HAUSDORFF DIMENSION AND COUNTABLE BOREL EQUIVALENCE RELATIONS

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Abstract. We show that if E is a countable Borel equivalence relation on $\mathbb{R}^n$, then there is a closed subset $A \subseteq [0,1]^n$ of Hausdorff dimension n so that $E \restriction A$ is smooth. More generally, if $\leq_Q$ is a locally countable Borel quasi-order on 2^ω and g is any gauge function of lower order than the identity, then there is a closed set A so that A is an antichain in $\leq_Q$ and $H^g(A) > 0$.

1. Introduction

Descriptive set theory has provided a general setting for comparing the relative difficulty of classification problems in mathematics, formalized as the study of Borel reducibility among equivalence relations. If E and F are equivalence relations on standard Borel spaces X and Y, say that E is **Borel reducible** to F if there is a Borel function $f\colon X \to Y$ such that for all $x_0, x_1 \in X$, we have $x_0 \mathrel{E} x_1 \iff f(x_0) \mathrel{F} f(x_1)$. Especially interesting in the theory are non-classifiability results. If $E \not\leq_B F$, then there does not exist any concretely definable (i.e. Borel) way to use elements of F as invariants to classify E. For example, Hjorth's theory of turbulence has given a general tool for proving such non-classifiability results, showing that many natural equivalence relations in mathematics cannot be classified by the isomorphism relation of any type of countable structure [H].

A well-studied subclass of Borel equivalence relations are the **countable Borel equivalence relations**, meaning those whose equivalence classes are all countable. To date, all known non-trivial results showing that $E \not\leq_B F$ for countable Borel equivalence relations E and F use measure theoretic techniques and Borel probability measures. See for example, the cocycle rigidity results used to prove nonreducibility results in [AK] and [T]. An important problem in the theory of countable Borel equivalence relations is to find new tools beyond just Borel probability measures for proving non-reducibility results. Such new tools seem to be needed to solve many open questions in the subject like the problem of whether every countable Borel equivalence relation is Borel bounded [BJ], whether every amenable countable Borel equivalence relation is hyperfinite [JKL, 6.2.(B)], the increasing union problem for hyperfinite Borel equivalence relations [DJK, p 194], or the universal vs measure universal problem [MSS, Question 3.13]. Those questions

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are all known to have positive answers modulo a null set with respect to any Borel probability measure, but we suspect these questions to have negative answers in general.

There are several promising candidates for new tools that could prove such non-reducibility results such as Martin's conjecture [DK], forcing [Sm], the $\mathcal{L}_{\omega_1,\omega}$ model theory of countable structures [CK], and the study of topological realizations of countable Borel equivalence relations [FKSV]. There are also several results showing certain tools *cannot* prove new non-reducibility results. These results are often in the context where this tool has an associated σ-ideal I, and we show that every countable Borel equivalence relation becomes simple after discarding a set in this ideal, or after restricting to an I-positive set. For example, we have the following well-known theorem of generic hyperfiniteness:

Theorem 1.1 (Hjorth-Kechris, Sullivan-Weiss-Wright, Woodin (see [KM, Theorem 12.1]))**.** *If E is a countable Borel equivalence relation on a Polish space X, then there is a comeager invariant Borel set $C \subseteq X$ so that $E \restriction C$ is hyperfinite.*

Recall here that a countable Borel equivalence relation E is hyperfinite if and only if $E \leq_B E_0$, where E_0 is the equivalence relation of eventual equality on infinite binary sequences [DJK, Theorem 7.1]. So no simple Baire category argument can be used to prove nonhyperfiniteness results that $E \not\leq_B E_0$ for any countable Borel equivalence relation E.

We have an analogous result to generic hyperfiniteness in the context of the ideal of Ramsey null subset of $[\omega]^\omega$, except that we only have hyperfiniteness on an I-positive set for the Ramsey null ideal:

Theorem 1.2 (Mathias and Soare [M, So] (see [KSZ, Theorem 8.17]))**.** *If E is a countable Borel equivalence relation on $[\omega]^\omega$, then there is an $A \in [\omega]^\omega$ so that $E \restriction [A]^\omega$ is hyperfinite.*

Recently, Panagiotopoulos and Wang have similarly analyzed the dual Ramsey ideal:

Theorem 1.3 ([PW, Theorem 1.2])**.** *If E is a countable Borel equivalence relation on $(\omega)^\omega$, then there is an $A \in (\omega)^\omega$ so that $E \restriction (A)^\omega$ is smooth.*

Recall here that an Borel equivalence relation E is **smooth** if $E \leq_B =_\mathbb{R}$ where $=_\mathbb{R}$ is the equivalence relation of equality on $\mathbb{R}$. Similar canonization theorems to the above are also known for certain other Ramsey-type ideals by work of Kanovei-Sabok-Zapletal [KSZ, Theorem 8.1].

The present paper investigates whether Hausdorff measures and Hausdorff dimension can be used to prove new non-reducibility results between Borel equivalence relations. We know that Lebesgue measure on 2^ω can be used to prove many interesting non-Borel-reducibility results (such as Slaman and Steel's proof [SS] that Turing equivalence on 2^ω is not hyperfinite, or the result that the shift action of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$ is not hyperfinite [K91]). If we take s-dimensional Hausdorff measure on 2^ω for $s < 1$, as $s \to 1$, these measures "approach" Lebesgue measure. More generally, we can take arbitrary gauge measures for gauge functions g with $\lim_{t\to 0} g(t)/t = \infty$, and let g approach the identity function $g(t) = t$ which corresponds to the case of Lebesgue measure. Our hope was that the spectrum of complexities of Borel equivalence relations that can be "seen" by these s-dimensional Hausdorff measures or gauge measures becomes more and more complex as $s \to 1$.

Unfortunately, this is not the case. Our main theorem shows that any gauge measure H^g with the above-mentioned property trivializes every countable Borel equivalence relation to be smooth on a set of positive H^g-measure. So our main result is another in the line of work of Theorems 1.1, 1.2, and 1.3.

Theorem 1.4. *Suppose $g\colon [0,\infty) \to [0,\infty)$ is a gauge function of lower order than the identity and that E is a countable Borel equivalence relation on 2^ω. Then there is a closed set $A \subseteq 2^\omega$ such that $E \restriction A$ is smooth, and $H^g(A) > 0$. In particular, there is a closed set of Hausdorff dimension 1 such that $E \restriction A$ is smooth.*

We note that in contrast, the arguments of [SS] and [K91] show that Turing equivalence or the orbit equivalence relation of the shift action of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$ are both non-hyperfinite on any positive measure set with respect to Lebesgue measure.

By using an appropriate bijection between 2^ω and $[0,1]^n$ we also show that every countable Borel equivalence relation on $\mathbb{R}^n$ is smooth on a set of Hausdorff dimension n (Corollary 4.3).

We also prove some generalizations of these results to locally countable Borel quasi-orders on $\mathbb{R}^n$ and 2^ω. For example, if $\leq_Q$ is any locally countable Borel quasi-order on 2^ω, then there is a closed set $A \subseteq 2^\omega$ so that A is an antichain under $\leq_Q$, and A has Hausdorff dimension 1.

2. Preliminaries

A quasi-order $\leq_Q$ on a space X is a reflexive transitive relation on X. We say that $\leq_Q$ is **locally countable** if for every $y \in X$, $\{x \in X \colon x \leq_Q y\}$ is countable. We say that $\leq_Q$ is Borel if is Borel as a subset of X^2. Among the examples of locally countable Borel quasi-orders are countable Borel equivalence relation – equivalence relations on X whose classes are all countable. A reference for the theory of countable Borel equivalence relations and locally countable Borel quasi-orders is the recent survey paper [K24] of Kechris. Note that by Lusin-Novikov uniformization [K95, 18.10], if $\leq_Q$ is a countable Borel quasi-order, then there are countably many Borel functions $(f_i : X \to X)_{i\in\omega}$ so that $y \leq_Q x$ iff there exists an $i \in \omega$ so that $f_i(x) = y$.

Our conventions surrounding Hausdorff dimension and gauge measures follow those of Rogers [R]. Recall that gauge measures generalize the idea of Hausdorff measures and Hausdorff dimension to arbitrary gauge functions. A **gauge function** $g\colon [0,\infty) \to [0,\infty)$ is an increasing function that is continuous on the right, $g(0) = 0$, and $g(t) > 0$ for $t > 0$. If (X,d) is a metric space, then recall that we define the g-measure H^g on subsets of X as follows: For every $\delta > 0$, let

$$H^g_\delta(A) = \inf\{\sum_{i=0}^{\infty} g(\mathrm{diam}(U_i))\colon (U_i) \text{ is an open cover of } A \text{ by sets of diameter} < \delta\}.$$

Then the **g-measure** H^g is defined as $\lim_{\delta\to 0^+} H^g_\delta$.

Definition 2.1. Suppose that f and g are gauge functions. We write $f \prec g$ if $\lim_{t\to 0^+} g(d)/f(d) = 0$ (or equivalently $\lim_{t\to 0^+} f(d)/g(d) = \infty$) and say that g has **higher order** than f.

Below, we work with gauge measures on the Cantor space 2^ω of infinite binary sequences equipped with the metric $d(x,y) = 2^{-n}$ where n is least such the nth bit of x and y differ: $x(n) \neq y(n)$. We will also work with the spaces $\mathbb{R}^n$ with the

Euclidean metric. In a metric space, we let $B_r(x)$ denote the open ball of radius r around a point x.

The s-dimensional Hausdorff measure is the gauge measure given by the power functions $g(t) = t^s$. Here if $g(t) = t$, then H^g is Lebesgue measure on 2^ω, and if $g(t) = t^n$, then H^g is Lebesgue measure on $\mathbb{R}^n$. We will often write H^s for the s-dimensional Hausdorff measure for $s \in \mathbb{R}^+$. Then the **Hausdorff dimension** of a set A is

$$\dim(A) = \inf\{s : H^s(A) = 0\} = \sup\{s : H^s(A) = \infty\}.$$

We let $2^{<\omega}$ denote the set of finite binary strings, and we use the letters s, t for its elements. We let $|s|$ denote the length of s and $s(n)$ is the nth bit of s. Finally, if $s, t \in 2^{<\omega}$, we let $s^\frown t$ denote the concatenation of s and t.

3. Proof of the main theorem

First, we fix notation for describing a binary tree T where at each level, either all nodes at this level **split** (i.e. have two successors in T), or all nodes at this level have exactly one successor in T.

Definition 3.1. Given a set $A \subseteq \omega$, and a function $y\colon 2^{<\omega} \to 2$, let $T_{A,y} \subseteq 2^{<\omega}$ be the set of $t \in 2^{<\omega}$ such that for all $n < |t|$, if $n \in A$, then $t(n) = y(t \upharpoonright n)$.

That is, $T_{A,y}$ is the tree where if $t \in T_{A,y}$ and $|t| \notin A$, then both $t^\frown 0$ and $t^\frown 1$ are in $T_{A,y}$. However, if $|t| \in A$, then the only successor of t in $T_{A,y}$ is $t^\frown y(t)$.

We also fix notation for the uniform measure on $[T_{A,y}]$:

Definition 3.2. Let $\mu_{A,y}$ be the uniform measure on $[T_{A,y}]$, so that if $t \in T_{A,y}$ is a splitting node, then both its successors have equal measure, i.e., $\mu_{A,y}([t^\frown 0]) = \mu_{A,y}([t^\frown 1])$.

Note that since all nodes in $T_{A,y}$ at a given level are either splitting nodes, or none are splitting nodes, this implies that if $s, t \in [T_{A,y}]$ have the same length, then $\mu_{A,y}([s]) = \mu_{A,y}([t])$, and indeed if $t \in T_{A,y}$ has length n, then $\mu_{A,y}([t]) = 2^{-n+|A\cap n|}$, since $|A \cap n|$ gives the number of non-splitting levels below n.

Our first lemma relates the rate at which elements appear in a set $A \subseteq \omega$ with the rate of convergence of gauge functions g such that all $\mu_{A,y}$-positive subsets of $[T_{A,y}]$ have positive g-measure H^g.

Lemma 3.3. *Suppose g is a gauge function with $g \prec \mathrm{id}$ and that $A \subseteq \mathbb{N}$ is such that*

$$(\dagger) \qquad |A \cap n| \leq \log_2\left(\frac{g(2^{-n})}{2^{-n}}\right)$$

for all but finitely many n. Then for all $y\colon 2^{<\omega} \to 2$, and all $B \subseteq [T_{A,y}]$ with $\mu_{A,y}(B) > 0$, we have $H^g(B) > 0$.

Proof. Our proof relies on the following claim, which is essentially one direction of Frostman's lemma.

Claim 3.3.1. *Suppose μ is a Borel probability measure on 2^ω such that for all $x \in 2^\omega$ and sufficiently small $r > 0$, $g(r) > \mu(B_r(x))$. Then $\mu(B) > 0$ implies $H^g(B) > 0$.*

Proof. Consider an open cover (U_i) of x by sets of sufficiently small diameter r. We may assume the cover is by open balls $U_i = B_{r_i}(x_i)$, since any set of diameter r in 2^ω is contained in an open ball of the same diameter. Then

$$\sum_i g(\operatorname{diam}(U_i)) \geq \sum_i g(r_i) \geq \sum_i \mu(B_{r_i}(x_i)) \geq \mu(B) > 0.$$

So, in particular $H^g_\delta(B) \geq \mu(B)$ for any $\delta < r$, and so $H^g(B) \geq \mu(B)$. ☐

It remains to show that $\mu_{A,y}$ satisfies the conditions of the claim. Now we have $\mu_{A,y}(B_{2^{-n}}(x)) = 2^{-n+|A\cap n|} < g(2^{-n})$ for all but finitely many n, where the last inequality follows from (†).

Finally, note that there are infinite sets A satisfying (†) since by assumption that $g \prec \mathrm{id}$, we have $\lim_{t\to 0^+} \frac{g(t)}{t} = \infty$. ☐

Next we show that if $\leq_Q$ is a locally countable Borel quasi-order and y is sufficiently generic, then $\mu_{A,y}$-a.e. $x \in [T_{A,y}]$ is not $\leq_Q$-above any other element of $[T_{A,y}]$. So there is a $\mu_{A,y}$-conull $\leq_Q$-antichain in $[T_{A,y}]$.

Lemma 3.4. *Suppose $A \subseteq \omega$ is infinite, and $(f_i)_{i\in\omega}$ is a countable set of Borel functions on 2^ω. Then for a comeager set of $y\colon 2^{<\omega} \to 2$, for $\mu_{A,y}$-a.e. $x \in [T_{A,y}]$, for all $i \in \omega$, if $f_i(x) \neq x$, then $f_i(x) \notin [T_{A,y}]$.*

Proof. If $f_i(x) \neq x$, then there is some $s \in 2^{<\omega}$ so that $x \supseteq s$ and $f_i(x) \not\supseteq s$. Fix such $i \in \omega$ and $s \in 2^{<\omega}$. It suffices to show that for comeagerly many y, the set of $x \in [T_{A,y}]$ such that $x \supseteq s$, $f_i(x) \not\supseteq s$ and $f_i(x) \in [T_{A,y}]$ is $\mu_{A,y}$-null. The argument will be by showing that as we build a generic y, it is dense to halve the measure of $x \in [T_{A,y}]$ so that $f_i(x) \in [T_{A,y}]$.

By definition of $T_{A,y}$, we have $f_i(x) \notin [T_{A,y}]$ is equivalent to $(\exists n \in A) f_i(x)(n) \neq y(f_i(x) \restriction n)$. Define

$$B_y = \{x \in [T_{A,y}] \colon x \supseteq s \wedge f_i(x) \not\supseteq s \wedge (\forall n)(n \in A \implies f_i(x)(n) = y(f_i(x) \restriction n))\}.$$

Elements in B_y are the "bad" elements of $[T_{A,y}]$ and we want to show that for comeagerly many y, $\mu_{A,y}(B_y) = 0$.

If p is a function from $2^k \to 2$, then let $B_{y,p} = \{x \in [T_{A,y}] \colon x \supseteq s \wedge f_i(x) \not\supseteq s \wedge (\forall n \leq k)(n \in A \implies f_i(x)(n) = p(f_i(x) \restriction n)\}$. The difference between B_y and $B_{y,p}$ is that the last y in the definition of B_y has become p in $B_{y,p}$. So, if $p_k = y \restriction 2^k$, then $B_y = \bigcap B_{y,p_k}$. We claim that given any $p\colon 2^k \to 2$, there is a dense set of $q \supseteq p$ such that for comeagerly many $y \in [q]$, we have $\lambda(B_{y,q}) \leq \frac{1}{2}\lambda(B_{y,p})$. This claim implies that for comeagerly many y, $\mu_{A,y}(B_y) = 0$, which will conclude the proof.

Suppose $k' > k$ and $p'\colon 2^{k'} \to 2$ extends p. We need to show that there is a q extending p' so that for comeagerly many $y \in [q]$, we have $\lambda(B_{y,q}) \leq \frac{1}{2}\lambda(B_{y,p})$. Suppose $n \in A$ is such that $n > k'$. Let $B^0_{y,p} = \{x \in B_{y,p} \colon f_i(x)(n) = 0\}$ and $B^1_{y,p} = \{x \in B_{y,p} \colon f_i(x)(n) = 1\}$, so $B_{y,p} = B^0_{y,p} \sqcup B^1_{y,p}$. Now consider $C = \{y \colon \lambda(B^0_{y,p}) < \frac{1}{2}\lambda(B_{y,p})\}$. This set is analytic and so it has the Baire property. First, consider the case that C is nonmeager in $[p']$. So there is some $q' \supseteq p'$ such that C is comeager in $[q']$. We may assume that $q'\colon 2^m \to 2$ where $m > n$. Let

$$q(t) = \begin{cases} 0 & \text{if } |t| = n \text{ and } t \not\supseteq s \\ q'(t) & \text{otherwise} \end{cases}$$

Since $q(t) = q'(t)$ for all t compatible with s, and since the set $B^0_{y,p}$ only depends on the values of $y(t)$ such that t is compatible with s, we have that C is also comeager in $[q]$. Finally, since $q(t) = 0$ if $|t| = n$ and $t \not\supseteq s$, we have that $B_{y,q} \subseteq B^0_{y,p}$, and so $\lambda(B_{p,q}) \leq \frac{1}{2}\lambda(B_{y,p})$.

If C is meager in $[p']$, then the set $\{y\colon \lambda(B^1_{y,p}) \leq \frac{1}{2}\lambda(B_{y,p})\}$ is comeager in $[p']$ (and in particular it is nonmeager). The argument in this case is identical to the above argument, just changing the roles of 0 and 1. This finishes the proof of the claim. □

Theorem 3.5. *If $\leq_Q$ is a locally countable Borel quasi-order on 2^ω, and g is a gauge function such that $g \prec \mathrm{id}$, then there is a closed $\leq_Q$-antichain $B \subseteq 2^\omega$ with $H^g(B) > 0$.*

Proof. By Lusin-Novikov uniformization, fix countably many Borel functions (f_i) generating $\leq_Q$ and $A \subseteq \omega$ that is sufficiently sparse as in Lemma 3.3. Let $y\colon 2^{<\omega} \to 2$ be such that for $\mu_{A,y}$-a.e. $x \in [T_{A,y}]$, if $f_i(x) \neq x$, then $f_i(x) \notin T_{A,y}$. Such a y exists since there is a comeager set of such y by Lemma 3.4. Thus, there is a $\mu_{A,y}$-conull set $C \subseteq [T_{A,y}]$ that forms a $\leq_Q$-antichain, so by inner regularity of $\mu_{A,y}$ there is a closed set $B \subseteq C$ with $\mu_{A,y}(B) > 0$ that is a $\leq_Q$-antichain. By Lemma 3.3 $H^g(B) > 0$.

□

Proof of Theorem 1.4. Suppose E is a countable Borel equivalence relation on 2^ω. Then viewing E as a locally countable Borel quasi-order, if $B \subseteq 2^\omega$ is a closed antichain for E so that $H^g(B) > 0$, then B meets each E-class in at most one point, so $E \restriction B$ is smooth.

To see the last part of the theorem, let $g_s = t^s$ be the gauge function defining the s-dimensional Hausdorff measure H^s. Choose a gauge function $g \prec \mathrm{id}$ so that $g_s \prec g$ for all $s \preceq 1$, for instance $g(t) = t^{1-\frac{1}{t}}$. Then $H^g(B) > 0$ implies $H^s(B) > 0$ for all $s < 1$, so $\dim(B) = 1$. □

By an analogous argument to the proof of Theorem 1.4, we get a similar result for locally countable Borel quasi-orders.

Corollary 3.6. *Suppose that Q is a locally countable Borel quasi-order on 2^ω. Then there is a closed Q-antichain of Hausdorff dimension 1.*

4. Results on $\mathbb{R}^n$

We can transfer all our results above from the space 2^ω to the space $\mathbb{R}^n$. This is because there are Borel bijections between 2^ω and $[0,1]$ which preserve the property of having positive gauge measure. To show this we begin with a proposition about functions between gauge measures on different metric spaces.

Proposition 4.1. *Suppose (X_1, d_1) and (X_2, d_2) are metric spaces, g is a gauge function, and $h\colon [0,\infty) \to [0,\infty)$ is continuous and strictly increasing with $h(0) = 0$. Suppose $f\colon X_1 \to X_2$ has the property that for all sets $A \subseteq X_1$, $f(A)$ can be covered by at most k sets of d_2-diameter $h(\mathrm{diam}_{d_1}(A))$. Then for any B, we have $H^{g\circ h^{-1}}(f(B)) \leq kH^g(B)$.*

Proof. Given any cover (U_i) of $B \subseteq X_1$ by sets of diameters less than δ, the sets $f(U_i)$ cover $f(B)$, and we can cover each set $f(U_i)$ by k sets $V_{i,1}, \ldots V_{i,k}$ of diameter at most $h(\operatorname{diam}(U_i))$. Hence, $H^{g\circ h^{-1}}_{h(\delta)}(f(B)) \leq kH^g_\delta(B)$. The proposition follows by taking the limit as $\delta \to 0$ since h is continuous on the right. □

We will mostly apply this proposition below with h equal to the identity. In this case, the statement of the proposition becomes the following: suppose for all sets $A \subseteq X_1$, $f(A)$ can be covered by at most k sets of d_2-diameter $\operatorname{diam}_{d_1}(A)$. Then for any B, we have $H^g(f(B)) \leq kH^g(B)$.

Proposition 4.1 is related to a classical result in fractal geometry that relates the Hausdorff measures, and thus Hausdorff dimensions, of sets and their images along Hölder continuous functions: If $A \subseteq R^n$ is any set and $f : R^n \to R^m$ is Hölder continuous with exponent $\alpha \in \mathbb{R}^+$ and multiplicative constant c, then for any $s \in \mathbb{R}^+$, $H^{s/\alpha}(f(A)) \leq c^{s/\alpha}H^s(A)$ [F14, Proposition 3.1]. However, the hypothesis of Proposition 4.1 can be satisfied by functions that are not Hölder continuous. A prime example of such functions are the bijections between 2^ω and $[0,1]^n$ we will construct now.

Proposition 4.2. *There is a Borel bijection $f\colon 2^\omega \to [0,1]^n$ so that for all sets A, and all gauge functions g, $H^g(A) > 0$ if and only if $H^{g\circ h^{-1}}(f(A)) > 0$ where $h = t^{1/n}$.*

Proof. We begin by proving the case $n = 1$. Let $f\colon 2^\omega \to [0,1]$ map each infinite binary sequence x to the real number given by the binary expansion of x. Since the dyadic rationals have both a finite and an infinite binary expansion, this map is not injective. However, f is a bijection between $\{x \in 2^\omega : x \text{ is not eventually constant}\}$ and the complement of the dyadic rationals. Both these sets are co-countable in 2^ω and $[0,1]$ respectively. Hence their complements have H^g-measure 0 for every gauge function g and we can redefine f on this countable set so that it is a bijection from $2^\omega \to [0,1]$, and hence ignore these countable sets in what follows.

Now any set $A \subseteq 2^\omega$ can be covered by a basic open set of the same diameter. So suppose $s \in 2^{<\omega}$ is a finite binary sequence of length $n = |s|$. Then the basic open set $[s] = \{x \in 2^\omega : x \supseteq s\}$ of diameter 2^{-n} is mapped by f to an interval of the form $(p/2^n, (p+1)/2^n)$, which also has diameter 2^{-n} in the Euclidean metric. So f has the property that for all B, $H^g(f(B)) \leq H^g(B)$ by Proposition 4.1 letting h be the identity and k being 1.

Now we argue similarly for f^{-1}. Any set in $[0,1]$ can be covered by a closed interval of the same diameter. Suppose $[a,b]$ is a closed interval. Let m be the integer so that $1/2^m < \operatorname{diam}([a,b]) \leq 1/2^{m-1}$. There is a unique dyadic rational of the form $p/2^m$ in (a,b) where p is an integer. Hence $[a,b] \subseteq [p/2^m - 1/2^{m-1}, p/2^m + 1/2^{m-1}]$. So $[a,b]$ is covered by four dyadic intervals of length $1/2^m$: $[p/2^m - 1/2^{m-1}, p/2^m - 1/2^m], \ldots, [p/2^{m-1} + 1/2^m, p/2^m + 1/2^{m-1}]$. All of these intervals are the images of basic open sets in 2^ω of diameter $1/2^m$ which is less than $\operatorname{diam}([a,b])$. So for any set B, we have $H^g(B) \leq 4H^g(f(B))$ by Proposition 4.1 applied to f^{-1}. So combining with the above paragraph, we have that for all B, $\frac{1}{4}H^g(B) \leq H^g(f(B)) \leq H^g(B)$.

Now we prove the case $n > 1$. Let d_∞ be the metric on $(2^\omega)^n$ defined by $d_\infty((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \sup_i d(x_i,y_i)$ where d is the usual metric on 2^ω. Consider the function $j_n\colon (2^\omega, d) \to ((2^\omega)^n, d_\infty)$ defined by $j(x) = (y_1,\ldots,y_n)$ where $y_i(j) = x(jn+i)$ so, y_i is all the bits of x that are i mod n in order. Then if $d(x,y) = 2^{-k}$, then $d(j(x),j(y)) = 2^{-\lfloor k/n \rfloor}$. So using $h(t) = t^{1/n}$ and Proposition 4.1 on j_n and j_n^{-1}, we conclude there are constants k_1 and k_2 so that $k_1H^g(B) < H^{g\circ h}(f(B)) < k_2H^g(B)$.

Finally, let $f_n : 2^\omega \to [0,1]^n$ be defined by $f_n(x) = (f(y_1), \dots, f(y_n))$ where $j_n(x) = (y_1, \dots, y_n)$, and f is the function from the case $n = 1$ defined above. Then apply Proposition 4.1 and note that if d_∞ is the sup metric on $[0,1]^n$, and d is the usual Euclidean metric on $[0,1]^n$, then any set $A \subseteq [0,1]^n$ of d-diameter r has d_∞-diameter at most r. Conversely, there is a constant c_n so that any set $A \subseteq [0,1]^n$ of d_∞ diameter r can be covered by c_n sets of d-diameter r. □

Now we can obtain a version of Proposition 1.4 for $\mathbb{R}^n$.

Corollary 4.3. *Suppose $g: [0,\infty) \to [0,\infty)$ is a gauge function of lower order than $t \mapsto t^n$ and that E is a countable Borel equivalence relation on $\mathbb{R}^n$. Then there is a closed set $A \subseteq [0,1]$ such that $E \restriction A$ is smooth, and $H^g(A) > 0$. In particular, there is a closed set of Hausdorff dimension n such that $E \restriction A$ is smooth.*

Proof. Let $g'(t) = g(t^{1/n})$. Note that g' has lower order than the identity if and only if g has lower order than t^n. Let $f: 2^\omega \to [0,1]^n$ be the Borel bijection from Proposition 4.2. Given E on $\mathbb{R}^n$, define E' on 2^ω by $x \, E' \, y$ if $f(x) \, E \, f(y)$. We can apply Theorem 1.4 to E' to obtain a closed set A' with $H^{g'}(A') > 0$ and such that $E' \restriction A'$ is smooth. Now $f(A') \subseteq [0,1]$ has positive H^g measure by Proposition 4.2 and is Borel since an injective image of a Borel set under a Borel function is Borel. So $E \restriction f(A')$ is smooth since f is a bijection and $E' \restriction A'$ is smooth. To finish, let $A \subseteq f(A')$ be closed with $H^g(A) > 0$.

To see the last part of the corollary, recall that Theorem 1.4 allows us to take A' with $\dim(A') = 1$ such that $E' \restriction A'$ is smooth. Thus, by the above arguments we can get an $A \subseteq f(A')$ with $\dim(A) = n$ so that $E \restriction A$ is smooth. □

We finish by noting that the same arguments used to prove Corollaries 4.3 can be used to obtain an analogue of this result for locally countable Borel quasi-orders.

Corollary 4.4. *Suppose that Q is a locally countable Borel quasi-order on $\mathbb{R}^n$. Then there is a closed Q-antichain of Hausdorff dimension n.*

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