BOREL DIMENSION GROWTH AND HYPERFINITENESS

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ABSTRACT. We show that every increasing union of Borel graphs of pointwise volume growth at most $\exp(O(r^\gamma))$ is hyperfinite, where $\gamma \approx 0.1523$ is the unique real root of $(1-\gamma)^3 - 4\gamma = 0$. This implies a positive answer to the special case of Weiss's question on the hyperfiniteness of Borel actions of countable amenable groups, for countable amenable groups locally of volume growth $\exp(O(r^\gamma))$ for γ as above. We also show that every bounded degree Borel graph of subexponential volume growth has Borel Følner tilings, improving a result of Downarowicz and Zhang for graphs generated by free Borel actions of groups of subexponential volume growth. Our main tools are the development of a Borel analogue of the two-parameter dimension growth function of a metric space as introduced by Dranishnikov and Sapir, and ball carving algorithms from theoretical computer science. We end with a pair of conjectures relating Borel amenability, Borel subexponential dimension growth, and hyperfiniteness, which would imply a positive answer to Weiss's question.

1. INTRODUCTION

This paper is a contribution to the study of amenability, hyperfiniteness, and almost finiteness of countable Borel equivalence relations. The study of Borel reducibility of equivalence relations arose in descriptive set theory as a way of studying the relative difficulty of classification problems in mathematics. By the Glimm-Effros dichotomy of Harrington, Kechris, and Louveau [HKL90], the simplest nontrivial class of Borel equivalence relations is the class of hyperfinite Borel equivalence relations. However, many fundamental questions about hyperfinite Borel equivalence relations remain open. One important open problem is Weiss's question from 1984 of whether every Borel action of a countable amenable group generates a hyperfinite orbit equivalence relation [Wei84]. That is, does group theoretic tameness exactly correspond to descriptive set theoretic-simplicity?

Regular progress has been made on Weiss's problem over the last several decades for various classes of countable amenable groups: the groups $\mathbb{Z}^n$ (Weiss, unpublished), finitely generated groups of polynomial volume growth [JKL02], abelian groups [GJ15], locally nilpotent groups [SS24], and polycyclic groups [CJMST23].

The work in the papers [GJ15] and [SS24] was quite technical and required a great deal of information about the internal geometry of the groups in question. This style of attack seemed ill suited to understanding Weiss's problem in general, since we have a very poor understanding of the geometry of arbitrary amenable groups. In the past few years, some effort has been made to use softer methods for approaching this problem. An advance here was made in the paper [CJMST23] which introduced tools from the study of asymptotic dimension to attack the problem, and greatly simplified those earlier results at the same time as generalizing them.

Another line of progress has come from trying to generalize these techniques for proving hyperfiniteness of group actions to proving hyperfiniteness of Borel graphs with various growth bounds or amenability properties. This immediately forces one to confront the problem discussed above since arbitrary graphs have no nice or "regular" geometry. In [CJMST20, page 879] it is mentioned as an open question whether every Borel graph of uniformly subexponential growth is hyperfinite, and [Mar23, Problem 3.5] asked whether every Borel graph of uniformly bounded polynomial growth is hyperfinite. We note here that Tessera [Tes08, Proposition 3.5] showed in 2008 that every Borel graph of uniformly subexponential growth on a standard probability space (X,μ) is amenable and hence hyperfinite modulo a μ-null set by

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Date: August 23, 2026.

2020 *Mathematics Subject Classification.* 03E15, 05C63, 20F69, 43A07, 51F30.

JG and VR were supported by the Czech Science Foundation (GA ČR), project No. 26-23599M.

AM was supported in part by NSF grant DMS-2348788.

the Connes-Feldman-Weiss theorem [CFW81] (see also [Kas25, Theorem 18]). So the measure-theoretic analogue of the above problem on subexponential growth has a positive answer, analogously to how the Ornstein-Weiss theorem [OW80] gives a positive measure-theoretic answer to Weiss's problem.

Recently, Bernshteyn and Yu [BY25] made a significant advance here, showing that every Borel graph of polynomial growth is hyperfinite. Their result uses techniques from local algorithms in theoretical computer science and a Borel version of the Lovász Local Lemma to do a randomized ball carving construction to find a type of "padded decomposition" of the space, witnessing that it has finite Borel asymptotic dimension. Hyperfiniteness then follows from [CJMST23]. This use of the theory of distributed algorithms in descriptive set theory is one of many such fruitful connections that have emerged in the past few years between these fields growing out of the work in [Ber23].

Our main result is a generalization of the result of Bernshteyn and Yu, showing that increasing unions of Borel graphs of sufficiently slow intermediate growth are hyperfinite:

Theorem A. *Let γ be the unique real root of $(1-\gamma)^3 - 4\gamma = 0$, so $\gamma \approx 0.1523$. Suppose X is a standard Borel space and $\rho_0 \geq \rho_1 \geq \ldots$ is a decreasing sequence of Borel extended metrics on X so that for every n, ρ_n has pointwise volume growth $\exp(O(r^\gamma))$. That is, for every $n \in \mathbb{N}$ and $x \in X$, there exist $C, N > 0$ such that $|B_r^{\rho_n}(x)| \leq \exp(Cr^\gamma)$ for all $r \geq N$. Then $\bigcup_n E_{\rho_n}$ is hyperfinite, where E_{ρ_n} is the finite distance equivalence relation for ρ_n.*

Since all the tools in our proofs are from dimension theory and metric geometry, we have stated our results in general for Borel extended metric spaces. One important example of such a decreasing sequence of Borel extended metrics $\rho_0 \geq \rho_1 \geq \ldots$ is where $G_0 \subseteq G_1 \subseteq \ldots$ is an increasing sequence of locally countable Borel graphs on a standard Borel space X, and ρ_n is the path length metric on G_n. Hence, by applying Theorem A to these graph metrics, the theorem shows that an increasing union of Borel graphs, each of pointwise volume growth $\exp(O(r^\gamma))$, is hyperfinite.

Another important example of a locally finite Borel extended metric space arises when $\Gamma \curvearrowright X$ is a Borel action of a countable group Γ on a standard Borel space X. If Γ has a finite symmetric generating set S, then the word metric ρ_S on the action is a Borel extended metric, where $\rho_S(x, x')$ is the least length of a word w in the generators S so that $w \cdot x = x'$. The finite distance equivalence relation for ρ_S is the orbit equivalence relation of the action $\Gamma \curvearrowright X$. More generally, if Γ is not finitely generated, let $S_0 \subseteq S_1 \subseteq \ldots \subseteq \Gamma$ be a sequence of finite symmetric sets so that $\bigcup_n S_n$ generates Γ. Then the associated locally finite word metrics form a decreasing sequence $\rho_{S_0} \geq \rho_{S_1} \geq \ldots$. By applying Theorem A to such metrics, this implies that Weiss's problem has a positive answer for any countable group whose finitely generated subgroups all have volume growth $\exp(O(r^\gamma))$. However, we note that the existence of groups of that growth which are not of locally polynomial growth (and hence already known to have hyperfinite actions by [SS24]) is an open problem. In particular, Grigorchuk's gap conjecture [Gri91] states that any group of growth slower than $\exp(r^{0.5})$ has polynomial growth. The group with the currently known slowest volume growth is the first Grigorchuk group, for which Erschler and Zheng proved that its volume growth is $\exp(r^{\alpha+o(1)})$ where $\alpha \approx 0.7674$ [EZ20]. Thus, it remains an interesting open problem to increase the exponent γ in Theorem A. However, we do note that there are many interesting Borel graphs and non-free Borel actions of countable groups of intermediate volume growth bounded by $\exp(r^\alpha)$ for $\alpha \leq \gamma$, which are therefore hyperfinite by Theorem A. See, for example, [KW25; BCDN12].

A problem related to hyperfiniteness has been the study of Følner tilings or almost finiteness [Ele18] of graphs and metric spaces. Recall that a finite set $A \subseteq X$ in an extended metric space (X, ρ) is (r, ε)-Følner if $|B_r(A)| \leq (1+\varepsilon)|A|$, where $B_r(A)$ is the closed ball of radius r around A. An (r, ε)-Følner tiling of X is a partition of the space X into (r, ε)-Følner sets of uniformly bounded diameter. Recently Downarowicz, Huczek, and Zhang showed the existence of such tilings of finitely generated amenable groups [DHZ19], and [CJKMST18] studied measurable tilings of free actions of amenable groups. Using our techniques we also generalize a theorem of Downarowicz and Zhang [DZ23] (see also [BBW26]), who showed that every free Borel action of a group of subexponential growth has a Borel Følner tiling[1]. Our result generalizes

[1]The Borel result follows from their topological result for free continuous actions on zero-dimensional spaces since given a Borel action of a countable group on a Polish space X, one can change the topology of X to a finer zero-dimensional Polish topology with the same collection of Borel sets, where this action is continuous. See [Kec95, Exercise 13.5] and the discussion in [BBW26]

this theorem both to non-free actions of groups of subexponential growth and, more generally, to extended metric spaces of subexponential growth:

Theorem B. *Suppose (X,ρ) is a Borel extended metric space of uniformly subexponential volume growth $\mathrm{vol}(r) \in e^{o(r)}$. Then for every $r, \varepsilon > 0$, there is a Borel (r,ε)-Følner tiling of (X,ρ).*

Our proofs use two main new tools. First, following [BY25], we use ideas from ball-carving algorithms in theoretical computer science to find ways of efficiently subdividing a space of subexponential growth into finite pieces. However, we do a simpler deterministic ball-carving construction which does not require any use of the Lovász Local Lemma. These ideas are contained in Section 3, where we introduce these ideas in an abstract form which we call **Følner layerings**, and show they exist in all Borel metric spaces of subexponential growth. The existence of such Følner layerings can already be used to easily prove our Theorem B on Følner tilings, and we construct these tilings in Section 5. The existence of Borel Følner layerings is in fact equivalent to the existence of Borel Følner tilings for Følner Borel extended metric spaces, and we show this in Section 5.1. However, these Følner layerings are easier to construct inductively and to analyze.

Our second main tool is to generalize some techniques from the study of finite Borel asymptotic dimension and its connection to hyperfiniteness. The one-parameter dimension growth of a metric space–the function giving its dimension at each "scale"–originated in work of Gromov [Gro99] and has been well studied. We introduce a Borel analogue of Dranishnikov and Sapir's two-parameter dimension growth function $\dim^X(s,d)$ from their paper [DS12b]. Roughly, the Borel dimension $\dim_B^X(s,d)$ of (X,ρ) is the least n so there is a cover of X by $n+1$ families of Borel sets $\mathcal{U}_0,\dots,\mathcal{U}_n$ of diameter at most d so that any two distinct sets in $\mathcal{U}_i$ have distance strictly greater than s (see Definition 2.2). So the limit of this Borel dimension function gives Borel asymptotic dimension of X:

$$\mathrm{asdim}_B(X) = \lim_{s\to\infty}\lim_{d\to\infty} \dim_B^X(s,d).$$

We show that certain conditions on the growth of this Borel dimension function $\dim_B^X(s,d)$ imply hyperfiniteness. More generally, we give a hyperfiniteness condition for a countable sequence of decreasing Borel extended metrics, generalizing the work from [CJMST23, Theorem 7.3] on the hyperfiniteness of increasing unions of equivalence relations of finite asymptotic dimension:

Theorem C. *Suppose X is a standard Borel space, and $\rho_0 \geq \rho_1 \geq \dots$ is a decreasing sequence of locally finite Borel extended metrics on X. Suppose there exist sequences $(s_n)_{n\in\mathbb{N}}$, $(d_n)_{n\in\mathbb{N}}$ of positive real numbers with $s_n \to \infty$ such that for every m, for all sufficiently large n,*

$$24\dim_B^{\rho_m}(s_{n+1},d_{n+1})d_{n-1} \leq s_n \leq d_n.$$

Then $\bigcup_n E_{\rho_n}$ is hyperfinite.

In particular, suppose $a \geq 1$ and $b \geq 0$ and for every m, there is a constant C_m such that

$$\dim_B^{\rho_m}(s, C_m s^a) \in O(s^b).$$

Then $\bigcup_n E_{\rho_n}$ is hyperfinite provided $ab \leq 1/4$.

We note that there is a reasonable definition of the Borel dimension also for locally countable Borel extended metric spaces X, using covers where the sets in $\mathcal{U}_i$ may be infinite, but they must still have bounded diameter and the associated equivalence relation $E_{\mathcal{U}_i}$ whose classes are the elements of $\mathcal{U}_i$ must be smooth. Using that definition of Borel dimension, Theorem C above also remains true for locally countable Borel extended metric spaces. See Remark 4.4.

In Section 3 we use our results on Følner layerings to compute the following upper bound on the Borel dimension of a space of volume growth $\exp(O(r^\alpha))$:

Theorem D. *Suppose X is a Borel extended metric space of volume growth bounded by*

$$\mathrm{vol}(r) \in \exp(O(r^\alpha))$$

for some $\alpha \in (0,1)$. Then there exists a constant $C > 0$ such that

$$\dim_B^X(s, Cs^{1/(1-\alpha)}) \in O(s^{\alpha/(1-\alpha)^2}).$$

Together, Theorems C and D imply Theorem A for spaces with uniformly bounded volume growth $\exp(O(r^\gamma))$. We set $a = 1/(1-\alpha)$ and $b = \alpha/(1-\alpha)^2$ from the bounds in Theorem D, and then the condition $ab \leq 1/4$ from Theorem C gives hyperfiniteness for all $\alpha \leq \gamma$ where γ is as in Theorem A. The stronger version of Theorem A with only pointwise volume growth bounds then follows from the uniformly bounded case by defining a new collection of metrics with uniformly bounded volume growth whose finite distance relations have the same union.

We note that recently Jing Yu has announced an improvement of Theorem D showing that under the same assumptions, $\dim_B^X(s, Cs^{1/(1-\alpha)}) \in O(s^{\alpha/(1-\alpha)})$. The proof uses the Lovász Local Lemma and randomized ball carving methods from [BY25]. By combining this result with Theorem C as above, Yu's result improves the bound on γ in Theorem A to be the least root of $4\gamma = (1-\gamma)^2$ so $\gamma = 3-2\sqrt{2} \approx 0.1716$.

We remark that one fruitful aspect of our approach here is that it solves a difficult issue in earlier work on Weiss's problem. Several previous partial results on Weiss's problem, particularly [GJ15] and [CJMST23], proved those results for particular classes of countable amenable groups by first proving positive results for free actions of these groups, then generalizing this to all actions by using tricks around classifying possible stabilizers and the complexity of the conjugacy equivalence relations of these groups to handle the non-free case. In particular, [CJMST23] uses a theorem of Schneider and Seward [SS24, Theorem 5.1] that if all free Borel actions of countable polycyclic groups have hyperfinite orbit equivalence relations, then all Borel actions of polycyclic groups have hyperfinite orbit equivalence relations to overcome this difficulty. Indeed, one result in [CJMST23] is that free Borel actions of the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ are hyperfinite, but it is left open whether all (in particular non-free) Borel actions of that group are hyperfinite. As described in [SS24, Section 8], this difficulty in handling non-freeness is a significant issue in continuing work on Weiss's problem. However, this issue is handled easily in the present work. Because we are only using upper bounds on growth as our assumption, and this upper bound passes to subspaces and to quotient groups, our techniques work for all Borel actions of groups with bounded growth of the type we are considering, whether or not the action is free. We hope the ideas here continue to be useful in approaching the non-free case of Weiss's problem.

In Section 6, we discuss how dimension growth is related to measure-theoretic hyperfiniteness. We introduce the Borel separation index function $\mathrm{si}_B^X(s)$ of a Borel extended metric space X, as a quantitative version of the asymptotic separation index of [CJMST23]. Precisely, $\mathrm{si}_B^X(s)$ is the least n so that there is a cover of X by $n+1$ families of Borel sets $\mathcal{U}_0, \ldots, \mathcal{U}_n$ all of finite diameter (but with no uniform bound on their diameter) so that any two distinct sets in $\mathcal{U}_i$ have distance strictly greater than s. So the limit of this Borel separation index is the asymptotic separation index from [CJMST23]: $\mathrm{asi}_B(X) = \lim_{s\to\infty} \mathrm{si}_B^X(s)$. If μ is a Borel probability measure on X, we also define the μ-measurable separation index $\mathrm{si}_\mu^X(s)$ where we may discard an invariant nullset, and $\mathrm{asi}_\mu(X) = \lim_{s\to\infty} \mathrm{si}_\mu^X(s)$. We then formulate a metric analogue of Elek's [Ele12] result (refining [Kai97]) characterizing μ-hyperfiniteness of Borel graphs, see Lemma 6.1. We then use this to show Theorem E from the introduction, refining a result of Weilacher [Wei21] who proved the equivalence of (1) and (2) below, building on work of Conley and Miller [CM16]:

Theorem E. *If (X, ρ) is a locally finite Borel extended metric space, and μ is any Borel probability measure on X, then the following are equivalent:*

(1) *E_ρ is μ-hyperfinite*
(2) $\mathrm{asi}_\mu(X) \leq 1$.
(3) si_μ^X *has non-exponential growth: there is no $b > 1$ such that $\mathrm{si}_\mu^X(s) \geq b^s$ for all sufficiently large s.*

Finally, we make some conjectures concerning a Borel version of the single variable Borel dimension growth function $\mathrm{sdim}_B^X(s) = \lim_{d\to\infty} \dim_B^X(s, d)$ which was originally introduced by Gromov [Gro99]. We will call this single-variable Borel dimension growth function $\mathrm{sdim}_B^X(s)$ **the Borel dimension growth of** X. As a corollary of Theorem E, if every measure hyperfinite Borel equivalence relation is hyperfinite, then every locally finite Borel extended metric space of subexponential Borel dimension growth is hyperfinite since $\mathrm{si}_\mu^X(s) \leq \mathrm{si}_B^X(s) \leq \mathrm{sdim}_B^X(s)$ for all Borel probability measures μ on X. Since it is an open problem whether every measure-hyperfinite Borel equivalence relation is hyperfinite, we see there is a large gap between the current rates of dimension growth which imply hyperfiniteness, and the exponential dimension growth rates which are present in all known non-hyperfinite CBERs which are not measure hyperfinite.

We end the paper with a pair of conjectures on Borel dimension growth which would imply a positive solution to Weiss's problem.

Conjecture F.

(1) *Suppose $\Gamma \curvearrowright X$ is a Borel action of a finitely generated amenable group Γ on a standard Borel space X, and ρ is the word metric on this action. Then (X, ρ) has subexponential Borel dimension growth.*

(2) *If (X, ρ) is a locally finite Borel extended metric space which has subexponential Borel dimension growth, then E_ρ is hyperfinite.*

The following considerations motivate Conjecture F and provide partial evidence for its two assertions. First, part (1) is suggested by the conjectured equivalence between subexponential dimension growth and property A in the classical setting (see [DS12b, Question 2.13] and [Opp14, Page 922]). Second, Elek and Timár's work on Følner property A, strong Følner hyperfiniteness, and uniform Borel amenability provides related structural evidence [ET25; ET24]. Third, the close connection between measure-theoretic hyperfiniteness and subexponential separation-index growth in Theorem E provides evidence for part (2). Finally, several results in [CJMST23] and in the present paper establish special cases of the two assertions, both by proving Borel dimension-growth bounds for actions of certain amenable groups and by showing that sufficiently slow dimension growth implies hyperfiniteness.

Part (1) of Conjecture F would already require substantial progress in the classical study of the dimension growth of countable groups. It is an open question whether every finitely generated amenable group has subexponential dimension growth, and (1) would be a strengthening of this. Indeed, it is already an open problem whether the amenable group $\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})$ has subexponential dimension growth [DS12b, Section 4][2]. However we note that the conjecture that a group has property A if and only if it has subexponential dimension growth would imply that this is true, since every amenable group has property A. A key difficulty in this investigation is the open question of how wreath products affect dimension growth (see [DS12b, Section 4]). We note here that in parallel to this, wreath products have been a key difficulty in further progress on Weiss's question; whether there is a positive solution to Weiss's question for the group $\mathbb{Z} \wr \mathbb{Z}$ has been the key next "test problem" after the work in [CJMST23].

Part (2) of Conjecture F would require a significant new advance beyond the methods in Section 4. Those methods originate in the work of Gao and Jackson [GJ15] and have been a central tool for hyperfiniteness proofs since they were introduced. But we have pushed them as far as possible here; in Remark 4.8 we show rigorously that the condition $ab \leq 1/4$ in Theorem C cannot be improved further using the methods of Section 4. However, we remark that even modest progress here would already be very interesting. For example, even showing that part (2) is true for Borel extended metric spaces of sublinear dimension growth would substantially increase the exponent in our main Theorem A. Further, since the dimension growth of spaces whose volume growth is $\exp(O(r^\alpha))$ for some fixed $\alpha < 1$ is polynomially bounded with polynomial control by our Theorem D, proving this polynomial special case of Conjecture F would imply that all Borel extended metric spaces whose volume growth is $\exp(O(r^\alpha))$ for some fixed $\alpha < 1$ are hyperfinite.

1.1. **Acknowledgements.** The authors would like to thank Anton Bernshteyn, Atticus Cull, Felix Weilacher, and Jing Yu for helpful discussions about the material of the paper.

GPT 5.6 Pro was used to proofread a draft of this paper.

2. Preliminaries

2.1. **Extended metric spaces.** An extended metric space is one where the metric may take the value ∞. Formally, an **extended metric** on a set X is a function $\rho : X^2 \to [0, \infty]$ that is symmetric, satisfies the triangle inequality, and where $x = x' \leftrightarrow \rho(x, x') = 0$ for all $x, x' \in X$. An **extended metric space** (X, ρ) is a set X equipped with an extended metric. One example of an extended metric space is the shortest path metric on a graph that has more than one connected component, so points in different connected components are at infinite distance. We will typically use the uppercase letters X, Y, Z to denote metric

[2] This problem seems highly related to the open problem in Dranishnikov-Sapir [DS12b, Question 3.9] of understanding the function $k \mapsto \mathrm{sdim}^{\mathbb{Z}^k}(2)$ of the dimension growth of $\mathbb{Z}^k$ at scale 2 with the ℓ^1 metric.

spaces, omitting the metric ρ when it is understood. We will typically use x, y, z for points in these spaces, and A, B, C for subsets of them. If (X, ρ) is an extended metric space and $X' \subseteq X$ is a subset of X, we can equip X' with the restriction $\rho \upharpoonright X'$ of the metric ρ to X' to obtain the sub-metric space $(X', \rho \upharpoonright X')$ of (X, ρ).

For the remainder of this section, assume (X, ρ) is an extended metric space. We define the distance from $x \in X$ to a set $A \subseteq X$ by $\rho(x, A) \coloneqq \inf_{y \in A} \rho(x, y)$, and if $A, B \subseteq X$ their distance is $\rho(A, B) \coloneqq \inf_{x \in A, y \in B} \rho(x, y)$. The **diameter** of A is $\operatorname{diam}(A) \coloneqq \sup_{x,y \in A} \rho(x, y)$. If $r \in [0, \infty)$ and $x \in X$, we let

$$B_r(x) \coloneqq \{x' \in X : \rho(x, x') \leq r\}$$

denote the **closed r-ball around** x. Note that the cardinality of any 0-ball is 1 since $B_0(x) = \{x\}$. If $A \subseteq X$ is a set, we define $B_r(A) \coloneqq \{x' \in X : \rho(x', A) \leq r\}$ to be the set of all points of distance at most r from A. In case we need to specify the metric or space in our definitions (such as when we have more than one metric on a space, or a point or set is contained in multiple different metric spaces), we add ρ or X as a superscript to our notations such as $B_r^X(A)$ or $\operatorname{diam}^\rho(A)$ to emphasize the metric space we are working in.

We say an extended metric space (X, ρ) is **locally finite** if $B_r(x)$ is finite for every $x \in X$ and $r \in [0, \infty)$, and we say it is **locally countable** if $B_r(x)$ is countable for every $x \in X$ and $r \in [0, \infty)$. We define the **(uniform upper) volume growth** of the space X to be the function giving the supremum of the cardinality of all r-balls in X for each r:

$$\operatorname{vol}(r) \coloneqq \sup_{x \in X} |B_r(x)|.$$

We say that X has uniformly bounded volume growth if $\operatorname{vol}(r) < \infty$ for all $r > 0$.

Some important classes of growth are as follows. We say that a real valued function f has **subexponential** growth if for every $b > 1$, $f(x) < b^x$ for all sufficiently large x. Equivalently, $\log(f(x)) \in o(x)$. One important growth bound that implies subexponential growth is the existence of $C > 0$ and $\alpha < 1$ such that $f(x) \leq \exp(Cx^\alpha)$ for all sufficiently large x. We will write $f \in \exp(O(x^\alpha))$ to mean this. This condition is equivalent to the weaker condition that there is $C, D > 0$ such that $f(x) \leq D \exp(Cx^\alpha)$ for all sufficiently large x. A final condition on growth that is weaker than having subexponential growth is non-exponential growth, i.e. not having an exponential function as a lower bound. We say that f has **non-exponential** growth if there does not exist any $b > 1$ such that $f(x) \geq b^x$ for all sufficiently large x.

If $A \subseteq X$, we define the **external r-boundary** of A to be:

$$\partial_r(A) \coloneqq B_r(A) \setminus A.$$

We say that a nonempty finite set A is **(r, ε)-Følner** if $|\partial_r(A)| \leq \varepsilon |A|$, or equivalently if $|B_r(A)| \leq (1+\varepsilon)|A|$. We say that A is simply **ε-Følner** if it is $(1, \varepsilon)$-Følner.

We will often deal with families $\mathcal{U} \subseteq \mathcal{P}(X)$ of subsets of X, where $\mathcal{P}(X)$ denotes the powerset of X, and we will use uppercase calligraphic letters like $\mathcal{A}, \mathcal{B}, \mathcal{U}$ to denote such collections. All such families of sets we consider in this paper will be collections of sets that all have finite diameter. The **support of $\mathcal{U}$**, denoted

$$\operatorname{supp}(\mathcal{U}) \coloneqq \bigcup_{A \in \mathcal{U}} A$$

is the union of all its elements. $\mathcal{U} \subseteq \mathcal{P}(X)$ is said to be a **cover** of X if $\operatorname{supp}(\mathcal{U}) = X$. We say that $\mathcal{U}$ is a **disjoint family** if all distinct $A, B \in \mathcal{U}$ are disjoint. We say that $\mathcal{U}$ has **diameter uniformly bounded by** r if $\operatorname{diam}(A) \leq r$ for all $A \in \mathcal{U}$. We say that $\mathcal{U}$ is **s-separated** if $\rho(A, B) > s$ for all distinct $A, B \in \mathcal{U}$. Hence, a 0-separated family is disjoint. We let

$$B_r(\mathcal{U}) \coloneqq \{B_r(A) : A \in \mathcal{U}\}$$

denote the collection consisting of the closed r-balls around every $A \in \mathcal{U}$.

Suppose $\mathcal{U} = (\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_n)$ is a finite sequence of families of subsets of X, where $\mathcal{U}_i \subseteq \mathcal{P}(X)$ for every i. We define the support of this sequence

$$\operatorname{supp}(\mathcal{U}) = \operatorname{supp}(\mathcal{U}_0, \ldots, \mathcal{U}_n) \coloneqq \operatorname{supp}(\mathcal{U}_0) \cup \ldots \cup \operatorname{supp}(\mathcal{U}_n)$$

to be the union of the supports of the $\mathcal{U}_i$. We will say that $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_n$ **covers** X if $\operatorname{supp}(\mathcal{U}_0, \ldots, \mathcal{U}_n) = X$.

2.2. **Descriptive set theory.** Our standard reference for basic descriptive set theory is [Kec95]. Our descriptive set-theoretic definitions and conventions follow that book. A **Borel extended metric space** (see [CJMST23, Section 2.3]) is a standard Borel space X equipped with an extended metric $\rho\colon X^2 \to [0,\infty]$ such that the function ρ is Borel.

The **finite distance equivalence relation** on (X,ρ), denoted E_ρ, is the equivalence relation on X where $x \mathrel{E_\rho} x'$ if and only if $\rho(x,x') < \infty$. Note that if (X,ρ) is a locally countable extended metric space, then E_ρ is a **countable Borel equivalence relation** or CBER, meaning the relation E_ρ is Borel as a subset of $X \times X$, and all of its equivalence classes are countable. A CBER E is **hyperfinite** if it is an increasing union $E = \bigcup_n F_n$ of CBERs $F_0 \subseteq F_1 \subseteq \dots$ with finite classes. See [Kec25] for a recent survey of the theory of countable Borel equivalence relations, which is our standard reference for concepts of hyperfiniteness of CBERs, and basic properties about them.

Most of the families of sets $\mathcal{F} \subseteq \mathcal{P}(X)$ that we will use in the paper will be disjoint families of sets. We will call such a family **Borel** if $\operatorname{supp}(\mathcal{F})$ is Borel, and the equivalence relation $E_{\mathcal{F}}$ on $\operatorname{supp}(\mathcal{F})$ whose classes are exactly the sets in $\mathcal{F}$ is a Borel equivalence relation on $\operatorname{supp}(\mathcal{F})$.

In this paper, we largely work in the setting where the metric space (X,ρ) is locally finite, and our families $\mathcal{F} \subseteq \mathcal{P}(X)$ have bounded diameter, and thus each $A \in \mathcal{F}$ is finite. In this case, we could equivalently work with the standard Borel space $[X]^{<\infty}$ of finite subsets of X, in which case $\mathcal{F}$ is Borel as a subset of that space iff it is Borel in the sense of the above definition (i.e. if $\operatorname{supp}(\mathcal{F})$ and $E_{\mathcal{F}}$ are Borel).

We omit some routine verifications that some of our constructions are Borel in our proofs, since they are defined by “local rules” and are hence Borel by [Pik21, Lemma 5.17], or where the Borelness follows from Lusin-Novikov uniformization.

2.3. **Dimension growth.** Recall asymptotic dimension was introduced by Gromov [Gro93], and one of its equivalent definitions uses the following notion we will call an (s,d)-cover, following [YZ24]:

Definition 2.1 ([YZ24])**.** Suppose (X,ρ) is an extended metric space and $s,d \in [0,\infty)$. An (s,d)**-cover** $\mathcal{U} = (\mathcal{U}_0,\dots,\mathcal{U}_n)$ of X is a cover where each $\mathcal{U}_i$ is s-separated and has diameter uniformly bounded by d. We say that the (s,d)-cover $\mathcal{U} = (\mathcal{U}_0,\dots,\mathcal{U}_n)$ has $n+1$ **elements** or **colors**.

Equivalently, one may color the points of X by $n+1$ colors and consider the equivalence relation generated by pairs of points of the same color that are at distance at most s, which is called the s-walk equivalence relation. Then the existence of an (s,d)-cover is equivalent to the statement that the s-walk equivalence relation has classes of diameter at most d.

Using this definition of an (s,d)-cover, X has **asymptotic dimension** $n \in \mathbb{N}$ if n is the least such that for every $s \geq 0$ there exists d for which there is an (s,d)-cover of X with $n+1$ elements. (See [BD08] for a discussion of many other equivalent definitions).

More generally, it is natural to consider the two-parameter function that records, for each separation bound s and diameter bound d, the least n such that there is an (s,d)-cover of X with $n+1$ elements. This was first formally defined in the paper of Dranishnikov-Sapir [DS12b].[3] Precisely, the (s,d)**-dimension** of a metric space X is:

$$\dim^X(s,d) \coloneqq \inf\{n\colon \text{there is an } (s,d)\text{-cover of } X \text{ with } n+1 \text{ elements}\}.$$

Note that $\dim^X(s,d)$ is increasing in the first coordinate, and decreasing in the second coordinate. The limit of $\dim^X$ gives the usual asymptotic dimension of X:

$$\operatorname{asdim}(X) \coloneqq \lim_{s\to\infty}\lim_{d\to\infty}\dim^X(s,d).$$

The **single-variable dimension growth function**

$$\operatorname{sdim}^X(s) = \lim_{d\to\infty}\dim^X(s,d)$$

has also been studied extensively, and was also originally defined by Gromov [Gro99, Section 6.F], who called it the dimension of X on the scale s. It is an open question whether a finitely generated group has

[3]We emphasize that these definitions come from the expanded and updated 2012 arXiv v4 of the paper and not the 2011 published version of the paper [DS11]. Some errors in that version were corrected in a corrigendum [DS12a], prompting the expanded arXiv v4: the “final version” of the paper.

property A if and only if it has subexponential dimension growth (see [DS12b, Question 2.13] and [Opp14, Page 922], and see also [Oza12]). We mention also that if $d\colon [0,\infty) \to [0,\infty)$ is an increasing function, then **the d-controlled dimension of X** is the function $s \mapsto \dim^X(s, d(s))$. For example, [DS12b] proves many bounds on the d-controlled dimension growth for various groups and choices of functions d.

We remark that an alternate definition of asymptotic dimension based on "uncolored covers" is as follows. Say the r-multiplicity of a cover $\mathcal{U} \subseteq \mathcal{P}(X)$ of X is $\sup_{x\in X} |\{U \in \mathcal{U} \colon U \cap B_r(x) \neq \emptyset\}|$, that is the supremum of the number of elements of $\mathcal{U}$ that are met by some r-ball. Then $\operatorname{asdim}(X) \leq n$ if and only if for every r there exists d and a cover $\mathcal{U} \subseteq \mathcal{P}(X)$ of X by sets of diameter at most d so that the r-multiplicity of $\mathcal{U}$ is $\leq n+1$ [BD08, Theorem 19]. Based on this equivalent definition of asymptotic dimension, one could define "uncolored" dimension growth functions. For example, let $\operatorname{udim}^X(r,d) = \inf\{n\colon$ there exists a cover $\mathcal{U} \subseteq \mathcal{P}(X)$ of X by sets of diameter at most d so that the r-multiplicity of $\mathcal{U}$ is $n+1\}$. We note that $\operatorname{udim}^X(s/2,d) \leq \dim^X(s,d)$ simply by taking the union of the elements of a "colored" cover to obtain an uncolored one. Dranishnikov has defined yet another related dimension growth function based on Lebesgue numbers [Dra06]. The exact relationship between all these colored vs uncolored dimension growth functions is an open problem [DS12b, Question 1.5]. We will not study these "uncolored" dimension growth functions; they do not correspond to the original definition of dimension growth given by Gromov, and they are less related to our main concerns.

This paper studies a Borel version of Dranishnikov-Sapir's two-parameter dimension growth function, which we define for locally finite metric spaces.

Definition 2.2. Let (X,ρ) be a locally finite Borel extended metric space. A **Borel (s,d)-cover** of X is an (s,d)-cover $\mathcal{U} = (\mathcal{U}_0, \ldots, \mathcal{U}_n)$ where each $\mathcal{U}_i$ is Borel. We define the two-parameter Borel dimension function of X by:

$$\dim_B^X(s,d) \coloneqq \inf\{n \in \mathbb{N}\colon \text{there is a Borel } (s,d)\text{-cover of } X \text{ with } n+1 \text{ elements}\}.$$

We define the associated single-variable Borel dimension function of X by:

$$\operatorname{sdim}_B^X(s) \coloneqq \lim_{d\to\infty} \dim_B^X(s,d).$$

Note that the Borel asymptotic dimension of X as defined in [CJMST23] for locally finite metric spaces is related to its Borel dimension growth by

$$\operatorname{asdim}_B(X) = \lim_{s\to\infty} \lim_{d\to\infty} \dim_B^X(s,d) = \lim_{s\to\infty} \operatorname{sdim}_B^X(s).$$

We note that one key upper bound on $\dim_B^X(s,d)$ is $\operatorname{vol}(s)$, just as it is classically (see [DS12b, Lemma 2.3]).

Proposition 2.3. *Suppose (X,ρ) is a Borel extended metric space of uniformly bounded volume growth. Then $\dim_B^X(s,d) \leq \operatorname{vol}(s) - 1$ for all d, and so $\operatorname{sdim}_B^X(s) < \operatorname{vol}(s)$.*

Proof. Let G be the graph on X where distinct x and y are adjacent if $\rho(x,y) \leq s$. So each vertex has at most $\operatorname{vol}(s)-1$ neighbors. Then G has a Borel $\operatorname{vol}(s)$-coloring $f\colon X \to \{0,\ldots,\operatorname{vol}(s)-1\}$ by [KST99, Proposition 4.6]. So letting $\mathcal{U}_i = \{\{x\}\colon f(x) = i\}$ be the singletons consisting of points of color i, each family $\mathcal{U}_i$ is Borel, s-separated, and has diameter 0, and hence $\mathcal{U}_0, \ldots, \mathcal{U}_{\operatorname{vol}(s)-1}$ is an $(s,0)$-cover of X. □

Hence, two particularly important classes of Borel extended metric spaces always have $\dim_B^X(s,d)$ and hence $\operatorname{sdim}_B^X(s)$ bounded above by exponential functions in s, since they have exponential volume growth: the path length metric on bounded degree Borel graphs, and the word metric on a Borel action of a finitely generated group. Hence, a key dividing line here will be between those spaces with exponential dimension growth and non-exponential dimension growth.

In Section 6 we will also consider the Borel separation index of X, by relaxing the condition that the collections $\mathcal{U}_i$ of sets in the covers of X have uniformly bounded diameter to the requirement that these sets each have bounded diameter:

Definition 2.4. The Borel separation index of a locally finite Borel extended metric space (X,ρ) is the function defined by $\operatorname{si}_B^X(s) \coloneqq \inf\{n\colon$ there is a Borel cover $\mathcal{U}_0,\ldots,\mathcal{U}_n$ of X by $n+1$ families of sets of bounded diameter such that each $\mathcal{U}_i$ is s-separated$\}$. If μ is a finite Borel measure on X, we also define the μ-measurable separation index function: $\mathrm{si}_\mu^X(s) \coloneqq \min\{\mathrm{si}_B^{X'}(s) \colon X' \subseteq X \text{is an } E_\rho\text{-invariant } \mu\text{-conull Borel set}\}$. Finally, define the μ-measurable asymptotic separation index by $\mathrm{asi}_\mu(X) \coloneqq \lim_{s\to\infty} \mathrm{si}_\mu^X(s)$.

Note the Borel asymptotic separation index $\mathrm{asi}_B(X)$ of X as defined in [CJMST23] is the limit of its Borel separation index function:

$$\mathrm{asi}_B(X) = \lim_{s\to\infty} \mathrm{si}_B^X(s).$$

Note that $\mathrm{si}_B^X(s) \leq \mathrm{sdim}_B^X(s)$, though the inequality may be strict since a sequence $\mathcal{U}_0, \dots, \mathcal{U}_n$ witnessing $\mathrm{si}_B^X(s) = n$ may consist of finite sets that are not of uniformly bounded diameter. For example, by the results of [CJMST23], a free Borel action of $\mathbb{Z}^k$ on a standard Borel space X with the usual word metric gives an extended metric space X with $\mathrm{asdim}_B(X) = k$, but $\mathrm{asi}_B(X) = 1$. So if $k \geq 2$, then $\mathrm{si}_B^X(s) = 1 < k = \mathrm{sdim}_B^X(s)$ for all sufficiently large s.

We finish by briefly noting how rescaling a metric changes its volume growth and dimension function.

Proposition 2.5. *Suppose (X, ρ) is an extended metric space and $c > 0$. If $c\rho$ is the rescaling of the metric ρ by c, then for all $r, s, d > 0$,*

$$\mathrm{vol}^{c\rho}(cr) = \mathrm{vol}^\rho(r) \quad \textit{and} \quad \dim^\rho(s, d) = \dim^{c\rho}(cs, cd).$$

Hence, if (X, ρ) is a Borel locally finite extended metric space, then

$$\dim_B^\rho(s, d) = \dim_B^{c\rho}(cs, cd).$$

Proof. This follows since $B_{cr}^{c\rho}(x) = B_r^\rho(x)$, and a set has $c\rho$-diameter cd iff it has ρ-diameter d and a family of sets is cs-separated for $c\rho$ iff it is s-separated for ρ. □

We note one consequence of rescaling: if $\alpha > 0$, $C > 0$, $b > 1$, and $\mathrm{vol}^\rho(r) \leq Cb^{r^\alpha}$ for all $r \in [0, \infty)$, then $\mathrm{vol}^{c\rho}(r) \leq Cb^{(r/c)^\alpha} = C\big(b^{c^{-\alpha}}\big)^{r^\alpha}$. Hence, by rescaling the metric, we can change the base in this exponential volume growth bound to be any real number greater than 1. Hence, the base is unimportant in our results, and we typically use the base of the natural logarithm e in the statement of our theorems.

3. Følner layering and Ball carving

The key idea we use to analyze the dimension growth of spaces of subexponential growth will be a certain way of decomposing such a space inductively using finitely many maximal r-separated disjoint collections of (r, ε)-Følner sets. These ideas originate from ball-carving algorithms from distributed computing in theoretical computer science (see [Roz24, Theorem 1.10]). The following definition will be key to this analysis:

Definition 3.1. Let $r, \varepsilon > 0$. An (r, ε)**-Følner layering** of an extended metric space X is a finite sequence $\mathcal{F}_0, \dots, \mathcal{F}_{n-1}$ of families with associated subspaces $X_0 = X$ and $X_{i+1} \coloneqq X_i \setminus B_r(\mathrm{supp}(\mathcal{F}_i))$ for $i < n$, so that each $\mathcal{F}_i$ has uniformly bounded diameter, and so that for all $i < n$,

(1) $\mathcal{F}_i \subseteq \mathcal{P}(X_i)$ is a family of (r, ε)-Følner subsets of X_i.
(2) $\mathcal{F}_i$ is r-separated,
(3) $B_r(\mathrm{supp}(\mathcal{F}_0, \dots, \mathcal{F}_{n-1})) = X$.

A ε**-Følner layering** of X is an $(1, \varepsilon)$-Følner layering.

In item (1) in Definition 3.1 we emphasize that we mean a Følner subset of the subspace X_i and not a Følner subset of X. That is, for all $A \in \mathcal{F}_i$, $A \subseteq X_i$, and $|B_r^{X_i}(A)| \leq (1 + \varepsilon)|A|$.

Note that the union $\mathcal{F}_0 \cup \ldots \cup \mathcal{F}_{n-1}$ of all the sets in an (s, ε)-Følner layering is s-separated, since each $\mathcal{F}_i$ is s-separated and $\mathcal{F}_j$ is disjoint from $B_s(\mathrm{supp}(\mathcal{F}_i))$ for $j > i$ by the definition of X_j and the condition that $\mathcal{F}_j \subseteq \mathcal{P}(X_j)$. We will eventually use these s-separated families coming from Følner layerings to make (s, d)-covers which we use to bound the dimension growth of metric spaces of subexponential growth.

Our first goal will be to show that every Borel extended metric space of subexponential volume growth has an (r, ε)-Følner layering for every $r, \varepsilon > 0$. It will suffice to prove theorems just in the case $r = 1$, and then by rescaling the metric we can obtain similar results for all r. Hence most of our results below will just be about ε-Følner layerings.

We begin with a well known proposition showing some ball must be an ε-Følner set in any space of uniformly subexponential growth.

Proposition 3.2. *Suppose (X, ρ) is an extended metric space, $\varepsilon > 0$, and r is a positive integer such that $\mathrm{vol}^X(r+1) \leq (1+\varepsilon)^{r+1}$. Then for every $x \in X$, there is an ε-Følner set F containing x such that $F \subseteq B_r(x)$.*

Proof.

$$\frac{|B_1(x)|}{|B_0(x)|}\frac{|B_2(x)|}{|B_1(x)|}\cdots\frac{|B_{r+1}(x)|}{|B_r(x)|} = \frac{|B_{r+1}(x)|}{|B_0(x)|} \leq \mathrm{vol}(r+1) \leq (1+\varepsilon)^{r+1},$$

so there is some $s \leq r$ such that $\frac{|B_{s+1}(x)|}{|B_s(x)|} \leq 1+\varepsilon$. So $F = B_s(x)$ is the desired ε-Følner set. □

Note that the condition $\mathrm{vol}^X(r+1) \leq (1+\varepsilon)^{r+1}$ will hold for sufficiently large r if X has subexponential volume growth. Now we can show that a Borel ε-Følner layering exists in a space with this volume growth bound:

Lemma 3.3. *Suppose $\varepsilon > 0$, r is a fixed positive integer, and X is a locally finite Borel extended metric space such that $\mathrm{vol}^X(r+1) \leq (1+\varepsilon)^{r+1}$. Then there is a Borel ε-Følner layering of X by sets of diameter uniformly bounded by $2r$.*

Proof. Set $X_0 = X$. Given X_i, let $\mathcal{F}_i$ be a Borel maximal 1-separated family of ε-Følner sets in X_i of diameter bounded by $2r$, and then let $X_{i+1} = X_i \setminus B_1(\mathrm{supp}(\mathcal{F}_i))$. To see that such a $\mathcal{F}_i$ exists, consider the Borel graph on all $2r$-bounded ε-Følner subsets of X_i where distinct A, B are adjacent if $\rho(A, B) \leq 1$. This graph is locally finite since ρ is locally finite. Hence it has a Borel maximal independent set by [KST99, Prop 4.2, 4.5].

We claim that after $n = \mathrm{vol}(r)$ steps of this process, the 1-balls around the sets $\mathcal{F}_0, \ldots, \mathcal{F}_{n-1}$ cover the space: $B_1(\mathrm{supp}(\mathcal{F}_0, \ldots, \mathcal{F}_{n-1})) = X$, and hence $\mathcal{F}_0, \ldots, \mathcal{F}_{n-1}$ is the desired ε-Følner layering. Since X_i is a subspace of X, its volume growth is bounded by $\mathrm{vol}^{X_i}(r+1) \leq (1+\varepsilon)^{r+1}$. Hence, by Proposition 3.2 applied to the space X_i, every $x \in X_i$ is contained in some ε-Følner set $A \subseteq B_r^{X_i}(x)$. Maximality of $\mathcal{F}_i$ implies that A meets $B_1^{X_i}(F)$ for some $F \in \mathcal{F}_i$. Hence, since $X_{i+1} = X_i \setminus B_1(\mathrm{supp}(\mathcal{F}_i))$, we have shown

$$|B_r(x) \cap X_{i+1}| < |B_r(x) \cap X_i|$$

for every $x \in X_i$. This implies X_n is empty since if $x \in X_n$, then $|B_r(x) \cap X_i|$, for $0 \leq i \leq n$, would be a decreasing sequence of positive integers, but $|B_r(x) \cap X_0| = |B_r(x)| \leq \mathrm{vol}(r) = n$. □

As a corollary, every Borel extended metric space of uniformly bounded subexponential volume growth has Følner layerings, by a rescaling argument. We will use this fact in Section 5.

Corollary 3.4. *Suppose (X, ρ) is a Borel extended metric space with uniformly bounded subexponential volume growth. Then for every $r, \varepsilon > 0$, X has a Borel (r, ε)-Følner layering.*

Proof. Let $\rho' = r^{-1}\rho$. This rescaled metric still has uniformly bounded subexponential volume growth (see Proposition 2.5). Hence, for some positive integer d, $\mathrm{vol}^{\rho'}(d+1) \leq (1+\varepsilon)^{d+1}$, and so by Lemma 3.3, (X, ρ') has a Borel $(1, \varepsilon)$-Følner layering with respect to ρ'. Since a family is 1-separated with respect to ρ' if and only if it is r-separated with respect to ρ, and uniformly bounded diameters remain uniformly bounded under rescaling, this layering is a Borel (r, ε)-Følner layering with respect to ρ. □

If we remove the sets in a Følner layering from a space X, we can bound the volume growth of the subspace that remains in terms of the volume growth of X as follows:

Lemma 3.5. *Suppose X is a locally finite extended metric space, and $\mathcal{F}_0, \ldots, \mathcal{F}_{n-1}$ is an ε-Følner layering of X of diameter bounded by $d \geq 0$. Let $Y = X \setminus \mathrm{supp}(\mathcal{F}_0, \ldots, \mathcal{F}_{n-1})$. Then for every finite set $A \subseteq Y$, we have $|A| \leq \varepsilon |B_{d+1}^X(A)|$. Hence,*

$$\mathrm{vol}^Y(r) \leq \varepsilon \, \mathrm{vol}^X(r+d+1) \tag{1}$$

for all $r \geq 0$.

Proof. Since $A \subseteq Y = X \setminus \mathrm{supp}(\mathcal{F}_0, \ldots, \mathcal{F}_{n-1})$ and $X = B_1(\mathrm{supp}(\mathcal{F}_0, \ldots, \mathcal{F}_{n-1}))$ by the definition of a Følner layering, every $y \in A$ is contained in $\partial_1^{X_i}(C) = B_1^{X_i}(C) \setminus C$ for some $i < n$ and $C \in \mathcal{F}_i$. Let $\mathcal{C}_i = \{C \in \mathcal{F}_i \colon \partial_1^{X_i}(C) \cap A \neq \emptyset\}$ be the elements of $\mathcal{F}_i$ whose 1-boundary intersects A. So $A \subseteq$

$\operatorname{supp}(\partial_1^{X_0}(\mathcal{C}_0), \ldots, \partial_1^{X_{n-1}}(\mathcal{C}_{n-1}))$. Since the sets in $\mathcal{F}_i$ have diameter bounded by d, $\operatorname{supp}(\mathcal{C}_0, \ldots, \mathcal{C}_{n-1}) \subseteq B^X_{d+1}(A)$. Note that $|\operatorname{supp}(\mathcal{C}_0, \ldots, \mathcal{C}_{n-1})| = |\operatorname{supp}(\mathcal{C}_0)| + \ldots + |\operatorname{supp}(\mathcal{C}_{n-1})|$ since the sets $\operatorname{supp}(\mathcal{C}_i)$ are pairwise disjoint. Finally, note that since every $C \in \mathcal{C}_i$ is an ε-Følner set in X_i, and a finite union of disjoint ε-Følner sets is ε-Følner, $\operatorname{supp}(\mathcal{C}_i)$ is an ε-Følner subset of X_i. Hence,

$$|A| \leq |\operatorname{supp}(\partial_1^{X_0}(\mathcal{C}_0))| + \ldots + |\operatorname{supp}(\partial_1^{X_{n-1}}(\mathcal{C}_{n-1}))| \leq \varepsilon|\operatorname{supp}(\mathcal{C}_0)| + \ldots + \varepsilon|\operatorname{supp}(\mathcal{C}_{n-1})| = \varepsilon|\operatorname{supp}(\mathcal{C}_0, \ldots, \mathcal{C}_{n-1})| \leq \varepsilon|B^X_{d+1}(A)|.$$

Finally, Equation (1) follows by applying the above to the sets $A = B^Y_r(x)$ for each x, since then $B^X_{d+1}(A) \subseteq B^X_{r+d+1}(x)$, so $|B^Y_r(x)| \leq \varepsilon|B^X_{r+d+1}(x)|$. □

We will use the above lemma to show that in a metric space of subexponential growth, if we repeatedly remove Følner layerings from the space, after finitely many steps, the Følner layerings must cover the entire space. Hence this gives an upper bound on the Borel dimension $\dim^X_B(s, d)$ of Borel extended metric spaces of subexponential growth.

Lemma 3.6. *Suppose $\varepsilon > 0$ and r, n are positive integers. Then every locally finite Borel extended metric space X satisfying $\operatorname{vol}^X(r + 1) \leq (1 + \varepsilon)^{r+1}$ and $\operatorname{vol}^X((2r + 1)n) < \frac{1}{\varepsilon^n}$ has $\dim^X_B(1, 2r) < n$.*

Proof. Let $Y_0 = X$. For each $i \geq 0$, by Lemma 3.3, let $\mathcal{F}_i = (\mathcal{F}_{i,0}, \ldots, \mathcal{F}_{i,n_i-1})$ be a Borel ε-Følner layering of Y_i of diameter bounded by $2r$, and let $\mathcal{U}_i = \mathcal{F}_{i,0} \cup \ldots \cup \mathcal{F}_{i,n_i-1}$ be the union of all the sets in this layering, so $\mathcal{U}_i$ is 1-separated and has diameter bounded by $2r$. Let $Y_{i+1} = Y_i \setminus \operatorname{supp}(\mathcal{U}_i)$. It suffices to show that Y_n is empty, since then $\mathcal{U}_0, \ldots, \mathcal{U}_{n-1}$ is a Borel $(1, 2r)$-cover of X.

After deleting each layering, note that $\operatorname{vol}^{Y_{i+1}}(t) \leq \varepsilon \operatorname{vol}^{Y_i}(t + 2r + 1)$ for every $t \geq 0$, by applying Lemma 3.5 to Y_i. Hence, by induction $\operatorname{vol}^{Y_n}(0) \leq \varepsilon^n \operatorname{vol}^{Y_0}((2r + 1)n) < 1$, and so Y_n is empty. □

Remark 3.7. We remark that Proposition 3.2, Lemma 3.3, Lemma 3.5, and Lemma 3.6 together encapsulate the simple deterministic ball carving algorithm from the theory of local algorithms originally introduced in [AGLP89], see also [Roz24, Section 1.5]. In that language, a (n, d)-network decomposition of a graph is a cover of the vertices by families $\mathcal{C}_0, \ldots, \mathcal{C}_{n-1}$ (called clusters) of subsets where each $\mathcal{C}_i$ is 1-separated, and has diameter uniformly bounded by d. In our language, this is a $(1, d)$-cover with n elements. The ball carving algorithm makes each cluster $\mathcal{C}_i$ by sequentially starting at each point x of the space and taking the least radius s for which the ball $B_s(x)$ is ε-Følner, as in Proposition 3.2. Then one adds this set to the cluster and removes both it and the 1-ball around it from the space to make a collection of 1-separated sets. This cluster corresponds to what we are calling a Følner layering, where each layer in our Følner layering roughly corresponds to a single step in this clustering process which takes place at far enough apart distance in the graph to not interfere. The ball carving algorithm then removes the points in $\operatorname{supp}(\mathcal{C}_i)$ from the space, and does the same process again to make the next cluster $\mathcal{C}_{i+1}$. Then the analysis in Lemma 3.5 and Lemma 3.6 shows a bound on the total number of clusters made this way that are needed to cover the space.

We record a corollary of the above, which comes from a careful choice of ε in the above lemma.

Corollary 3.8. *If X is a locally finite Borel extended metric space, satisfying*

$$\left(\operatorname{vol}(r + 1)^{1/(r+1)} - 1\right)^n \operatorname{vol}((2r + 1)n) < 1,$$

for $r, n \in \mathbb{N}$, then $\dim^X_B(1, 2r) < n$.

Proof. This follows by applying Lemma 3.6 with $\varepsilon \coloneqq \operatorname{vol}(r + 1)^{1/(r+1)} - 1$. □

We record another corollary which uses a rescaling argument to bound dimension growth for all separations, and allows real number parameters for both the separation and diameter.

Lemma 3.9. *Suppose X is a locally finite Borel extended metric space. Then for all $s > 0$, $d \geq 2s$, and positive integers n, if $\left(\operatorname{vol}(d)^{2s/d} - 1\right)^n \operatorname{vol}(2dn) < 1$ then $\dim^X_B(s, d) < n$.*

Proof. Define $r := \lfloor \frac{d}{2s} \rfloor$, and note that $r \geq 1$. So $s(r+1) \leq s(2r) \leq d$, and $r+1 \geq \frac{d}{2s}$, hence

$$\left(\mathrm{vol}(s(r+1))^{1/(r+1)} - 1\right)^n \leq \left(\mathrm{vol}(d)^{2s/d} - 1\right)^n,$$

since vol is nondecreasing. We also have $s(2r+1)n \leq s(4r)n \leq 2dn$, and so

$$\mathrm{vol}(s(2r+1)n) \leq \mathrm{vol}(2dn).$$

And hence

$$\left(\mathrm{vol}(s(r+1))^{1/(r+1)} - 1\right)^n \mathrm{vol}(s(2r+1)n) < 1.$$

If we rescale the metric ρ by the factor s^{-1}, then $\mathrm{vol}^{s^{-1}\rho}(r) = \mathrm{vol}^{\rho}(sr)$ by Proposition 2.5. So by Corollary 3.8

$$\dim_B^{\rho}(s,d) = \dim_B^{s^{-1}\rho}(1, d/s) \leq \dim_B^{s^{-1}\rho}(1, 2r) < n.$$

□

We note that this result already gives a simpler proof of the Bernshteyn-Yu result [BY25, Corollary 3.16] that Borel metric spaces of polynomial growth $O(r^k)$ have Borel asymptotic dimension at most k, and hence their Theorem from [BY25] that every Borel graph of polynomial growth is hyperfinite [BY25, Corollary 1.16].

Corollary 3.10. *Suppose X is a Borel extended metric space and $\mathrm{vol}^X(r) \in O(r^k)$ for $k \in \mathbb{N}$. Then $\dim_B^X(s, s^{k+2}) \leq k$ for all sufficiently large $s > 0$, hence $\mathrm{asdim}_B(X) \leq k$, since $\dim_B^X(s,d)$ is decreasing in d and increasing in s.*

Proof. Let s be sufficiently large, let $d := s^{k+2}$, and note that $d \geq 2s$ for sufficiently large s. We will apply Lemma 3.9 with $n = k+1$. Since $\mathrm{vol}(r) \in O(r^k)$, taking k as a fixed constant, and computing growth as a function of s we have:

$$\frac{2s}{d}\log\left(\mathrm{vol}(d)\right) = \frac{2s}{d}\log\left(\mathrm{vol}(s^{k+2})\right) \in O(s^{-k-1}\log s) \to 0 \text{ as } s \to \infty.$$

Hence $\mathrm{vol}(d)^{2s/d} = \exp(O(s^{-k-1}\log s))$, and since $\exp(x) \leq 3x+1$ for sufficiently small $x \in (0,1)$, we have $\mathrm{vol}(d)^{2s/d} - 1 = O(s^{-k-1}\log s)$.

Note also $\mathrm{vol}(2d(k+1)) = \mathrm{vol}(O(s^{k+2})) \in O(s^{k(k+2)})$. Therefore

$$\begin{aligned}\left(\mathrm{vol}(d)^{2s/d} - 1\right)^{k+1}\mathrm{vol}(2d(k+1)) = O\left(\left(s^{-k-1}\log s\right)^{k+1} s^{k(k+2)}\right) &= O\left(s^{-(k+1)^2 + k(k+2)}(\log s)^{k+1}\right) \\ &= O\left(\frac{(\log s)^{k+1}}{s}\right) \to 0\end{aligned}$$

as $s \to \infty$. Thus, by Lemma 3.9, $\dim_B^X(s,d) < k+1$ for all sufficiently large s and hence $\dim_B^X(s,d) \leq k$. □

We can now bound the Borel dimension growth of any Borel extended metric space of growth $\mathrm{vol}(r) \in \exp(O(r^\alpha))$ for some $\alpha \in (0,1)$, which gives Theorem D from the introduction:

Theorem 3.11. *Suppose X is a Borel extended metric space of volume growth bounded by*

$$\mathrm{vol}(r) \in \exp(O(r^\alpha))$$

for some $\alpha \in (0,1)$. Then there exists a constant $C > 0$ such that

$$\dim_B^X(s, Cs^{1/(1-\alpha)}) \in O(s^{\alpha/(1-\alpha)^2}).$$

Proof. Fix $b > 1$ such that $\mathrm{vol}(r) \leq b^{r^\alpha}$ for sufficiently large r. Fix $C > 0$ so that $\delta := b^{2/C^{1-\alpha}} - 1 < 1$, and fix $D > 0$ so that $\delta b^{2^\alpha C^\alpha / D^{1-\alpha}} < 1$. Define

$$d(s) := Cs^{1/(1-\alpha)} \text{ and } c(s) := \left\lceil Ds^{\alpha/(1-\alpha)^2} \right\rceil + 1.$$

For sufficiently large s, if we let $d = d(s)$ and $c = c(s) - 1$, then we have $d \geq 2s$, and

$$\mathrm{vol}(d)^{2s/d} \leq (b^{d^\alpha})^{2s/d} = b^{2s/d^{1-\alpha}} = b^{2/C^{1-\alpha}} = \delta + 1$$

and

$$\mathrm{vol}(2dc)^{1/c} \le (b^{(2dc)^\alpha})^{1/c} = b^{2^\alpha d^\alpha / c^{1-\alpha}} \le b^{2^\alpha C^\alpha / D^{1-\alpha}} < \delta^{-1}.$$

Hence

$$(\mathrm{vol}(d)^{2s/d} - 1)^c \, \mathrm{vol}(2dc) \le \delta^c (b^{2^\alpha C^\alpha / D^{1-\alpha}})^c < 1.$$

So by Lemma 3.9, $\dim_B^X(s, d(s)) < c + 1 = c(s)$. □

4. Hyperfiniteness

In this section, we begin by proving Theorem C that shows sufficiently slow Borel dimension growth implies hyperfiniteness. Our result generalizes [CJMST23, Theorem 7.3] which itself generalized the main construction in Gao-Jackson [GJ15]. Our proof is a much more careful quantitative refinement of the same basic idea from [CJMST23, Theorem 7.3].

We make a few definitions for the purposes of the proof:

Definition 4.1. Suppose X is a set, and $\mathcal{U} = (\mathcal{U}_0, \ldots, \mathcal{U}_n)$ is a finite sequence of collections $\mathcal{U}_i \subseteq \mathcal{P}(X)$. If $A \subseteq X$, we define the **star of A with respect to $\mathcal{U}$** to be the union of all the sets from the $\mathcal{U}_i$ that it intersects:

$$A * \mathcal{U} = \bigcup \{B \colon \exists i (B \in \mathcal{U}_i) \wedge B \cap A \neq \emptyset\}.$$

If $\mathcal{A} \subseteq \mathcal{P}(X)$ is a family of subsets of X, we similarly define the **star of $\mathcal{A}$ with respect to $\mathcal{U}$** to be the collection of the stars of all these sets with $\mathcal{U}$:

$$\mathcal{A} * \mathcal{U} = \{A * \mathcal{U} \colon A \in \mathcal{A}\}.$$

If $\mathcal{V}$,$\mathcal{U}$ are collections of sets, we say $\mathcal{V}$ is **generated from $\mathcal{U}$** if $\mathcal{V} = \mathcal{A} * \mathcal{U}$ for some $\mathcal{A} \subseteq \mathcal{P}(X)$. In particular, every set in $\mathcal{V}$ is a union of sets from $\mathcal{U}$.

So for example, if $\mathcal{U}$ is a cover of X, then for all $A \subseteq X$, $A \subseteq A * \mathcal{U}$. And so if $\mathcal{V} \subseteq \mathcal{P}(X)$ is a cover of X, then $\mathcal{V} * \mathcal{U}$ is also a cover of X. We also note here how the star operation affects the diameter and separation of sets, since we'll want to calculate how it changes the parameters s and d of (s, d)-covers. If sets in $\mathcal{U}$ have diameter bounded by d, then $\mathrm{diam}(A * \mathcal{U}) \le \mathrm{diam}(A) + 2d$, and if $A, B \subseteq X$ have $\rho(A, B) > s$, then $\rho(A * \mathcal{U}, B * \mathcal{U}) > s - 2d$.

Following [CJMST23, Section 4], we also use the notion of a family of sets dividing a pair:

Definition 4.2. If $\{x, x'\} \subseteq X$ and $\mathcal{F} \subseteq \mathcal{P}(X)$ we say that $\mathcal{F}$ **divides** $\{x, x'\}$ if there is an $A \in \mathcal{F}$ such that both A and $X \setminus A$ meet $\{x, x'\}$. Similarly, we say $\mathcal{U} = (\mathcal{U}_0, \ldots, \mathcal{U}_n)$ divides $\{x, x'\}$ if there is some i so that $\mathcal{U}_i$ divides $\{x, x'\}$. We define $E_\mathcal{U}$ to be the equivalence relation on X where $x \mathrel{E_\mathcal{U}} x'$ if and only if $\{x, x'\}$ is not divided by $\mathcal{U}$. Equivalently, $x \mathrel{E_\mathcal{U}} x'$ if and only if $x \in A \leftrightarrow x' \in A$ for all i and $A \in \mathcal{U}_i$.

If $\mathcal{U} = (\mathcal{U}_0, \ldots, \mathcal{U}_n)$ is a cover of X, then note that $[x]_{E_\mathcal{U}}$ will be contained in the intersection of all the sets from the $\mathcal{U}_i$ that contain x. So for example, if $\mathcal{U}$ is a cover of X by sets of uniformly bounded diameter, then the $E_\mathcal{U}$ classes will have uniformly bounded diameter.

Below we will use two different ways of controlling how dividing is related in different collections $\mathcal{A}, \mathcal{B}$ of sets. If $\mathcal{U}$ is a cover of X and $\mathcal{A}$ is generated by $\mathcal{U}$, then if $\mathcal{A}$ divides $\{x, x'\}$, then $\mathcal{U}$ must divide $\{x, x'\}$. This is since if $A \in \mathcal{A}$ and its complement meet $\{x, x'\}$, then since $\mathcal{A}$ is generated by $\mathcal{U}$, there is some $U \in \mathcal{U}$ with $U \subseteq A$ that meets $\{x, x'\}$ and the complement of U must also meet $\{x, x'\}$. However, if $\mathcal{A}$ is a $2r$-separated collection of sets that divides $\{x, x'\}$ and $\rho(x, x') \le r$, then $B_r(\mathcal{A})$ will not divide $\{x, x'\}$, since if $A \in \mathcal{A}$ meets $\{x, x'\}$, then $B_r(A)$ must contain both points x, x'. More generally, any enlargements of the sets in $B_r(\mathcal{A})$ which remain disjoint cannot divide $\{x, x'\}$.

Now we can give a general lemma showing that if $\rho_0 \ge \rho_1 \ge \ldots$ is a decreasing sequence of locally finite Borel extended metrics with sufficiently slow dimension growth, then the union of their finite distance equivalence relations $\bigcup_n E_{\rho_n}$ is hyperfinite. The statement of this lemma tries to give the most general possible version of this construction. We will state some simpler corollaries afterwards which are easier to apply in practice.

Lemma 4.3. *Suppose X is a standard Borel space, $\rho_0 \geq \rho_1 \geq \dots$ is a decreasing sequence of locally finite Borel extended metrics on X, $(s_n)_{n\in\mathbb{N}}$, $(d_n)_{n\in\mathbb{N}}$ are sequences of positive real numbers so that $d_n \to \infty$.*

Define $(d'_n)_{n\geq 0}$ recursively by $d'_0 = d_0$ and

$$d'_n \coloneqq d_n + 4(\dim_B^{\rho_{n+1}}(s_{n+1}, d_{n+1}) + 2)d'_{n-1} \text{ for } n \geq 1,$$

and assume that for all $n \geq 1$,

$$s_n \geq 4(\dim_B^{\rho_{n+1}}(s_{n+1}, d_{n+1}) + 2)d'_{n-1}.$$

Then $\bigcup_n E_{\rho_n}$ is hyperfinite.

Proof. We will take a collection of Borel (s_n, d_n)-covers for (X, ρ_n) and build from them another collection of (s'_n, d'_n)-covers $\mathcal{V}^n$ for (X, ρ_n), where we define s'_n below, so that if $\rho_m(x, x') < \infty$ for some m, then $\mathcal{V}^n$ will divide $\{x, x'\}$ for at most finitely many n. Hence, the tail intersections of the equivalence relations $E_{\mathcal{V}^n}$ will witness the hyperfiniteness of $\bigcup_n E_{\rho_n}$.

Let $c_n = \dim_B^{\rho_n}(s_n, d_n)$, set $s'_0 = s_0$, and, for $n \geq 1$, set $s'_n = s_n - 4(c_{n+1} + 2)d'_{n-1}$. (The c_n are finite by our equations above, dropping an initial term if needed). Note that our equation above on s_n implies that $s'_n \geq 0$. For each n, fix a Borel (s_n, d_n)-cover $\mathcal{W}^n = (\mathcal{W}^n_0, \dots, \mathcal{W}^n_{c_n})$ of (X, ρ_n). We will build from each of these covers a "matrix" $\mathcal{U}^n_{ij}$ of sets $\mathcal{U}^n_{ij} \subseteq \mathcal{P}(X)$ where $0 \leq i \leq c_{n+1} + 1$ and $0 \leq j \leq c_n$ with the following properties:

1. The ith row $\mathcal{U}^n_i = (\mathcal{U}^n_{ij})_{j\leq c_n}$ will be an (s'_n, d'_n)-cover of X for every i and n.
2. For every $n \geq 1$, the jth column $\mathcal{U}^n_{*,j} = (\mathcal{U}^n_{ij})_{i\leq c_{n+1}+1}$ will satisfy that if $\rho_n(x, x') \leq d'_{n-1}$, then $\{x, x'\}$ is divided by at most one family $\mathcal{U}^n_{ij}$ in this column.
3. For every $n \geq 1$, each $\mathcal{U}^n_{ij}$ is generated from $\mathcal{U}^{n-1}_j$.

We'll define the sets $\mathcal{U}^n_{ij}$ by recursion on n and verify their properties (1)-(3) inductively. For the base case, let $\mathcal{U}^0_{ij} = \mathcal{W}^0_j$ for every $0 \leq i \leq c_1 + 1$ and $0 \leq j \leq c_0$. So we are copying the same cover in every row of the 0th matrix. Item (1) then is satisfied, and properties (2) and (3) are vacuously true.

For $n \geq 1$, we first define the $i = 0$ row by:

$$\mathcal{U}^n_{0j} \coloneqq \mathcal{W}^n_j * (\mathcal{U}^{n-1}_j),$$

so $\mathcal{U}^n_0$ is a Borel $(s_n - 2d'_{n-1}, d_n + 2d'_{n-1})$-cover and hence also a Borel $(s_n - 4d'_{n-1}, d_n + 4d'_{n-1})$-cover. For subsequent rows $0 < i \leq c_{n+1} + 1$, we take a ball of radius d'_{n-1} around the previous row and then star with $\mathcal{U}^{n-1}_j$.

$$\mathcal{U}^n_{ij} \coloneqq B_{d'_{n-1}}(\mathcal{U}^n_{(i-1)j}) * (\mathcal{U}^{n-1}_{jk})_{k\leq c_{n-1}}.$$

So inductively, for each i, $\mathcal{U}^n_i$ is a $(s_n - 4(i+1)d'_{n-1}, d_n + 4(i+1)d'_{n-1})$-cover of X, since at each step we expand around each set by a ball of radius d'_{n-1}, and then apply the star operation with a cover with sets of diameter at most d'_{n-1} which both enlarges the diameter and shrinks the separation of our sets by $4d'_{n-1}$. Then item (3) is clearly true.

All that remains is to check property (2). Suppose $\rho_n(x, x') \leq d'_{n-1}$, and $\{x, x'\}$ is divided by $A \in \mathcal{U}^n_{ij}$. Then both $x, x' \in B_{d'_{n-1}}(A)$, and thus $\{x, x'\}$ is not divided by $\mathcal{U}^n_{i'j}$ for all $i' > i$, since both x, x' will be contained in some set in $\mathcal{U}^n_{i'j}$, and since $\mathcal{U}^n_{i'j}$ is a collection of disjoint sets by the assumption that $s'_n \geq 0$.

We now have the following claim:

Claim: Suppose $(\mathcal{U}^n_{ij})_{i\leq c_{n+1}+1, j\leq c_n}$ is a collection of sets with the above properties and suppose $\mathcal{V}^n = \mathcal{U}^n_{c_{n+1}+1}$ is the (s'_n, d'_n)-cover from the last row of each matrix. Then if $\rho_m(x, x') < \infty$ for some m, then there are at most finitely many n so that $\mathcal{V}^n$ divides $\{x, x'\}$.

To prove the claim, fix k and (x, x') so that $\rho_k(x, x') < \infty$. Since $d_n \to \infty$ and hence $d'_n \to \infty$, we can choose $m \geq k$ so large that $\rho_k(x, x') < d'_{n-1}$ for every $n > m$. Then $\rho_n(x, x') < d'_{n-1}$ whenever $n > m$, since $\rho_n \leq \rho_k$ for $n \geq k$.

Now assume $n > m$. By assumption (2), for each j, $\{x, x'\}$ can be divided by at most one family $\mathcal{U}^n_{ij}$ in the jth column. Furthermore, if $\{x, x'\}$ is divided by some $\mathcal{U}^n_{ij}$ in the jth column of the nth matrix, where $j \leq c_n$, then it is also divided by some $\mathcal{U}^{n-1}_{jk}$ in the jth row $\mathcal{U}^{n-1}_j$ of the $(n-1)$st matrix since each $\mathcal{U}^n_{ij}$ in the jth column is generated by $\mathcal{U}^{n-1}_j$ by assumption (3) and so by the observation above. Hence, there is an injection from those pairs of indices (i, j) for which $\mathcal{U}^n_{ij}$ divides $\{x, x'\}$ into the indices (i, j) from the previous matrix for which $\mathcal{U}_{ij}^{n-1}$ divides $\{x,x'\}$. This injection never maps into the last row of the $n-1$st matrix $\mathcal{U}_{ij}^{n-1}$ since the columns of $\mathcal{U}_{ij}^n$ are indexed by $j \le c_n$, but the rows of $\mathcal{U}^{n-1}$ are indexed by $i \le c_n+1$.

Now we count the total number of divisions. Let $u_n = |\{(i,j)\colon \mathcal{U}_{ij}^n \text{ divides } \{x,x'\}\}|$ be the number of indices so that $\mathcal{U}_{ij}^n$ divides $\{x,x'\}$ and let $v_n = |\{j\colon \mathcal{U}_{c_{n+1}+1,j}^n \text{ divides } \{x,x'\}\}|$ be the number of indices in the last row of this matrix dividing $\{x,x'\}$. So $u_n \ge v_n$ for every n. Note $v_n = 0$ if and only if $\mathcal{V}^n$ does not divide $\{x,x'\}$. For $n \ge m$, our injection defined above gives $u_n - v_n \ge u_{n+1}$, and so by induction $u_m \ge u_{n+1} + \sum_{k=m}^n v_k$. Therefore $\sum_{k=m}^\infty v_k \le u_m < \infty$. Hence $v_k = 0$ for all but finitely many k, and so $\mathcal{V}^n$ divides $\{x,x'\}$ for only finitely many n. □Claim

So if $\rho_m(x,x') < \infty$, then $x \ E_{\mathcal{V}^n} \ x'$ for all but finitely many n. Hence the equivalence relations $F_m \coloneqq \bigcap_{n\ge m} E_{\mathcal{V}^n}$ are increasing and witness the hyperfiniteness of $\bigcup_n E_{\rho_n}$.

□

Remark 4.4. We have proved the above lemma for locally finite Borel extended metric spaces. However, we remark that the same proof works unchanged for locally countable Borel extended metric spaces, if we modify the definition of $\dim_B^X(s,d)$ to require not only that the collections of sets in our (s,d)-covers be uniformly bounded, but also that they be smooth. Suppose (X,ρ) is a locally countable Borel extended metric space. Then say that a **smooth Borel** (s,d)**-cover** of X is an (s,d)-cover $\mathcal{U} = (\mathcal{U}_0,\dots,\mathcal{U}_n)$ where each $\mathcal{U}_i$ is Borel, and the equivalence relation $E_{\mathcal{U}_i}$ is smooth. Then define the smooth Borel 2-parameter dimension function of X by:

$$\dim_B^X(s,d) \coloneqq \inf\{n \in \mathbb{N}\colon \text{there is a smooth Borel } (s,d)\text{-cover of } X \text{ with } n+1 \text{ elements}\}.$$

With this definition of $\dim_B^X(s,d)$ for locally countable spaces, we note that Lemma 4.3 and all the rest of the results in this section are true for all locally countable Borel extended metric spaces. This follows from the fact that a CBER is hyperfinite iff it is hypersmooth [DJK94, Theorem 5.1], and that the operations of taking balls and the star operations preserve smoothness. Precisely, if $\mathcal{U} = (\mathcal{U}_0,\dots,\mathcal{U}_n)$ is a smooth Borel (s,d)-cover, then $B_r(\mathcal{U}_0),\dots,B_r(\mathcal{U}_n)$ is a smooth Borel $(s-2r,d+2r)$-cover for any $r \le s/2$. Furthermore if $\mathcal{V} \subseteq \mathcal{P}(X)$ is a smooth Borel s'-separated collection of sets and $s' \ge 2d$, then $\mathcal{V} * \mathcal{U}$ is also a smooth Borel $s'-2d$-separated collection of sets. The smoothness of these collections follows from Lusin-Novikov uniformization.

This same remark that our results can be generalized to locally countable metrics applies to Corollary 4.5 and Theorem 4.7 below.

A typical way of verifying the hypotheses of Lemma 4.3 is to arrange that $d_n' \le 2d_n$. In that case, we have the following simplified criterion for checking hyperfiniteness. We also include another simplifying assumption that our recurrence about $\dim_B^{\rho_m}(s_{n+1},d_{n+1})$ only needs to hold for all sufficiently large n.

Corollary 4.5. *Suppose X is a standard Borel space, $\rho_0 \ge \rho_1 \ge \dots$ is a decreasing sequence of locally countable Borel extended metrics on X, and $(s_n)_{n\in\mathbb{N}}$, $(d_n)_{n\in\mathbb{N}}$ are sequences of positive real numbers with $s_n \to \infty$ so that for every m,*

$$24\dim_B^{\rho_m}(s_{n+1},d_{n+1})d_{n-1} \le s_n \le d_n \tag{2}$$

for all sufficiently large n. Then $\bigcup_n E_{\rho_n}$ is hyperfinite.

Proof. Suppose first there are infinitely many m such that $\dim_B^{\rho_m}(s_{n+1},d_{n+1}) = 0$ for infinitely many n. Then $\bigcup_n E_{\rho_n}$ is trivially hyperfinite as follows: Choose strictly increasing sequences $(m_k)_{k\in\mathbb{N}}$ and $(n_k)_{k\in\mathbb{N}}$ of integers such that $\dim_B^{\rho_{m_k}}(s_{n_k+1},d_{n_k+1}) = 0$, and let $\mathcal{U}^k$ be the corresponding (s_{n_k+1},d_{n_k+1})-cover of X which consists only of a single family of sets. Note that $\mathcal{U}^k$ is s_{n_k+1}-separated. Then the equivalence relations $F_n = \bigcap_{k\ge n} E_{\mathcal{U}^k}$ witness the hypersmoothness, and hence the hyperfiniteness, of $\bigcup_n E_{\rho_n}$. This is since if $\rho_m(x,x') < \infty$, then for some sufficiently large n, $\rho_{m_k}(x,x') \le s_{n_k+1}$ for all $k \ge n$ since the metrics ρ_n are decreasing, n_k and m_k are increasing, and $s_n \to \infty$. Then $\mathcal{U}^k$ does not divide $\{x,x'\}$ for all such $k \ge n$, since the sets in $\mathcal{U}^k$ are s_{n_k+1}-separated. Hence $x \ E_{\mathcal{U}^k} \ x'$ for all $k \ge n$.

So we may assume that for all but finitely many m,

$$\dim_B^{\rho_m}(s_{n+1},d_{n+1}) \ge 1 \tag{3}$$

holds for all sufficiently large n. Next we use some reindexing to simplify our assumptions. We will replace our decreasing sequence $\rho_0 \geq \rho_1 \geq \dots$ of metrics with the sequence $\rho_{i_0} \geq \rho_{i_1} \geq \dots$ where $(i_n)_{n\in\mathbb{N}}$ is a slow-growing nondecreasing sequence of integers such that $i_n \to \infty$ as $n \to \infty$. Note that since the equivalence relations E_{ρ_n} are increasing with n, the union $\bigcup_{n\geq 0} E_{\rho_n} = \bigcup_{n\geq 0} E_{\rho_{i_n}}$ does not change. By choosing i_0 sufficiently large, removing some finite initial segment from the beginning of the sequences $(d_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$, and choosing (i_n) to be sufficiently slow growing, we may assume without loss of generality that both Equation (3) and Equation (2) hold for all $n \geq 1$ and $m = n+1$.

Now we will show that the hypotheses of Lemma 4.3 hold. We claim that if we define $d'_0 \coloneqq d_0$ and $d'_n \coloneqq d_n + 4(\dim_B^{\rho_{n+1}}(s_{n+1}, d_{n+1}) + 2)d'_{n-1}$, then $d'_n \leq 2d_n$ for all n by induction. The base case is clear. For the induction step, note that Equation (3) implies that $4(\dim_B^{\rho_{n+1}}(s_{n+1}, d_{n+1}) + 2) \leq 12 \dim_B^{\rho_{n+1}}(s_{n+1}, d_{n+1})$. Then

$$d'_n \leq d_n + 12 \dim_B^{\rho_{n+1}}(s_{n+1}, d_{n+1})d'_{n-1} \leq d_n + 24 \dim_B^{\rho_{n+1}}(s_{n+1}, d_{n+1})d_{n-1} \leq 2d_n$$

where at the penultimate step we have used the induction hypothesis. Finally, this means that

$$4(\dim_B^{\rho_{n+1}}(s_{n+1}, d_{n+1}) + 2)d'_{n-1} \leq 12 \dim_B^{\rho_{n+1}}(s_{n+1}, d_{n+1})d'_{n-1} \leq 24 \dim_B^{\rho_{n+1}}(s_{n+1}, d_{n+1})d_{n-1} \leq s_n$$

by our assumption. □

Remark 4.6. For the sake of keeping our proofs simple, we have not optimized the constants in Lemma 4.3 and Corollary 4.5. For example, the factor of 4 in both of our equations in the statement of Lemma 4.3 could be improved to $2+\varepsilon$ for any ε by replacing the d'_{n-1}-balls in the construction above with $\frac{\varepsilon}{2}d'_{n-1}$-balls. However, this sort of constant-factor improvement would not yield an improvement to the exponent in Theorem A. Similarly, the constant in Corollary 4.5 could be improved from 24 to $8+\varepsilon$ for any ε by using that improved version of Lemma 4.3, and by replacing Equation (3) with the equation $\dim_B^{\rho_m}(s_{n+1}, d_{n+1}) \geq 1/\varepsilon$ by using the fact that if this fails for infinitely many m, then $\bigcup_{n\geq 0} E_{\rho_n}$ is an increasing union of spaces of finite asymptotic dimension and is hence hyperfinite by [CJMST23, Theorem 7.3]. However, as above this constant improvement would yield no improvement to our exponent in Theorem A. We also remark that Corollary 4.5 is still true with the weaker assumption that $24 \dim_B^{\rho_{n+1}}(s_{n+1}, d_{n+1})d_{n-1} \leq s_n$, and $24 \dim_B^{\rho_{n+1}}(s_{n+1}, d_{n+1})d_{n-1} \leq d_n$ (with no relationship between s_n and d_n), assuming that $d_n \to \infty$. However in all cases of interest, we will have that $s_n \leq d_n$, so we have stated the inequalities this way to make the statement simpler.

Recall that in Theorem 3.11 we proved an estimate of the form $\dim_B^X(s, Cs^a) \in O(s^b)$ (where $C > 0$ is a constant, and $a = 1/(1-\alpha)$ and $b = \alpha/(1-\alpha)^2$) for graphs of growth $\mathrm{vol}(r) \in \exp(O(r^\alpha))$. So we are interested in cases where one can use Lemma 4.3 to prove hyperfiniteness for spaces with this type of polynomially bounded and controlled dimension growth.

Theorem 4.7. *Suppose $a \geq 1$, $b \geq 0$, and $ab \leq 1/4$. Suppose $\rho_0 \geq \rho_1 \geq \dots$ is a decreasing sequence of locally countable Borel extended metrics on a standard Borel space X such that for every m, there is a constant C_m such that*

$$\dim_B^{\rho_m}(s, C_m s^a) \in O(s^b).$$

Then $\bigcup_n E_{\rho_n}$ is hyperfinite.

Proof. Let

$$\beta \coloneqq 2a,\ \ s_n \coloneqq \exp(\beta^n \log n) \text{ and } d_n \coloneqq \log\log(s_n)s_n^a$$

for $n \geq 2$. We will verify the hypotheses of Corollary 4.5. So we need to show that for sufficiently large n,

$$24 \dim_B^{\rho_m}(s_{n+1}, d_{n+1})d_{n-1} \leq s_n \leq d_n.$$

Clearly $s_n \leq d_n$ for sufficiently large n, so to finish it suffices to show that

$$\frac{\dim_B^{\rho_m}(s_{n+1}, d_{n+1})d_{n-1}}{s_n} \to 0 \text{ as } n \to \infty.$$

Note first that for sufficiently large n, $d_n \geq C_m s_n^a$, and so for sufficiently large n, $\dim_B^{\rho_m}(s_n, d_n) \leq \dim_B^{\rho_m}(s_n, C_m s_n^a) \in O(s_n^b)$. Note also that $\log\log(s_n) = \log(\beta^n \log n) = n\log(\beta) + \log\log n \in O(n)$.

Hence,

$$\begin{aligned}\frac{\dim_B^{\rho_m}(s_{n+1},d_{n+1})d_{n-1}}{s_n} &= \frac{O(s_{n+1}^b)\log\log(s_{n-1})s_{n-1}^a}{s_n}\\ &= O(n)\frac{s_{n+1}^b s_{n-1}^a}{s_n}\\ &\leq O(n)\exp\left(\beta^n(\frac{1}{2}\log(n+1)+\frac{1}{2}\log(n-1)-\log(n))\right)\\ &\leq O(n)\exp\left(\frac{\beta^n}{2}\log\left(\frac{(n+1)(n-1)}{n^2}\right)\right)\\ &\leq O(n)\exp\left(\frac{\beta^n}{2}\log\left(1-\frac{1}{n^2}\right)\right)\leq O(n)\exp\left(-\frac{\beta^n}{2n^2}\right)\to 0 \text{ as } n\to\infty.\end{aligned}$$

□

Corollary 4.5 and Theorem 4.7 combined prove Theorem C from the introduction.

Remark 4.8. We remark that the assumption that $0 < ab \leq 1/4$ in Theorem 4.7 is the optimal result where we can obtain hyperfiniteness by directly applying Lemma 4.3 for spaces where $\dim_B^X(s, Cs^a)$ has growth of order s^b. That is, suppose (X,ρ) is a Borel extended metric space and there are constants $C, D > 0$ such that $\dim_B^\rho(s,d) \geq Ds^b$ for all sufficiently large s and $d \geq Cs^a$. Suppose also we have sequences $(s_n)_{n\geq 0}$ and $(d_n)_{n\geq 0}$ where $d_n \geq Cs_n^a$ and $d_n \to \infty$ satisfying the conditions in Lemma 4.3, where $\rho_n = \rho$ is the constant sequence. Then we claim that $4ab \leq 1$.

This is since we must satisfy $s_n \geq 4(\dim_B^{\rho_{n+1}}(s_{n+1}, d_{n+1}) + 2)d'_{n-1}$ by assumption, so we have that $s_{n+1}^b d'_{n-1} \in O(s_n)$. Then since $d'_{n-1} \geq d_{n-1} \geq Cs_{n-1}^a$, this implies $s_{n+1}^b s_{n-1}^a \in O(s_n)$. Let $x_n := \log s_n$. Then taking logarithms gives

$$ax_{n-1} + bx_{n+1} \leq x_n + O(1).$$

Assuming $4ab > 1$, choose $0 < a' < a$ and $0 < b' < b$ such that $4a'b' > 1$. Now $x_n \to \infty$ because $s_n \geq 8d'_{n-1} \geq 8d_{n-1}$ and $d_n \to \infty$ as $n \to \infty$. This implies $a'x_{n-1} + b'x_{n+1} < x_n$ for sufficiently large n. By the AM-GM inequality, this implies $x_n > 2\sqrt{a'b'x_{n-1}x_{n+1}}$ so $x_n^2 > 4a'b'x_{n-1}x_{n+1}$, and $\frac{x_n}{x_{n-1}} > 4a'b'\frac{x_{n+1}}{x_n}$. Thus there is an N such that $\frac{x_{m+1}}{x_m} \leq \frac{1}{(4a'b')^{m-N}}\frac{x_{N+1}}{x_N}$ for all $m \geq N$. Hence if $4a'b' > 1$, then $\frac{x_{m+1}}{x_m} \to 0$ as $m \to \infty$, contradicting $x_n \to \infty$.

We can now prove that a decreasing sequence of locally countable Borel extended metrics of growth at most $\exp(O(r^\gamma))$ is hyperfinite, where γ is as in the abstract:

Theorem 4.9. *Let $\gamma \approx 0.1523$ be the unique real root of $(1-\gamma)^3 - 4\gamma = 0$, and suppose X is a standard Borel space and $\rho_0 \geq \rho_1 \geq \dots$ is a decreasing sequence of locally countable Borel extended metrics on X so that for every n, ρ_n has volume growth bounded by $\mathrm{vol}^{\rho_n}(r) \in \exp(O(r^\gamma))$. Then $\bigcup_n E_{\rho_n}$ is hyperfinite.*

Proof. By Theorem 3.11, for each n there is a constant $C_n > 0$ such that $\dim_B^{\rho_n}(s, C_n s^{1/(1-\gamma)}) \in O(s^{\gamma/(1-\gamma)^2})$, and hence by Theorem 4.7, it is hyperfinite if $4\gamma \leq (1-\gamma)^3$, which holds by the definition of γ. □

We can remove the uniform volume growth assumption and weaken it to a pointwise volume growth bound by passing to a new decreasing sequence of metrics to which we can apply Theorem 4.9. This gives Theorem A from the introduction:

Theorem 4.10. *Let γ be the unique real root of $(1-\gamma)^3 - 4\gamma = 0$, so $\gamma \approx 0.1523$. Suppose X is a standard Borel space and $\rho_0 \geq \rho_1 \geq \dots$ is a decreasing sequence of Borel extended metrics on X so that for every n, ρ_n has pointwise volume growth $\exp(O(r^\gamma))$. That is, for every $n \in \mathbb{N}$ and $x \in X$, there exist $C, N > 0$ such that $|B_r^{\rho_n}(x)| \leq \exp(Cr^\gamma)$ for all $r \geq N$. Then $\bigcup_n E_{\rho_n}$ is hyperfinite, where E_{ρ_n} is the finite distance equivalence relation for ρ_n.*

Proof. We will define a new decreasing sequence of Borel extended metrics ρ'_n with uniformly bounded volume growth $\exp(O(r^\gamma))$ so that the union of the connectedness relations is unchanged: $\bigcup_n E_{\rho'_n} = \bigcup_n E_{\rho_n}$, and then apply Theorem 4.9 to these new metrics ρ'_n.

For $m, n \in \mathbb{N}$, let $A_{m,n}$ be the set of $x \in X$ such that $|B_r^{\rho_m}(x)| \leq \exp(nr^\gamma)$ for every $r \geq n$. So the sets $A_{m,n}$ are Borel and $\bigcup_{n\in\mathbb{N}} A_{m,n} = X$. Note that the restriction of ρ_m to $A_{m,n}$ has uniformly bounded volume growth $\exp(O(r^\gamma))$ for every n. Hence, for a single fixed extended metric ρ_m, we could show that E_{ρ_m} is hyperfinite by applying Theorem 4.9 to the decreasing sequence of metrics $\sigma_0 \geq \sigma_1 \geq \ldots$ defined by setting $\sigma_n(x, x') \coloneqq \rho_m(x, x')$ if $x, x' \in A_{m,n}$, $\sigma_n(x, x') = 0$ if $x = x'$, and $\sigma_n(x, x') = \infty$ otherwise. To handle the countably many metrics ρ_n, we will use a more elaborate version of this sort of idea.

For $n \in \mathbb{N}$, define the metric $\rho'_n(x, y)$ to be the infimum of $\sum_{i<k} \rho_{m_i}(x_i, x_{i+1})$ for all sequences $x = x_0, \ldots, x_k = y$ and $m_0, \ldots, m_{k-1} \leq n$ such that $x_i, x_{i+1} \in A_{m_i,n}$ for all $i < k$. So ρ'_n is the infimum of the lengths of "labeled" paths from x to y, where each step from x_i to x_{i+1} is labeled by m_i where both $x_i, x_{i+1} \in A_{m_i,n}$, and we measure distance using the metric ρ_{m_i}. Note that $\rho'_{n+1} \leq \rho'_n$ for all n by definition, and the union of the finite distance relations is unchanged: $\bigcup_n E_{\rho'_n} = \bigcup_n E_{\rho_n}$ as required, since $\bigcup_{n\in\mathbb{N}} A_{m,n} = X$.

We now compute a uniform upper bound on the volume growth of the metrics ρ'_n. We begin by noting there is a finite bound on the number of steps we need to consider in these walks defining ρ'_n. Suppose an allowed walk $x_0, \ldots, x_k$ has two steps $i < j < k$ where $m_i = m_j$ and every step strictly between them has label less than m_i. Then the part of the walk from $x_i, \ldots, x_{j+1}$ can be replaced by a single step directly from x_i to x_{j+1} with label m_i. Thus, we may assume all walks contain their largest label m_i at most once. Thus, by induction (splitting the walk before and after this label), we may consider only walks with at most $S(n)$ steps, where $S(n)$ satisfies the recurrence $S(0) = 1$ and $S(n) = 2S(n-1) + 1$, since we have 1 step of largest label in the walk, and then the walk before and after that has length at most $S(n-1)$.

Now fix n and suppose $r \geq n$. If $y \in B_r^{\rho'_n}(x)$, then there is such a walk $x = x_0, \ldots, x_k = y$ with $k \leq S(n)$ steps, labels $m_0, \ldots, m_{k-1}$, and total length at most r. Each step has distance $\rho_{m_i}(x_i, x_{i+1}) \leq r$ in the metric ρ_{m_i}. Since $x_i \in A_{m_i,n}$, we have $|B_r^{\rho_{m_i}}(x_i)| \leq \exp(nr^\gamma)$. Since there are $n+1$ possible choices for the value of m_i, at each step of the walk, and then at most $|B_r^{\rho_{m_i}}(x_i)|$ choices of the next point to walk to given this label m_i, we therefore have

$$|B_r^{\rho'_n}(x)| \leq ((n+1)\exp(nr^\gamma))^{S(n)} \in \exp(O(r^\gamma)).$$

Here n is fixed. □

We remark that the method of making metrics with uniformly bounded volume growth in the above proof is not special to this sort of intermediate growth; it just uses that the growth bounds we are using have sufficient closure. In particular, that class of functions $\exp(O(r^\gamma))$ is closed under taking constant powers and constant multiples.

5. Følner tilings and layerings

In this section, we will show that we can use Følner layerings to obtain Følner tilings. We begin by recalling the definition of a Følner tiling:

Definition 5.1. Let $r, \varepsilon > 0$. An (r, ε)**-Følner tiling** of an extended metric space X is a cover $\mathcal{U}$ of X by disjoint (r, ε)-Følner sets of uniformly bounded diameter. An ε**-Følner tiling** is a $(1, \varepsilon)$-Følner tiling.

Equivalently, an (r, ε)-Følner tiling is a uniformly bounded-diameter equivalence relation on X whose classes are (r, ε)-Følner.

Recall also the following definition of a setwise Følner metric space:

Definition 5.2 ([ET25])**.** A locally finite extended metric space (X, ρ) is **setwise Følner** if for every $r, \varepsilon > 0$, there exists some $R > 0$ such that for every nonempty finite set $A \subseteq X$ there is an (r, ε)-Følner set F such that $A \subseteq F \subseteq B_R(A)$.

Recall that X is **Følner** if for every $r, \varepsilon > 0$, there exists $R > 0$ such that for every $x \in X$ there is an (r, ε)-Følner set F so that $x \in F \subseteq B_R(x)$. We note for context that Elek and Timár [ET25] have shown that every bounded degree graph is Følner if and only if it is setwise Følner, and their proof extends straightforwardly to show that an extended metric space of uniformly bounded volume growth is Følner if and only if it is setwise Følner. However, we will not need this fact for any of our arguments.

Metric spaces of subexponential growth are well-known to be setwise Følner by an easy argument similar to Proposition 3.2:

Proposition 5.3. *Suppose (X,ρ) is an extended metric space of uniformly subexponential volume growth. Then (X,ρ) is setwise Følner.*

Proof. Fix $r,\varepsilon>0$. By subexponential volume growth, choose a positive integer t such that $\mathrm{vol}(r(t+1))\le(1+\varepsilon)^{t+1}$. Since $B_r(A)=\bigcup_{x\in A}B_r(x)$, we have that

$$\frac{|B_r(A)|}{|B_0(A)|}\frac{|B_{2r}(A)|}{|B_r(A)|}\cdots\frac{|B_{(t+1)r}(A)|}{|B_{tr}(A)|}=\frac{|B_{(t+1)r}(A)|}{|B_0(A)|}\le\frac{\mathrm{vol}(r(t+1))|A|}{|A|}\le(1+\varepsilon)^{t+1},$$

so there is some $s\le t$ such that $\frac{|B_{r(s+1)}(A)|}{|B_{rs}(A)|}\le 1+\varepsilon$. So $F=B_{rs}(A)$ is the desired (r,ε)-Følner set since $B_r(F)=B_r(B_{rs}(A))\subseteq B_{r(s+1)}(A)$. □

To finish, we show that if X is setwise Følner and has Borel Følner layerings, then it has Borel Følner tilings:

Lemma 5.4. *Suppose (X,ρ) is a locally finite Borel extended metric space of uniformly bounded volume growth that is setwise Følner, and suppose X has a Borel (r,ε)-Følner layering for every $r,\varepsilon>0$. Then X has a Borel (r,ε)-Følner tiling for every $r,\varepsilon>0$.*

Proof. Suppose $r,\varepsilon>0$, and define $\delta:=\varepsilon/(1+\mathrm{vol}(r))$. We will construct a (r,ε)-Følner tiling.

Since X is setwise Følner, choose a sufficiently large R so that for every nonempty finite A, there is a (r,δ)-Følner set $\phi(A)$ such that $A\subseteq\phi(A)\subseteq B_R(A)$. By Lusin-Novikov uniformization, we can choose such a function ϕ such that ϕ is Borel, since there are finitely many (r,δ)-Følner sets satisfying $A\subseteq F\subseteq B_R(A)$. Without loss of generality we may assume $R\ge r/2$.

Let $\mathcal{F}=(\mathcal{F}_0,\dots,\mathcal{F}_{n-1})$ be a $(2R,\delta)$-Følner layering of X. The sets $\{\phi(F)\colon F\in\mathcal{F}_i \text{ for some } i<n\}$ are disjoint, and are all (r,δ)-Følner by definition of ϕ. To finish the proof, we will enlarge each of these sets a small amount to make a tiling that covers all of X, while keeping the property that these new enlarged sets are (r,ε)-Følner.

Now $X\setminus\mathrm{supp}(\mathcal{F})$ is covered by $\{\partial_{2R}^{X_i}(F)\colon F\in\mathcal{F}_i\wedge i<n\}$. If A is (r,δ)-Følner and $|B|\le\delta|A|$, then $A\cup B$ is $(r,(\mathrm{vol}(r)+1)\delta)$-Følner, since $|A|\le|A\cup B|$, and $|\partial_r(A\cup B)|\le|\partial_r(A)|+|\partial_r(B)|\le\delta|A|+\mathrm{vol}(r)\delta|A|$. So we can finish by enlarging the sets $\phi(F)$ to include part of $\partial_{2R}^{X_i}(F)$ so that they form a tiling, and these sets will still be (r,ε)-Følner.

Precisely, let ψ be a Borel function on $X\setminus\mathrm{supp}(\{\phi(F)\colon F\in\mathcal{F}_i \text{ for some } i<n\})$ such that if $\psi(x)=(i,F)$, then $F\in\mathcal{F}_i$ and $x\in\partial_{2R}^{X_i}(F)$. So ψ selects for each remaining x some $F\in\mathcal{F}_i$ whose $2R$-boundary contains it. Then our desired tiling is $\{\phi(F)\cup\{x\colon\psi(x)=(i,F)\}\colon F\in\mathcal{F}_i \text{ for some } i<n\}$. □

Putting together Corollary 3.4, Proposition 5.3 and Lemma 5.4 proves Theorem B from the introduction:

Theorem 5.5. *Suppose (X,ρ) is a Borel extended metric space of uniformly subexponential volume growth. Then for every $r,\varepsilon>0$, there is a Borel (r,ε)-Følner tiling of (X,ρ).* □

Remark 5.6. We remark that the proof of Lemma 5.4 can be easily modified to show analogous clopen tilings and topological almost finiteness results in the setting of topological dynamics, since our constructions are entirely "local". The modifications of our constructions in Lemma 3.3 and Lemma 5.4 required to prove the topological dynamics analogues exactly follow similar modifications discussed in [CJMST23, Section 10], showing that many of the constructions of that paper easily adapt to the realm of topological dynamics.

5.1. **An equivalence between layerings and tilings.** To finish this section, we expand on the above, and prove an equivalence between tilings and layerings in arbitrary Borel extended metric spaces of uniformly bounded volume growth.

We can now show that the existence of Borel Følner tilings and Følner layerings are equivalent:

Theorem 5.7. *Suppose X is a Borel extended metric space of uniformly bounded volume growth. Then X has a Borel (r,ε)-Følner tiling for every $r,\varepsilon>0$ iff X is setwise Følner and has a Borel (r,ε)-Følner layering for every $r,\varepsilon>0$.*

Proof. We have already proved the reverse implication in Lemma 5.4. So we prove the two forward implications. Suppose X has a Borel (r,ε)-Følner tiling for every $r,\varepsilon>0$. Given such a Borel (r,ε)-Følner tiling $\mathcal{F}$ of X, of diameter bounded by d, if $A\subseteq X$ is nonempty and finite, let $F=A*\mathcal{F}$ be the union of the sets in $\mathcal{F}$ meeting A. Then F is (r,ε)-Følner since it is a union of disjoint (r,ε)-Følner sets, and $F\subseteq B_d(A)$. So X is setwise Følner. So it remains to prove the existence of Følner layerings.

Fix $r,\varepsilon>0$, and choose δ sufficiently small so that $\frac{1+\delta}{1-\mathrm{vol}(r)\delta}\leq 1+\varepsilon$. Let $\mathcal{A}$ be a Borel (r,δ)-Følner tiling of X. Fix d bounding the diameters of the members of $\mathcal{A}$. The Borel graph on $\mathcal{A}$ in which distinct $A,A'\in\mathcal{A}$ are adjacent when $\rho(A,A')\leq r$ has bounded degree (in particular bounded by $\mathrm{vol}(d+r)-1$), and hence has a Borel n-coloring for some n. Thus, we can decompose $\mathcal{A}$ into sets

$$\mathcal{A}=\mathcal{A}_0\sqcup\cdots\sqcup\mathcal{A}_{n-1}$$

such that each $\mathcal{A}_i$ is r-separated.

We now recursively define a (r,ε)-Følner layering $\mathcal{F}_0,\ldots,\mathcal{F}_{n-1}$ of X. Define $X_0=X$ and, for $i<n$, $\mathcal{F}_i\coloneqq\{A\cap X_i : A\in\mathcal{A}_i \text{ and } A\cap X_i\neq\emptyset\}$ and $X_{i+1}\coloneqq X_i\setminus B_r(\mathrm{supp}(\mathcal{F}_i))$. So each $\mathcal{F}_i$ has diameter bounded by d and is r-separated.

We verify that for each $A\in\mathcal{A}_i$, the set $F\coloneqq A\cap X_i$ is (r,ε)-Følner. Now $A\setminus F=A\setminus X_i=A\cap\bigcup_{j<i}B_r(\mathrm{supp}(\mathcal{F}_j))\subseteq B_r^X\left(\partial_r^X(A)\right)$ and so

$$|F|\geq|A|-\mathrm{vol}(r)|\partial_r^X(A)|\geq(1-\mathrm{vol}(r)\delta)|A|.$$

Since $|B_r^X(F)|\leq|B_r^X(A)|\leq(1+\delta)|A|\leq\frac{1+\delta}{1-\mathrm{vol}(r)\delta}|F|$, we have that F is (r,ε)-Følner in X, and hence (r,ε)-Følner in X_i.

To finish, note that $B_r\left(\mathrm{supp}(\mathcal{F}_0,\ldots,\mathcal{F}_{n-1})\right)=X$, since if $x\in X$, then $x\in A$ for some $A\in\mathcal{A}$, and let i be such that $A\in\mathcal{A}_i$. Then either $x\in A\cap X_i$ and hence $x\in\mathrm{supp}(\mathcal{F}_i)$ by definition of $\mathcal{F}_i$, or $x\in A\setminus X_i=A\cap\bigcup_{j<i}B_r(\mathrm{supp}(\mathcal{F}_j))$, so $x\in B_r(\mathrm{supp}(\mathcal{F}_0,\ldots,\mathcal{F}_{n-1}))$. □

6. Separation index growth, subexponential dimension growth, and μ-hyperfiniteness

Now we turn to the setting of CBERs on standard probability spaces, and we begin translating a standard characterization of μ-hyperfiniteness of locally finite Borel graphs into a characterization of μ-hyperfiniteness of the equivalence relations E_ρ of locally finite Borel extended metric spaces (X,ρ), where μ is a Borel probability measure on X. Recall that we say that a CBER E is μ-hyperfinite if and only if there is a conull Borel set $X'\subseteq X$ such that $E\upharpoonright X'$ is hyperfinite. Equivalently, if there is a conull E-invariant Borel set $X''\subseteq X$ such that $E\upharpoonright X''$ is hyperfinite. We will then use our characterization below to prove that if E_ρ is not μ-hyperfinite, then (X,ρ) has exponential μ-measurable separation index growth.

If X is a locally finite Borel extended metric space and μ is a finite Borel measure on X, we say that X is an (s,ε)**-expander** with respect to μ if for every Borel $2s$-separated family $\mathcal{A}\subseteq\mathcal{P}(X)$ of finite sets, we have $\mu(B_s(\mathrm{supp}\,\mathcal{A}))\geq(1+\varepsilon)\mu(\mathrm{supp}\,\mathcal{A})$.

We have the following characterization of μ-hyperfiniteness for locally finite extended metric spaces, which is an easy adaptation of similar well-known lemmas for group actions and Borel graphs. Item (3) in this characterization is due to Elek [Ele12], refining an earlier result of Kaimanovich [Kai97].

Lemma 6.1. *Suppose ρ is a locally finite Borel extended metric on a standard probability space (X,μ). Then the following are equivalent:*

1. *E_ρ is μ-hyperfinite*
2. *For every $\varepsilon>0$ and every $s>0$, there is a s-separated Borel family $\mathcal{A}\subseteq\mathcal{P}(X)$ of finite sets such that $\mu(\mathrm{supp}(\mathcal{A}))\geq 1-\varepsilon$.*
3. *For every Borel subset $X'\subseteq X$ of positive measure and every $\varepsilon>0$ and $s>0$, X' is not an (s,ε)-expander.*

Proof. (1) ⇒ (3): Suppose $(F_n)_{n\in\mathbb{N}}$ are finite Borel equivalence relations witnessing that E_ρ is hyperfinite on a conull set. Without loss of generality, we may assume X' is contained in this conull set. By restricting to X', these equivalence relations also witness that $E_{\rho\restriction X'}$ is hyperfinite. Pick $r > 2s$, and for each n, let

$$\mathcal{A}_n \coloneqq \{\{y \in X' \colon B_r^{X'}(y) \subseteq [x]_{F_n}\} \colon x \in X'\}.$$

So $\mathcal{A}_n$ consists of the r-interiors of the $F_n \restriction X'$-classes. Now the collection $\mathcal{A}_n$ is r-separated and hence has separation greater than $2s$, since each F_n-class meets at most one element $A \in \mathcal{A}_n$, and $B_r(A)$ is contained in this F_n-class. Every $x \in X'$ is in $\operatorname{supp}(\mathcal{A}_n)$ for some n, since $B_r^{X'}(x)$ is a finite set of points which must all eventually be F_n-related to x. Hence $\mu(\operatorname{supp}(\mathcal{A}_n)) \to \mu(X')$ as $n \to \infty$, and so taking $\mathcal{A}_n$ with $\mu(\operatorname{supp}(\mathcal{A}_n)) > \frac{\mu(X')}{1+\varepsilon}$ witnesses that X' is not an (s,ε)-expander.

(3) ⇒ (2): Fix $s > 0$ and $\varepsilon > 0$, and choose $\delta > 0$ such that $\delta/(1+\delta) < \varepsilon$. Suppose that for all positive measure sets $X' \subseteq X$, X' is not an (s,δ)-expander. Let $\mathcal{A}$ be a maximal (modulo μ-null sets) Borel family of s-separated finite sets such that $\mu(B_s(\operatorname{supp}(\mathcal{A}))) \le (1+\delta)\mu(\operatorname{supp}(\mathcal{A}))$. Such a collection must exist by a measure exhaustion argument[4]. We claim that $B_s(\operatorname{supp}(\mathcal{A}))$ is conull, and hence (2) is satisfied. Otherwise, let $X' = X \setminus B_s(\operatorname{supp}(\mathcal{A}))$. Then we can find some Borel s-separated family of finite sets $\mathcal{A}' \subseteq \mathcal{P}(X')$ whose support has positive measure such that $\mu(B_s^{X'}(\operatorname{supp}(\mathcal{A}'))) \le (1+\delta)\mu(\operatorname{supp}(\mathcal{A}'))$. Note by the definition of X', every point of $\operatorname{supp}(\mathcal{A}')$ has distance greater than s from $\operatorname{supp}(\mathcal{A})$. So $B_s(\operatorname{supp}(\mathcal{A}' \cup \mathcal{A})) \subseteq B_s(\operatorname{supp}(\mathcal{A})) \cup B_s^{X'}(\operatorname{supp}(\mathcal{A}'))$, and $\mathcal{A}' \cup \mathcal{A}$ is s-separated, contradicting the maximality of $\mathcal{A}$.

(2) ⇒ (1): For each n, let $\mathcal{A}_n$ be a Borel $2n+1$-separated family such that $\mu(\operatorname{supp}(\mathcal{A}_n)) \ge 1 - \frac{1}{2^n}$, and let $\mathcal{A}'_n = B_n(\mathcal{A}_n)$, so $\mathcal{A}'_n$ is 1-separated. Then let E_n be the equivalence relation whose classes are the elements of $\mathcal{A}'_n$, combined with singletons for every $x \notin \operatorname{supp}(\mathcal{A}'_n)$. So $x \mathrel{E_n} y$ if either $x = y$, or there is an $A \in \mathcal{A}'_n$ such that $x, y \in A$. Note E_n is a finite Borel equivalence relation on X.

Define $F_n = \bigcap_{k \ge n} E_k$ to be their tail intersections, so F_n is an increasing sequence of finite Borel equivalence relations. The equivalence relations F_n will witness that E_ρ is μ-hyperfinite by a Borel-Cantelli argument. Fix N. For all $n \ge N$, the μ-measure of the set of x such that there is some $y \in B_N(x)$ such that $x \not\mathrel{E_n} y$ is at most $\frac{1}{2^n}$. Hence, by the union bound, for all $m \ge N$, the measure of the set of x such that there is some $y \in B_N(x)$ such that $x \not\mathrel{F_m} y$ is at most $\sum_{n=m}^{\infty} \frac{1}{2^n} = \frac{1}{2^{m-1}}$. Hence, for a conull set of x, for all $y \in B_N(x)$ we have $x \bigcup_n F_n \, y$. Since this is true for every N, $(F_n)_{n\in\mathbb{N}}$ witnesses the μ-hyperfiniteness of E_ρ. □

We have the following corollary: if E_ρ is not μ-hyperfinite, then si_B^X has exponential growth, and hence so does sdim_B^X. Indeed, we can characterize μ-hyperfiniteness for locally finite Borel extended metric spaces via their separation index growth, which extends a characterization due to Felix Weilacher [Wei21] who proved the equivalence of (1) and (2) below. This is Theorem E from the introduction:

Corollary 6.2. *Suppose ρ is a locally finite Borel extended metric on a standard probability space (X,μ). Then the following are equivalent:*

(1) *E_ρ is μ-hyperfinite.*
(2) $\operatorname{asi}_\mu(X) \le 1$.
(3) si_μ^X *has non-exponential growth.*

Proof. Weilacher [Wei21] has shown that (1) implies (2), and (2) implies (3) since a bounded function has subexponential growth. Hence it suffices to prove that (3) implies (1), which we do by contraposition.

Suppose E_ρ is not μ-hyperfinite. By Lemma 6.1, there is a Borel subset $X' \subseteq X$ of positive measure and $s, \varepsilon > 0$ such that X' is an (s,ε)-expander. Clearly $\operatorname{si}_\mu^X(s) \ge \operatorname{si}_\mu^{X'}(s)$ by restricting covers of X to X', so it suffices to prove that X' has an exponential lower bound on its separation index growth.

[4]Recall the principle of measure exhaustion: there cannot be uncountably many disjoint sets of positive measure in a Borel probability space (see e.g. [Fre03, Section 215]). This is true because any uncountable sequence of positive real numbers has infinite sum; otherwise it would contain only finitely many numbers greater than $1/n$ for each positive integer n. So any wellordered chain of Borel sets that is strictly increasing in measure in a σ-finite measure space must be countable, and so its union is Borel.

Suppose $\mathcal{A} \subseteq \mathcal{P}(X')$ is a family of finite sets that is $2ns$-separated. Then since $B^{X'}_{is}(\mathcal{A})$ has separation greater than $2(n-i)s \geq 2s$ for every $i < n$ and X' is an (s,ε)-expander, we have $\mu(\operatorname{supp}(B^{X'}_{(i+1)s}(\mathcal{A}))) \geq (1+\varepsilon)\mu(\operatorname{supp}(B^{X'}_{is}(\mathcal{A})))$, and so $\mu(\operatorname{supp}(B^{X'}_{ns}(\mathcal{A}))) \geq (1+\varepsilon)^n \mu(\operatorname{supp}\mathcal{A})$. Hence $\mu(\operatorname{supp}\mathcal{A}) \leq \frac{\mu(X')}{(1+\varepsilon)^n}$. Thus, if we cover a conull subset of X' by families of finite sets of separation greater than $2ns$, since the total measure of the supports of all the families must be at least $\mu(X')$, we must have at least $(1+\varepsilon)^n$ families of sets. So $\operatorname{si}^{X'}_\mu(2ns) \geq (1+\varepsilon)^n - 1$ for all integers n. So $\operatorname{si}^{X'}_\mu(r) > b^r$ for any $1 < b < (1+\varepsilon)^{1/(2s)}$ for sufficiently large real numbers $r > 0$. □

Note in particular since $\operatorname{sdim}^X_B(s) \geq \operatorname{si}^X_B(s) \geq \operatorname{si}^X_\mu(s)$ for any Borel probability measure μ, this means that for any locally finite Borel extended metric (X,ρ) with subexponential Borel dimension growth, E_ρ is μ-hyperfinite for every Borel probability measure μ on X. So if (X,ρ) has subexponential Borel dimension growth, but E_ρ is *not* hyperfinite, then it would solve the hyperfiniteness vs measure hyperfiniteness problem in the negative [Kec25, Problem 7.29]. Stated another way, all locally finite Borel extended metric spaces (X,ρ) that are known to be non-hyperfinite have exponential dimension growth. Hence, there is currently a large gap between these exponential dimension growths in all known non-hyperfinite spaces, and the dimension growths which are known to imply hyperfiniteness by Theorem C.

7. Borel subexponential dimension growth conjectures

We begin by restating our conjectures on Borel dimension which would imply a positive solution to Weiss's question.

Conjecture F.

(1) *Suppose $\Gamma \curvearrowright X$ is a Borel action of a finitely generated amenable group Γ on a standard Borel space X, and ρ is the word metric on this action. Then (X,ρ) has subexponential Borel dimension growth.*
(2) *If (X,ρ) is a locally finite Borel extended metric space which has subexponential Borel dimension growth, then E_ρ is hyperfinite.*

These conjectures imply a positive answer to Weiss's question, even though part (1) of the conjecture is only about finitely generated groups. This follows from a standard way of combining a decreasing sequence of locally finite metrics into a single locally finite metric.

Proposition 7.1. *If Conjecture F is true, then Weiss's question has a positive answer.*

Proof. Suppose $\Gamma \curvearrowright X$ is a Borel action of a countable amenable group on a standard Borel space. We need to show the action is hyperfinite. Let $\Gamma = \bigcup_n S_n$ where $S_0 \subseteq S_1 \subseteq \ldots$ are an increasing sequence of finite symmetric subsets of Γ, and let ρ_n be the word metric arising from the restriction of this action to the finitely generated subgroup $\langle S_n \rangle$ generated by S_n, so $\rho_0 \geq \rho_1 \geq \ldots$. Then by Conjecture F.(1) each ρ_n has subexponential dimension growth.

Now let $f\colon \mathbb{N} \to \mathbb{N}$ be a sufficiently fast growing increasing function with $f(0) = 0$ and $f(n) \to \infty$, and define

$$\rho_f(x,x') \coloneqq \inf_n \max(f(n), \rho_n(x,x')).$$

So $\rho_f(x,x') = \max(f(n), \rho_n(x,x'))$ where n is least such that $\rho_n(x,x') < f(n+1)$. To see that ρ_f satisfies the triangle inequality, suppose $\rho_f(x_0,x_1), \rho_f(x_1,x_2) < \infty$, choose n_0, n_1 realizing the corresponding infima, and let $n = \max(n_0,n_1)$. Then $\rho_n(x_0,x_1) \leq \rho_f(x_0,x_1)$, $\rho_n(x_1,x_2) \leq \rho_f(x_1,x_2)$, and $f(n) \leq \max(\rho_f(x_0,x_1), \rho_f(x_1,x_2))$. Hence

$$\rho_f(x_0,x_2) \leq \max(f(n), \rho_n(x_0,x_2)) \leq \rho_f(x_0,x_1) + \rho_f(x_1,x_2).$$

So ρ_f is an extended metric, and it has uniformly bounded volume growth.

By definition of ρ_f, and since the metrics ρ_n are decreasing,

(1) if $\rho_n(x,x') \geq f(n)$, then $\rho_f(x,x') \leq \rho_n(x,x')$, and
(2) if $\rho_n(x,x') < f(n+1)$, then $\rho_f(x,x') \geq \rho_n(x,x')$.

So if $\mathcal{U}_0, \ldots, \mathcal{U}_l$ is an (s,d)-cover of (X, ρ_n), where $s < f(n+1)$ and $d \geq f(n)$, then these sets $\mathcal{U}_i$ have diameter bounded by d by (1) above, and are s-separated by (2). Hence $\dim_B^{\rho_f}(s,d) \leq \dim_B^{\rho_n}(s,d)$ for all $s < f(n+1)$ and $d \geq f(n)$. Hence $\mathrm{sdim}_B^{\rho_f}(s) \leq \mathrm{sdim}_B^{\rho_n}(s)$ for all $s < f(n+1)$. Thus, since f is sufficiently fast growing and $\mathrm{sdim}_B^{\rho_n}(s)$ has subexponential growth for every n, $\mathrm{sdim}_B^{\rho_f}$ has subexponential Borel dimension growth.

Hence by Conjecture F.(2), the orbit equivalence relation of the action $\Gamma \curvearrowright X$ is hyperfinite, since it is equal to E_{ρ_f}. $\square$

7.1. **Borel Følner property A.** One might hope that a stronger version of part (1) of Conjecture F is true for all Borel extended metric spaces of uniformly bounded volume growth satisfying a suitable amenability property. We make a more speculative conjecture that this is true for all such spaces satisfying a Borel version of Elek and Timár's Følner property A [ET25]. We recall the definition of this property:

Definition 7.2. Say that an extended metric space (X, ρ) of uniformly bounded volume growth has **Følner property A** if for every $r, \varepsilon > 0$, there is an $R > 0$ and a function $\Theta \colon X \times X \to [0,1]$ such that

(1) For every $x \in X$, $y \notin B_R(x)$ implies $\Theta(x,y) = 0$, and $\sum_{y \in X} \Theta(x,y) = 1$.
(2) If $\rho(x,x') \leq r$, then $\sum_{y \in X} |\Theta(x,y) - \Theta(x',y)| < \varepsilon$.
(3) For all $x \in X$, $\sum_{y \in X} \sum_{z \in B_r(y)} |\Theta(x,y) - \Theta(x,z)| < \varepsilon$.

If X is a locally finite Borel extended metric space, we say that X has **Borel Følner property A** if there are Borel functions Θ as above for each $r, \varepsilon > 0$.

Note items (1) and (2) above just give the usual definition of property A. That is, defining $\Theta_x(y) = \Theta(x,y)$, we have that Θ_x is a probability measure supported on $B_R(x)$, and if x, x' are distance at most r, then the distance between Θ_x and $\Theta_{x'}$ is less than ε in the 1-norm. Item (3) is the additional Følnerness condition: the scale-r ℓ^1 gradient of Θ: $\mathcal{E}_r(\Theta_x) = \sum_{y \in X} \sum_{z \in B_r(y)} |\Theta_x(y) - \Theta_x(z)|$ is at most ε, so the values of this probability measure Θ_x do not change much on average when moving distance at most r in the space.

The word metric associated to any Borel action of a countable amenable group has Følner property A [ET25, Proposition 7.4]. That is, suppose $\Gamma \curvearrowright X$ is a Borel action of a countable amenable group with finite symmetric generating set S. Let ρ be the word metric where $\rho(x,x')$ is the least n so that there exists $s_1, \ldots, s_n \in S$ such that $s_1 s_2 \cdots s_n \cdot x = x'$. Then (X, ρ) has Borel Følner property A. To see this, given $\varepsilon, r > 0$, using the existence of Følner sets in Γ, let $K = \{s_1 \cdots s_k \colon k \leq r \wedge s_i \in S\}$ be all group elements of word length at most r, and let $F \subseteq \Gamma$ be a two-sided Følner set such that $\frac{1}{|F|} \sum_{\gamma \in K} (|\gamma F \triangle F| + |F\gamma \triangle F|) < \varepsilon$. Then defining $\Theta(x,y) = \frac{|\{\gamma \in F \colon \gamma \cdot x = y\}|}{|F|}$, it is easy to check that Θ is as required.

Hence, the following conjecture generalizes part (1) of Conjecture F. By a graph G having Borel Følner property A or subexponential dimension growth below, we mean that the path length metric on G has these properties.

Conjecture 7.3. *Suppose G is a bounded degree Borel graph on a standard Borel space X. Then if G has Borel Følner property A, then G has subexponential Borel dimension growth.*

We remark that a generalization of the above conjecture to all Borel extended metric spaces of uniformly bounded volume growth is not true. For example, let $(n_i)_{i \geq 1}$ be a sufficiently fast growing sequence of natural numbers, and let X_i be the metric space $\mathbb{Z}^{n_i}$ with the metric $\rho_i(g,h) = i\|g-h\|_\infty$ of i multiplied by the usual ℓ_∞ metric. Then consider the disjoint union $X = \bigsqcup_i X_i$ of all these spaces, with the metric where $\rho(x,x') = \rho_i(x,x')$ if $x, x' \in X_i$ for some i, and $\rho(x,x') = \infty$ otherwise. Note that X has uniformly bounded volume growth. Since $\mathrm{sdim}^{\mathbb{Z}^n}(1) = n$, it follows that $\mathrm{sdim}^X(i) \geq n_i$, so the dimension growth of X can be arbitrarily fast (and hence not subexponential), but it is easy to show that this space has Følner property *A*. Since these spaces are countable, the Borel and classical dimension agree.

This counterexample reflects the fact that the right setting here is perhaps that of Borel coarse geometry: given a metric ρ, the key object to analyze is really the metric generated by walks of steps of length at most s. That is, the associated metric ρ_s where $\rho_s(x,x')$ is the infimum length of a ρ-path from x to x' where each step in the path has distance at most s. (Or in the language of coarse geometry, the relations of having distance at most s for each s generate a natural collection of the entourages for ρ). The correct metric generalization of Conjecture 7.3 is that if ρ is a Borel extended metric with uniformly bounded volume growth and Borel Følner property A, then the metric ρ_s has subexponential Borel dimension growth for every s. Likewise, a construction similar to that in 7.1 shows that if the entourages of ρ all have subexponential dimension growth, then one can similarly make a stretched version ρ_f of the metric keeping the same connectedness relation so that ρ_f has subexponential Borel dimension growth.

References

[AGLP89] Baruch Awerbuch, Andrew V. Goldberg, Michael Luby, and Serge A. Plotkin. "Network decomposition and locality in distributed computation". In: *Proceedings of the 30th Annual Symposium on Foundations of Computer Science*. IEEE Computer Society, 1989, pp. 364–369. DOI: 10.1109/SFCS.1989.63504.

[BBW26] Anton Bernshteyn, Matt Bowen, and Felix Weilacher. *Measurable matchings in unbalanced graphs*. Preprint, arXiv:2606.11558 [math.LO]. 2026.

[BCDN12] Ievgen Bondarenko, Tullio Ceccherini-Silberstein, Alfredo Donno, and Volodymyr Nekrashevych. "On a family of Schreier graphs of intermediate growth associated with a self-similar group". In: *European J. Combin.* 33.7 (2012), pp. 1408–1421. DOI: 10.1016/j.ejc.2012.03.006.

[BD08] G. Bell and A. Dranishnikov. "Asymptotic dimension". In: *Topology Appl.* 155.12 (2008), pp. 1265–1296. DOI: 10.1016/j.topol.2008.02.011.

[Ber23] Anton Bernshteyn. "Distributed algorithms, the Lovász Local Lemma, and descriptive combinatorics". In: *Inventiones Mathematicae* 233.2 (2023), pp. 495–542. DOI: 10.1007/s00222-023-01188-3.

[BY25] Anton Bernshteyn and Jing Yu. "Large-scale geometry of Borel graphs of polynomial growth". In: *Adv. Math.* 473 (2025), Paper No. 110290, 51. DOI: 10.1016/j.aim.2025.110290.

[CFW81] A. Connes, J. Feldman, and B. Weiss. "An amenable equivalence relation is generated by a single transformation". In: *Ergodic Theory Dynam. Systems* 1.4 (1981), pp. 431–450. DOI: 10.1017/s014338570000136x.

[CJKMST18] Clinton T. Conley, Steve C. Jackson, David Kerr, Andrew S. Marks, Brandon Seward, and Robin D. Tucker-Drob. "Følner tilings for actions of amenable groups". In: *Math. Ann.* 371.1-2 (2018), pp. 663–683. DOI: 10.1007/s00208-017-1633-0.

[CJMST20] Clinton Conley, Steve Jackson, Andrew Marks, Brandon Seward, and Robin Tucker-Drob. "Hyperfiniteness and Borel combinatorics". In: *J. Eur. Math. Soc. (JEMS)* 22.3 (2020), pp. 877–892. DOI: 10.4171/jems/935.

[CJMST23] Clinton T. Conley, Steve C. Jackson, Andrew S. Marks, Brandon M. Seward, and Robin D. Tucker-Drob. "Borel asymptotic dimension and hyperfinite equivalence relations". In: *Duke Math. J.* 172.16 (2023), pp. 3175–3226. DOI: 10.1215/00127094-2022-0100.

[CM16] Clinton T. Conley and Benjamin D. Miller. "A bound on measurable chromatic numbers of locally finite Borel graphs". In: *Math. Res. Lett.* 23.6 (2016), pp. 1633–1644. DOI: 10.4310/MRL.2016.v23.n6.a3.

[DHZ19] Tomasz Downarowicz, Dawid Huczek, and Guohua Zhang. "Tilings of amenable groups". In: *J. Reine Angew. Math.* 747 (2019), pp. 277–298. DOI: 10.1515/crelle-2016-0025.

[DJK94] R. Dougherty, S. Jackson, and A. S. Kechris. "The structure of hyperfinite Borel equivalence relations". In: *Trans. Amer. Math. Soc.* 341.1 (1994), pp. 193–225. DOI: 10.2307/2154620.

[Dra06] A. N. Dranishnikov. "Groups with a polynomial dimension growth". In: *Geom. Dedicata* 119 (2006), pp. 1–15. DOI: 10.1007/s10711-005-9026-z.

[DS11] Alexander Dranishnikov and Mark Sapir. "On the dimension growth of groups". In: *J. Algebra* 347 (2011), pp. 23–39. DOI: 10.1016/j.jalgebra.2011.09.004.

[DS12a] Alexander Dranishnikov and Mark Sapir. "Corrigendum to "On the dimension growth of groups" [J. Algebra 347 (1) (2011) 23–39] [MR2846394]". In: *J. Algebra* 370 (2012), pp. 407–409. DOI: 10.1016/j.jalgebra.2012.07.048.

[DS12b] Alexander Dranishnikov and Mark Sapir. *On the dimension growth of groups.* arXiv:1008.3868v4 [math.GR]. 2012.

[DZ23] Tomasz Downarowicz and Guohua Zhang. "Symbolic extensions of amenable group actions and the comparison property". In: *Mem. Amer. Math. Soc.* 281.1390 (2023), pp. vi+95. DOI: 10.1090/memo/1390.

[Ele12] Gábor Elek. "Finite graphs and amenability". In: *J. Funct. Anal.* 263.9 (2012), pp. 2593–2614. DOI: 10.1016/j.jfa.2012.08.021.

[Ele18] Gábor Elek. *Qualitative graph limit theory. Cantor Dynamical Systems and Constant-Time Distributed Algorithms.* Preprint, arXiv:1812.07511 [math.DS]. 2018.

[ET24] Gábor Elek and Ádám Timár. *Uniform Borel Amenability.* Preprint, arXiv:2408.12565v5 [math.DS]. 2024.

[ET25] Gábor Elek and Ádám Timár. "Strong almost finiteness". In: *J. Funct. Anal.* 289.10 (2025), Paper No. 111116, 47. DOI: 10.1016/j.jfa.2025.111116.

[EZ20] Anna Erschler and Tianyi Zheng. "Growth of periodic Grigorchuk groups". In: *Invent. Math.* 219.3 (2020), pp. 1069–1155. DOI: 10.1007/s00222-019-00922-0.

[Fre03] D. H. Fremlin. *Measure theory. Vol. 2.* Broad foundations, Corrected second printing of the 2001 original. Torres Fremlin, Colchester, 2003, 563+12 pp. (errata).

[GJ15] Su Gao and Steve Jackson. "Countable abelian group actions and hyperfinite equivalence relations". In: *Invent. Math.* 201.1 (2015), pp. 309–383. DOI: 10.1007/s00222-015-0603-y.

[Gri91] Rostislav I. Grigorchuk. "On growth in group theory". In: *Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990).* Math. Soc. Japan, Tokyo, 1991, pp. 325–338.

[Gro93] M. Gromov. "Asymptotic invariants of infinite groups". In: *Geometric group theory, Vol. 2 (Sussex, 1991).* Vol. 182. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 1993, pp. 1–295.

[Gro99] M. Gromov. "Endomorphisms of symbolic algebraic varieties". In: *J. Eur. Math. Soc. (JEMS)* 1.2 (1999), pp. 109–197. DOI: 10.1007/PL00011162.

[HKL90] L. A. Harrington, A. S. Kechris, and A. Louveau. "A Glimm-Effros dichotomy for Borel equivalence relations". In: *J. Amer. Math. Soc.* 3.4 (1990), pp. 903–928. DOI: 10.2307/1990906.

[JKL02] S. Jackson, A. S. Kechris, and A. Louveau. "Countable Borel equivalence relations". In: *J. Math. Log.* 2.1 (2002), pp. 1–80. DOI: 10.1142/S0219061302000138.

[Kai97] Vadim A. Kaimanovich. "Amenability, hyperfiniteness, and isoperimetric inequalities". In: *C. R. Acad. Sci. Paris Sér. I Math.* 325.9 (1997), pp. 999–1004. DOI: 10.1016/S0764-4442(97)89093-3.

[Kas25] Alexander Sebastien Kastner. *Borel Graphs: Measurable Consequences of their Geometry and Complexity of Labeling Problems.* UCLA Electronic Theses and Dissertations. Thesis (Ph.D.)–University of California, Los Angeles; ProQuest ID: Kastner_ucla_0031D_24215; ProQuest dissertation no. 32114497; Merritt ID: ark:/13030/m5nm5547. ProQuest LLC, Ann Arbor, MI, 2025, p. 64.

[Kec25] Alexander S. Kechris. *The theory of countable Borel equivalence relations.* Vol. 234. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 2025, pp. xiii+161.

[Kec95] Alexander S. Kechris. *Classical descriptive set theory.* Vol. 156. Graduate Texts in Mathematics. Springer-Verlag, New York, 1995, pp. xviii+402. DOI: 10.1007/978-1-4612-4190-4.

[KST99] A. S. Kechris, S. Solecki, and S. Todorcevic. "Borel chromatic numbers". In: *Adv. Math.* 141.1 (1999), pp. 1–44. DOI: 10.1006/aima.1998.1771.

[KW25] George Kontogeorgiou and Martin Winter. "(Random) trees of intermediate volume growth". In: *Random Structures Algorithms* 67.4 (2025), Paper No. e70035, 20. DOI: 10.1002/rsa.70035.

[Mar23] Andrew S. Marks. "Measurable graph combinatorics". In: *ICM—International Congress of Mathematicians. Vol. 3. Sections 1–4.* EMS Press, Berlin, 2023, pp. 1488–1502. DOI: 10.4171/ICM2022/157.

[Opp14] Izhar Oppenheim. "An intermediate quasi-isometric invariant between subexponential asymptotic dimension growth and Yu's property A". In: *Internat. J. Algebra Comput.* 24.6 (2014), pp. 909–922. DOI: 10.1142/S0218196714500416.

[OW80] Donald S. Ornstein and Benjamin Weiss. "Ergodic theory of amenable group actions. I. The Rohlin lemma". In: *Bull. Amer. Math. Soc. (N.S.)* 2.1 (1980), pp. 161–164. DOI: 10.1090/S0273-0979-1980-14702-3.

[Oza12] Narutaka Ozawa. "Metric spaces with subexponential asymptotic dimension growth". In: *Internat. J. Algebra Comput.* 22.2 (2012), pp. 1250011, 3. DOI: 10.1142/S0218196711006777.

[Pik21] Oleg Pikhurko. "Borel combinatorics of locally finite graphs". In: *Surveys in combinatorics 2021*. Vol. 470. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2021, pp. 267–319.

[Roz24] Václav Rozhoň. *Invitation to Local Algorithms*. Preprint, arXiv:2406.19430 [cs.DC]. 2024.

[SS24] Scott Schneider and Brandon Seward. "Locally nilpotent groups and hyperfinite equivalence relations". In: *Math. Res. Lett.* 31.2 (2024), pp. 511–578. DOI: 10.4310/mrl.241024235042.

[Tes08] Romain Tessera. "Quantitative property A, Poincaré inequalities, L^p-compression and L^p-distortion for metric measure spaces". In: *Geom. Dedicata* 136 (2008), pp. 203–220. DOI: 10.1007/s10711-008-9286-5.

[Wei21] Felix Weilacher. *Measure Asymptotic Separation Index and Hyperfiniteness*. Preprint. https://math.berkeley.edu/~weilacher/measure_asi_and_hyperfiniteness.pdf. 2021.

[Wei84] Benjamin Weiss. "Measurable dynamics". In: *Conference in modern analysis and probability (New Haven, Conn., 1982)*. Vol. 26. Contemp. Math. Amer. Math. Soc., Providence, RI, 1984, pp. 395–421. DOI: 10.1090/conm/026/737417.

[YZ24] Jing Yu and Xingyu Zhu. *Asymptotic-Type Dimension Bounds through Combinatorial Approaches*. arXiv:2411.16660 [math.MG]. 2024.

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