STRUCTURE IN COMPLETE SECTIONS OF THE SHIFT ACTION OF A RESIDUALLY FINITE GROUP

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If Γ is a countable discrete group, and X is a standard Borel, we let Γ act on X^Γ via the left shift where for all $\alpha, \beta \in \Gamma$ and $x \in X^\Gamma$ we have $(\alpha \cdot x)(\beta) = x(\alpha^{-1}\beta)$. We let $\mathrm{Free}(X^\Gamma)$ be the set of $x \in X^\Gamma$ such that for all nonidentity $\gamma \in \Gamma$, we have $\gamma \cdot x \neq x$. Recall that a countable group Γ is *residually finite* if for every nonidentity $\gamma \in \Gamma$ there is a homomorphism $h\colon \Gamma \to \Delta$ from Γ to a finite group Δ such that $h(\gamma) \neq 1$.

We show the following:

Theorem 1. *Suppose Γ is a residually finite group, and $A \subseteq \mathrm{Free}(2^\Gamma)$ is a Borel complete section of the shift action of Γ. Then there is a finite index normal subgroup $\Gamma' \leq \Gamma$ and an $x \in A$ such that $\Gamma' \cdot x \subseteq A$.*

This generalizes previous results of the author for $\mathbb{F}_2$, and of Gao-Jackson-Seward for $\mathbb{Z}^n$.

Definition 2. If Γ is a countable group and X is an arbitrary set, then we say a function $p\colon \Gamma \to X$ is a *forgetful homomorphism* if we can put a group structure $(X, *)$ on X such that p is a group homomorphism.

Now we have the following lemma:

Lemma 3. *If $p\colon \Gamma \to X$ is a forgetful homomorphism, then for all $\alpha \in \Gamma$, $\alpha \cdot p$ is a forgetful homomorphism.*

Proof. Given the group structure $(X, *)$ on X making $p\colon \Gamma \to X$ a group homomorphism, define a new group structure $(X, \odot)$ on X, where we define $g_1 \odot g_2 = g_1 * p(\alpha) * g_2$ for all $g_1, g_2 \in X$. Then the identity in $(X, \odot)$ is $p(\alpha^{-1})$, the inverse of g in $(X, \odot)$ is $p(\alpha^{-1}) * g^{-1} * p(\alpha^{-1})$, and

$$\begin{aligned}(\alpha \cdot p)(\beta\gamma) &= p(\alpha^{-1}\beta\gamma) = p(\alpha^{-1}) * p(\beta) * p(\gamma)\\ &= (p(\alpha^{-1}) * p(\beta)) * p(\alpha) * (p(\alpha^{-1}) * p(\gamma))\\ &= (p(\alpha^{-1}) * p(\beta)) \odot (p(\alpha^{-1}) * p(\gamma) = (\alpha \cdot p)(\beta) \odot (\alpha \cdot p)(\gamma)\end{aligned}$$

so $\alpha \cdot p$ is a group homomorphism from Γ to $(X, \odot)$. $\square$

We now define a forcing partial order:

Definition 4. Given a countable group Γ, let $\mathbb{P}_\Gamma$ be the partial order of forgetful homomorphisms of the form $p\colon \Gamma \to \omega^n$ for some natural number

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n, such that $\operatorname{ran}(p)$ is finite, and where if $p\colon \Gamma \to \omega^n$ and $q\colon \Gamma \to \omega^m$ are conditions, then $q \leq_{\mathbb{P}_\Gamma} p$ if $m \geq n$ and $\forall i \leq n \forall \gamma \in \Gamma$ we have $p(\gamma)(i) = q(\gamma)(i)$.

We now note several facts about $\mathbb{P}_\Gamma$.

Lemma 5. *If $\alpha \in \Gamma$, then the map $p \mapsto \alpha \cdot p$ is an automorphism of $\mathbb{P}_\Gamma$. Thus, x is $\mathbb{P}_\Gamma$-generic iff $\alpha \cdot x$ is $\mathbb{P}_\Gamma$-generic.*

Proof. That $\alpha \cdot p \in \mathbb{P}_\Gamma$ follows from Lemma 3. The map $p \mapsto \alpha \cdot p$ is an automorphism of $\mathbb{P}_\Gamma$ since if $p\colon \Gamma \to \omega^n$, then $q \leq p$ iff $\forall i \leq n \forall \gamma \in \Gamma$ we have $p(\gamma)(i) = q(\gamma)(i)$ iff $\forall i \leq n \forall \gamma \in \Gamma$ we have $\alpha \cdot p(\gamma)(i) = \alpha \cdot q(\gamma)(i)$ iff $\alpha \cdot p \leq \alpha \cdot q$. $\square$

Lemma 6. *If $p \in \mathbb{P}_\Gamma$, then there is a finite set $S \subseteq \Gamma$ such that for all $\alpha \in \Gamma$, there exists a $\gamma \in S$ such that $\alpha \cdot p = \gamma \cdot p$.*

Proof. Let S consist of one element in the preimage of each element of $\operatorname{ran}(p)$. Since $\operatorname{ran}(p)$ is finite, S is finite. Let γ be such that $p(\gamma) = p(\alpha)$. Then since p is a forgetful homomorphism, for all $\beta \in \Gamma$ we have

$$\alpha \cdot p(\beta) = p(\alpha^{-1}\beta) = p(\alpha^{-1})p(\beta) = p(\gamma^{-1})p(\beta) = \gamma \cdot p(\beta).$$

$\square$

Lemma 7. *Suppose Γ is residually finite. Then a generic filter for $\mathbb{P}_\Gamma$ yields some $x\colon \Gamma \to \omega^\omega$ such that $x \in \operatorname{Free}((\omega^\omega)^\Gamma)$.*

Proof. Since Γ is residually finite, for each nonidentity $\gamma \in \Gamma$ there is a homomorphism $h\colon \Gamma \to \Delta$ to a finite group Δ with $h(\gamma) \neq 1$. Let $k = |\Delta|$ and $\hat{h}\colon \Gamma \to k$ be obtained by forgetting the group structure on Δ. Then given any condition $p\colon \Gamma \to \omega^n$ in $\mathbb{P}_\Gamma$, there is an extension $q\colon \Gamma \to \omega^{n+1}$ of p where $q(\gamma)(n) = \hat{h}(\gamma)$ for all $\gamma \in \Gamma$. Since $h(\gamma) \neq h(1)$, we also have that $q(\gamma) \neq q(1)$. Thus, the set of $q \in \mathbb{P}_\Gamma$ such that $q(\gamma) \neq q(1)$ is dense. Hence, a generic filter yields some generic $x \in \operatorname{Free}((\omega^\omega)^\Gamma)$. $\square$

It is worth noting that when Γ is finitely generated and residually finite, then $\mathbb{P}_\Gamma$ is a countable partial order. We're now ready to prove Theorem 1.

Proof of Theorem 1. Since Seward and Tucker-Drob have shown for all countable Γ there is a Γ-equivariant continuous function $\operatorname{Free}((\omega^\omega)^\Gamma) \to \operatorname{Free}(2^\Gamma)$, it suffices to prove the analogous theorem for $\operatorname{Free}((\omega^\omega)^\Gamma)$.

Suppose $A \subseteq \operatorname{Free}((\omega^\omega)^\Gamma)$ is a Borel complete section of the shift action. Then for some $\mathbb{P}_\Gamma$-generic $y \in \operatorname{Free}((\omega^\omega)^\Gamma)$ there is some $\alpha \in \Gamma$ such that $\alpha \cdot y \in A$. Let $x = \alpha \cdot y$. Now x is also $\mathbb{P}_\Gamma$-generic by Lemma 5. Since $x \in A$ there must be some p in the filter associated to x such that $p \Vdash x \in A$. Now since p is a forgetful homomorphism to a finite group, the kernel Γ' of p is a normal subgroup of finite index. Since for all $\gamma \in \Gamma'$ we have $\gamma \cdot p = p$, we see that for all $\gamma \in \Gamma'$, we have that $\gamma \cdot x$ is a $\mathbb{P}_\Gamma$-generic real extending p and hence $\gamma \cdot x \in A$. $\square$

We mention one other easy corollary of our forcing. Gao, Jackson, and Seward have shown that for every finitely generated group Γ with elements $\gamma_0, \gamma_1, \ldots$, every function $f : \omega \to \omega$ tending to infinity, and every sequence $\{A_i\}_{i \in \omega}$ of Borel complete sections for the shift action of Γ on $\mathrm{Free}(2^\Gamma)$, there exists some $x \in \mathrm{Free}(2^\Gamma)$ such that for infinitely many n, there exists some $i \leq f(n)$ such that $\gamma_i \cdot x \in A_n$. Our forcing can be used to see the special case of their theorem when Γ is residually finite.

Indeed, when Γ is residually finite, any generic for $\mathbb{P}_\Gamma$ has this property. To see this, take any $p \in \mathbb{P}$, and let S be the associated finite set as in Lemma 6. Let n be large enough that $S \subseteq \{\gamma_0, \gamma_1, \ldots, \gamma_{f(n)}\}$. Let x be a $\mathbb{P}_\Gamma$-generic extending p. Since A_n is a complete section, there must be some $\alpha \in \Gamma$ such that $\alpha \cdot x \in A_n$. Further, since $\alpha \cdot x$ is $\mathbb{P}_\Gamma$-generic, there must be some $q \leq \alpha \cdot p$ such that $q \Vdash x \in A_n$. However, since $q \leq \alpha \cdot p$, we must have that $q \leq \gamma \cdot p$ for some $\gamma \in S$ by Lemma 6. Thus, the condition $\gamma^{-1} \cdot q \leq p$ forces that there is some $i \leq f(n)$ such that a generic x extending $\gamma^{-1} \cdot q$ has $\gamma_i \cdot x \in A_n$. Hence, for each m, there is a dense set of conditions forcing that there is some $n > m$ and $i \leq f(n)$ such that a generic x has $\gamma_i \cdot x \in A_n$.