Voronoi Cells of Varieties

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Voronoi Cells

Definition. Every subset X of $\mathbb{R}^n$ defines a Voronoi decomposition of the ambient Euclidean space. Let X be a real algebraic variety of codimension c and y a smooth point on X. Its *Voronoi cell* consists of all points whose closest point in X is y, i.e.

$$\mathrm{Vor}_X(y) \;:=\; \left\{ u \in \mathbb{R}^n : y \in \underset{x\in X}{\arg\min} \|x-u\|^2 \right\}.$$

The Voronoi cell $\mathrm{Vor}_X(y)$ is a convex semialgebraic set of dimension c, living in the normal space $N_X(y)$ to X at y. Its boundary consists of the points in X that have at least two closest points in X, including y.

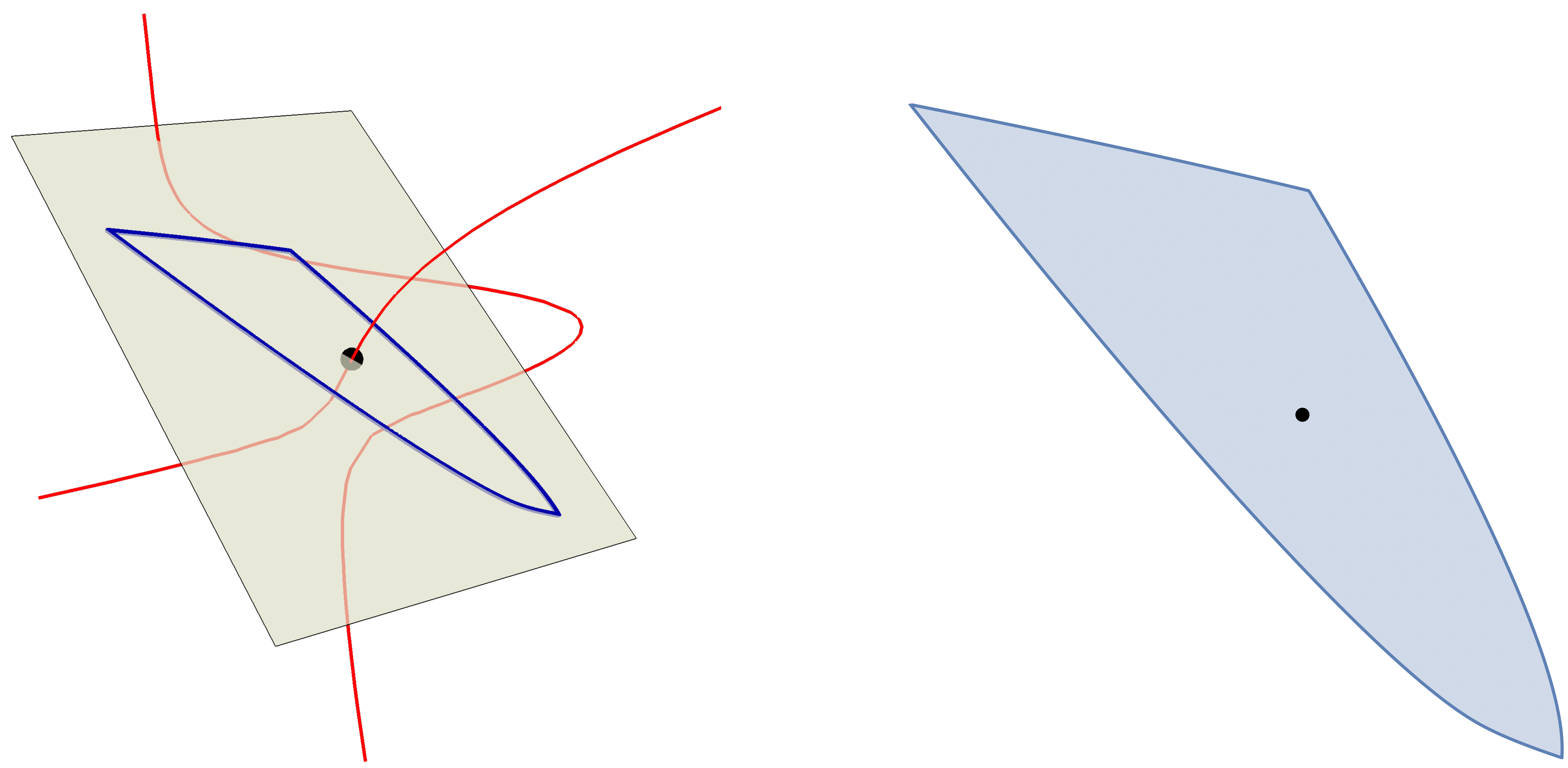

Figure 1: A quartic space curve, shown with the Voronoi cell in one of its normal planes.

Voronoi Degree

Definition. The algebraic boundary of the Voronoi cell $\mathrm{Vor}_X(y)$ is a hypersurface in the normal space to X at y. Its degree $\delta_X(y)$ is called the *Voronoi degree* of X at y.
Theorem. Let $X \subset \mathbb{P}^n$ be a curve of degree d and geometric genus g with at most ordinary multiple points as singularities. The Voronoi degree at a general point $y \in X$ equals

$$\delta_X(y) \;=\; 4d + 2g - 6,$$

provided X is in general position in $\mathbb{P}^n$.
Example. If X is a rational curve of degree d, then $g = 0$ and hence $\delta_X(y) = 4d - 6$. If X is an elliptic curve, so the genus is $g = 1$, then we have $\delta_X(y) = 4d - 4$. The space curve depicted above with $d = 4$ and $g = 1$ has Voronoi degree $\delta_X(y) = 12$.
Theorem. Let $X \subset \mathbb{P}^n$ be a smooth surface of degree d. Then its Voronoi degree equals

$$\delta_X(y) \;=\; 3d + \chi(X) + 4g(X) - 11,$$

provided the surface X is in general position in $\mathbb{P}^n$ and y is a general point on X. Here $\chi(X) := c_2(X)$ is the topological Euler characteristic, which equals the degree of the second Chern class of the tangent bundle, and $g(X)$ is the genus of the curve obtained by intersecting X with a general smooth quadratic hypersurface in $\mathbb{P}^n$.

Low Rank Matrices

Fix the space $\mathbb{R}^{m\times n}$ of real $m \times n$ matrices. Two natural norms are the *Frobenius norm* $\|U\|_F := \sqrt{\sum_{ij} U_{ij}^2}$ and the *spectral norm* $\|U\|_2 := \max_i \sigma_i(U)$ which extracts the largest singular value. Let X denote the variety of real $m \times n$ matrices of rank $\le r$.
Fix a rank r matrix V in X. Let $U \in \mathrm{Vor}_X(V)$ and let $U = \Sigma_1\, D\, \Sigma_2$ be its singular value decomposition. Let $D^{[r]}$ be the matrix that is obtained from D by replacing all singular values except for the r largest ones by zero. By the Eckart-Young Theorem, we have $V = \Sigma_1 \cdot D^{[r]} \cdot \Sigma_2$. The Eckart-Young Theorem works for both norms, so both give the same Voronoi cell $\mathrm{Vor}_X(V)$.
Theorem. Let V be an $m \times n$-matrix of rank r. The Voronoi cell $\mathrm{Vor}_X(V)$ is congruent up to scaling to the unit ball in the spectral norm on the space of $(m-r) \times (n-r)$-matrices.