Finite Difference Methods

Definition 1 Difference approximation
Approximate derivatives of function $y(x)$ using values of the function at specified points.

Example

$$y'(x_j) \approx \frac{y(x_{j+1}) - y(x_j)}{x_{j+1} - x_j}$$

Definition 2 F.D.M
Use difference approximations to BVP at all mesh points simultaneously with boundary conditions to approximate BVP directly.

Special Case. Linear 2nd order BVP

$$L y(x) \equiv -y'' + p(x)y' + q(x)y + rx = 0$$

$$a_0 y(a) - a_1 y'(a) = \alpha, \quad b_0 y(b) + b_1 y'(b) = \beta,$$

where $a_0, a_1, b_0, b_1 \geq 0, \ a_0 + b_0 \neq 0$.

Consider at first $L y(x) = 0, y(a) = \alpha, y(b) = \beta$.
Assume $p, q, r$ are continuous on $x \in [a, b]$, s.t., $q(x) \geq Q > 0, |p(x)| \leq P$. Then by Corollary to Thm 2, the BVP has a unique solution.

Devide interval $[a, b]$ into $N$ steps of constant size $h$, — called "uniform" mesh.

$$h = \frac{b - a}{N}, \quad x_j = a + jh, \quad x_{j+1} = x_j + h, \quad \text{for } j = 0, 1, \ldots, N - 1.$$

Then at each point $x_j$, we approximate $L y = 0$ by finite difference approximations and at $x_0, x_N$ we apply boundary conditions

$$y(x_0) = \alpha, \quad y(x_N) = \beta.$$
Assume \( y \), the true solution, is sufficiently differentiable. Use centered difference schemes to approximate \( y'(x_j) \),

\[
y'(x_j) \approx \frac{y(x_{j+1}) - y(x_{j-1})}{2h}
\]

Use Taylor’s expansion to analyze the order of accuracy of this approximation:

\[
\frac{y(x_{j+1}) - y(x_{j-1})}{2h} = \frac{1}{2h}[y(x_j) + hy'(x_j) + \frac{h^2}{2} y''(x_j) + \frac{h^3}{3!} y'''(\eta_1)] - \frac{1}{2h}[y(x_j) - hy'(x_j) + \frac{h^2}{2} y''(x_j) - \frac{h^3}{3!} y'''(\eta_2)]
\]

where \( \eta_1 \in (x_j, x_{j+1}) \), \( \eta_2 \in (x_{j-1}, x_j) \).

This is a \( O(h^2) \) approximation.

To approximate \( y''(x_j) \), use a centered scheme:

\[
\frac{y(x_{j+1}) - 2y(x_j) + y(x_{j-1})}{h^2} = \frac{1}{h^2}[y(x_j) + hy'(x_j) + \frac{h^2}{2} y''(x_j) + \frac{h^3}{3!} y'''(x_j) + \frac{h^4}{4!} y^{(4)}(\eta_1)] - \frac{1}{h^2}[y(x_j) - hy'(x_j) + \frac{h^2}{2} y''(x_j) - \frac{h^3}{3!} y'''(x_j) + \frac{h^4}{4!} y^{(4)}(\eta_2)]
\]

where \( \eta \in (x_{j-1}, x_{j+1}) \).

This is a \( O(h^2) \) approximation.
Summarize:

\[
\begin{align*}
\frac{y(x_{j+1}) - y(x_j)}{2h} &= y'(x_j) + \frac{h^2}{6}y''(\eta) \\
\frac{y(x_{j+1}) - 2y(x_j) + y(x_{j-1})}{h^2} &= y''(x_j) + \frac{h^2}{12}y^{(4)}(\eta)
\end{align*}
\]

are \(O(h^2)\) approximations.

Approximate BVP \(Ly(x) \equiv -y'' + py' + qy + r = 0\) by FDM scheme

\[
L_h y_j = \frac{-y_{j+1} + 2y_j - y_{j-1}}{h^2} + p(x_j)\frac{y_{j+1} - y_{j-1}}{2h} + q(x_j)y_j + r(x_j) = 0,
\]

\(j = 0, 1, \ldots, N-1\), where \(y_j \approx y(x_j)\), with \(y_0 = \alpha, y_N = \beta\).

We’ll try to solve all those \(N - 1\) linear equations simultaneously.

Truncation Error:

\[
\tau_j = L_h(y(x_j))
\]

\[
= \frac{1}{h^2}[-y(x_{j+1}) + 2y(x_j) - y(x_{j-1})] + p(x_j)\frac{y(x_{j+1}) - y(x_{j-1})}{2h} + q(x_j)y_j + r(x_j)
\]

\[
= -y''(x_j) - \frac{h^2}{12}y^{(4)}(\eta) + p(x_j)(y'(x_j) + \frac{h^2}{6}y'''(\tilde{\eta})) + q(x_j)y_j + r(x_j)
\]

\[
= \frac{h^2}{12}y^{(4)}(\eta) + \frac{h^2}{6}p(x_j)y'''(\tilde{\eta}).
\]

\(\Rightarrow\)

\[
\|\tau_j\| \leq \frac{h^2}{12}[M_4 + 2PM_3], \text{ where } |p(x)| \leq P,
\]

\[
M_4 = \max_{\eta \in [a,b]} |y^{(4)}(\eta)|, \quad M_3 = \max_{\eta \in [a,b]} |y^{(3)}(\eta)|.
\]

Find difference approximation to approximate BVP to \(O(h^2)\).
To solve the equations, write in matrix form, to get linear system to solve.

Unknowns: \[
\begin{pmatrix}
y_1 \\
\vdots \\
y_{N-1}
\end{pmatrix}
= \vec{y}.
\]

Denote \( p_j = p(x_j), q_j = q(x_j), r_j = r(x_j). \)

Rearrange difference equations, multiply by \( h^2 \) to get

\[-(1 + \frac{h}{2}p_j)y_{j-1} + (2 + h^2q_j)y_j - (1 - \frac{h}{2}p_j)y_{j+1} = -h^2r_j,\]

for \( j = 1, 2, \ldots, N - 1. \)

Note that for \( j = 1, N - 1, \) the equations can be written as

\[-(1 + \frac{h}{2}p_1)y_1 - (1 - \frac{h}{2}p_1)y_2 = -h^2r_1 + (1 + \frac{h}{2}p_1)y_0,\]

\[-(1 + \frac{h}{2}p_{N-1})y_{N-2} + (2 + h^2q_{N-1})y_{N-1} = -h^2r_{N-1} + (1 - \frac{h}{2}p_{N-1})y_N.\]

Write in matrix form

\[A\vec{y} = \vec{d}\]

where \( \vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_{N-1} \end{pmatrix}, \vec{d} = \begin{pmatrix} d_1 \\ \vdots \\ d_{N-1} \end{pmatrix}.\)

And \( A \) is tridiagonal,

\[A = \begin{bmatrix}
 b_1 & c_1 & & \\
 a_2 & b_2 & c_2 & \\
 & \ddots & \ddots & \ddots \\
 & & a_{N-2} & b_{N-2} & c_{N-2} \\
 & & & a_{N-1} & b_{N-1}
\end{bmatrix}\]

where \( b_j = 2 + h^2q_j, a_j = -(1 + \frac{h}{2}p_j), c_j = -(1 - \frac{h}{2}p_j), d_j = -h^2r_j \) for \( j = 2, \ldots, N - 2. \)

And \( d_1 = (1 + \frac{h}{2}p_1)\alpha - h^2r_1, d_{N-1} = (1 - \frac{h}{2}p_{N-1})\beta - h^2r_{N-1}. \)

Convention: \( a_1 = 0, c_{N-1} = 0. \)

Questions:

(1) \( A\vec{y} = \vec{d} \) Existence/Uniqueness of the solution.
   How do we find it efficiently?

(2) Do solutions \( y_j \) of the matrix equations converge to the solution of BVP, \( y(x_j) \)
   as \( h \to 0? \) I.e., \( |y_j - y(x_j)| \overset{?}{\to} 0 \) as \( h \to 0, j \to \infty \) for \( x_j = a + jh \) fixed.
To answer Q(1):

**Theorem 6** Let \( p, q, r \) be continuous on \( x \in [a, b] \), s.t., \(|p| \leq P, q \geq Q > 0\). Then if \( h \leq \frac{2}{P} \), then the matrix problem has a unique solution.

**Proof**

A is strictly diagonal-dominant, since

\[ |b_j| > |a_j| + |c_j| \quad j = 1, \ldots, N - 1, \]

under the conditions of the Thm, hence non-singular.

Since by Gershgorian-circle Thm, the eigenvalues \( \lambda_i \) of \( A \) are in disks

\[ |z - b_j| \leq |a_j| + |c_j|. \]

Disks all lie in right half plane, since \( b_j > 0 \), and |radius| < distance from origin to centers.

\( \Rightarrow \lambda_i \neq 0, \forall i, \)

\( \Rightarrow A \) nonsingular.

To show \(|b_j| > |a_j| + |c_j|\), use assumptions:

\[ |a_j| = \left| 1 + \frac{h}{2p_j} \right|, \quad |c_j| = \left| 1 - \frac{h}{2p_j} \right|. \]

But

\[ \frac{h}{2}|p_j| \leq \frac{h}{2}P \leq 1, \]

\( \Rightarrow \)

\[ |a_j| + |c_j| = 1 - \frac{h}{2}p_j + 1 + \frac{h}{2}p_j = 2. \]

And

\[ |b_j| \geq 2 + h^2Q > 2, \]

\( \Rightarrow A \) is strictly diagonal-dominant.

\( \Rightarrow \) Matrix equations have a unique solution.

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