ON A QUESTION OF SLAMAN AND STEEL

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Abstract. We consider an old question of Slaman and Steel: whether Turing equivalence is an increasing union of Borel equivalence relations none of which contain a uniformly computable infinite sequence. We show this question is deeply connected to problems surrounding Martin's conjecture, and also in countable Borel equivalence relations. In particular, if Slaman and Steel's question has a positive answer, it implies there is a universal countable Borel equivalence relation which is not uniformly universal, and that there is a $(\equiv_T, \equiv_m)$-invariant function which is not uniformly invariant on any pointed perfect set.

1. Introduction

This paper is a contribution to the study of problems surrounding Martin's conjecture on Turing invariant functions and countable Borel equivalence relations. Our central focus is an old open question of Slaman and Steel which they posed [SS] in reaction to their proof in the same paper that Turing equivalence is not hyperfinite. The question they asked is whether Turing equivalence can be expressed as a union of Borel equivalence relations E_n where $E_n \subseteq E_{n+1}$ for all n and so that no E_n-class $[x]_{E_n}$ contains an infinite sequence of reals uniformly computable from x. While this seems to be a very specific question about computability, we show (Theorem 3.5) that it is equivalent to a much more general question of whether every countable Borel equivalence relation is what we call hyper-Borel-finite (see Definition 3.1).

This question of Slaman and Steel has been completely unstudied since the 1988 paper where it was posed, and it remains open. However, we show that it is deeply connected to problems in both Borel equivalence relations, and problems surrounding Martin's conjecture. In particular, we show (Corollary 5.6.(1)) that if Slaman and Steel's question has a positive answer, then there is a Borel invariant function from Turing equivalence to many-one equivalence which is not uniformly invariant on any pointed perfect set. (In Section 2 we discuss some open problems concerning invariant functions from Turing equivalence to many-one equivalence which are suggested by Kihara-Montalbán's recent work [KM]). We also show (Corollary 5.6.(2)) that if Slaman and Steel's question has a positive answer, then many-one equivalence on 2^ω is a universal countable Borel equivalence relation. Since many-one equivalence on 2^ω is not uniformly universal [M, Theorem 1.5.(5)], this implies that

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if Question 3.4 has a positive answer, the conjecture of the second author that every universal countable Borel equivalence relation is uniformly universal ([M, Conjecture 1.1]) is false.

Our main construction is given in Theorem 5.5. This is the first result constructing a non-uniform function between degree structures in computability theory from any sort of hypothesis.

Suppose we want to construct a counterexample to part I of Martin's conjecture. That is, we want to build a Turing invariant function $f\colon 2^\omega \to 2^\omega$ such that the Turing degree of f is not constant on a cone, and $f(x) \not\geq_T x$ on a cone. An obvious strategy is to build f in countably many stages. At stage n, we determine some partial information about $f(x)$ in order to diagonalize against $f(x)$ computing x via the nth Turing reduction. At stage n we also specify how to "code" $f(y)$ into $f(x)$ for some of the y such that $y \equiv_T x$ (to ensure that at the end of the construction, f is Turing invariant). Now consider the relation E_n where $x \mathrel{E_n} y$ if $f(y)$ has been coded into $f(x)$ by the nth stage of the construction. Clearly E_n is an equivalence relation, $E_n \subseteq E_{n+1}$ for all n, and Turing equivalence is the union of these equivalence relations: $\equiv_T = \bigcup_n E_n$.

A problem in attempts to construct counterexamples to Martin's conjecture is that we know essentially nothing about the ways in which Turing equivalence can be written as an increasing union, apart from Slaman and Steel's original theorem that Turing equivalence is not hyperfinite. In particular, it is open whether every way of writing Turing equivalence as an increasing union $\equiv_T = \bigcup_n E_n$ must be trivial in the sense that there is some n and some pointed perfect set P where E_n is already equal to Turing equivalence, i.e. $E_n \restriction P = (\equiv_T \restriction P)$ (see Conjecture 6.1). If Conjecture 6.1 is true, attempts to build counterexamples to Martin's conjecture in the way indicated above seem hopeless.

In the authors' opinion, understanding how Turing equivalence may be expressed as an increasing union, and Slaman and Steel's Question 3.4 seem to be a vital steps towards understanding Martin's conjecture. If Question 3.4 has a positive answer, one can hope to improve on the construction in Theorem 5.5 to give a counterexample to Martin's conjecture. If Question 3.4 has a negative answer, perhaps Conjecture 6.1 is true, and there is no nontrivial way of approximating Turing equivalence from below in countably many stages.

1.1. **Preliminaries.** Our conventions and notation are largely standard. For background on Martin's conjecture, see [MSS]. For a recent survey of the field of countable Borel equivalence relations, see [K19].

We use lowercase x, y, z to denote elements of 2^ω, and f, g for functions on 2^ω. If $x \in 2^\omega$, we use $\overline{x}$ to denote the real obtained by flipping all the bits of x (or the complement of x, viewing x as a subset of ω). If $f\colon 2^\omega \to 2^\omega$, we similarly use $\overline{f}$ to denote the function where $\overline{f}(x) = \overline{f(x)}$ for all x. If $A \subseteq \omega$ and $x \in 2^\omega$, we let $x \restriction A$ denote the restriction of the function x to A. Equivalently, viewing elements of 2^ω as subsets of ω, $x \restriction A$ is $x \cap A$. Provided $y \in 2^\omega$ is not the constant sequence of all 1s, if A is computable, then $x \restriction A \leq_m y$ iff there is a computable function $\rho\colon A \to \omega$ so that for all $n \in A$, $x(n) = y(\rho(n))$. This is because given such a $\rho\colon A \to \omega$, we can fix n_0 so $y(n_0) = 0$, and define $\rho'\colon \omega \to \omega$ by $\rho'(n) = \rho(n)$ if $n \in A$ and $\rho'(n) = n_0$ otherwise. Then ρ' gives a many-one reduction of $x \cap A$ to y.

Fix a computable bijection $\langle \cdot, \cdot \rangle \colon \omega^2 \to \omega$. We will assume that for all i, j we have $\langle i, j \rangle \geq i$ and $\langle i, j \rangle \geq j$. If $A \subseteq \omega$, the *ith column of* A is $A^{[i]} = \{\langle i, j \rangle \in A \colon j \in \omega\}$.

2. Versions of Martin's conjecture for invariant functions from Turing to many-one degrees

In [KM], Kihara and Montalbán study uniformly degree invariant functions from Turing degrees to many-one degrees. One of our main results is that if Slaman and Steel's question has a positive answer, then there is a degree invariant function from Turing degrees to many-one degrees which is not uniformly Turing invariant on any pointed perfect set. In this section, we briefly discuss some open problems around such functions which are suggested by Kihara-Montalbán's work.

Recall a function $f\colon 2^\omega \to 2^\omega$ is $(\equiv_T, \equiv_m)$-*invariant* if $x \equiv_T y$ implies $f(x) \equiv_m f(y)$. (In the terminology of Borel equivalence relations, we would say f is a *homomorphism* from $\equiv_T$ to $\equiv_m$.) A function $f\colon 2^\omega \to 2^\omega$ is *uniformly* $(\equiv_T, \equiv_m)$-*invariant* if there is a function $u\colon \omega^2 \to \omega^2$ so that if $x \equiv_T y$ via the programs (i,j), then $f(x) \equiv_m f(y)$ via the programs $u(i,j)$. If $c \in 2^\omega$, then $x \leq_m^c y$ if there is a function $\rho\colon \omega \to \omega$ computable from c so that $x(n) = y(\rho(n))$ for all n. If $f, g\colon 2^\omega \to 2^\omega$, then we write $f \leq_m^\triangledown g$ if there is a Turing cone of x with base c so that $f(x) \leq_m^c g(x)$.

Kihara and Montalbán show that uniformly $(\equiv_T, \equiv_m)$-invariant functions are well-quasi-ordered by $\leq_m^\triangledown$ and are in bijective correspondence with Wadge degrees via a simply defined map which they give [KM]. It follows from this bijection with Wadge degrees that the smallest uniformly $(\equiv_T, \equiv_m)$-invariant functions which are not constant on a cone are the Turing jump: $x \mapsto x'$ and its complement $x \mapsto \overline{x'}$, which are easily seen to correspond to the maps associated to universal open and closed sets; the lowest nontrivial classes in the Wadge hierarchy.

Implicit in Kihara-Montalbán's work are obvious analogues of Martin's conjecture [SS, Conjecture I, II] and Steel's conjecture [SS, Conjecture III] for $(\equiv_T, \equiv_m)$-invariant functions. We state these conjectures:

Conjecture 2.1 (Martin's conjecture for $(\equiv_T, \equiv_m)$-invariant functions)**.** *Assume* $\mathsf{AD} + \mathsf{DC}$. *Then*

I. *If $f\colon 2^\omega \to 2^\omega$ is $(\equiv_T, \equiv_m)$-invariant and the many-one degree $[f(x)]_m$ of f is not constant on a Turing cone of x, then $f \geq_m^\triangledown j$, or $f \geq_m^\triangledown \overline{j}$, where $j(x) = x'$ is the Turing jump.*

II. *If $f, g\colon 2^\omega \to 2^\omega$ are $(\equiv_T, \equiv_m)$-invariant, then $f \geq_m^\triangledown g$ or $\overline{g} \geq_m^\triangledown f$. Furthermore, the order $\leq_m^\triangledown$ well-quasi-orders the functions on 2^ω that are $(\equiv_T, \equiv_m)$-invariant.*

Conjecture 2.2 (Steel's conjecture for $(\equiv_T, \equiv_m)$-invariant functions)**.** *Suppose* $\mathsf{AD}{+}\mathsf{DC}$*, and suppose $f\colon 2^\omega \to 2^\omega$ is $(\equiv_T, \equiv_m)$-invariant. Then there is a uniformly $(\equiv_T, \equiv_m)$-invariant function g so that $f \equiv_m^\triangledown g$.*

Conjecture 2.2 implies Conjecture 2.1 by Kihara-Montalbán's work in [KM].

There is an important relationship between Turing invariant functions and $(\equiv_T, \equiv_m)$-invariant functions. Since $x \leq_T y$ if and only if $x' \leq_m y'$, any Turing invariant function can be turned into a $(\equiv_T, \equiv_m)$-invariant function by applying the Turing jump. However, because of the parameter c in the definition of $\leq_m^\triangledown$, it is not true that if $f' \geq_m^\triangledown g'$, then $f(x) \geq_T g(x)$ on a Turing cone of x. In particular, we do not know whether Conjecture 2.1 and Conjecture 2.2 imply Martin's conjecture and Steel's conjecture. However, if we strengthen Conjecture 2.2 to use the relation "$\leq_m$ on a cone" rather than $\leq_m^\triangledown$, then we do obtain a strengthening of Steel's conjecture [SS, Conjecture III].

Conjecture 2.3. *Suppose* AD*, and suppose* f *is* $(\equiv_T, \equiv_m)$*-invariant. Then there is a uniformly* $(\equiv_T, \equiv_m)$*-invariant function* g *so that* $f(x) \equiv_m g(x)$ *on a Turing cone of* x*.*

A standard argument (see the first footnote in [MSS]) shows that if f is $(\equiv_T, \equiv_m)$-invariant, then $f(x) \equiv_m g(x)$ on a cone for some uniformly $(\equiv_T, \equiv_m)$-invariant function g if and only if f is itself uniformly $(\equiv_T, \equiv_m)$-invariant on a pointed perfect set.

Proposition 2.4. *Conjecture 2.3 implies Steel's conjecture,* [SS, Conjecture III]*.*

Proof. Suppose $f \colon 2^\omega \to 2^\omega$ is Turing invariant. Then by Conjecture 2.3, the map $x \mapsto f(x)'$, is uniformly $(\equiv_T, \equiv_m)$-invariant on a pointed perfect set. Hence f is uniformly Turing invariant on the same pointed perfect set. □

Kihara and Montalbán's work is more generally stated for functions to the space $\mathcal{Q}^\omega$, where $\mathcal{Q}$ is a better-quasi-order. One can more generally ask about the analogues of the above conjectures for functions to $\mathcal{Q}^\omega$. We have the following observation due to Kihara-Montalbán that the relation $\leq_m^\triangledown$ cannot be replaced with "$\leq_m$ on a cone" in their work when $\mathcal{Q} \neq 2$:

Proposition 2.5 (Kihara-Montalbán, private communication)**.** *Suppose* AD*. Then the* $(\equiv_T, \equiv_m)$*-invariant functions from* 2^ω *to* 3^ω *which are not constant on a cone are not well-quasi-ordered by the relation "*$\leq_m$ *on a cone".*

Proof. By [M, Theorem 3.6], many-one reducibility on 3^ω is a uniformly universal countable Borel equivalence relation. Letting $=_\mathbb{R}$ denote equality on the real numbers, there is hence a uniform Borel reduction $f \colon 2^\omega \times \mathbb{R} \to 3^\omega$ from $\equiv_T \times =_\mathbb{R}$ to many-one reducibility on 3^ω. For each $y \in \mathbb{R}$, the function $f_y(x) = f(x, y)$ is thus a uniformly $(\equiv_T, \equiv_m)$-invariant function. Note that if $y \neq y'$, then $f_y(x)$ and $f_{y'}(x)$ are not $\equiv_m$-equivalent on a cone of x, nor are they constant on a cone (since f is a Borel reduction).

Thus, the relation on Borel functions "$\leq_m$ on a cone" cannot be a well-quasi-order on the Borel uniformly $(\equiv_T, \equiv_m)$-invariant functions from $2^\omega \to 3^\omega$, since then it would therefore give a well-quasi-order of $\mathbb{R}$. □

In fact, it is easy to see from the proof of [M, Theorem 3.6] that for all y, y' and all z, $f_y(z) \not\geq_m f_{y'}(z)$. So all the functions f_y constructed above are incomparable under $\leq_m$.

It is an open question whether the relation $\leq_m^\triangledown$ can be replaced with "$\leq_m$ on a cone" in Kihara-Montalbán's theorem on the space 2^ω.

Question 2.6. *Assume* $\mathsf{AD} + \mathsf{DC}$*. Is there is an isomorphism between the Wadge degrees and the degrees of the uniformly* $(\equiv_T, \equiv_m)$*-invariant functions under the relation "*$\leq_m$ *on a cone"? If* f *is uniformly* $(\equiv_T, \equiv_m)$*-invariant and the many-one degree* $[f(x)]_m$ *of* f *is not constant on a Turing cone of* x*, then is* $f(x) \geq_m j(x)$ *on a cone, or* $f(x) \geq_m \overline{j(x)}$ *on a cone, where* $j(x) = x'$ *is the Turing jump?*

3. Slaman and Steel's Question

The following notion is essentially due to Slaman and Steel:

Definition 3.1 ([SS]). Suppose $(f_i)_{i\in\omega}$ is a countable sequence of Borel functions $f_i\colon X \to X^\omega$. Say that a countable Borel equivalence relation F on X is $(f_i)_{i\in\omega}$-*finite* if there is no $i \in \omega$ and $x \in X$ such that the set $\{f_i(x)(j)\colon j \in \omega\}$ is infinite and $\{f_i(x)(j)\colon j \in \omega\} \subseteq [x]_F$. That is, no $f_i(x)$ is a sequence of infinitely many different elements in the F-class of x. Say that E is *hyper-*$(f_i)_{i\in\omega}$*-finite* if there is an increasing sequence $F_0 \subseteq F_1 \subseteq \ldots$ of Borel subequivalence relations of E such that F_n is $(f_i)_{i\in\omega}$-finite for every n, and $\bigcup_n F_n = E$. Finally, say that E is *hyper-Borel-finite* if for every countable collection of Borel functions $(f_i)_{i\in\omega}$ where $f_i\colon X \to X^\omega$, E is *hyper-*$(f_i)_{i\in\omega}$*-finite.*

Here we can think of each set $\{f_i(x)\}_{i\in\omega}$ as being a potential witnesses that some F-class is infinite, which we would like to avoid.

Clearly every hyperfinite Borel equivalence relation is hyper-Borel-finite. It is an open problem to characterize the hyper-Borel-finite equivalence relations.

Question 3.2. *Is there a non-hyperfinite countable Borel equivalence relation that is hyper-Borel-finite?*

Question 3.3. *Is every countable Borel equivalence relation hyper-Borel-finite?*

Slaman and Steel consider the special case of Definition 3.1 where the function $f_i\colon 2^\omega \to (2^\omega)^\omega$ gives the columns from the real given by the ith Turing reduction $\Phi_i(x)$:

$$f_i(x)(j) = \begin{cases} \{n\colon \langle j, n\rangle \in \Phi_i(x)\} & \text{if } \Phi_i(x) \text{ is total} \\ x & \text{otherwise.} \end{cases}$$

We say that Turing equivalence is *hyper-recursively-finite* if $\equiv_T$ is hyper-$(f_i)_{i\in\omega}$-finite for the above functions $(f_i)_{i\in\omega}$. Slaman and Steel posed the question of whether Turing equivalence is hyper-recursively-finite in [SS, Question 6], though in the setting of AD rather than just for Borel functions. We work in the Borel setting because it makes the statements of some of our theorems more straightforward. However, all the arguments of the paper can be adapted to the setting of AD as usual.

Question 3.4 ([SS]). *Is Turing equivalence hyper-recursively-finite?*

This problem about Turing equivalence is equivalent to the more general problem of whether every countable Borel equivalence relation is hyper-Borel-finite. This self-strengthening property of hyper-recursive-finiteness of $\equiv_T$ will be an essential ingredient in our proof of Theorem 5.5.

Theorem 3.5. *The following are equivalent:*

1. $\equiv_T$ *is hyper-recursively-finite.*
2. *Every countable Borel equivalence relation E is hyper-Borel-finite.*

Proof. (1) is a special case of (2), and is hence implied by it. We prove that (1) implies (2). Fix a witness $F_0 \subseteq F_1 \subseteq \ldots$ that $\equiv_T$ is recursively finite. We wish to show that every countable Borel equivalence relation E is hyper-Borel-finite. We may assume that E is a countable Borel equivalence relation on 2^ω. We may further suppose that E is Δ^1_1 and $(f_i)_{i\in\omega}$ is uniformly Δ^1_1; our proof relativizes.

Since E is a Δ^1_1 relation with countable vertical sections, and $(f_i)_{i\in\omega}$ is uniformly Δ^1_1, there is some computable ordinal notation α such that for all $x \in 2^\omega$ and for all $y \mathrel{E} x$, $x^{(\alpha)} \geq_T y$, and $x^{(\alpha)} \geq_T \bigoplus_{i\in\omega} f_i(x)$. Now if we let $\beta = \omega \cdot \alpha$, then $x^{(\alpha)} \geq_T y$ implies $x^{(\beta)} \geq_T y^{(\beta)}$. Hence, if $x \; E \; y$, then $x^{(\beta)} \equiv_T y^{(\beta)}$, and $x^{(\beta)} \geq_T \left(\bigoplus_{i\in\omega} f_i(x)\right)^{(\beta)}$. Note that the function $x \mapsto x^{(\beta)}$ is injective.

Define E_k by

$$x \; E_k \; y \iff xEy \wedge x^{(\beta)} F_k y^{(\beta)}.$$

We claim that $(E_k)_{k\in\omega}$ witness that E is hyper-(f_i)-finite. Suppose not. Then there exists E_k, x and i such that $\{f_i(x)(j) \colon j \in \omega\}$ is infinite and $x \; E_k \; f_i(x)(j)$ for all $j \in \omega$. This implies $x^{(\beta)} \; F_k \; (f_i(x)(j))^{(\beta)}$ for all j by definition of E_k. Now the sequence $\left((f_i(x)(j))^{(\beta)}\right)_{j\in\omega}$ is uniformly recursive in $x^{(\beta)}$ since $x^{(\beta)} \geq_T (f_i(x))^{(\beta)}$. The set $\left\{(f_i(x)(j))^{(\beta)} \colon j \in \omega\right\}$ is still infinite since the jump operator $x \mapsto x^{(\beta)}$ is injective. This contradicts that $(F_k)_{k\in\omega}$ is a witness that $\equiv_T$ is hyper-recursively-finite. □

The key in the above proof is that given any countable Borel equivalence E on X and Borel functions (f_i) from $X \to X^\omega$, we can find an injective Borel homomorphism h from E to $\equiv_T$ so that the image of each f_i under h is a computable function. Similar theorems to Theorem 3.5 are true for other weakly universal countable Borel equivalence relations, and collections of "universal" functions with respect to them. For example, let E_∞ be the orbit equivalence relation of the shift action of the free group $\mathbb{F}_\omega = \langle \gamma_{i,j} \rangle_{i,j\in\omega}$ on $\omega^{\mathbb{F}_\omega}$ (so we are indexing the generators of $\mathbb{F}_\omega$ by elements of ω^2). Let $f_i(x)(j) = (\gamma_{i,j} \cdot x)$. Then E_∞ is hyper-(f_i)-finite if and only if every countable Borel equivalence relation is hyper-Borel-finite.

Boykin and Jackson have introduced the class of Borel bounded equivalence relations [BJ]. For these equivalence relations it is an open problem whether there is some non-hyperfinite Borel bounded equivalence relation, and also whether all Borel equivalence relations are Borel bounded. Similarly both these problems are open for the hyper-Borel-finite Borel equivalence relations. We pose the question of whether there is a relationship between E being hyper-Borel-finite and being Borel bounded.

Question 3.6. *Is every Borel bounded countable Borel equivalence relation hyper-Borel-finite?*

Straightforward measure theoretic and Baire category arguments cannot prove that any countable Borel equivalence relation is not hyper-Borel-finite. This follows for Baire category from generic hyperfiniteness. To analyze hyper-Borel-finiteness in the measure theoretic setting, we first need an easy lemma about functions selecting subsets of a finite set. Below, $\operatorname{Prob}(X)$ indicates the probability of an event X.

Lemma 3.7. *Suppose (X, μ) is a standard probability space, $k \leq n$, Y is a finite set where $|Y| = n$, and $g \colon X \to [Y]^k$ is any measurable function associating to each $x \in X$ a subset of Y of size k. Then for any $m \geq 1$, there is a set $S \subseteq Y$ with $|S| \leq m$ such that* $\operatorname{Prob}(g(x) \cap S \neq \emptyset) \geq 1 - (1 - k/n)^m$.

The point of the lemma for us is the case where $0 \ll k \ll n$, and $m = \lceil \frac{n}{\sqrt{k}} \rceil$. Think of g as being a probabilistic process for choosing k elements out of our set Y of size n. Then the lemma says we can choose $S \subseteq Y$ of size $|S| \leq m$ such that $\operatorname{Prob}(S \cap g(x) \neq \emptyset)$ is close to 1. That is, we can find a "small" S (of size much less than $|Y| = n$) so that with very high probability, one of the k elements we choose using the process g comes from S. This is because $(1 - k/n)^{n/k} \approx 1/e$, so $(1 - k/n)^m \approx (1/e)^{\sqrt{k}} \approx 0$.

Proof. If we select i from Y uniformly at random, and x from X at random (wrt μ), then $\mathrm{Prob}(i \notin g(x)) = 1 - k/n$, since $g(x)$ has k elements. So if we pick m elements $i_1, \dots, i_m$ from Y uniformly at random (allowing repetitions in the list), and let $S = \{i_1, \dots, i_m\}$, then $\mathrm{Prob}(S \cap g(x) = \emptyset) = (1 - k/n)^m$. Hence, there must be some fixed set $S = \{i_1, \dots, i_m\}$ such that $\mathrm{Prob}(g(x) \cap S = \emptyset) \leq (1 - k/n)^m$, and so $\mathrm{Prob}(g(x) \cap S \neq \emptyset) \geq 1 - (1 - k/n)^m$. (It is possible that $|S| < m$ if we have repetitions). □

We now have the following theorem analyzing hyper-Borel-finiteness in the measure theoretic setting:

Theorem 3.8. *Suppose E is a countable Borel equivalence relation on a standard Borel space X, $(f_i)_{i\in\omega}$ are Borel functions from X to X^ω, and μ is a Borel probability measure on X. Then there is a μ-conull Borel set B so that $E \restriction B$ is hyper-(f_i)-finite.*

Proof. We claim that for any $\epsilon > 0$, and any single Borel function $f\colon X \to X^\omega$, there is a Borel set $A \subseteq X$ with $\mu(A) > 1 - \epsilon$ such that $E \restriction A$ is f-finite. (By f-finite for a single f, we mean that no $E \restriction A$-class contains an infinite set of the form $\{f(x)(j) \colon j \in \omega\}$).

The theorem follows easily from this claim. Choose a sequence of positive real numbers $(a_{i,n})_{i,n\in\omega}$ so that $\sum_{i,n} a_{i,n} < \infty$. Then for each i and n, let $A_{i,n} \subseteq X$ be a Borel set so that $E \restriction A_{i,n}$ is hyper-f_i-finite (just for the single function f_i), and $\mu(A_i) > 1 - a_{i,n}$. Then let $B_m = \bigcap_{n\geq m \wedge i\in\omega} A_{i,n}$. Since $B_m \subseteq A_{i,m}$ for every i, $E \restriction B_m$ is $(f_i)_{i\in\omega}$-finite (for the entire sequence of $(f_i)_{i\in\omega}$). The B_m are increasing sets. We have $\mu(B_m) > 1 - \sum_{n\geq m\wedge i\in\omega} a_{i,n}$, so $\mu(B_m) \to 1$. Let $A = \bigcup_m B_m$. Then $E \restriction A$ is hyper-(f_i)-finite as witnessed by $E \restriction B_m$.

We prove the claim. Fix a Borel function $f\colon X \to X^\omega$. Without loss of generality we may assume that $\{f(x)(j) \colon j \in \omega\}$ is infinite for every x. The idea here is to use Lemma 3.7 to find a set A of measure $\mu(A) > 1 - \epsilon$ such that for every $x \in A$, there is some j such that $f(x)(j) \notin A$.

We may first assume by the Borel isomorphism theorem that $X = 2^\omega$. Consider the function $U_l(x) = \{N_s \colon s \in 2^l \wedge (\exists j) f(x)(j) \in N_s\}$. That is, $U_l(x)$ is the collection of basic open neighborhoods N_s, where s has length l, such that N_s contains some element of the sequence $f(x)$. Since the neighborhoods N_s separate points, for every x we have $|U_l(x)| \to \infty$ as $l \to \infty$. Letting $X_{l,k} = \{x \in X \colon |U_l(x)| \geq k\}$, we may choose a sufficiently large l so that $\mu(X_{l,k}) > 1 - \epsilon$.

Now by picking $l \gg k \gg 0$ sufficiently large and applying Lemma 3.7 to the function selecting the least k elements of $U_l(x)$, we can choose a set $S \subseteq \{N_s \colon s \in 2^l\}$ of size $|S| < 2^l/\sqrt{k}$ so that $\mu(\{x \in X_{l,k} \colon U_l(x) \cap S \neq \emptyset\}\})$ is arbitrarily close to $\mu(X_{l,k})$. Note that $\mu(\bigcup S) < \frac{1}{\sqrt{k}}$. Let

$$A = \{x \in X_{l,k} \setminus \bigcup S \colon \exists i f(x)(i) \in \bigcup S\}$$

The claim follows. □

The above proof is trivial in the sense that the subequivalence relations witnessing hyper-(f_i)-finiteness are simply the original equivalence relation restricted to some Borel subset of X. This style of witness that an equivalence relation is hyper-Borel-finite cannot work in general to show that an equivalence relation is hyper-Borel-finite. For example, there is no increasing sequence of Borel sets

$(A_k)_{k\in\omega}$ such that $2^\omega = \bigcup_k A_k$, and the equivalence relations $\equiv_T \restriction A_k$ witness that $\equiv_T$ is hyper-recursively finite. To see this, note that some A_n must contain a pointed perfect set, and hence $\equiv_T \restriction A_n$ must contain a uniformly computable infinite sequence.

4. Strengthenings of the Kuratowski-Mycielski theorem

Two often used constructions in computability theory are

(1) There is a Borel function $f\colon 2^\omega \to 2^\omega$ so that if $x_0, \ldots, x_n$ are distinct, then $f(x_0), \ldots f(x_n)$ are mutually 1-generic.
(2) There is a Borel function $f\colon 2^\omega \to 2^\omega$ so that for all x, $f(x)$ is x-generic.

(1) is true since there is a perfect tree whose infinite paths are mutual 1-generics (hence f in (1) may be continuous). (2) is true since x' can compute an x-generic real uniformly, and so f in this case may be Baire class 1 (i.e. $\mathbf{\Sigma}^0_2$-measurable).

It is impossible to have a function f with both properties (1) and (2):

Proposition 4.1. *There is no Borel function $f\colon 2^\omega \to 2^\omega$ so that:*

(1) *If $x_0, \ldots, x_n$ are distinct, then $f(x_0), \ldots f(x_n)$ are mutually 1-generic.*
(2) *For all x, $f(x)$ is x-generic.*

Proof. If (2) holds, then $\mathrm{ran}(f)$ is nonmeager. This is true because if $\mathrm{ran}(f)$ is meager, the complement of $\mathrm{ran}(f)$ is comeager and hence it would contain a dense G_δ set A which is coded by some real z. But since $f(z)$ is z-generic, $f(z) \in A$, and so $f(z) \notin \mathrm{ran}(f)$.

Now $\mathrm{ran}(f)$ is $\mathbf{\Sigma}^1_1$ and so it has the Baire property. Since $\mathrm{ran}(f)$ is nonmeager, it is therefore comeager in some basic open set N_s. But this implies that $\mathrm{ran}(f)$ contains two elements $f(x_0) \neq f(x_1)$ which are equal mod finite and hence are not mutually 1-generic. □

The point of this section is to prove Lemma 4.2 where we make (1) above compatible with a weakening of (2). Instead of $f(x)$ being x-generic, we can make $f(x)$ and x a minimal pair. The precise lemma we will need is the following, which will be an essential ingredient in the proof of Theorem 5.5.

Lemma 4.2. *Suppose E is a countable Borel equivalence relation on X. Then there is a Borel function $f\colon 2^\omega \to 2^\omega$ such that*

(1) *If $x_0, \ldots, x_n$ are distinct, then $f(x_0), \ldots f(x_n)$ are mutually 1-generic.*
(2) *For all $x, y \in 2^\omega$ such that $x \mathrel{E} y$, there is no z so that $z \leq_m f(y)$ via a many-one reduction with infinite range, and $z \leq_m x$ or $\overline{z} \leq_m x$.*

Note that since $f(x)$ is 1-generic, if $z \leq_m f(x)$ via a many-one reduction with infinite range, then z is not computable.

This lemma follows easily from the more general Lemma 4.3:

Proof of Lemma 4.2: Apply Lemma 4.3 where $X = Y = Z = 2^\omega$, $C_n \subseteq (2^\omega)^n$ is the set of mutually 1-generic n-tuples, S_1 is the relation $\geq_m$ via a many-one reduction with infinite range, and $x \mathrel{R} z$ if $x \geq_m z$ or $x \geq_m \overline{z}$. Note that if $\rho\colon \omega \to \omega$ is a many-one reduction with infinite range, then there is no $z \in 2^\omega$ such that $z \leq_m y$ via ρ for comeagerly many y, since the set of y such that there exists an n such that $n \in z \iff \rho(n) \notin y$ is dense. □

We now prove the following strengthening of the Kuratowski-Mycielski theorem [K95, Theorem 19.1]. Say that a relation $R \subseteq X \times Y$ has *countable vertical sections* if for all $x \in X$ there are countably many $y \in Y$ such that $x \mathrel{R} y$.

Lemma 4.3. *Suppose E is a countable Borel equivalence relation on a Polish space X. Let Y, Z be Polish spaces and $R \subseteq X \times Z$ and $S_n \subseteq Y^n \times Z$ be Borel relations with countable vertical sections. Then for any collection $(C_n)_{n\in\omega}$ of comeager sets $C_n \subseteq Y^n$, there is a Borel injection $f\colon X \to Y$ such that*

(1) *For all $x_1, \ldots, x_n \in X$, $(f(x_1), \ldots, f(x_n)) \in C_n$.*
(2) *For all $x \in X$ and distinct $x_1, \ldots, x_n \in [x]_E$, if $x \mathrel{R} z$ and $(f(x_1), \ldots, f(x_n)) \mathrel{S_n} z$, then there is a nonmeager set of $\vec{y} \in Y^n$ such that $\vec{y} \mathrel{S_n} z$.*

Roughly this says that there is a Borel function f so that any finitely many elements of $\operatorname{ran}(f)$ are "mutually generic" (i.e. in C_n), and that if $x_1, \ldots, x_n \in [x]_E$, then x and $(f(x_1), \ldots, f(x_n))$ form a "minimal pair" (with respect to R and S_n).

Proof. Fix countable bases $\mathcal{B}_X, \mathcal{B}_Y, \mathcal{B}_Z$ of X, Y, and Z. Also fix a complete metric d generating the topology of Y. Say that an *approximation* p of f is a function $p\colon P \to \mathcal{B}_Y$ where P is a Borel partition of X into finitely many Borel sets. Say that an approximation $p'\colon P' \to \mathcal{B}_Y$ refines $p\colon P \to \mathcal{B}_Y$ if P' refines P, and if $A' \in P'$ and $A \in P$ are such that $A' \subseteq A$, then $p'(A') \subseteq p(A)$.

Suppose that $p_0, p_1, \ldots$ is a sequence of approximations where p_{n+1} refines p_n,

(a) $\max\{\operatorname{diam}(U)\colon U \in \operatorname{ran}(p_n)\} \to 0$ as $n \to \infty$, and
(b) for all n, there exists $m > n$, so that $A \in \operatorname{dom}(p_n)$, $A' \in \operatorname{dom}(p_m)$ and $A' \subseteq A$ implies $\operatorname{cl}(p_m(A')) \subseteq p_n(A)$, where cl denotes closure.

Then we can associate to this sequence the function $f\colon X \to Y$ where $f(x) = y$ if $\{y\} = \bigcap_n p_n(A_{x,n})$ where $A_{x,n}$ is the unique element of $\operatorname{dom}(p_n)$ such that $x \in A_n$. Conditions (a) and (b) ensure that $\bigcap_n p_n(A_{x,n})$ is a singleton for every x. We will construct f in this way, where the sequence $(p_i)_{i\in\omega}$ is a sufficiently generic sequence of approximations. Clearly (1) in the statement of the Lemma will be true for a sufficiently generic sequence. We give a density argument to justify why (2) will be true.

Since R, S_n have countable vertical sections, by Lusin-Novikov uniformization [K95, 18.5], there are Borel functions $(g_i)_{i\in\omega}$ and $(h_{n,i})_{i,n\in\omega}$ where $g_i\colon X \to Z$ and $h_{n,i}\colon Y^n \to Z$ such that $x \mathrel{R} z$ iff $g_i(x) = z$ for some i, and $\vec{y} \mathrel{S_n} z$ iff $h_{n,i}(\vec{y}) = z$ for some i. By perhaps refining the sets C_n, we may assume that the functions $h_{n,i}$ are continuous on C_n, since any Borel function is continuous on a comeager set [K95, Theorem 8.38]. By the Feldman-Moore theorem, we may fix a Borel action of a countable group Γ generating E. Let $\mathcal{G}$ be the set of $z \in Z$ such that for some n, there is a nonmeager set of $\vec{y} \in Y^n$ such that $\vec{y} \mathrel{S_n} z$.

Fix an approximation p, finitely many disjoint basic open sets $V_1, \ldots, V_n \subseteq X$ and group elements $\gamma_1, \ldots, \gamma_n \in \Gamma$, and $j, k \in \omega$. It suffices to show that we can refine p to an approximation p^* such that for all $x \in X$, if $\gamma_i \cdot x \in V_i$ for all $i \leq n$, then either

$$(h_{n,k} \restriction C_n)(p^*([\gamma_1 \cdot x]) \times \ldots \times p^*([\gamma_n \cdot x])) \in \mathcal{G}, \text{ or} \tag{*}$$

$$g_j(x) \notin (h_{n,k} \restriction C_n)(p^*([\gamma_1 \cdot x]) \times \ldots \times p^*([\gamma_n \cdot x])) \tag{**}$$

where by $[\gamma_i \cdot x]$ we mean the element of $\mathrm{dom}(p^*)$ that contains $\gamma_i \cdot x$. That is, the condition above is that if $\gamma_i \cdot x \in V_i$ for all $i \leq n$, then the value of $h_{n,k}(f(\gamma_1 \cdot x), \ldots, f(\gamma_n \cdot x))$ is "forced" by p^* to be in $\mathcal{G}$, or forced to be different from $g_j(x)$.

Let $B = \{x \colon (\forall i \leq n) \gamma_i \cdot x \in V_i\}$. These are the x for which me must ensure that either (*) or (**) holds. Let $P = \mathrm{dom}(p)$. By refining the domain of p, we may assume that every element of P is either contained in or disjoint from $\gamma_i \cdot B$ for every $i \leq n$. By similarly refining the domain, we may furthermore assume that if $A \in P$ is such that $A \subseteq \gamma_i \cdot B$, then $\gamma_{i'} \gamma_i^{-1} \cdot A \in P$ for all $i' \leq n$.

We now define p^*. For all $A \in P$ such that $A \not\subseteq \gamma_i \cdot B$ for all $i \leq n$, put $A \in \mathrm{dom}(p^*)$, and define $p^*(A) = p(A)$. Any remaining $A \in P$ belongs to a tuple $(A_1, \ldots, A_n)$ of elements of P where $A_i \subseteq \gamma_i \cdot B$ for all $i \leq n$ and $A_{i'} = \gamma_{i'} \cdot \gamma_i^{-1} \cdot A_i$ for all $i, i' \leq n$ (by our assumption on P from the previous paragraph). So for all x, if $\gamma_i \cdot x \in A_i$ for some $i \leq n$, then $\gamma_i \cdot x \in A_i$ for all $i \leq n$. We will define p^* on these A_i to satisfy (*) or (**). Letting $U_i = p(A_i)$ for every $i \leq n$, we ask if there are basic open sets $U_i', U_i'' \subseteq U_i$ and disjoint basic open sets $W', W'' \subseteq Z$ so that $(h_{n,k} \restriction C_n)(U_1', \ldots, U_n') \subseteq W'$ and $(h_{n,k} \restriction C_n)(U_1'', \ldots, U_n'') \subseteq W''$.

Case 1: if such W' and W'' do not exist, then put $A_i \in \mathrm{dom}(p^*)$ and define $p^*(A_i) = p(A_i) = U_i$ for every $i \leq n$. Since $h_{n,k} \restriction C_n$ is continuous, then $(h_{n,k} \restriction C_n)(U_1, \ldots, U_n)$ must be a singleton, which must therefore be in $\mathcal{G}$. So in this case (*) is satisfied for all x such that $\gamma_i \cdot x \in A_i$ for $i \leq n$.

Case 2: if such W' and W'' do exist, let $A_i' = \{x \colon g_j(\gamma_i^{-1} \cdot x) \in W'\}$, and for every $i \leq n$, put both A_i' and $A_i \setminus A_i'$ in $\mathrm{dom}(p^*)$, and define $p^*(A_i') = U_i''$, and $p^*(A_i \setminus A_i'') = U_i'$. Then by definition, (**) holds for every x such that $\gamma_i \cdot x \in A_i$ for $i \leq n$. $\square$

We remark that there are interesting open problems about the extent to which the Kuratowski-Mycielski theorem can be generalized. For example,

Question 4.4. *Does there exist a Borel function $g \colon 2^\omega \to 2^\omega$, so that for all distinct x, y with $x \leq_T y$, $g(x)$ and $g(y)$ are mutually x-generic?*

5. A nonuniform construction

In our main construction in the proof of Theorem 5.5, we will do coding using countably many computable injective strictly increasing functions $c_m \colon \omega \to \omega$ with disjoint ranges. Precisely, we will ensure that if $x \mathrel{E} y$, then $f(x) \leq_1 f(y)$ via one of these one-one reductions c_m. We begin this section with some definitions and lemmas related to the kind of coding we will do. The reader may want to read the first few paragraphs of the proof of Theorem 5.5 up to the definition of f and verification of (1) to motivate these definitions.

Definition 5.1. Suppose $(c_m)_{m \in \omega}$ is a sequence of strictly increasing functions $c_m \colon \omega \to \omega$ with disjoint ranges. We define the *decoding function* $d \colon \omega \to \omega^{<\omega}$ associated to $(c_m)_{m \in \omega}$ as follows:

$$d(n) = \begin{cases} \emptyset & \text{if } n \notin \mathrm{ran}(c_m) \text{ for any } m \\ m^\frown d(c_m^{-1}(n)) & \text{if } n \in \mathrm{ran}(c_m) \end{cases}$$

Where $\emptyset$ denotes the empty string and $\frown$ denotes concatenation of strings. Similarly, define $d_s\colon \omega \to \omega^{<\omega}$ in the same way but where we only use c_m where $m \leq s$.

$$d_s(n) = \begin{cases} \emptyset & \text{if } n \notin \operatorname{ran}(c_m) \text{ for any } m \leq s \\ m^\frown d_s(c_m^{-1}(n)) & \text{if } n \in \operatorname{ran}(c_m) \text{ and } m \leq s \end{cases}$$

Finally, define $b, b_s\colon \omega \to \omega$ as follows. Define $b(n) = (c_{m_0} \circ \ldots \circ c_{m_k})^{-1}(n)$ where $m_0, \ldots, m_k$ are such that $d(n) = (m_0, \ldots, m_k)$. Similarly, $b_s(n) = (c_{m_0} \circ \ldots \circ c_{m_k})^{-1}(n)$ where $m_0, \ldots, m_k$ are such that $d_s(n) = (m_0, \ldots, m_k)$.

We can think of d in the following way. Any $n \in \omega$ can be in the range of at most one c_m since the $(c_m)_{m\in\omega}$ have disjoint ranges. If n is in the range of some c_m, the number $c_m^{-1}(n)$ is strictly less than n since c_m is strictly increasing. Iterating this process, there is a unique longest sequence $m_0, \ldots, m_k$ so that $n \in \operatorname{ran}(c_{m_0} \circ \ldots \circ c_{m_k})$. This longest such sequence $(m_0, \ldots, m_k)$ is defined to be $d(n)$. The function d_s is defined the same way but where we restrict to only considering c_m with $m \leq s$. Finally, b and b_s are the functions which maps n to the number obtained by repeatedly taking the inverse image of n under $c_{m_0}, \ldots, c_{m_k}$ where $(m_0, \ldots, m_k)$ is either $d(n)$ or $d_s(n)$ respectively. Note that $d_s(n)$ is an initial segment of $d(n)$, and in fact $d(n) = d_s(n)^\frown d(b_s(n))$ for every n, s.

We now describe the functions $(c_m)_{m\in\omega}$ we will use in the proof of Theorem 5.5. Below if $t \in \omega^{<\omega}$ is a sequence, then $\max t$ denotes the largest number in the sequence t. We take the convention that $\max \emptyset = 0$. Recall that $A^{[i]} = \{\langle i, j\rangle \in A\colon j \in \omega\}$ is the ith column of A.

Lemma 5.2. *There is a sequence $(c_m)_{n\in\omega}$ of injective strictly increasing computable functions $c_m\colon \omega \to \omega$ with disjoint ranges and an infinite computable set $D_0 \subseteq \omega$ so that D_0 is disjoint from $\bigcup_{m\in\omega} \operatorname{ran}(c_m)$, and*

1. *For all computable $\rho\colon \omega \to \omega$, there exists an $s \in \omega$ so that either $b_s(\rho(\omega))$ is finite, or there exists a computable infinite set B so that for all $n \in B$, $\max d(\rho(n)) \leq s$, and $b_s(\rho(B))$ is infinite.*
2. *For all computable $\rho\colon \omega \to \omega$, there exists an $s \in \omega$ so that either for infinitely many i, $b_s(\rho(\omega^{[i]}))$ is finite, or there is a computable set B so that for all $n \in B$, $\max d(\rho(n)) \leq s$, and for all but finitely many i, $b_s(\rho(B^{[i]}))$ is infinite.*

Proof. Suppose $\rho\colon \omega \to \omega$ is computable and $\rho'\colon \omega \to \omega$ is defined by $\rho'(\langle i, j\rangle) = \rho(j)$, so ρ' copies the values of ρ on every column of ω. Then if (2) holds for ρ' then (1) holds for ρ. So we only need to verify property (2).

We construct the sequence $(c_m)_{m\in\omega}$ in countably many stages where at stage s we define the computable function c_s. We will also build a sequence $(D_m)_{m\in\omega}$ of subsets of ω where for all m, $D_m \supseteq D_{m-1}$, D_m is disjoint from $\operatorname{ran}(c_m)$, and $\operatorname{ran}(c_0) \cup \ldots \cup \operatorname{ran}(c_m) \cup D_m$ is coinfinite. Though each c_m and D_m will be computable, neither the sequence $(c_m)_{m\in\omega}$ nor $(D_m)_{m\in\omega}$ will be uniformly computable.

Note that since each c_m is computable and strictly increasing, $\operatorname{ran}(c_m)$ is computable. Hence for each s, the functions d_s and b_s will be computable.

Let c_0 be any computable strictly increasing function and $D_0 \subseteq \omega$ be any computable infinite set so that D_0 and $\operatorname{ran}(c_0)$ are disjoint and $D_0 \cup \operatorname{ran}(c_0)$ is coinfinite. Since we will ensure that $\operatorname{ran}(c_m)$ is disjoint from $D_m \supseteq D_0$ for every m, the required property that D_0 will be disjoint from $\cup_m \operatorname{ran}(c_m)$ will be true at the end of the construction.

At stage s, let $\rho\colon \omega \to \omega$ be the sth total computable function, and suppose we have defined c_s and D_s. We will define c_{s+1} and D_{s+1} so that (2) is true. We many assume that there is some k so that for all $i \geq k$, $b_s(\rho(\omega^{[i]}))$ is infinite. If this is not the case, then property (2) is already true for ρ and we may define $D_{s+1} = D_s$, and let c_{s+1} to be an arbitrary strictly increasing computable injection so that $\operatorname{ran}(c_{s+1})$ is disjoint from $\operatorname{ran}(c_0) \cup \ldots, \cup \operatorname{ran}(c_s) \cup D_{s+1}$ and so that $\operatorname{ran}(c_0) \cup \ldots, \cup \operatorname{ran}(c_{s+1}) \cup D_{s+1}$ is cofinite.

So fix k so that for all $i \geq k$, $b_s(\rho(\omega^{[i]}))$ is infinite. Now we can find a computable set $D_{s+1} \supseteq D_s$ so that for every $i \geq k$, $D_{s+1} \cap b_s(\rho(\omega^{[i]}))$ is infinite and disjoint from $\operatorname{ran}(c_0) \cup \ldots \cup \operatorname{ran}(c_s)$. We do this by at step n defining D_{s+1} on a large enough finite segment to ensure that there are at least n elements of $b_s(\rho(\omega^{[i]}))$ in D_{s+1} for every $k \leq i \leq n$. Note that every element of $\operatorname{ran}(b_s)$ is disjoint from $\operatorname{ran}(c_0) \cup \ldots \cup \operatorname{ran}(c_s)$ by definition of b_s. At step n we also choose n new elements not in $D_s \cup \operatorname{ran}(c_0) \cup \ldots \cup \operatorname{ran}(c_s)$ and promise that they will not be in D_{s+1} (so that at the end of the construction $D_{s+1} \cup \operatorname{ran}(c_0) \cup \ldots \cup \operatorname{ran}(c_s)$ is coinfinite.

Once we have defined D_{s+1} as above, we have that for every n such that $b_s(\rho(n)) \in D_{s+1}$, $\max d(\rho(n)) \leq s$. This is since $d(\rho(n)) = d_s(\rho(n))^\frown d(b_s(\rho(n))) = d_s(\rho(n))$ since $b_s(\rho(n))$ is not in the range of any c_m since it is in D_{s+1}. By definition of d_s, we have $\max d_s(m) \leq s$ for all m. Finally, the set $B = \{n \colon b_s(\rho(n)) \in D_{s+1}\}$ is computable (since D_{s+1} and b_s are computable) and is our desired computable set. $\square$

Of course, the range $\rho(\omega)$ of a computable function $\rho\colon \omega \to \omega$ is just a c.e. set, and we could equivalently state Lemma 5.2 to be about c.e. sets instead. For example, part (2) of Lemma 5.2 would become: if $(A_i)_{i\in\omega}$ is a uniformly c.e. family of subsets of ω, then either for infinitely many i, $b_s(A_i)$ is finite, or there is a computable set $C \subseteq \bigcup_i A_i$ so that $\max d(n) \leq s$ for all $n \in C$ and for all but finitely many i, $b_s(A_i \cap C)$ is infinite. Here $(A_i)_{i\in\omega}$ is $(\rho(\omega^{[i]}))_{i\in\omega}$, and B in the above lemma would be $\rho^{-1}(C)$. We stated the Lemma 5.2 in the above form since this is the way it will eventually be used, where ρ is some many-one reduction.

Two important ideals in the proof of Theorem 5.5 will be the first and second iterated Fréchet ideals on ω which we denote I_1 and I_2. We use I_2 when we are simultaneously analyzing all the columns of a many-one reduction.

Definition 5.3. Let $I_1 = \{A \subseteq \omega \colon A \text{ is finite}\}$. Let $I_2 = \{A \subseteq \omega \colon \text{ for all but finitely many } i,\ A^{[i]} \text{ is finite}\}$.

An important idea in our proof of Theorem 5.5 is captured by the following simple proposition. One should think here of a set not being in an ideal I on ω as a notion of largeness. For example for the Fréchet ideal I_1, $A \notin I_1$ iff A is infinite.

Proposition 5.4. *Suppose $S \subseteq \omega^{<\omega}$ is a finitely branching tree, $t\colon \omega \to S$ is an arithmetic function, and I is an arithmetically definable ideal on ω (such as I_1 or I_2). Let $T \subseteq S$ be defined by $T = \{s \in S \colon \{n \colon t(n) \supseteq s\} \notin I\}$. Then T is an arithmetically definable subtree of S. Furthermore any $s \in T$ with no extensions in T has $\{n \colon t(n) = s\} \notin I$. So by Kőnig's lemma, either T has an infinite branch and hence an arithmetically definable infinite branch, or there is some s so that $\{n \colon t(n) = s\} \notin I$.*

Proof. First we show T is closed downwards and is hence a tree. Suppose $s_1 \in T$, and $s_0 \subseteq s_1$. Then since $\{n\colon t(n) \supseteq s_0\} \supseteq \{n\colon t(n) \supseteq s_1\}$ and any superset of a set not in I is also not in I, we have $s_0 \in T$.

Now if $s \in T$, and $s_0, \ldots, s_k$ are the immediate extensions of s in S, then we can partition the set $\{n\colon t(n) \supseteq s\}$ which is not in I into finitely many sets: $\{n\colon t(n) = s\}$, and $\{n\colon t(n) \supseteq s_i\}$ for each $i \leq k$. At least one of theses sets must be not in I since a union of finitely many sets in I is in I. Hence, any $s \in T$ with no extensions in T has $\{n\colon t(n) = s\} \notin I$. $\square$

In the proof of Theorem 5.5 we will use same idea as the above proposition, but in a relativized form, and where t is a function to finitely branching tree in a different space (a tree made of elements of $[x]_E^{<\omega}$).

We are ready to prove our main theorem showing that a positive answer to Question 3.4 implies the existence of non-uniform invariant functions that are incomparable with the identity function.

Theorem 5.5. *Suppose E is a hyper-Borel-finite Borel equivalence relation on 2^ω. Then there exists an injective Borel function $f\colon 2^\omega \to 2^\omega$ such that for all $x_0, x_1 \in 2^\omega$*

1. *If $x_0 \mathrel{E} x_1$, then $f(x_0) \equiv_1 f(x_1)$*
2. *If $x_0 \not\mathrel{E} x_1$, then $f(x_0) \not\equiv_m f(x_1)$.*
3. *For every noncomputable x, $f(x)$ is $\leq_m$-incomparable with both x and $\overline{x}$.*
4. *For all $x \in 2^\omega$, there does not exist an infinite sequence $(x_i)_{i\in\omega}$ of distinct reals such that $\bigoplus_i f(x_i) \leq_m f(x)$.*

Proof. Let $\mathbb{F}_\omega \curvearrowright 2^\omega$ be a Borel action of the group $\mathbb{F}_\omega$ that generates the equivalence relation E. Let $(\gamma_i)_{i\in\omega}$ be a computable enumeration of the group $\mathbb{F}_\omega$ so that group multiplication is computable. Let $h_i\colon 2^\omega \to (2^\omega)^\omega$ be the Borel function where $h_i(x) \in (2^\omega)^\omega$ is the ith real arithmetically definable from $\bigoplus_{j\in\omega} \gamma_j \cdot x$ (using some computable bijection to identify 2^ω with $(2^\omega)^\omega$). Intuitively, $\bigoplus_{j\in\omega} \gamma_j \cdot x$ codes the entire orbit of the x under the group action. For example, for every $x \in 2^\omega$, the stabilizer of x (i.e. $\{i\colon \gamma_i \cdot x = x\}$) is arithmetically definable from $\bigoplus_{j\in\omega} \gamma_j \cdot x$. Since the function $x \mapsto \bigoplus_{j\in\omega} \gamma_j \cdot x$ is Borel, each h_i is Borel since it is the composition of a Borel function with an arithmetic function. Let $(E_j)_{j\in\omega}$ be a witness that E is hyper-$(h_i)_{i\in\omega}$-finite, so $E_0 \subseteq E_1 \subseteq \ldots$, and $E = \bigcup_{j\in\omega} E_j$.

Let $g\colon 2^\omega \to 2^\omega$ be a function as in Lemma 4.2, letting the relation E be our given equivalence relation E. Let $(c_m)_{m\in\omega}$ be as in Lemma 5.2. We define $f\colon 2^\omega \to 2^\omega$ by:

$$f(x)(n) = \begin{cases} f(\gamma_i \cdot x)(c_{\langle i,j\rangle}^{-1}(n)) & \text{if } \exists i,j \text{ so } n \in \operatorname{ran}(c_{\langle i,j\rangle}) \text{ and } x \mathrel{E_j} \gamma_i \cdot x \\ g(x)(n) \text{ otherwise.} \end{cases}$$

This definition is self-referential, but it is not circular. If $f(x)(n) = f(\gamma_{i_0} \cdot x)(n_0)$ where $n_0 = c_{\langle i_0,j_0\rangle}^{-1}(n)$, then $n_0 < n$ since $c_{\langle i_0,j_0\rangle}$ is strictly increasing. So after finitely many applications of the definition of f we will reach the base case of the definition and find a sequence $i_0, \ldots, i_k$ and n_k where $f(x)(n) = g(\gamma_{i_k} \cdots \gamma_{i_0} \cdot x)(n_k)$. These kinds of self-referential definitions where we code values of f into itself have been used before in the study of the Borel complexity of equivalence relations from computability theory (see e.g. [MSS, Theorem 2.5] and [M, Theorem 3.6]).

By the definition of f, part (1) of the theorem is true. Given any $x_0 \mathrel{E} x_1$, let i be such that $\gamma_i \cdot x_0 = x_1$. There is some j such that $x_0 \mathrel{E_j} x_1$. Then the function $c_{\langle i,j\rangle}$ is a one-one reduction witnessing $f(x_1) \leq_1 f(x_0)$. This is because for all $n_0 \in \omega$, $f(\gamma_i \cdot x_0)(n_0) = f(x_0)(c_{\langle i,j\rangle}(n_0))$ by letting $n = c_{\langle i,j\rangle}(n_0)$ in the definition of $f(x_0)$. Arguing symmetrically, we also have $f(x_0) \leq_1 f(x_1)$.

The idea of the proof is that f as generic as possible, given that we have to do coding to ensure that if $x_0 \mathrel{E} x_1$, then $f(x_0) \equiv_1 f(x_1)$. Intuitively, there are two types of bits n of $f(x)$. There are infinitely many "generic" bits n where $f(x)(n) = g(x)(n)$. The remaining bits are used for coding where we record the value of the bits of $f(\gamma_i \cdot x)$ for $i \in \omega$. This coding scheme is also chosen to be generic (as made precise by Lemma 5.2). Supposing $z \leq_m f(x)$, the crux of the proof is understanding how well this many-one reduction can iteratively decode this coding to find bits of $g(\gamma_i \cdot x)(n)$ for many different i and n. The high level idea of the proof is that if there is a many-one reduction whose range decodes to be values of $g(\gamma_i \cdot x)(n)$ for a "large" set of i and n (according to some ideal), then we get a contradiction to $(E_j)_{j\in\omega}$ being a hyper-$(h_i)_{i\in\omega}$-finiteness witness for E. But if a many-one reduction only uses values of $g(\gamma_i \cdot x)(n)$ for finitely many i on a large set, then since these $g(\gamma_i \cdot x)$ are mutually generic (and g has the stronger properties give in Lemma 4.2), then z cannot be x or values of $f(y)$ for $y \not\mathrel{E} x$.

Our next goal is to give a definition of $f(x)$ that is only in terms of the function g and is not self-referential. First we make a definition that describes when we recursively use the first clause $f(x)(n) = f(\gamma_i \cdot x)(c^{-1}_{\langle i,j\rangle}(n))$ of the definition of $f(x)(n)$ to "decode" it. Say a sequence $(\langle i_0, j_0\rangle, \langle i_1, j_1\rangle, \ldots, \langle i_k, j_k\rangle) \in \omega^{<\omega}$ is *x-valid* if

$$(\gamma_{i_{m-1}} \cdots \gamma_{i_0} \cdot x) \mathrel{E_{j_m}} (\gamma_{i_m} \cdots \gamma_{i_0} \cdot x)$$

for every $m \leq k$. Note that if a sequence is x-valid then every initial segment of it is x-valid.

Let $d_x(n)$ be the longest initial segment of $d(n)$ that is x-valid. So $d_x \colon \omega \to \omega^{<\omega}$. Hence if $d_x(n) = (\langle i_0, j_0\rangle, \langle i_1, j_1\rangle, \ldots, \langle i_k, j_k\rangle)$, then

$$f(x)(n) = f(\gamma_{i_0} \cdot x)(c^{-1}_{\langle i_0,j_0\rangle}(n))$$

by the definition of f since $n \in \operatorname{ran}(c_{\langle i_0,j_0\rangle})$ by the definition of d, and since $x \mathrel{E_{j_0}} \gamma_{i_0} \cdot x$ by the definition of being x-valid. Similarly, we have inductively that for every $m \leq k$,

$$\begin{aligned} &f(\gamma_{i_{m-1}} \cdots \gamma_{i_0} \cdot x)(c^{-1}_{\langle i_{m-1},j_{m-1}\rangle} \circ \ldots \circ c^{-1}_{\langle i_0,j_0\rangle}(n)) \\ &\quad = f(\gamma_{i_m} \cdots \gamma_{i_0} \cdot x)(c^{-1}_{\langle i_m,j_m\rangle} \circ \ldots \circ c^{-1}_{\langle i_0,j_0\rangle}(n)) \end{aligned}$$

again using the definition of f, the definition of d, and since $(\gamma_{i_{m-1}} \cdots \gamma_{i_0} \cdot x) \mathrel{E_{j_m}} (\gamma_{i_m} \cdots \gamma_{i_0} \cdot x)$ by the definition of being x-valid. Finally, either $d_x(n) = d(n)$ and so $c^{-1}_{\langle i_k,j_k\rangle} \circ \ldots \circ c^{-1}_{\langle i_0,j_0\rangle}(n) \notin \operatorname{ran}(c_m)$ for any m by the definition of d, or $d_x(n)$ is a proper initial segment of $d(n) = (\langle i_0, j_0\rangle, \langle i_1, j_1\rangle, \ldots, \langle i_k, j_k\rangle, \langle i_{k+1}, j_{k+1}\rangle, \ldots)$, so $c^{-1}_{\langle i_k,j_k\rangle} \circ \ldots \circ c^{-1}_{\langle i_0,j_0\rangle}(n) \in \operatorname{ran}(c_{\langle i_{k+1},j_{k+1}\rangle})$ but $(\gamma_{i_k} \cdots \gamma_{i_0} \cdot x) \not\mathrel{E_{j_{k+1}}} (\gamma_{i_{k+1}} \cdots \gamma_{i_0} \cdot x)$, since $d_x(n)$ is the longest initial segment of $d(n)$ that is x-valid. In either case, in the definition of $f(\gamma_{i_k} \cdots \gamma_{i_0} \cdot x)(c^{-1}_{\langle i_k,j_k\rangle} \circ \ldots \circ c^{-1}_{\langle i_0,j_0\rangle}(n))$ we use the second clause of the definition, and so

$$\begin{aligned} &f(\gamma_{i_k}\cdots\gamma_{i_0}\cdot x)(c^{-1}_{\langle i_k,j_k\rangle}\circ\ldots\circ c^{-1}_{\langle i_0,j_0\rangle}(n)) \\ &= g(\gamma_{i_k}\cdots\gamma_{i_0}\cdot x)(c^{-1}_{\langle i_k,j_k\rangle}\circ\ldots\circ c^{-1}_{\langle i_0,j_0\rangle}(n)). \end{aligned}$$

Putting together the above three displayed equations, we have shown that if $d_x(n) = (\langle i_0,j_0\rangle,\langle i_1,j_1\rangle,\ldots,\langle i_k,j_k\rangle)$, then we have the following explicit definition of $f(x)$ in terms of g.

$$f(x)(n) = g(\gamma_{i_k}\cdots\gamma_{i_0}\cdot x)(c^{-1}_{\langle i_0,j_0\rangle}\circ\ldots\circ c^{-1}_{\langle i_k,j_k\rangle}(n)).$$

To make this definition more compact, we introduce two more functions. Define $y_x\colon\omega\to[x]_E$ and $b_x\colon\omega\to\omega$ as follows. If $d_x(n) = (\langle i_0,j_0\rangle,\ldots,\langle i_k,j_k\rangle)$, then $y_x(n) = \gamma_{i_k}\cdots\gamma_{i_0}\cdot x$ and $b_x(n) = c^{-1}_{\langle i_0,j_0\rangle}\circ\ldots\circ c^{-1}_{\langle i_k,j_k\rangle}(n)$. Hence for all n,

$$f(x)(n) = g(y_x(n))(b_x(n)). \tag{*}$$

That is for all n, $f(x)(n)$ codes the bit $b_x(n)$ of $g(y_x(n))$. Note that for all n, $b_x(n)\geq b(n)$ since $d_x(n)$ is an initial segment of $d(n)$.

Similarly, we define $d_{s,x}\colon\omega\to\omega^{<\omega}$ by letting $d_{s,x}(n)$ be the longest initial segment of $d_s(n)$ that is x-valid. Note that $d_x(n) = d_{s,x}(n)$ for sufficiently large s (i.e. $s\geq\max d(n)$). Define also $y_{s,x}\colon\omega\to[x]_E$ and $b_{s,x}\colon\omega\to\omega$ as follows. If $d_{s,x}(n) = (\langle i_0,j_0\rangle,\ldots,\langle i_k,j_k\rangle)$, then $y_{s,x}(n) = \gamma_{i_k}\cdots\gamma_{i_0}\cdot x$ and $b_{s,x}(n) = c^{-1}_{\langle i_0,j_0\rangle}\circ\ldots\circ c^{-1}_{\langle i_k,j_k\rangle}(n)$. An identical kind of induction to the one above using the properties of being x-valid show that for all n and s,

$$f(x)(n) = f(y_{s,x}(n))(b_{s,x}(n)). \tag{**}$$

Note, though, that in this equation (**) we have f on the right hand side instead of g. This is because it is possible that $d_x(n)\supsetneq d_s(n)$ and so n needs to be further decoded using functions c_m for $m>s$.

Our analysis of reals that are many-one reducible to $f(x)$ will be based on analyzing a finitely branching tree built out of elements in $[x]_E$, which is related to (*) above. Let $[x]_E^{<\omega}$ be the set of finite sequences $(y_0,\ldots,y_l)$ so that $y_i\in[x]_E$ for all $i\leq l$. We define a function $t_x(n)\colon\omega\to[x]_E^{<\omega}$ as follows. Given $d(n) = (\langle i_0,j_0\rangle,\ldots,\langle i_k,j_k\rangle)$, consider the sequence $(x,\gamma_{i_0}\cdot x,\ldots,\gamma_{i_k}\cdots\gamma_{i_0}\cdot x)$. This sequence may contain elements that are repeated so we define $t_x(n)$ to be a "deduplicated" version of this sequence, so $t_x(n) = (y_0,\ldots,y_l)$ has the same elements as $(x,\gamma_{i_0}\cdot x,\ldots,\gamma_{i_k}\cdots\gamma_{i_0}\cdot x)$, but where each element occurs exactly once. Precisely, let $y_0 = x$ and y_{j+1} be the first element of the sequence $(x,\gamma_{i_0}\cdot x,\ldots,\gamma_{i_k}\cdots\gamma_{i_0}\cdot x)$ that is not equal to y_m for any $m\leq j$. Intuitively, if $t_x(n) = (y_0,\ldots,y_l)$, this means $y_0 = x$ and $f(y_0)(n)$ codes a bit of $f(y_1)$ which codes a bit of $f(y_2)$, $\ldots$, which codes a bit of $f(y_l)$ which is equal to a bit of $g(y_l)$, assuming that $d(n)$ is x-valid. Note that even if $d(n)$ is not x-valid, then $y_x(n)$ is an element of $t_x(n)$. One final fact we will often use about the relationship between $y_x(n)$ and $t_x(n)$ is that if $\max d(n)\leq s$, $r = (y_0,\ldots,y_l)$, $y_l\not E_s\, x$, and $t_x(n)\supseteq r$, then $y_x(n) = y_i$ for some $i\leq l$. That is, in this case even though $t_x(n)$ may contain many elements not in r, the value $y_x(n)$ must come from r. This is since any part of the sequence $d(n)$ that yields part of $t_x(n)$ that extends r cannot be x-valid since $y_l\not E_s\, x$, and $\max d(n)\leq s$.

Note that we are defining $t_x(n)$ using the function $d(n)$ instead of $d_x(n)$ because we want $t_x(n)$ to be arithmetically definable relative to $\bigoplus_{j\in\omega}\gamma_j\cdot x$. This is so we can use the idea of Proposition 5.4, relative to $\bigoplus_{j\in\omega}\gamma_j\cdot x$. (The definition of $d_x(n)$ depends on our hyper-Borel-finiteness witness $(E_j)_{j\in\omega}$ and we have no bound its complexity).

We will also define a similar function to t_x but using the function $d_s(n)$ instead of $d(n)$. Precisely, define $t_{s,x}(n)\colon \omega \to [x]_E^{<\omega}$ as follows. Given $d_s(n) = (\langle i_0, j_0\rangle, \dots, \langle i_k, j_k\rangle)$, let $t_{s,x}(n)$ be the de-duplicated version of the sequence $(x, \gamma_{i_0}\cdot x, \dots, \gamma_{i_k}\cdots\gamma_{i_0}\cdot x)$ as in the definition of t_x. Note that since $d_s(n) = d(n)$ for $s \geq \max d(n)$, we have that $t_x(n) = t_{s,x}(n)$ if $s \geq \max d(n)$. An important property of $t_{s,x}$ is that its values (unlike t_x) form a finitely branching tree. Precisely, If $t_{s,x}(n) = (y_0, \dots, y_l)$, we must have that for every $k \leq l$, $y_k = \gamma_i \cdot y_j$ for some $i \leq s$ and $j \leq k$. This is by definition of d_s and $t_{s,x}$. Hence, the downward closure of all the values of $t_{s,x}(n)$ forms a finitely branching tree in $[x]_E^{<\omega}$. Mostly (except at the end of Claim 3), using Lemma 5.2 we will work on sets $B \subseteq \omega$ where $\max d(\rho(n)) \leq s$, and hence $t_x(\rho(n)) = t_{s,x}(\rho(n))$ for all $n \in B$.

Because we have introduced many different functions, we briefly summarize:

- $g\colon 2^\omega \to 2^\omega$ is the generic function from Lemma 4.2 whose range is a set of mutual 1-generics, and so that if $x\ E\ y$ and $z \leq_m g(x)$ via a many-one reduction with infinite range, then $z \nleq_m y$ and $\overline{z} \nleq_m y$.
- $(E_j)_{j\in\omega}$ are the witness that E is hyper-$(h_i)_{i\in\omega}$–finite. The functions $(h_i)_{i\in\omega}$ are those that are arithmetically definable from $\bigoplus_j \gamma_j\cdot x$ (i.e. arithmetically definable from the orbit of x).
- $f\colon 2^\omega \to 2^\omega$ is the Borel reduction from E to $\equiv_m$ we're building. The definition of f in terms of g, $(E_j)_{j\in\omega}$ and $(c_m)_{m\in\omega}$ is given at the beginning of the proof.
- $(c_m)_{m\in\omega}$ are the "coding functions" used to ensure that if $x\ E\ y$, then $f(x) \leq_1 f(y)$. Precisely, if $x\ E_j\ \gamma_i\cdot x$, then $f(\gamma_i\cdot x) \leq_1 f(x)$ via $c_{\langle i,j\rangle}$. Each c_m is computable, injective, and increasing, and but the sequence $(c_m)_{m\in\omega}$ is not uniformly computable. The c_m have disjoint ranges. The sequence $(c_m)_{m\in\omega}$ is a "generic" such sequence and is constructed in Lemma 5.2. The functions $d, d_s\colon \omega \to \omega^{<\omega}$ and $b, b_s\colon \omega \to \omega$ are associated functions used for decoding and defined in Definition 5.1.
- The function $d_x\colon \omega \to \omega^{<\omega}$ is defined so that $d_x(n)$ is the longest substring of $d(n)$ that is x-valid, where we define x-valid sequences according to which clause of the definition of $f(x)(n)$ would be used to decode them. Using d_x, we then gave a definition (*) above of the function $f(x)$ just in terms of g: $f(x)(n) = g(y_x(n))(b_x(n))$, where $y_x\colon \omega \to [x]_E$, and $b_x\colon \omega\to\omega$ were defined in terms of $d_x(n)$.
- The function $t_x\colon \omega \to [x]_E^{<\omega}$ maps each bit n to the sequence of distinct $y_0, y_1, \dots, y_k$ where $y_0 = x$ and $f(x)(n)$ is a coded bit of $f(y_1)$ which is a coded bit of $f(y_2)\ \dots$, assuming $d(n)$ is x-valid. Note that $y_x(n)$ is an element of $t_x(n)$ for all n. The function $t_{s,x}$ is defined similarly to t_x, except where we use the sequence $d_s(n)$ instead of $d(n)$. We use this function $t_{s,x}$ because its values form a finitely branching tree. Typically below (except at the end of Claim 3) we will work on sets $B \subseteq \omega$ on which $\max d(n) \leq s$, and hence there is no difference in these functions: $t_x(n) = t_{s,x}(n)$ for all $n \in B$. Similarly, $y_{s,x}$, $b_{s,x}$ and $d_{s,x}$ are defined analogously to y_x, b_x, and d_x but using d_s instead of d.

Now d_x and b_x are not computable in general since d is not computable and the set of x-valid sequences is also not computable in general. However, there are certain subsets of ω on which d_x and b_x *are* computable.

Claim 1. *Suppose $\rho\colon \omega \to \omega$ is computable, $r = (y_0, \dots, y_l) \in [x]_E^{<\omega}$, y_i is an element of r, and $s \in \omega$. Then*

(1) *$A = \{n \in \omega\colon t_{s,x}(n) = r \wedge y_{s,x}(n) = y_i\}$ is computable, and $d_{s,x} \upharpoonright A$ is computable. Hence if $B \subseteq \omega$ is computable and $\max d(\rho(n)) \leq s$ for all $n \in B$, then $A' = \{n \in B\colon t_x(\rho(n)) = r \wedge y_x(\rho(n)) = y_i\}$ is computable and $b_x \circ \rho$ is computable on A'.*

(2) *If $y_l \not\!\!E_s\, x$, then $A = \{n\colon t_{s,x}(\rho(n)) \supseteq r \wedge y_x(\rho(n)) = y_i\}$ and $d_{s,x} \upharpoonright A$ are computable. Hence if $B \subseteq \omega$ is computable and $\max d(\rho(n)) \leq s$ for all $n \in B$, then $A' = \{n \in B\colon t_x(\rho(n)) \supseteq r \wedge y_x(\rho(n)) = y_i\}$ is computable and $b_x \circ \rho$ is computable on A'.*

Proof. The idea is that given r and s, there is a finite amount of information about how group elements γ_i for $i \leq s$ act between elements of r, and how elements of r are E_j related for $j \leq s$. From this we can compute all of the above.

More precisely, the set of tuples (i, j_0, j_1) such that $i \leq s$ and $j_0, j_1 \leq l$ and $\gamma_i \cdot y_{j_0} = y_{j_1}$ is finite. Suppose we are given $d_s(n) = (\langle i_0, j_0\rangle, \dots, \langle i_k, j_k\rangle)$, where $i_m, j_m \leq s$ for every $m \leq k$ by definition of d_s. Then for each $m \leq k$ we can iteratively compute which element of r is equal to $\gamma_{i_m} \cdots \gamma_{i_0} \cdot x$, provided all previous values of $\gamma_{i_{m'}} \cdots \gamma_{i_0} \cdot x$ for $m' < m$ have been elements of r. We can also similarly compute the least m so that $\gamma_{i_m} \cdots \gamma_{i_0} \cdot x$ is not an element of r.

Similarly, the set of tuples (i_0, i_1, j) such that $j \leq s$ and $i_0, i_1 \leq l$ so that $y_{i_0}\ E_j\ y_{i_1}$ is finite. From this information, if $t_{x,s}(n) = r$, we can determine what subsequences of $d_s(n)$ are x-valid, and hence compute $d_{s,x}(n) \upharpoonright A$ in case (1). In case (2), note that since $y_l \not\!\!E_s\, x$, the least m so that $(\langle i_0, j_0\rangle, \dots, \langle i_m, j_m\rangle)$ is not x-valid must have the property that $\gamma_{i_{m'}} \cdots \gamma_{i_0} \cdot x$ is an element of r for all $m' \leq m$. Hence in this case we can also compute $d_{s,x} \upharpoonright A$. The claim follows. □ Claim 1.

We will prove two main claims about $z \in 2^\omega$ such that $z \leq_m f(x)$. Recall that if $y, z \in 2^\omega$ and $A \subseteq \omega$ is computable, by $z \upharpoonright A \leq_m y$ we mean there is a computable function $\rho\colon A \to \omega$ so that for all $n \in A$, $z(n) = y(\rho(n))$.

Claim 2. *Suppose $x, z \in 2^\omega$ are such that $z \leq_m f(x)$, and z is incomputable. Then there is a computable infinite set $A \subseteq \omega$ and some $y\ E\ x$ so that $z \upharpoonright A \leq_m g(y)$ via a many-one reduction with infinite range.*

Proof. Let $\rho\colon \omega \to \omega$ be the many-one reduction witnessing $z \leq_m f(x)$. The idea of the proof is to make a finitely branching tree T of elements of $[x]_E$ where $(y_0, \dots, y_l) \in T$ means that a "large" (according to some ideal) number of bits $f(x)(\rho(n))$ code values of $f(y_0)$ which code values of $f(y_1)$ ... which code values of $f(y_l)$ (assuming the code is x-valid). If the tree is finite, a "large" number of bits of the many-one reduction can be many-one reduced to a single $g(y)$ for $y \in [x]_E$. If the tree is infinite, some finite branch r in the tree must be coded in a way that is not x-valid, otherwise we would contradict that hyper-(h_i)-finiteness of the $(E_j)_{j\in\omega}$ (since our tree will be arithmetically definable relative to $\bigoplus_{i\in\omega} \gamma_i \cdot x$). Then we can find a "large" set of incorrectly coded bits corresponding to extensions of r that reduce to a single $g(y)$. We will make this tree using the same idea as Proposition 5.4 using the function t_x.

We break into two cases depending on which case hold for ρ in Lemma 5.2.(1).

Case 1: there is a computable set B and an s so that $\max d(\rho(n)) \leq s$ for all $n \in B$ and $b_s(\rho(B))$ is infinite.

In this case, let I be the ideal on subsets of B where for $A \subseteq B$, we have $A \in I$ if $b_s(\rho(A))$ is finite. Let $T = \{r \in [x]_E^{<\omega} \colon \{n \in B \colon t_x(\rho(n)) \supseteq r\} \notin I\}$. Hence T is a finitely branching tree analogously to Proposition 5.4, and it is arithmetically definable relative to $\bigoplus_{i\in\omega} \gamma_i \cdot x$. (The reason we are using this ideal I rather than the Fréchet ideal I_1 is in order to make the proof that the many-one reduction has infinite range easier).

If T is finite, as in Proposition 5.4, there must be some $r \in T$ such that $\{n \in B \colon t_x(\rho(n)) = r\} \notin I$. Let $A = \{n \in B \colon t_x(\rho(n)) = r\}$. Let $r = (y_0, \ldots, y_l)$. Since $y_x(n)$ is an element of $t_x(n)$ for every n, we can partition A into the finitely many sets $A_i = \{n \in B \colon t_x(\rho(n)) = r \wedge y_x(\rho(n)) = y_i\}$ for each $i \leq l$. Hence, there must be some y_i so that the set $A_i \notin I$. Fix this i. Now for every $n \in A_i$, $f(x)(\rho(n)) = g(y_i)(b_x(\rho(n)))$ by (*). Since by Claim 1, $b_x \circ \rho$ is computable on A_i, we therefore have $z \restriction A_i \leq_m g(y_i)$. To see this many-one reduction has infinite range note first that $b_s(\rho(A_i))$ is infinite by definition of I, $b(\rho(n)) = b_s(\rho(n))$ for all $n \in A_i$ (since $\max d(\rho(n)) \leq s$ for all $n \in B$), and so $b(\rho(A_i))$ is infinite. Finally, $b_x(m) \geq b(m)$ for all m by definition of b_x, and so $b_x(\rho(A_i))$ is infinite.

Now suppose T is infinite. Then there is an infinite branch in T that is arithmetically definable from $\bigoplus_{i\in\omega} \gamma_i \cdot x$. Since each $t_x(n)$ contains no repeated elements by definition, the set of $y \in [E]_x$ in this branch is infinite. So there is some $r = (y_0, \ldots, y_l) \in T$ in this branch so that $y_l \not\!\!E_s\, x$. Otherwise, this would contradict that E_s is (h_i)-finite. Now for all n such that $t_x(n) \supseteq r$, we must have $y_x(n) = y_i$ for some $i \leq l$. So since $\{n \in B \colon t_x(\rho(n)) \supseteq r\} \notin I$, there must be some y_i so that $A = \{n \in B \colon t(\rho(n)) \supseteq r \wedge y_x(n) = y_i\} \notin I$. Since $f(x)(\rho(n)) = g(y_i)(b_x(\rho(n)))$ for all $n \in A$, we have $z \restriction A \leq_m g(y_i)$ by Claim 1 since b_x is computable on A. This many-one reduction has infinite range on A by the same argument as the above paragraph: $b_s(\rho(A))$ is infinite, $b_s(\rho(A) = b(\rho(A))$, and $b_x(m) \geq b(m)$ for all m.

Case 2: There is an $s \in \omega$ so that $b_s(\rho(\omega))$ is finite.

Let s' be larger than both s and $\max d(b_s(\rho(n)))$ for all $n \in \omega$. This is finitely many values since there are only finitely many values of $b_s(\rho(n))$. So $\max d(\rho(n)) \leq s'$ for all $n \in \omega$ since $d(n) = d_s(n)^\frown d(b_s(n))$ for every n, s. Let $T = \{r \in [x]_E^{<\omega} \colon \{n \colon t_x(\rho(n)) \supseteq r\}$ is infinite$\}$. T is a finitely branching tree as in Proposition 5.4 since $\max d(\rho(n)) \leq s'$ for all $n \in \omega$, and so $t_{s',x}(\rho(n)) = t_x(\rho(n))$ for all $n \in \omega$.

If T is finite, then for all but finitely many n, we have $t(\rho(n)) = r$ for some $r \in T$. For each $r = (y_0, \ldots, y_l) \in T$ and $i \leq l$, let $A_{r,i} = \{n \colon t_x(\rho(n)) = r \wedge y_x(\rho(n)) = y_i\}$. So all but finitely many $n \in \omega$ are in some $A_{r,i}$, and there are finitely many sets $A_{r,i}$. By Claim 1, every $A_{r,i}$ is computable and $z \restriction A_{r,i} \leq_m g(y_i)$ for each $A_{r,i}$. If all these many-one reductions have finite range, then z is computable, since there are finitely many A_{r,y_i}. This is a contradiction. So one of these many-one reductions $z \restriction A_i \leq_m g(y_i)$ has infinite range.

Now suppose T is infinite, and so there is an infinite branch in T that is arithmetically definable from $\bigoplus_{i\in\omega} \gamma_i \cdot x$. The infinite set of $y \in [E]_x$ that appear in this branch is arithmetically definable from $\bigoplus_{i\in\omega} \gamma_i \cdot x$. So there some $r = (y_0, \ldots, y_l)$ in this branch so that $y_l \not\!\!E_s\, x$. Otherwise, this would contradict that E_s is (h_i)-finite. Let $A = \{n \colon t_x(\rho(n)) \supseteq r\}$. Let $A_i = \{n \in A \colon y_x(\rho(n)) = y_i\}$, so $A_0, \ldots A_l$

partition A. Since $\{t_x(\rho(n))\colon n \in A\}$ is infinite since it includes our infinite branch, we can find $i \leq l$ so that $\{t_x(\rho(n))\colon n \in A_i\}$ is infinite. In particular, the lengths of these $|t_x(\rho(n))|$ where $n \in A_i$ are arbitrarily large. Then A_i is computable and $b_x \circ \rho$ is computable on A_i by Claim 1. Since $f(x)(\rho(n)) = g(y_i)(b_x(\rho(n)))$ by (*) we have that $z \restriction A_i \leq_m g(y_i)$.

We now show the many-one reduction $b_x \circ \rho$ witnessing $z \restriction A_i \leq_m g(y_i)$ has infinite range on A_i. For all $n \in A_i$, $t_x(\rho(n)) \supseteq r$, and the difference in their lengths is bounded by $|t_x(\rho(n))| - |r| \leq |d(\rho(n))| - |d_x(\rho(n))|$. This because the elements of $t_x(\rho(n))$ that are not in r must come from elements of $d(\rho(n))$ that are not x-valid (i.e. not in $d_x(\rho(n))$) since $y_l \not\!E_s x$. Finally $|d(\rho(n))| - |d_x(\rho(n))| \leq b_x(\rho(n))$, since $b(\rho(n)) \geq 0$ and $b(\rho(n))$ is obtained from $b_x(\rho(n))$ by taking additional inverse images of $b_x(\rho(n))$ by the elements of c_m that are in $d(\rho(n))$ but not in $d_x(\rho(n))$, and the c_m are strictly increasing. Hence, $b_x(n) \geq |t_x(\rho(n))| - |r|$ and since the lengths of $|t_x(\rho(n))|$ are unbounded on A_i, the values of $b_x(\rho(n))$ are also unbounded on A_i. $\square$ Claim 2.

To show f has property (2), we prove the prove the contrapositive. Suppose $f(y) \leq_m f(x)$ for some $x, y \in 2^\omega$. By Lemma 5.2 there is a computable infinite set D_0 so that D_0 is disjoint from $\operatorname{ran}(c_m)$ for every m, and hence $f(y) \restriction D_0 = g(y) \restriction D_0$, so $g(y) \restriction D_0 \leq_m f(x)$. Note that $g(y) \restriction D_0$ is incomputable, since any 1-generic restricted to a computable set is incomputable. By Claim 2, there is some $y' \mathrel{E} x$ and infinite $A \subseteq D_0$ so that $g(y) \restriction A \leq_m g(y')$. (By applying the Claim to $z = \{n\colon \text{the } n\text{th element of } D_0 \text{ is in } g(y)\}$. Note that $z \leq_m f(x)$.) We must have $y = y'$, otherwise a computable subset of $g(y)$ is many-one reducible to $g(y')$ contradicting their mutual 1-genericity. Hence $y = y' \mathrel{E} x$.

To show f has property (3), first note that $f(x) \not\leq_m x$ since there is an infinite computable set D_0 such that $f(x) \restriction D_0 = g(x) \restriction D_0$, and $g(x) \restriction D_0 \not\leq_m x$ by the properties of g from Lemma 4.2. Similarly, $f(x) \not\leq_m \overline{x}$. Next, suppose x is incomputable. We will show $x \not\leq_m f(x)$. Supposing $x \leq_m f(x)$ for a contradiction, by Claim 2 there is some computable infinite set $A \subseteq \omega$ and some $y \mathrel{E} x$ so that $x \restriction A \leq_m g(y)$ via a many-one reduction with infinite range. Let $z = x \restriction A$ so $z \leq_m x$ and $z \leq_m g(y)$ via a many-one reduction with infinite range. This contradicts the properties of g from Lemma 4.2. An identical argument replacing x with $\overline{x}$ shows that $\overline{x} \not\leq_m f(x)$.

Now we prove a similar result to Claim 2 above but where we analyze all the columns of a many-one reduction using the ideal I_2. This is required to prove part (4) of the theorem.

Claim 3. *Suppose $x, z \in 2^\omega$ are such that $z \leq_m f(x)$, and $z^{[n]}$ is incomputable for every n. Then there is a computable set $B \subseteq \omega$ with $B \notin I_2$ and some $y \mathrel{E} x$ so that $z \restriction B \leq_m g(y)$.*

Proof. Let $\rho\colon \omega \to \omega$ be the many-one reduction witnessing $z \leq_m f(x)$. We break into two cases depending on which case hold for ρ in Lemma 5.2.(2).

Case 1: There is a computable set B so that $\max d(\rho(n)) \leq s$ for all $n \in B$ and for all but finitely many i, and $b_s(\rho(B^{[i]}))$ is infinite.

In this case, we use a similar idea as in Claim 2. Note that $B \notin I_2$. Let $T = \{r \in [x]_E^{<\omega}\colon \{n \in B\colon t_x(\rho(n)) \supseteq r\} \notin I_2\}$. So as in Proposition 5.4, T is a finitely branching tree that is arithmetically definable relative to $\bigoplus_{i \in \omega} \gamma_i \cdot x$.

Suppose T is finite. Then as in Proposition 5.4, there must be some $r \in T$ such that $\{n \in B \colon t_x(\rho(n)) = r\} \notin I_2$. Let $A = \{n \in B \colon t_x(\rho(n)) = r\}$. Let $r = (y_0, \dots, y_l)$. We can partition A into finitely many sets $A_i = \{n \in A \colon y_x(n) = y_i\}$ for each $i \leq l$, and so there must be some y_i so that the set $A_i \notin I_2$. Now for every $n \in A_i$, $f(x)(\rho(n)) = g(y_i)(b_x(\rho(n)))$. Since by Claim 1, A_i is computable and $b_x \circ \rho$ is computable on A_i, we therefore have $z \restriction A_i \leq_m g(y_i)$.

Now suppose T is infinite, so there is an infinite branch in T that is arithmetically definable from $\bigoplus_{i\in\omega} \gamma_i \cdot x$. There must be some $r = (y_0, \dots, y_l)$ in this branch so that $y_l \not\!\!E_s\, x$. Otherwise, this would contradict that E_s is (h_i)-finite. Now for all n such that $t_x(n) \supseteq r$, we must have $y_x(n) = y_i$ for some $i \leq l$. So since $\{n \in B \colon t_x(\rho(n)) \supseteq r\} \notin I_2$, there must be some y_i so that $A = \{n \in B \colon t(\rho(n)) \supseteq r \wedge y_x(n) = y_i\} \notin I_2$. A is computable and $b_x \circ \rho \restriction A$ is computable by Claim 1. So since $f(x)(\rho(n)) = g(y_i)(b_x(\rho(n)))$ for all $n \in A$ we have $z \restriction A \leq_m g(y_i)$.

Case 2: There is an s so that for infinitely many i, $b_s(\rho(\omega^{[i]}))$ is finite. Let $B = \bigcup\{\omega^{[i]} \colon b_s(\rho(\omega^{[i]})) \text{ is finite}\}$. B is not necessarily computable, but it is arithmetical. Now let $T = \{r \in [x]_E^{<\omega} \colon \{n \in B \colon t_{s,x}(\rho(n)) \supseteq r\} \notin I_2\}$.

If T is infinite, then there must be some $r = (y_0, \dots, y_l) \in T$ so that $y_l \not\!\!E_s\, x$, otherwise there would be an infinite branch in T that is arithmetically definable from $\bigoplus_{i\in\omega} \gamma_i \cdot x$ and an infinite subset of E_s contradicting that $(E_j)_{j\in\omega}$ is a hyper-(h_i)-finiteness witness. So fix an $r \in T$ so that $\{n \in B \colon t_{s,x}(\rho(n)) \supseteq r\} \notin I_2$. Then the larger set $A = \{n \in \omega \colon t_{s,x}(\rho(n)) \supseteq r\}$ (where we have replaced B with ω) also has $A \notin I_2$. Finally, there must be some y_i with $i \leq l$ so that $A_i = \{n \in A \colon y_x(n) = y_i\}$ has $A_i \notin I_2$. This set A_i is computable by Claim 1, and $z \restriction A_i \leq g(y_i)$.

If T is finite and there is an i so that $b_s(\rho(\omega^{[i]}))$ is finite and all but finitely many $n \in \omega^{[i]}$ have $d_s(\rho(n))$ is x-valid, then we claim $z^{[i]}$ is computable, which is a contradiction. Now $f(x)(n) = f(y_{s,x}(n))(b_{s,x}(n))$ for all n by (**), and if $d_s(n)$ is x-valid, then $b_{s,x}(n) = b_s(n)$ and b_s is computable. So for each $r \in T$ and y_i in r, $A_{r,i} = \{n \colon t_{s,x}(\rho(n)) = r \wedge y_{s,x}(\rho(n)) = y_i\}$ is computable by Claim 1, and $z \restriction (\omega^{[i]} \cap A_{r,i}) \leq_m f(y_i)$ via a reduction that has finite range since $b_s(\rho(\omega^{[i]}))$ is finite. So since the finitely many sets $A_{r,i} \cap \omega^{[i]}$ are computable and disjoint, and their union is equal to $\omega^{[i]}$ mod finite, we have that $z \restriction \omega^{[i]} = z^{[i]}$ is computable since we can partition it mod finite into finitely many computable pieces.

Thus, for all i such that $\omega^{[i]} \subseteq B$, there are infinitely many $n \in \omega^{[i]}$ so $d_s(\rho(n))$ is not x-valid. So let $B' = \{n \in B \colon d_s(\rho(n)) \text{ is not } x\text{-valid}\}$. Then $B' \notin I_2$. Let $T' = \{r \in [x]_E^{<\omega} \colon \{n \in B' \colon t_{s,x}(\rho(n)) \supseteq r\} \notin I_2\}$. Then T' is finite since it is a subset of T, and there must be some $r \in T'$ and some $y_i \in r$ so that $\{n \in B' \colon t_{x,s}(\rho(n)) = r \wedge y_x(\rho(n)) = y_i\} \notin I_2$. Hence, the larger computable set: $A = \{n \colon t_{x,s}(\rho(n)) = r \wedge d_s(\rho(n)) \text{ is not } x\text{-valid} \wedge y_x(\rho(n)) = y_i\} \notin I_2$. Finally, $z \restriction A \leq_m g(y_i)$ by Claim 1. □ Claim 3.

Now to prove (4) given the above claim, let D_0 be a computable infinite set disjoint from $\operatorname{ran}(c_m)$ for every m. So $f(x) \restriction D_0 = g(x) \restriction D_0$. Then assuming that $\bigoplus_{i\in\omega} f(x_i) \leq_m f(x)$, we also have that $\bigoplus_{i\in\omega}(f(x_i) \cap D_0) \leq_m f(x)$, and hence $\bigoplus_{i\in\omega}(g(x_i) \cap D_0) \leq_m f(x)$. But then by the Claim 3, there is a single $y \in [x]_E$ and a computable infinite set $B \subseteq \bigoplus_{i\in\omega} D_0$ so $B \notin I_2$ so that $\bigoplus_{i\in\omega}(g(x_i) \cap D) \restriction B \leq_m g(y)$. Taking some i so that $B^{[i]}$ is infinite and $x_i \neq y$ gives a contradiction since g maps to a set of mutual 1-generics. □

Hence, we have the following corollaries

Corollary 5.6. *Suppose $\equiv_T$ is hyper-recursively-finite. Then*

(1) *There is a Borel homomorphism from $\equiv_T$ to $\equiv_m$ which is not uniform on any pointed perfect set. Hence, Conjecture 2.3 is false.*
(2) *$\equiv_m$ and $\equiv_1$ on 2^ω are universal countable Borel equivalence relations. Hence, there is a universal countable Borel equivalence relation which is not uniformly universal. So* [M, Conjecture 1.1] *is false.*

Proof. To prove (1), let f be as in Theorem 5.5 for the equivalence relation $E = \equiv_T$. Let $\Phi_e \colon 2^\omega \to 2^\omega$ be a total Turing functional with inverse $\Phi_d \colon 2^\omega \to 2^\omega$ such that $x, \Phi_e(x), \Phi_e(x)^2, \ldots$ are all distinct and have the same Turing degree. Then if f was uniformly $(\equiv_T, \equiv_m)$-invariant it would contradict condition (4) of Theorem 5.5.

To prove (2), note first that if every countable Borel equivalence relation E is hyper-Borel-finite, the function f given in Theorem 5.5 is a Borel reduction from E to many-one equivalence $\equiv_m$ and one-one equivalence $\equiv_1$ on 2^ω. However, $\equiv_m$ is not uniformly universal by [M, Theorem 1.5.(5)], so not every universal countable Borel equivalence relation is uniformly universal. □

It is open whether there is a counterexample to Martin's conjecture or Steel's conjecture assuming $\equiv_T$ is hyper-recursively-finite.

Question 5.7. *Assume $\equiv_T$ is hyper-recursively-finite. Is Martin's conjecture false? Is Steel's conjecture false?*

6. Open questions

We pose a conjecture which would give a negative answer to Question 3.4. It states in a strong way that Turing equivalence cannot be nontrivially written as an increasing union of Borel equivalence relations.

Conjecture 6.1. *Suppose we write Turing equivalence as an increasing union $(\equiv_T) = \bigcup_n E_n$ of Borel equivalence relations E_n where $E_n \subseteq E_{n+1}$ for all n. Then there exists a pointed perfect set P and some i so that $E_i \restriction P = (\equiv_T \restriction P)$.*

In the context of probability measure preserving equivalence relations, an analogous phenomenon of non-approximability has been proved by Gaboriau and Tucker-Drob [GTD], e.g. for pmp actions of property (T) groups.

We know that Conjecture 6.1 implies some consequences of Martin's conjecture. In particular, Conjecture 6.1 implies that Martin measure is E_0-ergodic in the sense of [T].

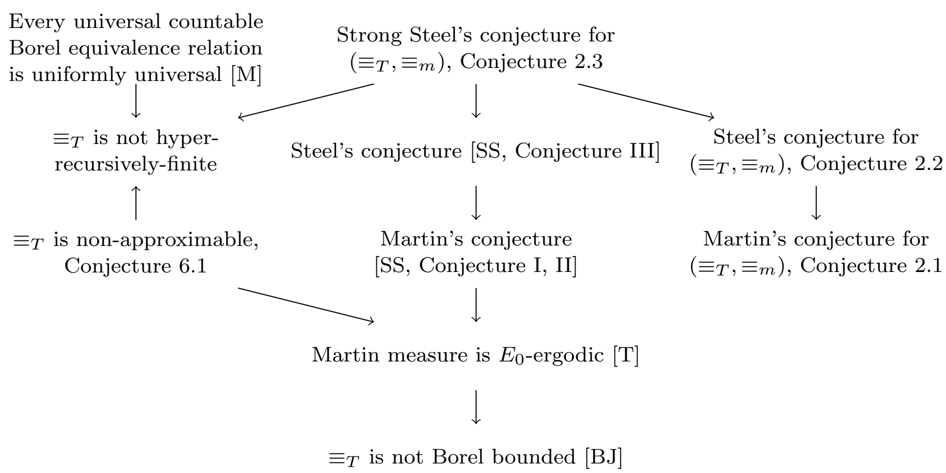
Proposition 6.2. *Suppose Conjecture 6.1 is true. Then if $f \colon 2^\omega \to 2^\omega$ is a Borel homomorphism from Turing equivalence to E_0, i.e. $x \equiv_T y \implies f(x) E_0 f(y)$, then the E_0-class of $f(x)$ is constant on a Turing cone.*

Proof. Let E_n be the subequivalence relation of $\equiv_T$ defined by $x \; E_n \; y$ if $x \equiv_T y$ and $\forall k \geq n (f(x)(k) = f(y)(k))$. That is the $f(x)$ and $f(y)$ are equal past the first n bits. By Conjecture 6.1, there is some i and some pointed perfect set P such that $E_i \restriction P = (\equiv_T \restriction P)$. Then by [MSS][Lemma 3.5] there is some pointed perfect set $P' \subseteq P$ such for $x, y \in P'$, if $x \equiv_T y$, then $f(x) = f(y)$. Define $f'(x) = f(y)$ if there is $y \in P'$ such that $x \equiv_T y$, and $f'(x) = \emptyset$ otherwise. Thus, $f' \colon 2^\omega \to 2^\omega$ is such that if $x \equiv_T y$, then $f'(x) = f'(y)$. Now any homomorphism from $\equiv_T$ to equality must be constant on a Turing cone, so f' is constant on a Turing cone. This implies the E_0-class of f is constant on a cone. □

It is open if Conjecture 6.1 implies Martin's conjecture.

Question 6.3. *Assume Conjecture 6.1 is true. Does this imply Martin's conjecture for Borel functions?*

The following is a diagram of some open questions surrounding Martin's conjecture. All relationships between these open problems which are not indicated by arrows are open.



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