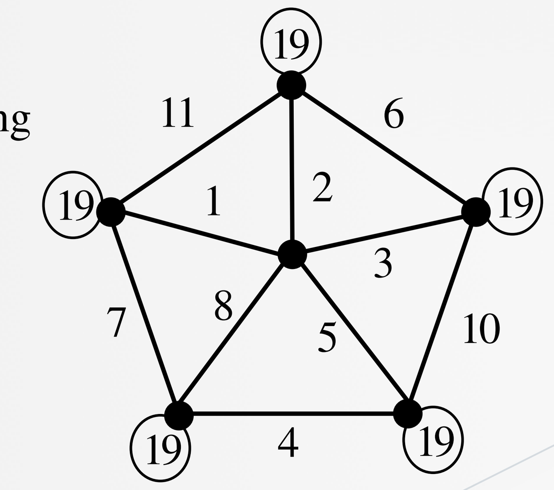
Labeling: of a graph G is an assignment $L: E \to \mathbb{Z}_{\geq 0}$ of a nonnegative integer $L(e)$ to each edge e of G

Magic

Magic Labeling:
- each edge label $\{0,1,\cdots,|E|\}$ is used exactly once
- the sums of the labels on all edges incident with a given node are equal to r

Example of Magic labeling on W_5 of index 19



$H_G(r)$: the number of magic labelings of G of index r

Theorem (Stanley 1973): *For a finite graph G, there exist polynomials $P_G(r)$ and $Q_G(r)$ such that $H_G(r) = P_G(r) + (-1)^r Q_G(r)$.*

Theorem (Stanley 1973): *If the graph G minus its loops is bipartite, then $H_G(r)$ is a polynomial of r.*

Quasi-polynomial of period at most 2

Generalization

Motivation

Birkhoff [1912] introduced the chromatic polynomial $\chi_G(k)$ (the number of proper k-colorings of a graph) in an attempt to prove the **Four Color Theorem:** $\chi_G(4) > 0$ if G is planar.

Our idea:

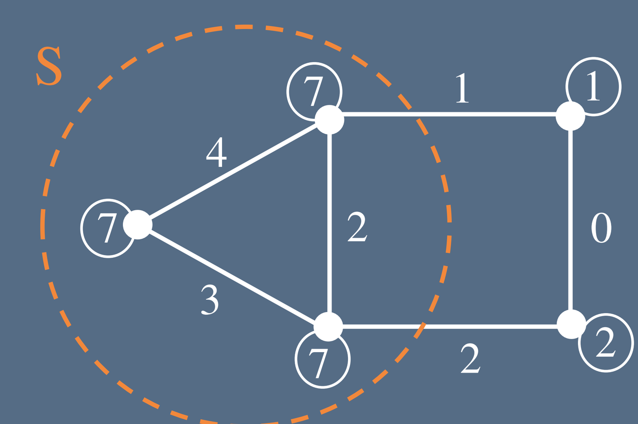
$A_G(k)$: the number of weakly antimagic k-labelings

- G has a weak antimagic labeling $\iff A_G(|E|) > 0$
- $A_G(k) = \sum_{\substack{S \subseteq V \\ |S| \geq 2}} c_S\, M_S(k)$ for some integers C_S
- Why weakly? The counting function of antimagic k-labelings is in general not a polynomial!

k −Labeling: each edge label is among $\{0,1,\cdots,k\}$

Partially Magic

Partially magic labeling of G over S ($S \subseteq V(G)$): is a labeling such that "magic occurs" just in S, that is, the sums of the labels on all edges incident with a given node in S are equal



$M_S(k)$: the number of partially magic k −labelings of G over S

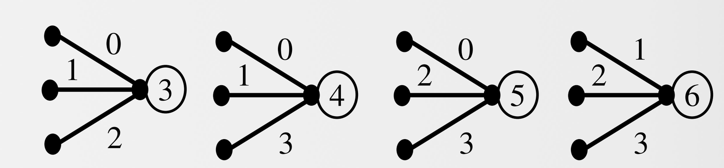
Theorem: *$M_S(k)$ is a quasi- polynomial in k with period at most 2.*

Theorem: *If the graph G minus its loops is bipartite, then $M_S(k)$ is a polynomial in k.*

Application

Example of $A_{K_{3,1}}(k)$:

Weakly antimagic 3-labelings, edge labels $\in \{0,1,2,3\}$

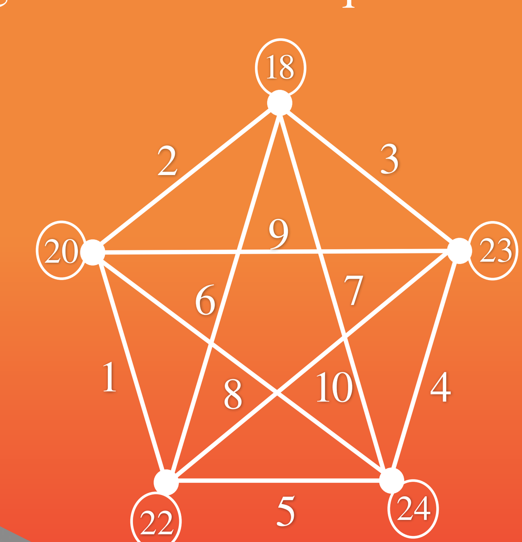


$A_{K_{3,1}}(3) = 4$

$A_{K_{3,1}}(k) = \binom{k+1}{3}$ Polynomial in k

Antimagic

Antimagic Labeling:
- each edge label $\{0,1,\cdots,|E|\}$ is used exactly once
- the sum of the labels on all edges incident with a given node is unique



Antimagic Graph Conjecture

Every connected graph except for k_2 admits an antimagic labeling.

Surprise: still open for trees !

Proved for:
- trees without vertices of degree 2
- connected graphs with minimum degree $\geq c \log|V|$
- k −regular graphs

$A_G(k)$: the number of **weakly** antimagic k −labelings

Main Theorem

Theorem: *$A_G(k)$ is a quasi-polynomial in k of period at most 2.*

Theorem: *If the graph G minus its loops is bipartite, then $A_G(k)$ is a polynomial in k.*

Corollary

Every bipartite graph without a K_2-component admits a weakly antimagic labeling.

Open Problems:
- Directed Antimagic Graph Conjecture
- Distinct Antimagic Counting

References

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