A SHORT PROOF OF THE CONNES-FELDMAN-WEISS THEOREM

ANDREW MARKS

Abstract. In this note, we give a proof of the theorem due to Connes, Feldman, and Weiss that a countable Borel equivalence relation on a standard probability space (X, μ) is μ-hyperfinite if and only if it is μ-amenable.

1 Introduction

A Borel equivalence relation E on a standard Borel space X is said to be *finite* if every equivalence class of E is finite, and *countable* if every equivalence class of E is countable. We will abbreviate countable Borel equivalence relation by CBER.

Definition 1.1. A CBER E on a standard Borel space X is *hyperfinite* if there is an increasing sequence $F_0 \subseteq F_1 \subseteq \ldots$ of finite Borel equivalence relations such that $E = \bigcup_i F_i$. If μ is a Borel probability measure on X, then E is said to be *μ-hyperfinite* if there is a μ-conull Borel set A such that $E \restriction A$ is hyperfinite.

By the Feldman-Moore theorem, every CBER arises as the orbit equivalence relation of a Borel action of a countable group.

Theorem 1.2 (Feldman-Moore, see [KM, Theorem 1.3])**.** *Suppose E is a CBER on a standard Borel space X. Then there is a Borel action $\Gamma \curvearrowright^a X$ of a countable discrete group Γ on X such that E is equal to the orbit equivalence relation E_a of this action where $x \mathrel{E_a} y$ if there exists $\gamma \in \Gamma$ such that $\gamma \cdot x = y$.*

Analogously to the theory of amenable groups (see [J] for an introduction to this theory), one can develop a theory of amenable equivalence relations.

Definition 1.3. Suppose E is a CBER on X. Then E is *amenable* if there is a Borel sequence $\lambda^n \colon E \to [0, 1]$ of functions assigning each pair of E-related points $(x, y) \in E$ to some $\lambda^n(x, y) \in [0, 1]$ such that if we let $\lambda^n_x(y) = \lambda^n(x, y)$, so λ^n_x is supported on $[x]_E$, then

(1) $\|\lambda^n_x\|_1 = \sum_{y \in [x]_E} \lambda^n_x(y) = 1$, and
(2) for all $(x, y) \in E$, we have $\|\lambda^n_x - \lambda^n_y\|_1 \to 0$ as $n \to \infty$.

If μ is a Borel probability measure on X, then we say E is *μ-amenable* if there is a conull Borel set A such that $E \restriction A$ is amenable.

The functions λ^n here are called Reiter functions, analogously to Reiter functions in the theory of amenable groups. Condition (1) above expresses that λ^n_x is a probability measure on the class $[x]_E$ of x. So amenability of E means there is a Borel way of assigning a countable sequence of probability measures on the E-class of each point x of the space so that these measures converge in ℓ^1 norm for all pairs of E-related points.

In this note, we give a short proof of the following fundamental theorem due to Connes, Feldman, and Weiss:

Theorem 1.4 ([CFW])**.** *Suppose E is a CBER on a standard probability space (X, μ). Then E is μ-hyperfinite iff E is μ-amenable.*

Nothing in this note is original, and these arguments and characterizations are all well-known and exist in the literature. Our aim is simply to organize them in a convenient way. The main difference between our proof and other arguments in the literature is that we avoid the use of Radon-Nikodym derivatives, following a suggestion of Robin Tucker-Drob. In Section 2 we give

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some examples of amenable equivalence relations. In Section 3 we recall some basic properties of amenability and hyperfiniteness. In Section 4 we recall the theory of graphings of CBERs, and give a characterization of μ-hyperfiniteness in terms of graphings. In Section 5 we recall another characterization of μ-hyperfiniteness in terms of isoperimetric constant due to Kaimanovich and Elek. In Section 6 we give our proof of the Connes-Feldman-Weiss theorem, adapted from a similar proof using Radon-Nikodym derivatives in [KM, Theorem 10.1].

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2 Examples

We briefly give several examples of amenable Borel equivalence relations. Recall that a transversal of an equivalence relation E on X is a subset $A \subseteq X$ that meets each equivalence class in exactly one point.

Proposition 2.1. *Suppose E is a CBER on a standard Borel space X which admits a Borel transversal $A \subseteq X$. Then E is amenable.*

Proof. This is witnessed by

$$\lambda^n(x,y) = \begin{cases} 1 & \text{if } y \in A \\ 0 & \text{if } y \notin A. \end{cases}$$

□

Proposition 2.2. *Suppose Γ is a countable amenable group and $\Gamma \curvearrowright^a X$ is a Borel action of Γ. Then E_a is amenable.*

Proof. Let $f_n \colon \Gamma \to [0,1]$ be a sequence of (right) Reiter functions for Γ, meaning that for every group element $\delta \in \Gamma$, $\sum_{\gamma \in \Gamma} |f_n(\gamma\delta) - f_n(\gamma)| \to 0$ as $n \to \infty$. Then the amenability of E_a is witnessed by the functions

$$\lambda^n(x,y) = \sum_{\{\gamma \in \Gamma : \gamma \cdot x = y\}} f_n(\gamma)$$

□

Proposition 2.3. *Suppose E is a hyperfinite Borel equivalence relation. Then E is amenable.*

Proof. Let the hyperfiniteness of E be witnessed by $F_0 \subseteq F_1 \subseteq \ldots$. Then the amenability of E is witnessed by the functions

$$\lambda^n(x,y) = \begin{cases} 0 & \text{if } y \notin [x]_{F_n} \\ \frac{1}{|[x]_{F_n}|} & \text{if } y \in [x]_{F_n} \end{cases}$$

□

The converse of Proposition 2.3 is an important open problem:

Open Problem 2.4 (See [KM, page 30])**.** *Suppose E is a CBER. If E is amenable, is E hyperfinite?*

3 Basic properties of hyperfiniteness and amenability, and quasi-invariant measures

In this section, we recall some basic closure properties of hyperfinite and amenable equivalence relations.

First, hyperfiniteness is closed downwards under taking subsets or subequivalence relations by simply restricting a hyperfiniteness witness appropriately. Suppose E, F are CBERs on X. We will say F is a subequivalence relation of E and write $F \subseteq E$ if $x\ F\ y \to x\ E\ y$ for all $x, y \in X$. That is, if the relation F is a subset of E, viewed as subsets of $X \times X$.

Proposition 3.1. *Suppose $E' \subseteq E$ are CBERs on X. If E is hyperfinite, then E' is hyperfinite. If $A \subseteq X$ is Borel and E is hyperfinite, then $E \upharpoonright A$ is hyperfinite.*

Proof. Suppose $F_0 \subseteq F_1 \subseteq \dots$ are finite Borel equivalence relations and $E = \bigcup_n F_n$. Then define the equivalence F'_n by $x\ F'_n\ y$ if $x\ E'\ y$ and $x\ F_n\ y$. Then $E' = \bigcup_n F'_n$. Similarly, $E \upharpoonright A$ is hyperfinite as witnessed by the equivalence relations $F_n \upharpoonright A$. $\square$

Amenability is similarly preserved under taking subequivalence relations or subsets, but we will not use this fact and we leave it as an exercise.

Hyperfiniteness and amenability are preserved under taking saturations of sets under equivalence relations. If E is an equivalence relation on X and $A \subseteq X$, let $[A]_E = \{x \in X : (\exists y \in A) x\ E\ y\}$ denote the *E-saturation* of A. We say that a set $A \subseteq X$ is *E-invariant* if $A = [A]_E$.

Lemma 3.2. *Suppose E is a CBER on a standard Borel space X, and $A \subseteq X$ is Borel.*

(1) $E \upharpoonright A$ is amenable if and only if $E \upharpoonright [A]_E$ is amenable.
(2) $E \upharpoonright A$ is hyperfinite if and only if $E \upharpoonright [A]_E$ is hyperfinite.

Proof. By the Feldman-Moore theorem, let $\Gamma \curvearrowright^a X$ be a Borel action of a countable group generating E, and let $\{\gamma_0, \gamma_1, \gamma_2, \dots\}$ be the elements of the group Γ, where γ_0 is the identity. Let $f : [A]_E \to A$ be the function where $f(x) = \gamma_i \cdot x$ where i is least such that $\gamma_i \cdot x \in A$.

First we prove (1). Suppose $E \upharpoonright A$ is amenable and $\lambda^n_x : A \to [0, 1]$ are Reiter functions for $E \upharpoonright A$. Then $E \upharpoonright [A]_E$ is amenable as witnessed by the Reiter functions $\zeta^n_x : [A]_E \to [0, 1]$ defined by $\zeta^n_x = \lambda^n_{f(x)}$. That is, we are using the function f to copy one of the Reiter functions $\lambda^n_{f(x)}$ on A for each $x \in [A]_E$.

Conversely, if $E \upharpoonright [A]_E$ is amenable and $\lambda^n_x : [A]_E \to [0, 1]$ are Reiter functions for $E \upharpoonright [A]_E$, then we can show that $E \upharpoonright A$ is amenable by taking the "pushforward" of these Reiter functions under f. Precisely, define $\zeta^n_x : A \to [0, 1]$ by $\zeta^n_x(y) = \sum_{z \in f^{-1}(y)} \lambda^n_x(z)$. Thinking of λ^n_x and ζ^n_x as measures on $[x]_E$, we have that ζ^n_x is the pushforward of λ^n_x under f. Then for all $x, y \in A$ with $x\ E\ y$, $\|\zeta^n_x - \zeta^n_y\|_1 \leq \|\lambda^n_x - \lambda^n_y\|_1$ by the triangle inequality, so $\|\zeta^n_x - \zeta^n_y\|_1 \to 0$ as $n \to \infty$.

Now we prove (2). Suppose $E \upharpoonright A$ is hyperfinite and $F_0 \subseteq F_1 \subseteq \dots$ are finite Borel equivalence relations on A such that $\bigcup_n F_n = E \upharpoonright A$. Our idea is to define new finite equivalence relations F'_n on $[A]_E$ which are equal to F_n when restricted to A. Define F'_n on $[A]_E$ by $x\ F'_n\ y$ if $x = y$ or if $f(x)\ F_n\ f(y)$, and there exist $i, j \leq n$ such that $\gamma_i \cdot f(x) = x$ and $\gamma_j \cdot f(y) = y$. Note $[x]_{F'_n} \subseteq \{x\} \cup \{\gamma_0, \dots, \gamma_n\} \cdot [f(x)]_{F_n}$, so F'_n has finite classes, and $\bigcup_n F'_n = E \upharpoonright [A]_E$.

Conversely, if $E \upharpoonright [A]_E$ is hyperfinite so $E \upharpoonright [A]_E = \bigcup_n F_n$ where F_n are increasing finite Borel equivalence relations on $[A]_E$, then the equivalence relations $F_n \upharpoonright A$ are finite and show that $E \upharpoonright A$ is hyperfinite. $\square$

We can restrict the types of measures we consider when proving the Connes-Feldman-Weiss theorem to just quasi-invariant measures.

Definition 3.3. Suppose E is a CBER on a standard Borel space X and μ is a Borel probability measure on X. Then we say that E is μ-quasi-invariant if for every Borel μ-null set A, the E-saturation $[A]_E$ of A is μ-null.

It is easy to show that if E is generated by a Borel action of a countable group $\Gamma \curvearrowright^a X$, then μ is E-quasi-invariant if and only if for all Borel sets A and $\gamma \in \Gamma$, $\mu(A) = 0$ if and only if $\mu(\gamma \cdot A) = 0$, if and only if for every partial Borel injection $f\colon X \to X$ such that $x \mathrel{E} f(x)$ for all $x \in \mathrm{dom}(f)$, $\mu(\mathrm{dom}(f)) > 0$ iff $\mu(\mathrm{ran}(f)) > 0$. See [KM, Proposition 8.1]. So another name for E being μ-quasi-invariant is E being *μ-measure class preserving* or mcp.

Note that if E is a CBER on X, and $E' \subseteq E$ is a subequivalence relation of E, and μ is a Borel probability measure on X and E is μ-quasi-invariant, then clearly E' is μ-quasi-invariant, since $[A]_{E'} \subseteq [A]_E$ for every set A.

Lemma 3.4. *Suppose E is a CBER on a standard Borel space X, and μ is a Borel probability measure on X. Then there is a Borel probability measure μ' on X such that E is μ'-quasi-invariant, and for all Borel E-invariant sets A, A is μ-null iff A is μ'-null. Hence, E is μ-amenable iff it is μ'-amenable, and E is μ-hyperfinite iff E is μ'-hyperfinite.*

Proof. By the Feldman-Moore theorem, let $\Gamma \curvearrowright^a X$ be a Borel action of a countable group Γ generating the equivalence relation E, and let $\{\gamma_0, \gamma_1, \ldots\}$ be the elements of the group Γ. Define the measure μ' on X by $\mu'(A) = \sum_{n=0}^{\infty} 2^{-n-1}\mu(\gamma_n \cdot A)$. So $\mu'(A)$ is the weighted average of μ over all translations of A by group elements in Γ.

Now if A is E-invariant, then $\gamma_n \cdot A = A$ for all γ_n, and so $\mu'(A) = \sum_{n\in\mathbb{N}} 2^{-n-1}\mu(\gamma_n \cdot A) = \sum_{n\in\mathbb{N}} 2^{-n-1}\mu(A) = \mu(A)$. So A is μ-null iff A is μ'-null. To show that E is μ'-quasi-invariant, suppose A is a Borel set which need not be E-invariant, and suppose that $\mu'(A) = 0$. Then by definition of μ', since $\mu'(A) = 0$, we have $\mu(\gamma_n \cdot A) = 0$ for all γ_n. But then using that $[A]_E$ is E-invariant, we have $\mu'([A]_E) = \mu([A]_E)$ and then $\mu([A]_E) = \mu(\bigcup_n \gamma_n \cdot A) \leq \sum_{n\geq 0} \mu(\gamma_n \cdot A) = 0$.

To finish we note that for any Borel probability measure μ, by Lemma 3.2, E is μ-amenable if and only if there is a μ-conull Borel set A such that $E \restriction A$ is amenable, if and only if there is an E-invariant Borel μ-conull set A such that $E \restriction A$ is amenable. Similarly, E is μ-hyperfinite if and only if there is a μ-conull Borel set A such that $E \restriction A$ is hyperfinite, if and only if there is an E-invariant Borel μ-conull set A such that $E \restriction A$ is hyperfinite. Hence, since an E-invariant Borel set is μ-conull iff it is μ'-conull, E is μ-amenable if and only if it is μ'-amenable, and E is μ-hyperfinite if and only if it is μ'-hyperfinite. $\square$

Thus, it suffices to prove the Connes-Feldman-Weiss Theorem 1.4 in the case of quasi-invariant measures, and we will sometimes assume our measures are quasi-invariant below.

We note one last nice property of quasi-invariant measures: conull Borel sets contain invariant conull Borel sets:

Proposition 3.5. *Suppose E is a CBER on a standard Borel space X, and μ is an E-quasi-invariant Borel probability measure on X. Then if $A \subseteq X$ is a conull Borel set, then there is an E-invariant conull Borel set $B \subseteq A$.*

Proof. Let $B = \{x \in A \colon [x]_E \subseteq A\}$ be the largest E-invariant subset of A. Now since A is μ-conull, $A_0 = [A]_E \setminus A$ is null, and so $[A_0]_E$ is also null since E is μ-quasi-invariant. But $B = A \setminus [A_0]_E$ since $[A_0]_E$ is all the E-classes inside which A is not E-invariant. $\square$

4 Graphings of countable Borel equivalence relations and hyperfiniteness

In the theory of CBERs, it is very useful to study *graphings*, that is, Borel graphs whose connectedness relation is equal to a given equivalence relation. Our proof will be based on this approach and will use graphings to understand and characterize μ-hyperfiniteness.

Definition 4.1. A (simple undirected) Borel graph G on a standard Borel space X is a graph whose set of vertices is X, and where the (symmetric, irreflexive) edge relation of G is Borel as a subset of $X \times X$. If $x, y \in X$, we will write $x \mathrel{G} y$ to denote that there is an edge from x to y (abusing notation here by identifying G with its edge relation). A graph is *bounded degree* if there is a natural number d so that every vertex has at most d neighbors, it is *locally finite* if every vertex has finitely many neighbors, and it is locally countable if every vertex has countably many neighbors. Let $N_G(x) = \{y \in X : x \mathrel{G} y\}$ denote the set of neighbors of x in G.

Proposition 4.2. *If G is a Borel graph on a standard Borel space X, then G is locally countable if and only if it is generated by countably many partial Borel injections. That is, there are Borel partial injections $(f_n)_{n\in\mathbb{N}}$ on X with Borel domains such that $x \mathrel{G} y$ if there exists some n such that $f_n(x) = y$.*

Proof. Suppose the edge relation of G is locally countable. The existence of such functions f_n follows from Lusin-Novikov uniformization [Ke, Theorem 18.10]. Conversely, if $f: X \to X$ is a Borel function, then its graph $\{(x, y): f(x) = y\}$ is Borel, and so the edge relation of the graph generated by the $(f_n)_{n\in\mathbb{N}}$ is Borel. □

The connectedness relation of a locally countable Borel graph is an important example of a CBER, as we use this associated equivalence relation to define amenability, hyperfiniteness, and quasi-measure-preservingness for locally countable Borel graphs:

Definition 4.3. Suppose G is a locally countable Borel graph on a standard Borel space X. Then we let E_G denote the connectedness relation of G, where $x \mathrel{E_G} y$ if there is a path from x to y in G. If G is a locally countable Borel graph, then E_G is a CBER relation which follows from Proposition 4.2. If E is a CBER on X, then we say that G is a *graphing* of E if $E_G = E$. Finally, we say that G is *hyperfinite* if its connectedness relation E_G is hyperfinite. If μ is a Borel probability measure on X, then we say that G is μ-hyperfinite if E_G is μ-hyperfinite, and we say G is μ-quasi-invariant if E_G is μ-quasi-invariant.

There is a more graph-theoretic definition of hyperfiniteness of a locally countable Borel graph: G is hyperfinite if and only if there are Borel graphs $G_0 \subseteq G_1 \subseteq \dots$ with finite connected components such that $G = \bigcup_n G_n$. We will not use this alternate characterization and we leave it as an exercise.

An important example of a class of Borel graphs are the following Schreier graphs of group actions. Suppose $\Gamma \curvearrowright^a X$ is a Borel action of the countable group Γ on a standard Borel space X, and $S \subseteq \Gamma$ is a symmetric set. Then let $\mathrm{Sh}(a, S)$ be the Borel graph on X where distinct $x, y \in X$ are adjacent iff $(\exists \gamma \in S)\gamma \cdot x = y$. These Schreier graphs and the Feldman-Moore theorem give one way of showing every CBER has a graphing G which is an increasing union of bounded degree Borel graphs. One could also prove this directly from Lusin-Novikov uniformization.

Proposition 4.4. *Suppose that E is a CBER on a standard Borel space X. Then there is an increasing sequence of bounded degree graphs $G_0 \subseteq G_1 \subseteq \dots$ such that the locally countable Borel graph $G = \bigcup_n G_n$ is a graphing of E.*

Proof. By the Feldman-Moore theorem, let $\Gamma \curvearrowright^a X$ be a Borel action of a countable group that generates E. Then choose an increasing sequence of finite symmetric sets $S_0 \subseteq S_1 \subseteq \dots \subseteq \Gamma$ whose union $\bigcup_n S_n$ generates Γ. Then each of the graphs $G_n = \mathrm{Sh}(a, S_n)$ is bounded degree (bounded by $|S_n|$). They are increasing since the sets S_n are increasing, and their union $G = \bigcup_n G_n$ is a locally countable Borel graph that is a graphing of E since $\bigcup_n S_n$ generates Γ. □

We can characterize μ-hyperfiniteness of such an increasing sequence of Borel graphs and hence the μ-hyperfiniteness of any CBER E as follows:

Lemma 4.5. *Suppose $G_0 \subseteq G_1 \subseteq \dots$ is an increasing sequence of locally finite Borel graphs on X, and let $G = \bigcup_n G_n$ be their union. Suppose μ is a Borel probability measure on X, and E_G is μ-quasi-invariant. Then G is μ-hyperfinite if and only if for every $n \in \mathbb{N}$ and $\epsilon > 0$ there exists a Borel set $Y \subseteq X$ with $\mu(Y) \geq 1 - \epsilon$ such that $G_n \upharpoonright Y$ has finite connected components.*

Proof. Suppose first that G is μ-hyperfinite. Let $X' \subseteq X$ be a μ-conull set such that $E_G \upharpoonright X'$ is hyperfinite. We may assume that X' is E_G-invariant by Proposition 3.5. Let $E_G \upharpoonright X' = \bigcup_m F_m$ where F_m are finite Borel equivalence relations on X'. Now fix n, and let $A_m = \{x \in X' : N_{G_n}(x) \subseteq [x]_{F_m}\}$, so $x \in A_m$ if its G_n-neighborhood is contained in the F_m-class of x. Note that since G_n is locally finite, and X' is E_G-invariant and hence E_{G_n}-invariant, every neighbor of x in G_n is in X', and thus must eventually be F_m related to x for sufficiently large m. Hence, for every $x \in X'$ there is some sufficiently large m such that $x \in A_m$, and so $\mu(A_m) \to 1$ as $m \to \infty$. So we can find a sufficiently large m such that $\mu(A_m) \geq 1 - \epsilon$. If $x \in A_m$, then the connected component of x in $G_n \restriction A_m$ is contained in $[x]_{F_m}$, and is hence finite. Let $Y = A_m$.

Conversely, suppose that for any $\epsilon > 0$ and every n, there exists a Borel set $Y \subseteq X$ with $\mu(Y) \geq 1 - \epsilon$ such that $G_n \restriction Y$ has finite connected components. Then for each n, find such a set A_n, with $\mu(A_n) \geq 1 - 2^{-n}$. Let $B_n = \bigcap_{k>n} A_k$ so $\mu(B_n) \geq 1 - \sum_{k>n} 2^{-k} = 1 - 2^{-n}$, and $B_0 \subseteq B_1 \subseteq \ldots$. Now $B = \bigcup_n B_n$ is a Borel set of μ-measure 1. Let B' be a μ-conull E_G-invariant Borel set contained in B by Proposition 3.5, and let $B'_n = B' \cap B_n$. Let F_n be the equivalence relation on B' where $x \; F_n \; y$ if $x = y$ or $x, y \in B'_n$ and there is a path from x to y in $G_n \restriction B'_n$. Then we claim that $E_G \restriction B'$ is the increasing union $\bigcup_n F_n$. This is since if $x, y \in B'$ and they are connected in G, then there is a path $x = x_0, x_1, \ldots, x_k = y$ in G from x to y. Since B' is G-invariant, each $x_i \in B'$ and hence is in B'_n for all sufficiently large n, and we also have that there is an edge from x_i to x_{i+1} in G_n for all sufficiently large n since the G_n are increasing and $G = \bigcup_n G_n$. □

The above characterization is interesting even in the case when the graphs $G_0 = G_1 = \ldots = G$ are all equal:

Corollary 4.6. *Suppose G is a locally finite Borel graph on X and suppose μ is a Borel probability measure on X and E_G is μ-quasi-invariant. Then G is μ-hyperfinite if and only if for every $\epsilon > 0$ there exists a Borel set $Y \subseteq X$ with $\mu(Y) \geq 1-\epsilon$ such that $G \restriction Y$ has finite connected components.* □

We note that the above characterization implies the theorem of Dye and Krieger that an increasing union of locally countable Borel graphs or CBERs is μ-hyperfinite if and only if each graph or CBER in the sequence is μ-hyperfinite. Though we will not use this result in the following sections. This is analogous to the theorem that a countable group Γ is amenable if and only if all its finitely generated subgroups are amenable.

Corollary 4.7 (Dye [D] and Krieger [Kr])**.** *Suppose X is a standard Borel space, and μ is a Borel probability measure on X.*

(1) If $G_0 \subseteq G_1 \subseteq \ldots$ are locally countable Borel graphs on X, then $G = \bigcup_n G_n$ is μ-hyperfinite if and only if G_n is μ-hyperfinite for every n.
(2) If $E_0 \subseteq E_1 \subseteq \ldots$ are CBERs on X, then $E = \bigcup_n E_n$ is μ-hyperfinite if and only if E_n is μ-hyperfinite for every n.

Proof. We begin by proving (1). The forward direction follows since hyperfiniteness passes to subequivalence relations by Proposition 3.1. To prove the reverse direction, first, we note that we may assume that μ is E_G-quasi-invariant by Lemma 3.4. Then let $(f_{n,i})_{i \in \mathbb{N}}$ be Borel partial injections generating G_n by Proposition 4.2. Then let H_m be the graph generated by the finitely many functions $\{f_{n,i} \colon n, i \leq m\}$, so H_m is bounded degree (degree bounded by $2(m+1)$), $H_m \subseteq G_m$ (since the graphs G_n are increasing), the H_m are increasing $H_0 \subseteq H_1 \subseteq \ldots$ (by definition), and finally $\bigcup_m H_m = G$. If G_m is μ-hyperfinite, then its subgraph H_m is μ-hyperfinite by Proposition 3.1, so for every $\epsilon > 0$ there exists a Borel Y with $\mu(Y) \geq 1 - \epsilon$ such that $H_m \restriction Y$ has finite connected components by Corollary 4.6. But then G is μ-hyperfinite by Lemma 4.5.

(2) follows from (1) by letting G_n be a locally countable Borel graphing of E_n for every n. Then the Borel graphs $G'_n = G_0 \cup \ldots \cup G_n$ are increasing, G'_n is a graphing of E_n, and $\bigcup_n G'_n$ is a graphing of E. □

5 Characterizing μ-hyperfiniteness with isoperimetric constant

If Γ is a finitely generated group, then one important characterization of amenability is that Γ is amenable if and only if the Cayley graph of Γ has isoperimetric constant 0 [J, Appendix A]. Analogously, we give a proof of a result due to Kaimanovich characterizing when a locally finite

Borel graph has a μ-hyperfinite connectedness relation in terms of a measure-theoretic version of isoperimetric constant.

Definition 5.1. If G is a locally finite Borel graph on a standard probability space (X, μ), then the μ-isoperimetric constant of G is the infimum of $\mu(\partial_G A)/\mu(A)$ over all Borel subsets $A \subseteq X$ of positive measure such that the induced subgraph $G \restriction A$ of G on A has finite connected components. Here, $\partial_G A$ denotes the set of vertices in $X \setminus A$ which are G-adjacent to a vertex in A.

For example, suppose Γ is a countable amenable group, $S \subseteq \Gamma$ is finite symmetric, and $\Gamma \curvearrowright^a (X, \mu)$ is a free μ-measure preserving Borel action of Γ. Then it is not too hard to show that $\mathrm{Sh}(a, S)$ has μ-isoperimetric constant 0 using the existence of Følner sets in Γ, and maximal independent sets of translates of these Følner sets with disjoint neighborhoods as in [KST].

Now we give a proof of the theorem of Kaimanovich characterizing when a locally finite graph G is μ-hyperfinite in terms of isoperimetric constant. Robin Tucker-Drob pointed out that Elek's proof [E] of Kaimanovich's theorem works in the greater generality in which the graph G is not assumed to be μ-invariant. The following presentation of Theorem 5.2 is due to Tucker-Drob, and is taken from [CGMTD, Section 2].

Theorem 5.2 (Kaimanovich [Ka], see also Elek [E])**.** *Let G be a locally finite Borel graph on a standard probability space (X, μ), and assume that E_G is μ-quasi-invariant. Then G is μ-hyperfinite if and only if for every positive measure Borel subset $X_0 \subseteq X$, the isoperimetric constant of the induced subgraph $G \restriction X_0$ of G on X_0 is 0.*

Proof. Suppose first that G is μ-hyperfinite. Then by Corollary 4.6, for every $\epsilon > 0$ there is a Borel set Y so that $G \restriction Y$ has finite connected components and $\mu(Y) \geq 1 - \epsilon$. But then $\partial_G Y$ is disjoint from Y and hence $\mu(\partial_G Y) \leq \epsilon$ and so $\mu(\partial_G Y)/\mu(Y) \leq \epsilon/(1-\epsilon)$. Since ϵ was arbitrary, this shows the isoperimetric constant of G is 0.

Now we prove the reverse direction. Assume now that for every positive measure Borel subset $X_0 \subseteq X$ the isoperimetric constant of $G \restriction X_0$ is 0. We will use Corollary 4.6, and show that for every $\epsilon > 0$ there exists a Borel set $Y \subseteq X$ with $\mu(Y) \geq 1 - \epsilon$ such that $G \restriction Y$ has finite connected components.

Given $\epsilon > 0$, by measure exhaustion we can find a maximal countable collection $\mathcal{A}$ of pairwise disjoint non-null Borel subsets of X such that

(i) $G \restriction \bigcup \mathcal{A}$ has finite connected components;
(ii) $\mu(\partial_G(\bigcup \mathcal{A})) \leq \epsilon \mu(\bigcup \mathcal{A})$;
(iii) If $A, B \in \mathcal{A}$ are distinct, then no vertex in A is adjacent to a vertex in B.

Let $Y = \bigcup \mathcal{A}$. We now claim that the set $X_0 = X - (Y \cup \partial_G Y)$ is null. Otherwise, by hypothesis we may find a Borel set $A_0 \subseteq X_0$ of positive measure such that $G \restriction A_0$ has finite connected components and $\mu(\partial_{G \restriction X_0} A_0) < \epsilon \mu(A_0)$. But then the collection $\mathcal{A}_0 = \mathcal{A} \cup \{A_0\}$ satisfies (i)-(iii) in place of $\mathcal{A}$, (i) and (iii) are clear and (ii) follows from the equality $\partial_G(\bigcup \mathcal{A}_0) = \partial_G Y \cup \partial_{G|X_0} A_0$ contradicting maximality of $\mathcal{A}$. Thus, $\mu(Y) = 1 - \mu(\partial_G Y) \geq 1 - \epsilon\mu(Y) \geq 1 - \epsilon$, and $G \restriction Y$ has finite connected components, which finishes the proof. $\square$

Remark 5.3. It is an open question whether every Δ^1_1 hyperfinite equivalence relation has a Δ^1_1 witness to its hyperfiniteness (see [JKL, 6.1(B)]). The proofs of Corollary 4.7 and Theorem 5.2 can both be seen to be effective, modulo an effective nullset. This can be used to show that if E is a μ-hyperfinite Δ^1_1 equivalence relation on $(2^{\mathbb{N}}, \mu)$, where μ is a Δ^1_1 Borel probability measure, then there is a Δ^1_1 μ-conull set $A \subseteq 2^{\mathbb{N}}$ and a Δ^1_1 sequence $F_0 \subseteq F_1 \subseteq \ldots$ of equivalence relations on A witnessing that $E \restriction A$ is hyperfinite. This result is due originally to M. Segal [S].

6 Characterizing μ-amenability

Namioka's trick is the standard way of transforming Reiter functions into Følner sets in amenable groups. We will use it as part of our proof of the Connes-Feldman-Weiss theorem.

Lemma 6.1 (Namioka's trick)**.** *For $a > 0$, let $1_{(a,\infty)}$ be the characteristic function of (a,∞). If $f, g\colon X \to [0,\infty)$ are in $\ell^1(X)$, then*

$$\int_0^\infty \|1_{(a,\infty)}(f) - 1_{(a,\infty)}(g)\|_1 \, da = \|f-g\|_1$$

Proof. For arbitrary $s, t \geq 0$,

$$\int_0^\infty |1_{(a,\infty)}(s) - 1_{(a,\infty)}(t)| \, \mathrm{d}a = |s-t|$$

Hence, for $f, g \in \ell^1(X)^+$ and every $x \in X$ we have

$$\int_0^\infty |1_{(a,\infty)}(f(x)) - 1_{(a,\infty)}(g(x))| \, \mathrm{d}a = |f(x)-g(x)|$$

and hence

$$\int_0^\infty \|1_{(a,\infty)}(f) - 1_{(a,\infty)}(g)\|_1 \, \mathrm{d}a = \|f-g\|_1$$

by Fubini's theorem. □

By Theorem 5.2, to show that every μ-amenable countable Borel equivalence relation is μ-hyperfinite, it suffices to show the following.

Theorem 6.2. *Suppose that E is a μ-amenable CBER on a standard probability space (X,μ), B is a Borel subset of X of positive measure, and $G \subseteq E$ is a bounded degree Borel graph on B. Then G has μ-isoperimetric constant 0.*

Proof. By discarding a nullset we may assume that E is amenable (instead of μ-amenable).

Let $\lambda^n\colon E \to [0,1]$ be Reiter functions witnessing the amenability of E. Fix $\epsilon > 0$. The rough idea of the proof is that we can use Namioka's trick to change the Reiter functions into sets that look like collections of Følner sets $\Pi \subseteq E$ in some sense. We will then find a set A witnessing that G has isoperimetric constant less than ϵ. The set A will be a union of sets of the form Π^z. These sets will have small boundary since the Følner sets are almost invariant.

To begin,

$$\lim_{n\to\infty} \int_B \sum_{y \in N_G(x)} \|\lambda^n_x - \lambda^n_y\|_1 \, \mathrm{d}\mu(x) = 0$$

by the dominated convergence theorem; since G has degree bounded by some d, for each n the integral on the LHS is bounded by $2d$. Hence, we can find a large enough n so that

$$(*) \qquad \int_B \sum_{y \in N_G(x)} \|\lambda^n_x - \lambda^n_y\|_1 \, \mathrm{d}\mu(x) < \epsilon\mu(B).$$

Let $\Lambda = \lambda^n$. As usual, let $1_{(a,\infty)}$ be the characteristic function of the interval (a,∞). Now by Namioka's trick, Fubini's theorem, and the fact that $\|\Lambda_x\|_1 = 1$, applied to the displayed equation above, we have

$$\int_0^\infty \int_B \sum_{y \in N_G(x)} \|1_{(a,\infty)}(\Lambda_x) - 1_{(a,\infty)}(\Lambda_y)\|_1 \, \mathrm{d}\mu(x) \, \mathrm{d}a < \epsilon\mu(B)$$

$$= \epsilon \int_B \|\Lambda_x\|_1 \, \mathrm{d}\mu(x) = \epsilon \int_0^\infty \int_B \|1_{(a,\infty)}(\Lambda_x)\|_1 \, \mathrm{d}\mu(x) \, \mathrm{d}a$$

Hence there is an $a > 0$ so that

$$\int_B \sum_{y \in N_G(x)} \|1_{(a,\infty)}(\Lambda_x) - 1_{(a,\infty)}(\Lambda_y)\|_1 \, \mathrm{d}\mu(x) < \epsilon \int_B \|1_{(a,\infty)}(\Lambda_x)\|_1 \, \mathrm{d}\mu(x).$$

Fix this a. Now $1_{(a,\infty)}(\Lambda)$ is the characteristic function of a subset of $E \subseteq X \times X$. Call this set R. For each $x \in X$, we have that $R_x = \{y : (x,y) \in R\}$ is finite. Indeed, $|R_x| \leq 1/a$. We have that $\|1_{(a,\infty)}(\Lambda_x)\|_1 = |R_x|$ and similarly $\|1_{(a,\infty)}(\Lambda_x) - 1_{(a,\infty)}(\Lambda_y)\|_1 = |R_x \triangle R_y|$, where $\triangle$ indicates symmetric difference. So rewriting,

$$(**) \qquad \int_B \sum_{y \in N_G(x)} |R_x \triangle R_y| \, \mathrm{d}\mu(x) < \epsilon \int_B |R_x| \, \mathrm{d}\mu(x)$$

By the Feldman-Moore theorem, let $\Gamma \curvearrowright^a X$ be a Borel action of a countable group generating E, and let $\{\gamma_0, \gamma_1, \gamma_2, \ldots\}$ be the elements of the group Γ. Let $R_n = \{(\gamma_i \cdot x, x) : x \in X \wedge i \leq n \wedge (\gamma_i \cdot x R x)\}$, so these relations are increasing: $R_0 \subseteq R_1 \subseteq \ldots$, and $R = \bigcup_n R_n$. Note that for every i and $z \in X$ we have that $R_i^z = \{x \colon (x, z) \in R_i\}$ is finite. By the dominated convergence theorem, there is some R_i so that the above inequality holds if we replace R by R_i. Set Π equal to this R_i. We now have

$$(***) \qquad \int_B \sum_{y \in N_G(x)} |\Pi_x \triangle \Pi_y| \, \mathrm{d}\mu(x) < \epsilon \int_B |\Pi_x| \, \mathrm{d}\mu(x).$$

Let H be the graph on X where distinct $z, z' \in X$ satisfy zHz' if $(\Pi^z \cup \partial_G \Pi^z) \cap (\Pi^{z'} \cup \partial_G \Pi^{z'}) \neq \emptyset$. For each $z \in X$, Π^z and $\partial_G \Pi^z$ are finite sets. Hence H is a locally finite Borel graph. We can find a Borel $\mathbb{N}$-coloring c of H by [KST, Proposition 4.5]. For each color k, let $A_k = \{x \in B : \exists z \in \Pi_x (c(z) = k)\}$. Then each connected component of $G \restriction A_k$ corresponds to some set of the form Π^z, and so $G \restriction A_k$ has finite connected components. We can use c to rewrite each side of (***) to get

$$\sum_k \int_B \sum_{y \in N_G(x)} |\{z \in \Pi_x \triangle \Pi_y : c(z) = k\}| \, \mathrm{d}\mu(x) < \epsilon \sum_k \int_B |\{z \in \Pi_x : c(z) = k\}| \, \mathrm{d}\mu(x)$$

so there is some k such that

$$\int_B \sum_{y \in N_G(x)} |\{z \in \Pi_x \triangle \Pi_y : c(z) = k\}| \, \mathrm{d}\mu(x) < \epsilon \int_B |\{z \in \Pi_x : c(z) = k\}| \, \mathrm{d}\mu(x)$$

But the right hand side of this equation is $\epsilon \int_B 1_{A_k} \, \mathrm{d}\mu = \epsilon \mu(A_k)$. Further $x \in \partial_G A_k$ implies there is some $y \in N_G(x)$ such that $\{z \in \Pi_x \triangle \Pi_y : c(z) = k\}$ is nonempty. Hence, the function integrated on the left hand side is greater than or equal to the characteristic function of $\partial_G A_k$. Hence,

$$\mu(\partial_G A_k) < \epsilon \mu(A_k)$$

□

From this we can conclude the Connes-Feldman-Weiss theorem.

Proof of Theorem 1.4. We have already shown that every μ-hyperfinite Borel equivalence relation is μ-amenable in Proposition 2.3. Now suppose E is μ-amenable. By Lemma 3.4 we may assume that μ is E-quasi-invariant. By Proposition 4.4, we can find an increasing sequence $G_0 \subseteq G_1 \subseteq \ldots$ of bounded degree graphs so that $\bigcup_n G_n$ is a graphing of E. Each graph G_i is μ-hyperfinite by combining Theorem 6.2 and Theorem 5.2. Hence E is μ-hyperfinite by Lemma 4.5. □

Historically, the Connes-Feldman-Weiss theorem was preceded by a theorem of Ornstein and Weiss [OW] that every Borel action of an amenable countable group Γ on a standard probability space (X, μ) induces a μ-hyperfinite countable Borel equivalence relation.

Combining this with Dye's theorem, we conclude that any two aperiodic ergodic actions of an amenable group are orbit equivalent.

Corollary 6.3 (Dye [D], Ornstein-Weiss [OW]). *Suppose $\Gamma \curvearrowright^a (X, \mu)$ and $\Delta \curvearrowright^b (Y, \nu)$ are aperiodic ergodic measure-preserving actions of amenable groups on standard probability spaces (X, μ) and (Y, ν). Then a and b are orbit equivalent.*

Proof. E_a is μ-amenable and E_b is ν-amenable by Proposition 2.2. Hence, by Theorem 1.4, E_a is μ-hyperfinite and E_b is ν-hyperfinite. Hence, both E_a and E_b are induced (modulo a nullset) by ergodic measure preserving Borel actions of $\mathbb{Z}$, which are orbit equivalent by Dye's theorem [D]. □

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