

Fixed point-free pseudo-Anosovs and the cinquefoil

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Knot detection has been one of the many classical questions in low-dimensional topology.

- genus, unknotting number, fundamental group, Alexander polynomial, Jones polynomial, ...
- more recent invariants: knot Floer homology, Khovanov homology, ...

Heegaard Floer homology (defined by Ozsváth-Szabó) and knot Floer homology (defined by Ozsváth-Szabó, Rasmussen) are powerful invariants of three-manifolds and knots that give many applications in low-dimensional topology.

Theorem (Ozsváth-Szabó, '04)

Knot Floer homology detects the unknot.

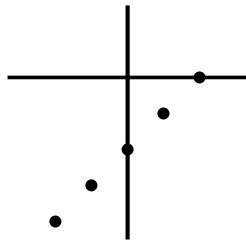
Theorem (Ghiggini, '06)

Knot Floer homology detects genus one fibered knots, i.e. trefoil and figure-eight.

Theorem (Ni, '06)

Knot Floer homology detects fiberedness of knots.

Natural question: does knot Floer homology detect the torus knot $T(2, 5)$?



Credit: *Encyclopedia Britannica*

Second to top grading

Question

The top Alexander grading of \widehat{HFK} tells us when the knot is fibered. What does the second to top grading tell us?

Answer

Second to top grading tells us information about the monodromy of the fibered knot.

Question

What do we know about monodromy of knots with $T(2, 5)$ knot Floer homology?

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What do we know about monodromy of knots with $T(2, 5)$ knot Floer homology?

By Thurston, a knot K can be either a torus knot, satellite knot, or hyperbolic. If K has the same Alexander polynomial as $T(2, 5)$, it cannot be a satellite knot [Baldwin-Hu-Sivek]. So this knot is either $T(2, 5)$ or a hyperbolic genus two fibered knot.

Indeed, there is an infinite family of hyperbolic genus two fibered knot with the same Alexander polynomial as $T(2, 5)$ [Misev].

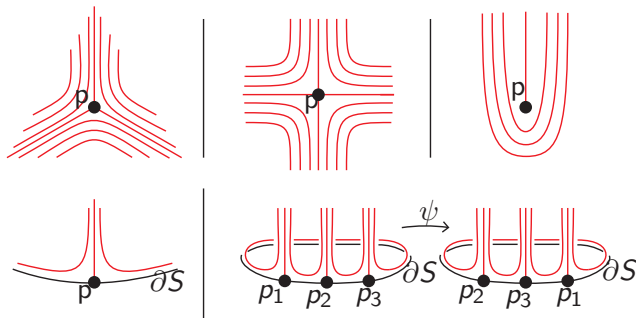
A hyperbolic fibered knot has pA monodromy

Definition

A *pseudo-Anosov* homeomorphism is a map $\psi : S \rightarrow S$ preserving a pair of transverse singular measured foliations (\mathcal{F}^u, μ^u) and (\mathcal{F}^s, μ^s) such that

$$\psi \cdot (\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda \mu^u), \text{ and } \psi \cdot (\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda^{-1} \mu^s)$$

for some fixed real number $\lambda > 1$, called the *dilatation* of ψ .



Fixed point counts of surface diffeomorphisms

Theorem (Baldwin-Hu-Sivek, '21)

Let $K \subset S^3$ be a hyperbolic knot such that $\widehat{HFK} \cong \widehat{HFK}(T(2,5))$ as bigraded vector spaces. Then the pseudo-Anosov representative of the monodromy of K has no fixed points in the interior.

Theorem (Ni, '21 and '22)

Let Y be a closed, oriented 3-manifold, and $K \subset Y$ be a hyperbolic fibered knot with monodromy ϕ . If $\text{rk}(\widehat{HFK}(Y, K, g-1)) = r$, then ϕ is freely isotopic to a pseudo-Anosov map with at most $r-1$ fixed points in the interior.

Theorem (Ghiggini-Spano, '22)

As vector spaces, $\widehat{HFK}(Y, K, g-1) \cong HF^\sharp(\phi)$ where ϕ is the monodromy associated to the fibered knot $K \subset Y$.

Slogan: count of fixed points gives rank of Floer homology!

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Theorem (Farber-Reinoso-W, '22)

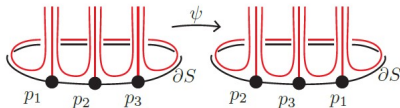
If $\widehat{HFK}(K; \mathbb{Q}) \cong \widehat{HFK}(T(2, 5); \mathbb{Q})$ as bi-graded vector spaces, then $K = T(2, 5)$.

In particular, $T(2, 5)$ is the only genus-two L-space knot in S^3 .

A hyperbolic fibered knot K is *fixed point-free* (FPF) if the pA representative of the monodromy has no fixed points in the interior.

Theorem (Farber-Reinoso-W, '22)

Let K be a hyperbolic, genus-two, fibered knot in S^3 . If the fractional Dehn twist coefficient $c(K) \neq 0$, then K is not FPF.



- Some words on c : measures the “twist” of the Thurston representative of the monodromy along each boundary component.
- Some words on $c = 0$: Ni showed that if $c(K) = 0$, then $\widehat{HFK}(K, g - 1) \geq 2$.

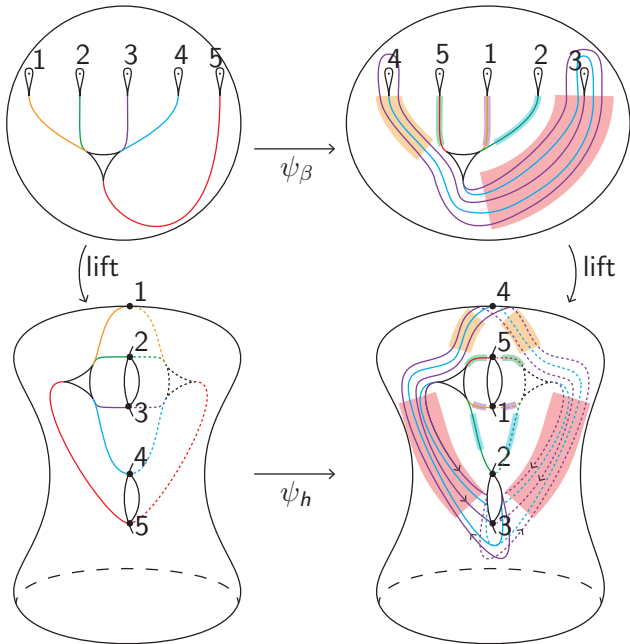
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Strategy to prove the FPF result

Step 1: Use Baldwin-Hu-Sivek to reduce to study pseudo-Anosov braids with certain singularity types and braid closures.

- If $c(K) \neq 0$, can cap off the boundary of the surface and extend the foliations to get a pseudo-Anosov map on the closed surface \widehat{S} .
- By the Euler-Poincaré formula $\sum_i^k (2 - p_i) = 2\chi(\widehat{S})$, among other arguments in Baldwin-Hu-Sivek (Prop 3.8) assuming FPF, obtain that the singularity types of interest are $(6; \emptyset; \emptyset)$ and $(4; \emptyset, 3^2)$.
- Furthermore, we know these FPF pAs on the genus two surface with one boundary component quotient to pA braids with unknot closures and singularity types $(3; 1^5; \emptyset)$ and $(2; 1^5; 3)$.



Strategy to prove the FPF result

Step 2: Show that there are no pAs (of the above singularity types) coming from the monodromy of a hyperbolic genus 2 fibered knot in S^3 .

- $(6; \emptyset; \emptyset)$
- $(4; \emptyset, 3^2)$.

$(6; \emptyset; \emptyset)$: the case with orientable foliation

This case can be ignored if one only cares about \widehat{HFK} detection.

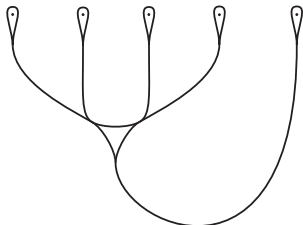
- Alexander polynomial of a fibered knot is the characteristic polynomial of action of monodromy on the first homology.
- Use Lefschetz fixed point theorem to constrain the trace and therefore coefficients of the possible Alexander polynomials.

$$\Delta_K(t) = t^4 - t^3 + ?t^2 - t + 1$$

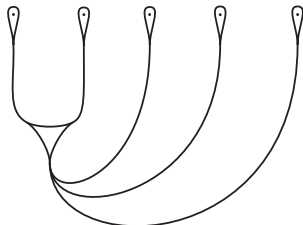
- $? = -1$
- This polynomial gives the minimal dilatation achieved by any pA on a genus two surface [Cho-Ham].
- This restricts our braid of interest to be conjugate to some full twists of $\sigma_1\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2$ [Lanneau-Thiffeault, e.g. given by Ham-Song]
- By FDTC consideration, we cannot have an unknot closure, so the double branched cover over the braid closure cannot be S^3 [Waldhausen].

$(4; \emptyset, 3^2)$: the “harder” case

- Obtain a set of “canonical tracks” that carry *all* possible conjugacy classes of pAs in this stratum.



The Peacock



The Snail

- Classify all possible fixed point free pAs on these “canonical” tracks.
- Show that the only fixed point free train track map possible does not correspond to a braid of unknot closure.

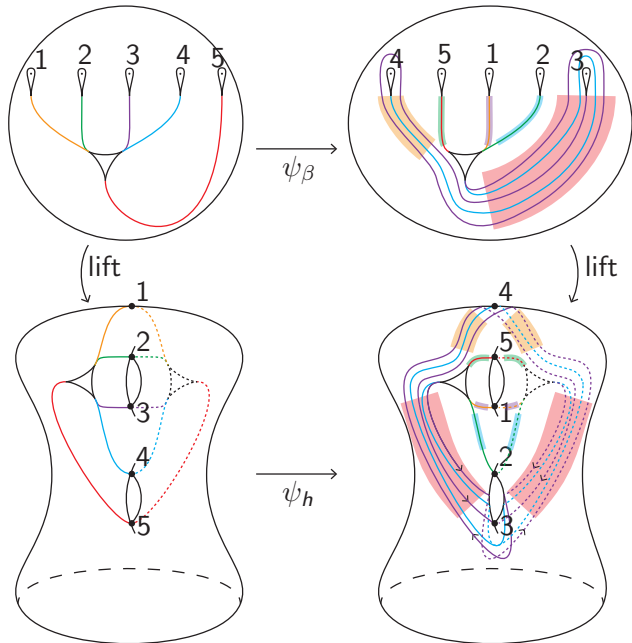
Train track and train track maps

Roughly speaking, a *train track* τ is a smoothed 1-dimensional graph that “encodes” information about particular pAs it can “carry”.

We will focus on *standardly embedded train tracks*. In particular, cusps only occur at vertices of infinitesimal polygons.

Definition

A *train track map* is a map $f : \tau \rightarrow \tau$ such that for any train path $g : I \rightarrow \tau$ the composition $f \circ g : I \rightarrow \tau$ is a train path.

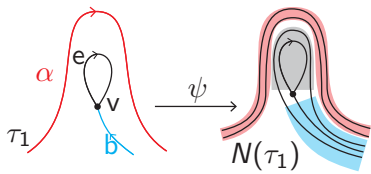
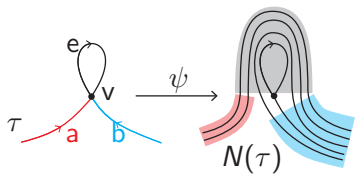
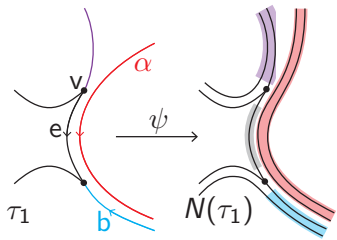
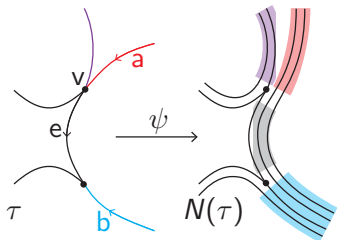


- Obtain a set of “canonical tracks” that carry *all* possible conjugacy classes of pAs in this stratum. This uses *tight splitting*.

Definition

Let $\tau \hookrightarrow S_{0,n}^1$ be a standardly embedded train track. Let v be a vertex of τ . Fix a train track map $f : \tau \rightarrow \tau$. We say we can *tight split* v if every appearance of an adjacent real edge a (resp. \bar{a}) in an image train path $f(\alpha)$ is in fact an appearance of the word aeb (resp. $\bar{b}\bar{e}\bar{a}$) for some real edge b .

Tight splitting



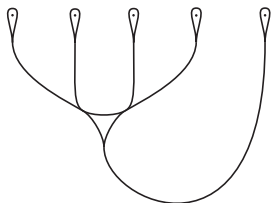
Case reduction by tight splitting

Definition

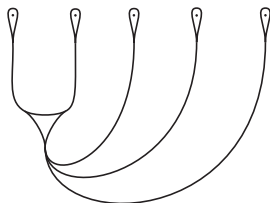
Let $\tau \hookrightarrow S_{0,n}^1$ be a standardly embedded train track. We say a loop switch $v \in \tau$ is a *joint* if it is adjacent to ≥ 2 real edges.

Theorem

Let ψ be a pseudo-Anosov on the punctured disk with at least one k -pronged singularity away from the boundary with $k \geq 2$. Then ψ is carried by a train track τ with no joints.



The Peacock



The Snail

Why no joint is good

Theorem (Los, Cotton-Clay)

If a pA is fixed point-free in the interior, then the trace of the incidence matrix is zero.

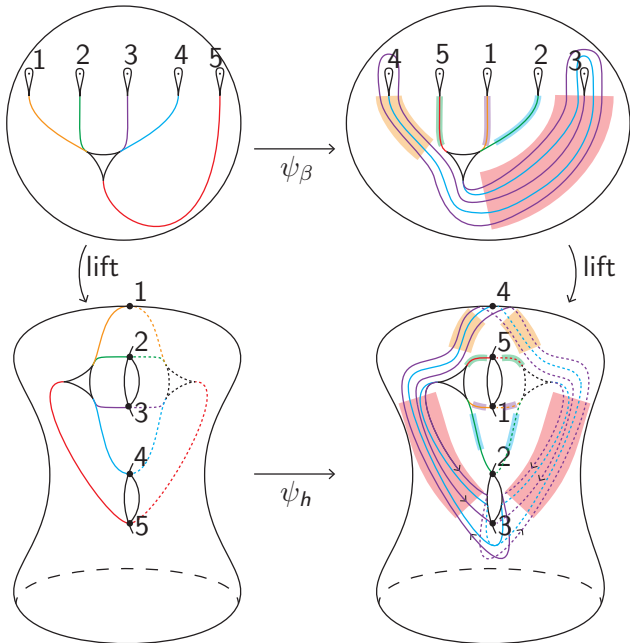
Remark

In our case of “no joint”, fixed point free upstairs in particular implies that the trace of the incidence matrix downstairs is also zero.

Lemma

Any pA carried by the Snail tightly splits to one carried by the Peacock.

So the remaining task is to examine all possible trace zero pA s downstairs on the Peacock track.

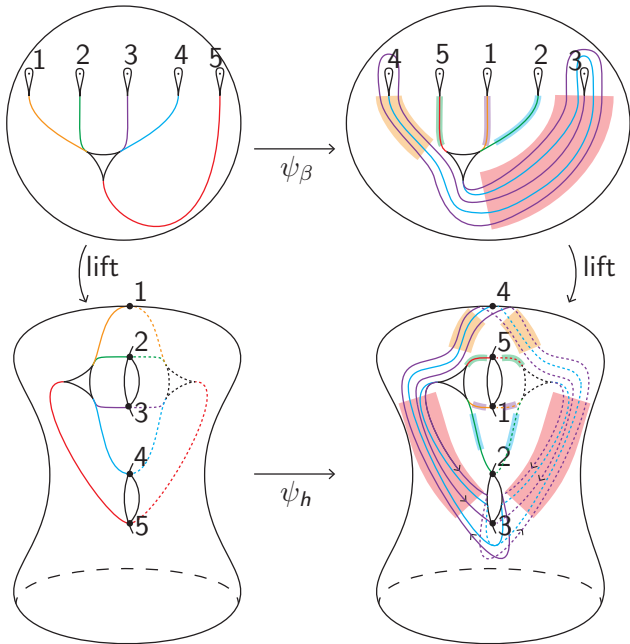


The Peacock track

(Optimistic) goal: Eliminate all potential trace zero pAs downstairs on the Peacock track in order to eliminate counterexamples of genus two pAs that are fixed point free. **This is impossible!**

Theorem

There is a unique family of trace zero pAs on the Peacock track, but none of the examples in this family can come from a hyperbolic fibered knot in S^3 .



The special FPF family

Example

Set $\beta_n = \sigma_1^{n+2} \sigma_2 \sigma_3 \sigma_4 \sigma_1 \sigma_2 \sigma_3 \sigma_4^2$ for $n \geq 0$. Then, β_n^{-1} is pseudo-Anosov, and conjugate to a braid carried by the Peacock track τ , which induces the train track map $f_n : \tau \rightarrow \tau$ defined by:

$$f_n(o) = p^\circ$$

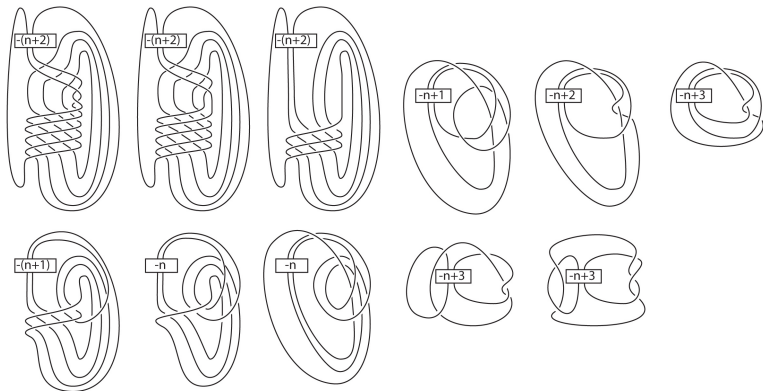
$$f_n(g) = b^\circ$$

$$f_n(r) = g^\circ$$

$$f_n(p) = \begin{cases} (r^- o^-)^{\frac{n}{2}+1} r^\circ \\ (r^- o^-)^{\frac{n+1}{2}} r^- o^\circ \end{cases} \quad f_n(b) = \begin{cases} (r^- o^-)^{\frac{n}{2}} r^- o^\circ \\ (r^- o^-)^{\frac{n+1}{2}} r^\circ \end{cases}$$

Proposition

The braid closure $\widehat{\Delta^{2k} \beta_n^{\pm 1}}$ is not an unknot for any $k, n \in \mathbb{Z}$.



Thanks for listening!