

RIGIDITY OF THE TURING DEGREES (AFTER OPENAI)

ANDREW MARKS

We give an exposition of the recent proof [4] that the Turing degrees are rigid.

Theorem 0.1 (OpenAI [4]). *There are no nontrivial automorphisms of the Turing degrees.*

By Slaman and Woodin's theorem [5, Theorem 6.3.1(2)], every automorphism π of the Turing degrees is induced by an arithmetic function $F: 2^\omega \rightarrow 2^\omega$, so $\pi([X]_T) = [F(X)]_T$ for all $X \in 2^\omega$.

It will suffice to prove that $F(X) \leq_T X$ for all $X \in 2^\omega$. That is, $\pi([X]_T) \leq_T [X]_T$ for all $X \in 2^\omega$. Since π was arbitrary, the same result applied to π^{-1} then gives $\pi([X]_T) \equiv_T [X]_T$, so π is the identity.

One fact that we will use is that for all $X, Y \in 2^\omega$, $F(X \oplus Y) \equiv_T F(X) \oplus F(Y)$, since automorphisms preserve least upper bounds.

We will use arithmetic Cohen forcing to prove the theorem. When we say a pair $(G, H) \in 2^\omega \times 2^\omega$ is arithmetically X -generic, we mean mutually arithmetically Cohen generic relative to X . If $F(X) \not\leq_T X$, then for all arithmetically X -generic reals (G, H) , we have $F(X) \not\leq_T X \oplus G \oplus H$. Hence, to prove the theorem, it suffices to prove the following lemma, which we can then apply to the arithmetic function representing π^{-1} :

Lemma 0.2. *For every $X \in 2^\omega$, if $(G, H) \in 2^\omega \times 2^\omega$ are arithmetically X -generic, then*

$$X \leq_T F(X) \oplus F(G) \oplus F(H).$$

Proof. The basic idea is to give a method to read off the bits of X one-by-one by uniformly computing a sequence of reals $F(G_n)$ from $F(X) \oplus F(G) \oplus F(H)$, where the $(G_n)_{n \in \omega}$ are all arithmetically X -generic, and examining a particular bit $F(G_n)(j)$ from them to recover information about X . In particular, we will use incompatible conditions forcing two different values of $F(\dot{G})(j)$ for some bit j and computable functions on 2^ω with dense orbits to be able to do this.

We'll use a group structure on 2^ω to help with this argument. The proof will in some ways be reminiscent of the proof that Baire measurable homomorphisms of Polish groups are automatically continuous (see [2, Theorem 9.10]). We'll use the symbol $+$ to denote the group operation of 2-adic addition $+$ on the infinite binary sequences 2^ω defined by adding their infinite binary expansions with standard binary carrying. We emphasize to the reader that $+$ here is different from the recursive join which we denote $\oplus$. The group identity here is the infinite sequence of all 0s. If $n \in \mathbb{N}$ is a natural number, we let nX be the sum of n copies of X , and we let $-X$ denote the inverse of X . Note that $2X$ is simply X with a 0 appended to the beginning. If the first bit of X is a 0, then we define $X/2$ to be X with this first bit removed, so $2(X/2) = X$.

Some properties of this group we will use are as follows. The operations of addition and inverse are computable. If $X, Y \in 2^\omega$ and $X \supseteq 0^k$ begins with k many 0s, then the first k bits of $X + Y$ are equal to the first k bits of Y . In particular, $nX \supseteq 0^k$ for all $n \in \mathbb{N}$. Finally, suppose $s = 0^k \frown 1 \in 2^{<\omega}$ is the finite binary sequence of k 0s followed by a single 1 where $\frown$ denotes concatenation. Then for any X extending s , the sequence $(nX)_{n \in \mathbb{N}}$ is dense in the basic clopen neighborhood $N_{0^k} = \{Y : Y \supseteq 0^k\}$ of all sequences beginning with 0^k . This last density property is key.

Claim. If (G, H) are arithmetically X -generic, then there is a k so that if $X', X'' \supseteq 0^k$ and $X', X'' \leq_T X$, then from $F(G + X') \oplus F(H - G) \oplus F(H + X') \oplus F(-H) \oplus F(H + X'') \oplus F(G + 2H)$, we can uniformly compute:

- (1) The infinite sequence $(F(G + nX'))_{n \in \mathbb{N}}$
- (2) $F(G + X'/2)$ if $X' \supseteq 0^{k+1}$.
- (3) $F(G + X' + X'')$

And the programs for these computations are independent of X' and X'' .

Proof of Claim. Suppose (G, H) are arithmetically X -generic. Note the pairs $(G, H - G)$, (G, H) , and $(G + H, -H)$ are also all arithmetically X -generic (this follows from the computability of addition and inverses which map arithmetic dense open sets to arithmetic dense open sets). Note similarly that if $X', X'' \leq_T X$, then (G, H) is arithmetically X -generic iff $(G + X', H + X'')$ is arithmetically X -generic.

Since 2-adic addition and division are computable, $H \leq_T G \oplus (H - G)$ and $G + H \leq_T G \oplus H$ and $G \leq_T (G + H) \oplus (-H)$ and $G + H \leq_T G \oplus (G + 2H)$. Here the last equation follows since $G + H = (G + (G + 2H))/2$. So since F is an automorphism, we have $F(H) \leq_T F(G) \oplus F(H - G)$ and $F(G + H) \leq_T F(G) \oplus F(H)$ and $F(G) \leq_T F(G + H) \oplus F(-H)$ and $F(G + H) \leq_T F(G) \oplus F(G + 2H)$. So by forcing equals truth, there is a condition $p \in 2^{<\omega} \times 2^{<\omega}$ extended by (G, H) and programs e_1, e_2, e_3, e_4 so that $p \Vdash "F(\dot{H}) \leq_T F(\dot{G}) \oplus F(\dot{H} - \dot{G}) \text{ via } e_1"$, $p \Vdash "F(\dot{G} + \dot{H}) \leq_T F(\dot{G}) \oplus F(\dot{H}) \text{ via } e_2"$, and $p \Vdash "F(\dot{G}) \leq_T F(\dot{G} + \dot{H}) \oplus F(-\dot{H}) \text{ via } e_3"$, and $p \Vdash "F(\dot{G} + \dot{H}) \leq_T F(\dot{G}) \oplus F(\dot{G} + 2\dot{H}) \text{ via } e_4"$.

Let k be longer than the length of p . The idea is that given such $X', X'' \supseteq 0^k$ and $X', X'' \leq_T X$, then $(G + X', H + X'')$ still extends p , and is arithmetically X -generic. Hence we can still use these programs $e_1, \dots, e_4$ for computing the above since forcing equals truth. So for all $n \in \mathbb{N}$, $F(H + X') \leq_T F(G + X') \oplus F(H - G)$ via e_1 , $F(G + H + (n+1)X') \leq_T F(G + nX') \oplus F(H + X')$ via e_2 , and $F(G + (n+1)X') \leq_T F(G + H + (n+1)X') \oplus F(-H)$ via e_3 . Starting from $F(G + X')$ and iteratively applying these reductions, we can uniformly compute the infinite sequence $(F(G + nX'))_{n \in \mathbb{N}}$ as desired, and the reduction does not depend on X' . (It is just composed of e_1, e_2, e_3). This yields (1).

For (2), if $X' \supseteq 0^{k+1}$, so $X'/2 \supseteq 0^k$, then $F(G + H + X'/2) \leq_T F(G + X') \oplus F(G + 2H)$ via e_4 , and then $F(G + X'/2) \leq_T F(G + H + X'/2) \oplus F(-H)$ via e_3 .

For (3), $F(G + H + X' + X'') \leq_T F(G + X') \oplus F(H + X'')$ via e_2 , and then $F(G + X' + X'') \leq_T F(G + H + X' + X'') \oplus F(-H)$ via e_3 , since both pairs $(G + X' + X'', H)$ and $(G + X', H + X'')$ are arithmetically X -generic and extend p . $\square$ Claim

Fix an arithmetically X -generic (G, H) , and k as in the Claim.

Let q be the first k bits of G . There must be some j so q has not forced the value of the j th bit of $F(G)$, that is, $F(G)(j)$. Otherwise F would be constant on all generics in this basic open neighborhood. Let $q_0 \subseteq G$ be of minimal length such that $q_0 \Vdash F(\dot{G})(j) = b$ where b is this bit. Let $q_0^- \subsetneq q_0$ be its immediate predecessor, and let q_1 be an extension of q_0^- incompatible with q_0 so that $q_1 \Vdash F(\dot{G})(j) = 1 - b$, forcing the opposite value at this bit. We will use these two different values of $F(\dot{G})(j)$ that could be forced for a generic G to decode the bits of X one by one, but using the uniform reductions from the Claim above.

Let $S: 2^\omega \rightarrow 2^\omega$ be the shift operation of deleting the first bit so $S(b \frown X) = X$. Let $l = |q_0^-|$ be the length of q_0^- . Since q_0 properly extends q , we have $l \geq k$. Let

$$X_m = 0^l \frown S^m(X)$$

be l many 0s followed by X with its first m bits removed. Fix the computable sequence Y to be the real that has a single 1 at position l . (So Y represents the 2-adic integer 2^l). By applying the claim above, we'll show that from $F(X) \oplus F(G) \oplus F(H)$, we can uniformly compute the infinite sequences $(F(G + X_m))_{m \in \mathbb{N}}$ and $(F(H + X_m))_{m \in \mathbb{N}}$, and also the m th bit of X for each m . Note that from $X \oplus G \oplus H$ we can compute $G + X_0$, $G + Y$, $H - G$, $H + X_0$, $H + Y$, and $-H$. So $F(X) \oplus F(G) \oplus F(H) \geq_T F(G + X_0) \oplus F(G + Y) \oplus F(H - G) \oplus F(H + X_0) \oplus F(H + Y)$.

Suppose we have computed $F(G + X_m)$ and $F(H + X_m)$. Then if the m th bit of X is 0, then X_m extends 0^{l+1} . So $G + nX_m$ is an X -generic real extending q_0 for all n , and so $F(G + nX_m)(j) =$

b for all n . However, if the m th bit of X is 1, then X_m extends $0^l \frown 1$. Hence, nX_m is dense in the basic clopen neighborhood N_{0^l} by our key property of the 2-adics on the first page. So there is some n so $G + nX_m$ extends q_1 , and so $F(G + nX_m)(j) = 1 - b$. So by the claim above from $F(G + X_m) \oplus F(H - G) \oplus F(H + X_m) \oplus F(-H)$ we can uniformly compute the sequence $(F(G + nX_m))_{n \in \omega}$, and wait to see if $F(G + nX_m)(j) = 1 - b$ for any n , and hence if the m th bit of X is 1.

So that this search doesn't go on forever, we also do a second search. Let $X'_m = X_m + Y$. Then if the m th bit of X is 0, then X'_m extends $0^l \frown 1$. So nX'_m is dense in the basic clopen neighborhood N_{0^l} , and hence there is some $n \in \mathbb{N}$ so that $G + nX'_m$ extends q_1 and hence $F(G + nX'_m)(j) = 1 - b$, whereas if the m th bit of X is 1, then X'_m extends $0^l \frown 0$ so $F(G + nX'_m)(j) = b$ for all n . Note that by (3) we can uniformly compute $F(G + X'_m)$ and $F(H + X'_m)$ and then uniformly compute the sequence $(F(G + nX'_m))_{n \in \omega}$ by (1).

Hence by executing both the above searches in parallel, we can find the m th bit of X . To finish, note that once we have determined the m th bit of X , then if this bit is 0, then $X_{m+1} = X_m/2$, and if the m th bit of X is 1, then $X_{m+1} = (X_m - Y)/2$. In either case, we can compute $F(G + X_{m+1})$ and $F(H + X_{m+1})$ using the uniform reductions in (2) and (3) from the claim. $\square$

OpenAI's result is the finishing touch on the proof that the Turing degrees are rigid. Slaman and Woodin's earlier paper on automorphisms and bi-interpretability [5] was a monumental piece of deep mathematical work. OpenAI's result now completes the argument. Some similar ideas as the Lemma above are found in the proof that any Baire homomorphisms between Polish groups must be continuous (see e.g. [2, Theorem 9.10]). Coding tricks somewhat reminiscent of the above are also in works of Groszek-Slaman [1] and later Lutz-Siskind [3].

AI acknowledgement: ChatGPT-Astra was used to proofread a draft of this note, and correct errors in it.

REFERENCES

- [1] Marcia J. Groszek and Theodore A. Slaman, *A basis theorem for perfect sets*, Bulletin of Symbolic Logic **4** (1998), no. 2, 204–209, DOI 10.2307/421023.
- [2] Alexander S. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995.
- [3] Patrick Lutz and Benjamin Siskind, *Part 1 of Martin's conjecture for order-preserving and measure-preserving functions*, Journal of the American Mathematical Society **38** (2025), no. 2, 319–367, DOI 10.1090/jams/1046.
- [4] OpenAI, *Rigidity of the Turing degrees*, September 24, 2026. Preprint, <https://github.com/openai/math/blob/main/preprints/Rigidity-of-the-Turing-degrees-September-24-2026/paper.pdf>.
- [5] Theodore A. Slaman and W. Hugh Woodin, *Definability in Degree Structures*, July 21, 2005. Unpublished manuscript, <https://math.berkeley.edu/~slaman/talks/sw.pdf>.