Optimistic equilibria in finitely additive mixed strategies

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Abstract

We are interested in analyzing economic games in which the underlying action spaces are possibly non-compact and/or the payoff functions are possibly discontinuous. Under these circumstances, there is no guarantee of the existence of a Nash equilibrium in mixed strategies. In fact, there are games for which no Nash equilibrium exists [SW57, BF68]. To attempt to restore equilibrium we allow each agent access to randomized strategies that are only finitely additive (i.e. not necessarily countably additive). This has the unfortunate side effect of introducing ambiguity into the players' payoff functions due to the failure of Fubini's theorem. By having players resolve this ambiguity optimistically at a candidate equilibrium profile and pessimistically at any deviation, we are able to recover a widely applicable equilibrium existence result.

1 Introduction

The standard method for proving the existence of a mixed strategy equilibrium in normal form games using the Glicksberg-Kakutani-Fan fixed point theorem does not work when the players' pure strategy sets are not compact or the payoff functions are discontinuous. There has been substantial work in finding conditions that ensure the existence of equilibria in discontinuous games [BM13, Car09, DM86a, MMT11, Pro11, Ren99], however there are not-particularly-pathological games that lack mixed equilibria [SW57, BF68]. We introduce a new type of equilibrium, called an *optimistic equilibrium*, which practically always exists. All that we require is that the strategy spaces be separable metric spaces and that the payoff functions be bounded and measurable.

As in [Mar97], we expand each player's mixed strategy space to the space of finitely additive probability measures. This has the well known, but unfortunate side effect of introducing ambiguity into the players' payoff functions due to the failure of Fubini's theorem for finitely additive measures. We say that a candidate strategy profile is an *optimistic equilibrium* if each player prefers the best-case payoff at the candidate profile to the worst-case payoff obtained from a deviation. We achieve our optimistic and pessimistic evaluations using upper- and lower-semicontinuous extensions of the players utility functions to the larger space. By considering these extensions *after* enlarging the strategy space, we are able to ensure that the payoff functions remain unchanged on the original space. That is, we are able to preserve all of the original character of the game.

In the first part of the paper, we find easy to describe optimistic equilibria in the Bertrand-Edgeworth model of duopoly pricing with limited supply; the Big Match; and the "game with no value" introduced in [SW57]. We then dive into the mathematics necessary to formally state and prove our equilibrium existence result. We find that iterated elimination of dominated strategies is a valid procedure for finding optimistic equilibria provided that domination is determined by comparing worst-case payoffs. We wrap up by proving that limits of ϵ-Nash equilibria will be optimistic equilibria in the expanded game provided ϵ tends to 0.

This paper is in many ways an extension of a paper of Marinacci [Mar97]. In [Mar97] players are given access to finitely additive mixed strategies. However, to avoid payoff ambiguity [Mar97] considers a restricted class of payoff functions for which Fubini's theorem does not fail. This restriction excludes many economically interesting games–especially those with discontinuities along diagonals.

2 The Bertrand-Edgeworth model of duopoly pricing with limited supply

We consider a model of duopoly pricing for a homogenous good in which each seller has a restricted quantity of the good to sell. We consider the same model as [DM86b]. It is proved that there exists a mixed equilibrium in this game, but, to our knowledge, no explicit equilibrium is known.

There are two sellers and a single good. Seller i has a stock S_i of the good to sell.

Each buyer is represented by a point on the unit interval and and total demand for the good at price p is given by $D(p)$. Each seller chooses a price $p_i \geq 0$. The buyers will choose to purchase from the seller with the lowest price. Any unserved buyers may then purchase from the second seller if they wish. If the sellers charge the same price, they capture the share of the market in proportion to their stock until their stock is exceeded.

In summary, the payoff function for the first (and analogously for the second) seller is

$$u_1(p_1, p_2) = \begin{cases} \min\{p_1 S_1, p_1 D(p_1)\} & \text{if } p_1 < p_2 \\ \min\left\{p_1 S_1, p_1 D(p_1)\frac{S_1}{S_1+S_2}\right\} & \text{if } p_1 = p_2 \text{ and } S_2 \geq \frac{D(p_1)S_2}{S_1+S_2} \\ p_1(D(p_1) - S_2) & \text{if } p_1 = p_2 \text{ and } S_2 < \frac{D(p_2)S_2}{S_1+S_2} \\ \max\left\{0, p_1 D(p_1)\frac{D(p_2)-S_2}{D(p_2)}\right\} & \text{if } p_1 > p_2. \end{cases}$$

We will make a few assumptions that are not made in [DM86b] to simplify our analysis. We will assume that D is continuous, strictly decreasing and that there exist $\underline{p}$ and $\overline{p}$ for which $D(\underline{p}) = S_1 + S_2$ and $D(\overline{p}) = 0$. We will also assume that the unrestricted monopoly profit function function $\Pi(p) = pD(p)$ is strictly concave on $[\underline{p}, \overline{p}]$.

We will write p^- for the vague strategy in which the seller plans to charge a price infinitesimally smaller than p.[1] If both sellers choose to play p^-, then there is some ambiguity about which is first-to-market. If each seller resolves the ambiguity in the most optimistic way possible, then each will believe that she will be first-to-market and that her price will be indistinguishable from p. That is, seller i believes that she will recognize the payoff $\min\{pS_i, \Pi(p)\}$ from the profile (p^-, p^-).

Theorem 1. *There are numbers $a, b \in \mathbb{R}$ defined in the proof below such that when $p \in [a, b]$, neither seller will have incentive to deviate from the profile (p^-, p^-) when they resolve their payoff ambiguity optimistically.*[2]

Proof. Let $\pi_1(p) = \min\{pS_1, \Pi(p))\}$. The function π_1 is seller 1's first-to-market payoff function. That is, $\pi_1(p)$ is the profit that seller 1 would achieve given that

[1] In the formalism of the sequel, p^- will stand for a finitely additive 0–1 measure that assigns probability 1 to every interval of the form $(p - \epsilon, p)$ with $\epsilon > 0$. See our remarks at the end of section 6.

[2] It is not always the case that $a \leq b$, in which case the theorem is vacuous. However, we are guaranteed that $a \leq b$ when the game is symmetric, so the theorem is not vacuous in general.

the other seller chooses a price higher than p. Let p_1^* be the unique point in $[\underline{p}, \overline{p}]$ at which π_1 achieves its maximum. The existence of this point is guaranteed by our assumptions on Π. Notice that seller 1's second-to-market profit is her first-to-market profit scaled down by the factor $(D(p_2) - S_2)/D(p_2)$. Notice also that the sellers will never choose to match prices exactly. Consequently, if seller 2 chooses a price $p_2 < p_1^*$, the price p_1^* will dominate all other prices that do not make seller 1 first-to-market. That is, seller 1 will respond either with a price that makes her first-to-market or with the price p_1^*.

Define $\hat{p}_1 \in (\underline{p}, p_1^*)$ by the equation

$$\pi_1(\hat{p}_1) = \pi_1(p_1^*)(D(\hat{p}_1) - S_2)/D(\hat{p}_1).$$

Our assumptions on D and Π guarantee that $\hat{p}_1$ is well defined. This definition of $\hat{p}_1$ ensures that

$$\pi_1(p) \geq p_1^* D(p_1^*)(D(p) - S_2)/D(p)$$

as long as $\hat{p}_1 \leq p \leq p_1^*$. That is, seller 1 has no incentive to deviate to a higher price from any price $p \in (\hat{p}_1, p_1^*)$ if she believes that she is first-to-market at the price p and that her deviation will leave her second-to-market facing the price p on the part of seller 2.

We see that if seller 1 believes that she is first-to-market with a price $p \in [\underline{p}, p_1^*]$, she will have no incentive to lower her price since π_1 is increasing on $[\underline{p}, p_1^*]$ and will have no incentive to increase her price if it means that she will be second-to-market.

We may repeat the same analysis for seller 2. Define $a = \max\{\hat{p}_1, \hat{p}_2\}$ and $b = \min\{p_1^*, p_2^*\}$. (In the symmetric case, $\hat{p}_1 = \hat{p}_2 \leq p_2^* = p_1^*$, so $a \leq b$.) Then, neither seller will have any incentive to deviate from the profile (p^-, p^-) when $p \in [a, b]$. □

This furnishes our first example of an optimistic equilibrium. The players are optimistic about their payoffs at the equilibrium profile, but are pessimistic about their payoffs when they consider deviations. Notice that the strategy profile (p, p), which closely resembles (p^-, p^-), is not an ϵ-Nash equilibrium for ϵ sufficiently close to 0.

We propose the following interpretation of this equilibrium. As long as prices fall within a certain range, sellers will tend to set similar prices, but each will try to gain an infinitesimal advantage so as to be first-to-market. This result is highly suggestive of a phenomenon known as Edgeworth price cycling [KRRS94]. An Edgeworth price cycle is a dynamic description of prices in which, as long as the prices fall in some range, we observe a price war and prices decrease together. As soon as prices are low enough that one seller is better off being second-to-market, that seller increases her price. This is quickly followed by a price increase by the other seller and the cycle starts anew. The optimistic equilibrium concept provides some justification for each seller's behavior during an Edgeworth price cycle. The sellers' pricing decisions at any point in time are quite reasonable provided each seller believes that she will get the advantage in the price war.

3 The Big Match

The Big Match is a finite stochastic game introduced by Blackwell and Ferguson [BF68] which they describe as follows.

> Every day player 2 chooses a number, 0 or 1, and player 1 tries to predict 2's choice, winning a point if he is correct. This continues as long as player 1 predicts 0. But if he ever predicts 1, all future choices for both players are required to be the same as that day's choices: if player 1 is correct on that day, he wins a point every day thereafter; if he is wrong on that day, he wins zero every day thereafter. The payoff to 1 is
>
> $$\limsup_{n\to\infty} \frac{a_1 + \cdots + a_n}{n},$$
>
> where a_m is the number of points he wins on the mth day.

Blackwell and Ferguson find that this game has no exact equilibrium, but that it has an ϵ-equilibrium for every $\epsilon > 0$.

Theorem 2 (Blackwell and Ferguson [BF68])**.** *The value of the big match is* $1/2$*. An optimal strategy for player 2 is to toss a fair coin every day. Player 1 has no optimal strategy, but for any non-negative integer* N *he can get*

$$V(N) = \frac{N}{2(N+1)}$$

by using strategy N*, defined as follows: having observed player 2's first* n *choices* $x_1, \ldots, x_n, n \geq 0$*, calculate the excess* k_n *of* 0*'s over* 1*'s among* $x_1, \ldots, x_n$*, and predict* 1 *with probability* $p(k_n + N)$ *where* $p(m) = 1/(m+1)^2$*.*

Let σ_N denote Player 1's Nth strategy as defined in the theorem. If we were advising Player 1, we would suggest that he "play σ_N for some very, very large N." There are finitely additive probability measures that assign probability 1 to all of $\mathbb{N}$, but assign probability 0 to each set of the form $\{0, 1, \cdots, K\}$ with $K \in \mathbb{N}$. We might model our suggestion by having Player 1 draw N from this finitely additive distribution and then use the strategy σ_N. We will denote any strategy of this form by σ_∞. It follows from our results below that the strategy profile in which Player 1 uses σ_∞ and Player 2 tosses a fair coin is an optimistic equilibrium.

How can we understand σ_∞? It is not obvious, but it is true that σ_N will eventually play 1 almost surely provided that $\limsup_{n\to\infty} k_n/n \le 0$. This implies that σ_∞ will also eventually play 1 almost surely when $\limsup_{n\to\infty} k_n/n \le 0$. However, for each partial history $(x_1, \ldots, x_n)$, the probability with which σ_∞ plays 1 on turn $n+1$ is given by $\lim_{N\to\infty} 1/(k_n + N + 1)^2 = 0$. That is, the probability that Player 1 will play 1 on any particular turn is 0. These two conclusions may seem contradictory, but are perfectly compatible in the context of finitely additive probability.

Putting our conclusions together gives σ_∞ an interpretation as a threat to Player 2: "Give me a payoff asymptotically greater than 1/2 by playing $\limsup_{n\to\infty} k_n/n > 0$. Otherwise, with no prior warning I will one day decide to play 1." The disadvantage that σ_N has relative to σ_∞ is that σ_N, by virtue of being countably additive, announces when Player 1 will have some positive probability of playing 1. This allows Player 2 the opportunity to play 0 and lock in a payoff of 1 for all time with positive probability.

4 A game with no value

Consider the following two-player, zero-sum game introduced by Sion and Wolfe [SW57]. Each player's action set is $[0, 1]$ and the payoff to player 1 is

$$u_1(x, y) = \begin{cases} 1 & \text{if } x > y \text{ or } x + \frac{1}{2} < y \\ 0 & \text{if } x = y \text{ or } x + \frac{1}{2} = y \\ -1 & \text{if } x < y < x + \frac{1}{2}. \end{cases}$$

It is shown in [SW57] that this game has no equilibrium in countably additive mixed strategies.

We allow each player access to strategies of the form x^+ and x^- where x^+ means some infinitesimal amount larger than x and x^- means some infinitesimal amount smaller than x. We may encounter payoff ambiguities when both players consider a strategy of this type. For example, $1, 0$ and -1 are all plausible payoffs at the strategy

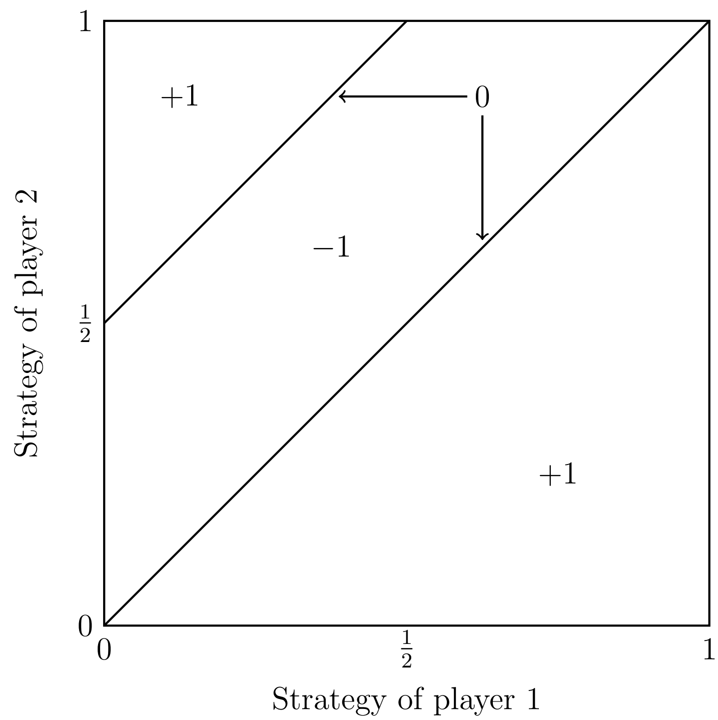

Figure 1: Payoff to player 1 in a game with no value.

profile $(1/2^-, 1/2^-)$. We will write $\overline{u}_1(x,y)$ for the highest plausible payoff at the strategy profile (x,y) and $\underline{u}_1(x,y)$ for the lowest. For example, $\overline{u}_1(1/2^-, 1/2^-) = 1$ and $\underline{u}_1(1/2^-, 1/2^-) = -1$.

We will say that the strategy x_1 worst-case dominates x_2 if $\underline{u}_1(x_1, y) \geq \underline{u}_1(x_2, y)$ for all strategies y. That is, we compare the worst-case outcomes when deciding that one strategy dominates another. We will find an optimistic equilibrium by eliminating worst-case dominated strategies. This procedure is justified in the sequel.

Theorem 3. *In this game, player 1 playing $x = 0$ with probability $1/3$ and $x = 1$ with probability $2/3$; and player 2 playing $y = 1/2^-$ with probability $1/3$ and $y = 1$ with probability $2/3$ is an equilibrium. The payoffs at this optimistic equilibrium are unambiguous, with player 1 achieving an expected payoff of $1/3$*

Proof. First, the strategy $x = 1$ on the part of player 1 dominates w for each $1/2 \leq w < 1$ (including strategies like $1/2^+$). This is because $\underline{u}_1(1, y) = 1$ if $y \neq 1$; $\underline{u}_1(1,1) = 0$; and $\underline{u}_1(w, 1) \leq 0$ for all $w \in [1/2, 1)$. After removing the worst-case dominated strategies, Player 1's remaining strategies are then $[0, 1/2) \cup \{1\}$.

Next, the strategy $y = 1/2^-$ of player 2 dominates each strategy z with $0 \leq z < 1/2^-$. Player 2's relevant payoff at $y = 1/2^-$ is

$$\underline{u}_2(w, 1/2^-) = \begin{cases} 1 & \text{if } 0 \leq w < 1/2^- \\ -1 & \text{if } w = 1/2^-, 1. \end{cases}$$

Each strategy $y \in [0, 1/2^-)$ does as poorly as $1/2^-$ in the worst case against $w = 1/2, 1/2^-, 1$. Player 2's remaining strategies are then $[1/2^-, 1]$.

Now, the strategy $x = 0$ on the part of player 1 dominates each strategy w with $0 < w < 1/2$. Player 1's payoff from $x = 0$ is

$$\underline{u}_1(0, z) = \begin{cases} -1 & \text{if } z = 1/2^- \\ 0 & \text{if } z = 1/2 \\ 1 & \text{if } z > 1/2. \end{cases}$$

However, every strategy $w \in (0, 1/2)$ has a worst-case payoff of -1 against $z = 1/2^-$ or $z = 1/2$. Notice that using the worst-case evaluation is necessary here to eliminate $w = 1/2^-$. Player 1 is then left with the strategy set $\{0, 1\}$.

Then, each strategy $y \in (1/2, 1)$ returns a payoff of -1 against each remaining strategy of player 1, so we may eliminate these. Player 2's strategy set is then $\{1/2^-, 1/2, 1\}$.

Then, player 2's strategy $y = 1/2^-$ does better than $z = 1/2$ in the worst case against $w = 0$. Both $y = 1/2^-$ and $z = 1/2$ perform equally poorly against $w = 1$ in the worst case, so we may eliminate $z = 1/2$. Player 2 is then left with the strategies $\{1/2^-, 1\}$.

We are left with the game with payoff matrix

	$\frac{1}{2}^-$	1
0	-1	1
1	1	0

This equilibrium in the statement of the theorem is an equilibrium in this game. □

5 Mathematical preliminaries

Let (X, Σ) be a measure space with X a separable metric space and Σ the Borel σ-algebra.

Definition 4. *We will say that* $\mu : \Sigma \to \mathbb{R}$ *is a* bounded, finitely additive measure *if*

(i) there is some $M > 0$ *such that* $|\mu(E)| < M$ *for all* $E \in \Sigma$*;*

(ii) $\mu\left(\bigcup_{j=1}^{K} E_j\right) = \sum_{j=1}^{K} \mu(E_j)$ *for every finite, pairwise disjoint collection* $E_1, \ldots, E_K$ *from* Σ*.*

Furthermore, we will say that μ *is* countably additive *if the last condition holds for every countable, pairwise disjoint collection* $\{E_j\}_{j=1}^{\infty}$*.*

The word measure without qualification will mean a bounded, finitely additive measure. The set of bounded, measurable functions on X will be denoted $\mathcal{F}(X)$. The set of bounded, finitely additive measures on X will be denoted $\mathrm{ba}(X)$. The weak* topology on $\mathrm{ba}(X)$ is the smallest (coarsest) topology that makes the map

$$\sigma \mapsto \int_X f \, d\sigma$$

continuous for each $f \in \mathcal{F}(X)$.

Definition 5. *We will call a measure* $\mu \in \mathrm{ba}(X)$ *a probability measure if* *(i)* $\mu(X) = 1$

(ii) $\mu(A) \geq 0$ *for all* $A \in \Sigma$.

Theorem 6 (Banach-Alaoglu)**.** *The set of probability measures on* X *is weak** *compact.*

Proof. See [AB06], theorem 6.25, for example. □

It is the Banach-Alaoglu theorem that makes the space of finitely additive probability measures useful as an extension of the space of countably additive probability measures.

Instead of thinking about extending the space of countably additive probability measures on X to the space of finitely additive probability measures, we could instead think about expanding X to a larger space and then considering countably additive measures on that space. A remarkable theorem of Yosida and Hewitt shows that these perspectives are equivalent [YH52], as we will now see.

Let $\Omega(X)$ be the set of finitely additive measures ω on X that take only the values 0 and 1[3] (i.e., $\omega(A) = 0$ or $\omega(A) = 1$ for all $A \in \Sigma$). For any $x \in X$, the point mass δ_x is in $\Omega(X)$, so we may regard X as a subset of $\Omega(X)$. We have assumed that X is a separable metric space precisely so that every countably additive 0–1 measure is a point mass.

We will regard $\Omega(X)$ as a topological space with the weak* topology that it inherits as a subset of $\mathrm{ba}(X)$. We have a description of this topology that is perhaps more intuitive. Given $A \in \Sigma$, define

$$\overline{A} = \{\omega \in \Omega(X) : \omega(A) = 1\}.$$

Each set $\overline{A}$ is open and sets of the form $\overline{A}$ generate the weak* topology on $\Omega(X)$.

Theorem 7. $\Omega(X)$ *is compact.*

Proof. This follows immediately from the Banach-Alaoglu theorem and the fact that $\Omega(X)$ is closed in the unit ball in $\mathrm{ba}(X)$. □

Theorem 8. X *is a dense open subset of* $\Omega(X)$*.*

[3] Some readers may prefer to think of these as ultrafilters on Σ.

Proof. The weak* topology is generated by sets of the form

$$\begin{aligned} W(\tau, \{B_j\}_{j=1}^k, \{\epsilon_j\}_{j=1}^k) &= \{\sigma \in \Omega(X) : |1_{B_j}(\sigma) - 1_{B_j}(\tau)| < \epsilon_j, j = 1, \ldots, k\} \\ &= \{\sigma \in \Omega(X) : \sigma(B_j) = \tau(B_j), j = 1, \ldots, k\} \end{aligned}$$

with $0 < \epsilon_j < 1$ and B_j measurable for all j. Replacing B_j by B_j^c as necessary, we may assume that $\tau(B_j) = 1$ for all j. From the formula

$$\tau(C \cap D) = \tau(C) + \tau(D) - \tau(C \cup D),$$

we see that the intersection of any two sets of τ-measure 1 must have measure 1. It follows that $\bigcap_{j=1}^k B_j$ is non-empty. Then,

$$\delta_x \in W(\tau, \{B_j\}_{j=1}^k, \{\epsilon_j\}_{j=1}^k)$$

for any $x \in \bigcap_{j=1}^k B_j$.

To see that X is open, note that $\overline{\{x\}} = \{x\}$ and $X = \bigcup_{x \in X} \{x\}$. $\square$

Given a bounded, measurable function $f : X \to \mathbb{R}$, we may extend it to a function on $\Omega(X)$ by defining $f : \Omega(X) \to \mathbb{R}$ by

$$f(\omega) = \int_X f \, d\omega.$$

Notice that $f(\delta_x) = f(x)$ for all $x \in X$. By definition of the weak* topology, the map $\omega \mapsto f(\omega)$ is continuous.

It turns out that every probability measure[4] on X may be represented as a *countably additive* measure on $\Omega(X)$.

Theorem 9 (Yosida-Hewitt [YH52], comment 4.5)**.** *Let σ be a probability measure on X. There is a regular, countably additive Borel probability measure $\overline{\sigma}$ on $\Omega(X)$ such that for every bounded, measurable function f on X,*

$$\int_X f \, d\sigma = \int_{\Omega(X)} f(\omega) \, d\overline{\sigma}(\omega) = \int_{\Omega(X)} \int_X f(x) \, d\omega(x) \, d\overline{\sigma}(\omega).$$

[4]We will state our results for probability measures since they are all that we need in the sequel. However, Yosida and Hewitt show in theorem 1.12 [YH52] that every measure may be written as a difference of nonnegative measures, which may be used to generalize the following results to all of ba(X).

We also know that every probability measure may be approximated by a linear combination of 0-1 measures.

Theorem 10 (Yosida-Hewitt [YH52], theorem 4.6)**.** *Every probability measure γ is the weak* limit of linear combinations of* 0-1 *measures, in the sense that for every $\epsilon > 0$ and every collection $\{f_j\}_{j=1}^{\ell}$ of bounded, measurable functions, there is a measure $\delta = \sum_{i=1}^{k} \alpha_i \omega_i$ with each $\omega_i \in \Omega(X)$ and each α_i a scalar such that*

$$\left| \int_X f_j \, d\gamma - \int_X f_j \, d\delta \right| < \epsilon$$

for all $j = 1, \ldots, \ell$.

6 Game theoretic setup

We consider a normal form game with N players. Each player has a strategy space A_i, which we assume to be a separable metric space. Let $\mathcal{A}_i$ be the Borel σ-algebra on A_i, let $A = \prod_{i=1}^{N} A_i$, and let $\mathcal{A}$ be the algebra on A generated by the measurable boxes. We will assume that each payoff function $u_i : A \to \mathbb{R}$ is measurable with respect to $\sigma(\mathcal{A})$, the smallest σ-algebra containing $\mathcal{A}$.

Let $P_i = \Omega(A_i)$, the space of 0-1 measures on A_i, and let $P = \prod_{i=1}^{N} P_i$. Finally, let F_i be the space of countably additive probability measures on P_i and let $F = \prod_{i=1}^{N} F_i$. Theorem 9 tells us that F_i is the same as the space of finitely additive probability measures on A_i. We would like to extend each u_i to a weak*-continuous function on P, but this is not generally possible.

Consider for example the zero-sum game in which each player chooses a natural number and the player who selects the larger number receives payment of 1. Let ν be any weak* limit of the sequence $\{\delta_n\}_{n \in \mathbb{N}}$. Let us try to figure out what $u_1(\nu, \nu)$ would be if u_1 extended continuously to $\Omega(\mathbb{N}) \times \Omega(\mathbb{N})$. On the one hand, we would have

$$u_1(\nu, \nu) = \lim_{n \to \infty} \lim_{m \to \infty} u_1(\delta_n, \delta_m) = -1.$$

On the other hand, we would have

$$u_1(\nu, \nu) = \lim_{m \to \infty} \lim_{n \to \infty} u_1(\delta_n, \delta_m) = 1.$$

This shows that our utility functions need not extend continuously to the expanded strategy space.

The only general condition on u_i that we know of that ensures that $u_i : A \to \mathbb{R}$ extends to a continuous function $u_i : P \to \mathbb{R}$ is that u_i be $\mathcal{A}$-measurable as in [Mar97]. We will focus instead on upper- and lower-semicontinuous extensions of the players' payoff functions.

Let Δ_i be the graph of $u_i : A \to \mathbb{R}$ regarded as a subset of $P \times \mathbb{R}$ and let $\overline{\Delta}_i$ be the closure of Δ_i. Define

$$\overline{u}_i(p) = \sup\{u : (p, u) \in \overline{\Delta}_i\}$$

and similarly

$$\underline{u}_i(p) = \inf\{u : (p, u) \in \overline{\Delta}_i\}.$$

We see immediately that $\overline{u}_i$ is upper-semicontinuous and $\underline{u}_i$ is lower-semicontinuous.

Next, define $\overline{V}_i : F \to \mathbb{R}$ and $\underline{V}_i : F \to \mathbb{R}$ by

$$\overline{V}_i(\sigma_1, \dots, \sigma_N) = \int_P \overline{u}_i \, d(\overline{\sigma}_1 \otimes \cdots \otimes \overline{\sigma}_N)$$

and

$$\underline{V}_i(\sigma_1, \dots, \sigma_N) = \int_P \underline{u}_i \, d(\overline{\sigma}_1 \otimes \cdots \otimes \overline{\sigma}_N).$$

Since $\overline{\sigma}_i$ is countably additive on P_i for each i, there is no issue with defining the product measure $\sigma_1 \otimes \cdots \otimes \sigma_N$. As a result, we will sometimes write $\overline{\sigma}$ for $\overline{\sigma}_1 \otimes \cdots \otimes \overline{\sigma}_N$ and similarly for $\overline{\sigma}_{-i}$.

Notice that the map $\overline{V}_i$ is upper-semicontinuous and $\underline{V}_i$ is lower-semicontinuous since the weak* topology on F_i corresponds to the topology of weak convergence on the space of countably additive measures on P_i.

Our extensions $\underline{V}_i$ and $\overline{V}_i$ agree when all of the players choose pure strategies (i.e. strategies in P) and at most one player takes advantage of a non-countably additive strategy (i.e. a strategy in $P_i \setminus A_i$).

Theorem 11. *Let $p \in P$ such that at most one p_i is not countably additive. That is, there is some j such that for each $i \neq j$, there exists $a_i \in A_i$ such that $p_i = \delta_{a_i}$. Then,*

$$\overline{V}_i(p) = \underline{V}_i(p) = \int_{A_j} u_i(a_1, \dots, a_N) \, dp_j(a_j).$$

Proof. Without loss of generality, assume $j = 1$. Suppose that we have a net

$$(\delta_{a_1^\alpha}, \dots, \delta_{a_N^\alpha}, u_i(a_1^\alpha, \dots, a_N^\alpha)) \to (p_1, \delta_{a_2}, \dots, \delta_{a_N}, u)$$

for some $u \in \mathbb{R}$.

We may choose $\beta \in A$ so that $\delta_{\alpha_i^N}(1_{a_i}) = 1$ for all $i \geq 2$ for all $\alpha \geq \beta$. This implies that

$$\begin{aligned}(\delta_{a_1^\alpha}, \dots, \delta_{a_N^\alpha}, u_i(a_1^\alpha, \dots, a_N^\alpha)) &= (\delta_{a_1^\alpha}, \delta_{a_2}, \dots, \delta_{a_N}, u_i(a_1^\alpha, \dots a_N)) \\ &= \left(\delta_{a_1^\alpha}, \delta_{a_2}, \dots, \delta_{a_N}, \int_{A_1} u_i(a_1, \dots a_N)\, d\delta_{a_1^\alpha}(a_1)\right) \\ &\to \left(p_1, \delta_{a_2}, \dots, \delta_{a_N}, \int_{A_1} u_i(a_1, \dots a_N)\, dp_1(a_1)\right)\end{aligned}$$

for all $\alpha \geq \beta$. It follows that

$$u = \int_{A_1} u_i(a_1, \dots a_N)\, dp_1(a_1). \qquad \square$$

In the event that the strategy spaces are one-dimensional it is often convenient to work with certain equivalence classes of elements of P. Suppose that each A_i is a Borel-measurable subset of $\mathbb{R}$. For any $a_i \in A_i$ we will write a_i^- for the set of measures in P_i that assign probability 1 to each of the sets $(a-\epsilon, a) \cap A_i$ with $\epsilon > 0$. Similarly, a_i^+ will denote the set of measures in P_i that assign probability 1 to each of the sets $(a, a+\epsilon) \cap A_i$ with $\epsilon > 0$. We will also abuse notation to write a_i for the set containing the probability measure δ_{a_i}. The sets $-\infty^+$ and ∞^- are defined analogously.

Our next result tells us that these sets partition A_i, so we may regard them as specifying an equivalence relation on A_i.

Lemma 12. *Suppose that A_i is a Borel-measurable subset of $\mathbb{R}$. Then, each element of p_i is a member of a_i^+, a_i^-, or a_i for some $a_i \in A_i \cup \{-\infty, \infty\}$.*

Proof. Let F be the cumulative distribution function of the probability measure $p_i \in P_i$. F is non-decreasing and takes (at most) the values 0 and 1. Let $a = \inf\{x : F(x) = 1\}$. If $\{x : F(x) = 1\}$ is empty, we set $a = \infty$ and see that p_i must assign probability 1 to each set of the form (M, ∞) with $M \in \mathbb{R}$, so $p_i \in \infty^-$. Similarly, if $a = -\infty$, we see that $p_i \in -\infty^+$. Suppose then that $a \in \mathbb{R}$. Let $\epsilon > 0$. Then,

$$p_i(a-\epsilon, a+\epsilon) = 1.$$

Written another way we see that

$$p_i(a-\epsilon, a) + p_i(\{a\}) + p_i(a, a+\epsilon) = 1.$$

Since p_i is a 0-1 measure, exactly one of these terms is 1 and the others are 0. Since the nonzero term must be the same for all $\epsilon > 0$, the result follows. □

In our introductory examples, we were evaluating our utility functions at equivalence classes by saying, for example,

$$\overline{u}_i(a_1^+, a_2^-) = \sup_{p_1 \in a_1^+, p_2 \in a_2^-} \overline{u}_i(p_1, p_2).$$

Using these equivalence classes greatly simplifies working with finitely additive measures in one dimensional games, but caution is necessary as there is no guarantee that the supremum will be achieved simultaneously by the *same* pair (p_1, p_2) for both $\overline{u}_1$ and $\overline{u}_2$. In many examples (including ours) this problem may be solved by picking a single finitely additive probability measure 0^+. We then interpret a^+ as the translation of this probability measure by a (i.e. $a^+(B) = 0^+(B - a)$), 0^- as the reflection of this measure across the origin, and a^- as the translation of 0^- by a. However, this cannot justify use of these equivalence classes in every situation.

7 Optimistic equilibria

We are now in a position to investigate the properties of optimistic equilibria.

Definition 13. *We will say that $\sigma \in F$ is an* optimistic equilibrium *if for each player i and each potential deviation $\tau_i \in F_i$,*

$$\overline{V}_i(\sigma) \geq \underline{V}_i(\tau_i, \sigma_{-i}).$$

Our most important result is that optimistic equilibria exist for a very large class of games.

Theorem 14. *Every game whose strategy spaces are separable metric spaces and whose payoff functions are bounded and measurable has an optimistic equilibrium.*

The proof follows the standard script. We will define a best response correspondence and observe that a fixed point corresponds to an optimistic equilibrium. We will then show that the best response correspondence satisfies the hypotheses of the Kakutani-Glicksberg-Fan fixed point theorem [Fan52, Gli52, K+41].

Definition 15. *We will say that a correspondence $\Gamma : X \to 2^X$ is* Kakutani *if each of the following hold* *(i)* X *is a subset of a topological vector space;*

(ii) Γ *is upper-hemicontinuous;*

(iii) $\Gamma(x)$ *is non-empty, compact and convex for all* $x \in X$.

Notice that if X is compact, to show that Γ is Kakutani, it suffices to show that Γ is upper-hemicontinuous and that $\Gamma(x)$ is non-empty and convex for all $x \in X$. Notice also that if $\{\Gamma^\alpha\}_{\alpha \in A}$ is any collection of Kakutani correspondences on a compact space X, then to show that $\bigcap_{\alpha \in A} \Gamma^\alpha$ is Kakutani, it suffices to show that $\bigcap_{\alpha \in A} \Gamma^\alpha(x)$ is non-empty for all $x \in X$.

Theorem 16 (Kakutani-Glicksberg-Fan)**.** *Let* X *be a non-empty, convex, compact subset of a topological vector space. If* $\Gamma : X \to 2^X$ *is Kakutani, then* Γ *has a fixed point.*

We now present the equilibrium existence proof.

Proof of theorem 14. For any $\gamma_i \in F_i$, define the correspondence

$$\mu \mapsto \mathrm{BR}_i(\mu, \gamma_i) = \{\tau \in F : \overline{V}_i(\tau_i, \mu_{-i}) \geq \underline{V}_i(\gamma_i, \mu_{-i})\}.$$

We claim that the correspondence $\mu \mapsto \mathrm{BR}_i(\mu, \gamma_i)$ is Kakutani. We will first show that this correspondence has a closed graph. Consider nets $\mu^\alpha \to \mu$ and $\tau^\alpha \to \tau$ with $\tau^\alpha \in \mathrm{BR}_i(\mu^\alpha, \gamma_i)$. Then, since $\overline{V}_i$ is upper-semicontinuous and $\underline{V}_i$ is lower-semicontinuous,

$$\overline{V}_i(\tau_i, \mu_{-i}) \geq \limsup_\alpha \overline{V}_i(\tau_i^\alpha, \mu_{-i}^\alpha) \geq \liminf_\alpha \underline{V}_i(\gamma_i, \mu_{-i}^\alpha) \geq \underline{V}_i(\gamma_i, \mu_{-i}).$$

That is, $\tau \in \mathrm{BR}_i(\mu, \gamma_i)$. Since F is compact, this implies that $\mu \mapsto \mathrm{BR}_i(\mu, \gamma_i)$ is upper-hemicontinuous.

To see that $\mathrm{BR}_i(\mu, \gamma_i)$ is non-empty, note that $(\gamma_i, \mu_{-i}) \in \mathrm{BR}_i(\mu, \gamma_i)$.

To show that $\mathrm{BR}_i(\mu, \gamma_i)$ is convex, suppose that $\sigma, \tau \in \mathrm{BR}_i(\mu, \gamma_i)$ and $\lambda \in (0, 1)$. We have

$$\overline{V}_i(\lambda\sigma_i + (1-\lambda)\tau_i, \mu_{-i}) = \lambda\overline{V}_i(\sigma_i, \mu_{-i}) + (1-\lambda)\overline{V}_i(\tau_i, \mu_{-i}) \geq \underline{V}_i(\gamma_i, \mu_{-i}).$$

It follows that $\mathrm{BR}_i(\mu, \gamma_i)$ is convex. We have shown that $\mu \mapsto \mathrm{BR}_i(\mu, \gamma_i)$ is Kakutani.

Define a new correspondence by

$$\mu \mapsto \mathrm{BR}_i(\mu) = \bigcap_{\gamma_i \in F_i} \mathrm{BR}_i(\mu, \gamma_i).$$

To show that BR_i is Kakutani, we need only show that $\mathrm{BR}_i(\mu)$ is non-empty. Since F is compact and $\mathrm{BR}_i(\mu, \gamma_i)$ is compact for each γ_i, it suffices to show that $\bigcap_{\ell=1}^{k} \mathrm{BR}_i(\mu, \gamma_i^\ell)$ is non-empty for every finite collection $\{\gamma_i^1, \ldots, \gamma_i^k\} \subset F_i$. Choose γ_i^j such that $\overline{V}_i(\gamma_i^j) \geq \overline{V}_i(\gamma_i^\ell)$ for all $\ell = 1, \ldots, k$. Then, $(\gamma_i^j, \mu_{-i}) \in \bigcap_{\ell=1}^{k} \mathrm{BR}_i(\mu, \gamma_i^\ell)$. It follows that BR_i is Kakutani.

As before, to show that $\mathrm{BR} = \bigcap_{i=1}^{N} \mathrm{BR}_i$ is Kakutani, it suffices to show that $\mathrm{BR}(\mu)$ is non-empty for all μ. This is clear from the definition of BR_i.

The result now follows from the Kakutani-Glicksberg-Fan theorem. □

Having proved that optimistic equilibria always exist, we will prove a few results to help us identify them. First, optimistic equilibria have the single deviation property.

Theorem 17. *A profile $\sigma \in F$ is an optimistic equilibrium if and only if for every i and every $p_i \in P_i$, $\overline{V}_i(\sigma) \geq \underline{V}_i(p_i, \sigma_{-i})$.*

Proof. Suppose that τ_i is a favorable deviation for player i: $\overline{V}_i(\sigma) < \underline{V}_i(\tau_i, \sigma_{-i})$. Then, since

$$\underline{V}_i(\tau_i, \sigma_{-i}) = \int_{P_i} \int_{P_{-i}} \underline{u}_i(p_i, p_{-i}) \, d\overline{\sigma}_{-i}(p_{-i}) \, d\overline{\tau}_i(p_i).$$

there must be at least one p_i for which

$$\underline{V}_i(p_i, \sigma_{-i}) = \int_{P_{-i}} \underline{u}_i(p_i, p_{-i}) \, d\overline{\sigma}_{-i} > \overline{V}_i(\sigma).$$

The other implication is part of the definition of an optimistic equilibrium. □

One of the most useful tools at our disposal for finding optimistic equilibria is iterated elimination of dominated strategies.

Definition 18. *We will say that $p_i \in P_i$* worst-case dominates *$q_i \in P_i$ if $\underline{u}_i(p_i, \gamma_{-i}) \geq \underline{u}_i(q_i, \gamma_{-i})$ for all $\gamma_{-i} \in P_{-i}$.*

Of course, we need to show that iterated elimination of worst-case dominated strategies is legitimate in the sense that an optimistic equilibrium obtained after eliminating strategies is an optimistic equilibrium in the original game.

Theorem 19. *Suppose that we remove some or all of the strategies worst-case dominated by some strategy $p_i \in P_i$ (other than p_i itself). If σ is an optimistic equilibrium in the resulting game then it is an optimistic equilibrium in the original game.*

Proof. Let σ be an optimistic equilibrium after some collection of strategies worst-case dominated by p_i have been removed. Suppose that q_i is a potential deviation for player i that was eliminated. Then,

$$\begin{aligned}
\underline{V}_i(q_i, \sigma_{-i}) &= \int_{P_{-i}} \underline{u}_i(q_i, p_{-i})\, d\overline{\sigma}_{-i}(p_{-i}) \\
&\leq \int_{P_{-i}} \underline{u}_i(p_i, p_{-i})\, d\overline{\sigma}_{-i}(p_{-i}) \\
&= \underline{V}_i(p_i, \sigma_{-i}) \\
&\leq \overline{V}_i(\sigma).
\end{aligned}$$

The result then follows from theorem 17. □

Our final result suggests another way of finding optimistic equilibria. They may appear as limits of ϵ-Nash equilibria as ϵ tends to 0.

Theorem 20. *Suppose that $\{\sigma^n\}_{n=1}^{\infty}$ is a sequence of countably additive strategy profiles such that σ^n is an ϵ^n-Nash equilibrium. Then, any limit point of $\{\sigma^n\}_{n=1}^{\infty}$ is an optimistic equilibrium.*

Proof. Let $\{\sigma^\alpha\}_{\alpha \in A}$ be a subnet of $\{\sigma^n\}_{n=1}^{\infty}$ that converges to σ. Let $p_i \in P_i$ be some potential deviation for player i. Pick a net $\{a_i^\alpha\}_{\alpha \in A}$ from A_i for which $a_i^\alpha \to p_i$.[5] Then,

$$\begin{aligned}
\underline{V}_i(p_i, \sigma_{-i}) &\leq \liminf_{\alpha \in A} \underline{V}_i(a_i^\alpha, \sigma_{-i}^\alpha) \\
&\leq \liminf_{\alpha \in A} \underline{V}_i(\sigma^\alpha) + \epsilon^\alpha \\
&\leq \overline{V}_i(\sigma).
\end{aligned}$$

□

[5] A priori, this net may have a different index set than $\{\sigma^\alpha\}_{\alpha \in A}$, but the product construction for nets allows us to find a single net $\{(\sigma^\beta, p_i^\beta)\}_{\beta \in B}$ for which $(\sigma^\beta, p_i^\beta) \to (\sigma, p_i)$.

8 Conclusion

In this paper, we defined an optimistic equilibrium to be an equilibrium in finitely additive mixed strategies in which players anticipate the best possible outcome at the equilibrium and the worst possible outcome from any deviation. We presented several games in which we were able to find optimistic equilibria where either no Nash equilibrium exists or where the optimistic equilibria provide new insight. We proved that every game whose strategy spaces are separable metric spaces and whose payoff functions are bounded and measurable possesses at least one optimistic equilibrium. We then showed that we may find optimistic equilibria using iterated elimination of worst-case dominated strategies. Finally, we found that a sequence of ϵ-Nash equilibria with decreasing ϵ converges to an optimistic equilibrium.

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