AN EXAMPLE OF CHEBOTAREV'S DENSITY THEOREM FOR INFINITE EXTENSIONS

LEONARD TOMCZAK

Let L/K be a finite Galois extension of number fields and $X \subseteq \mathrm{Gal}(L/K)$ closed under conjugation. Then the Chebotarev density theorem says that the set of places of K that are unramified in L with Frobenius $\mathrm{Frob}_{v,L/K}$ in X has density $\frac{\#X}{\#\mathrm{Gal}(L/K)}$. Here density can refer to either natural or Dirichlet density. This finite version easily implies a corresponding result for infinite Galois extensions [Ser89, 1 §1 Corollary 2]:

Theorem 1. *Let L/K be a Galois extension, possibly infinite, unramified outside a finite set. Let μ denote the Haar measure on $G = \mathrm{Gal}(L/K)$, normalized so that $\mu(G) = 1$. Let $X \subseteq G$ be closed under conjugation and assume $\mu(\partial X) = 0$. Then the set of places v of K that are unramified in L and with Frobenius $\mathrm{Frob}_{v,L/K}$ in X has density $\mu(X)$.*

Here ∂X denotes the boundary $\partial X = \overline{X} \setminus X^\circ$.

Remark. This is phrased more naturally as an equidistribution statement of Frobenius elements in G/conj, which concerns integrals of continuous functions. The assumption on $\mu(\partial X) = 0$ allows us by [Ser89, A.1 Proposition 1] to approximate $\mathbb{1}_X$ by continuous (class) functions and obtain the above result.

The goal of this note is to give an explicit example of this theorem for infinite extensions. We will compute the density of the set T of primes $p \neq 2, 5$ such that the decimal expansion of p^{-1} has odd period length. This example, in more generality, was studied in [Odo81] using an effective version of the Chebotarev density theorem. Our proof is simpler as we only compute the density and do not prove any error terms. We will show:

Theorem 2. *T has density $\frac{1}{3}$.*

Throughout, p denotes a prime not equal to 2 or 5. Note that the period length of the decimal expansion of p^{-1} is the multiplicative order of 10 modulo p. The first goal is to recast this condition in a form to which we can apply Theorem 1. To this end, define for any integer $j \geq 1$,

$$L_j = \mathbb{Q}(\zeta_{2^j}, \sqrt[2^j]{10}), \quad L'_j = \mathbb{Q}(\zeta_{2^{j+1}}, \sqrt[2^j]{10}).$$

Note that the only primes ramifying in L_j, L'_j are 2 and 5. The purpose of these fields becomes clear in the following lemma.

Lemma 3 ([Odo81, Proposition 2.2])**.** *Let $j \geq 1$. The following are equivalent:*

(1) 2^j is the largest power of 2 dividing $p-1$ and $x^{2^j} \equiv 10 \bmod p$ has a solution.
(2) p splits completely in L_j, but not in L'_j.

Proof. "(1) $\Rightarrow$ (2)" It is clear under the assumptions $\mathbb{Q}_p$ contains $\zeta_{2^j}, \sqrt[2^j]{10}$ so p splits completely in L_j. Since $p \neq 1 \bmod 2^{j+1}$, p does not split completely in L'_j. "(2) $\Rightarrow$ (1)" is similar by reversing the argument. □

Let $L = \bigcup_{j=1}^{\infty} L_j$. Note that also $L'_j \subseteq L$ for all j. let $\pi'_j : \mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{Gal}(L'_j/\mathbb{Q})$ denote the restriction map. Define

$$X_j = \{\sigma \in \mathrm{Gal}(L'_j/\mathbb{Q}) \colon \sigma \neq \mathrm{id}_{L'_j}, \sigma|_{L_j} = \mathrm{id}_{L_j}\},$$

$$X = \bigcup_{j\geq 1} \pi_j'^{-1}(X_j).$$

We note for later that the union defining X is a disjoint union. Also note that X is conjugation invariant since each X_j is. If 2^j is the largest power of 2 dividing $p-1$, then, since $(\mathbb{Z}/p\mathbb{Z})^\times$ is cyclic, 10 has odd order mod p if and only if $x^{2^j} \equiv 10 \bmod p$ has a solution. Thus, by the lemma

$$T = \{p \text{ prime} : p \neq 2, 5, \mathrm{Frob}_{p,L/\mathbb{Q}} \in X\}.$$

Thus, to finish the proof of Theorem 2, we only need to compute $\mu(X)$ and show $\mu(\partial X) = 0$. We achieve both of these at once by showing $\mu(\overline{X}) = \mu(X^\circ) = \frac{1}{3}$. For that we need the degrees of the fields:

Lemma 4. *The degree of L_j (resp. L_j') over $\mathbb{Q}$ is 2^{2j-1} (resp. 2^{2j}).*

Proof. It is easily seen, e.g. by considering ramification of the prime 5, that $\mathbb{Q}(\zeta_{2^{j+1}})$ and $\mathbb{Q}(\sqrt[2^j]{10})$ are linearly disjoint over $\mathbb{Q}$. The lemma then follows. □

This shows $\#X_j = 1$ and we obtain

$$\mu(X) = \sum_{j=1}^{\infty} \mu(\pi_j'^{-1}(X_j)) = \sum_{j=1}^{\infty} \frac{\#X_j}{\#\,\mathrm{Gal}(L_j'/\mathbb{Q})} = \sum_{j=1}^{\infty} 2^{-2j} = \frac{1}{3}.$$

Note that X is already open, so this shows the first half $\mu(X^\circ) = \frac{1}{3}$. For the closure, we introduce the sets $Y_j = \pi_j'(X)$. It is then easily seen $\overline{X} = \bigcap_{j\geq 1} \pi_j'^{-1}(Y_j)$ and the sets $\pi_j'^{-1}(Y_j)$ are decreasing, so

$$\mu(\overline{X}) = \lim_{j\to\infty} \mu(\pi_j'^{-1}(Y_j)) = \lim_{j\to\infty} \frac{\#Y_j}{\#\,\mathrm{Gal}(L_j'/\mathbb{Q})}$$

For $j \geq i$, let $\pi_{j,i}' : \mathrm{Gal}(L_j'/\mathbb{Q}) \to \mathrm{Gal}(L_i'/\mathbb{Q})$ denote the restriction map. Then $Y_j = \{\mathrm{id}_{L_j'}\} \cup \bigcup_{1\leq i\leq j} \pi_{j,i}'^{-1}(X_i)$ and the union is disjoint again, hence

$$\#Y_j = 1 + \sum_{i=1}^{j} \#X_i[L_j' : L_i'] = 1 + \sum_{i=1}^{j} 2^{2j-2i} = 1 + \frac{4^j - 1}{3}.$$

Combining this with $\#\,\mathrm{Gal}(L_j'/\mathbb{Q}) = 2^{2j}$ gives $\mu(\overline{X}) = \lim_{j\to\infty} \mu(\pi_j'^{-1}(Y_j)) = \frac{1}{3}$, as desired, finishing the proof of Theorem 2.

References

[Odo81] R. W. K. Odoni. "A Conjecture of Krishnamurthy on Decimal Periods and Some Allied Problems". In: *J. Number Theory* 13.3 (1981).

[Ser89] J.-P. Serre. *Abelian l-adic representations and elliptic curves.* Second. Advanced Book Classics. With the collaboration of Willem Kuyk and John Labute. Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1989.

Department of Mathematics, Evans Hall, University of California, Berkeley, CA 94720, USA
Email address: leonard.tomczak@berkeley.edu