Math 209: Von Neumann Algebras

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Contents

Part 1. General Theory of Von Neumann Algebras

1. Topologies on $\mathcal{B}(\mathcal{H})$

Let $\mathcal{H}$ denote a Hilbert space over $\mathbb{C}$. For us the inner product will be sesquilinear in the second argument. The space of bounded operators on $\mathcal{H}$ is denoted $\mathcal{B}(\mathcal{H})$.

We will consider the following topologies on $\mathcal{B}(\mathcal{H})$:

- norm topology.
- Weak operator topology (wo): Induced by the seminorms $p_{\xi,\eta}(T) = |\langle T\xi, \eta\rangle|$ for $\xi, \eta \in \mathcal{H}$. So a basis of neighborhoods of 0 is given by $\{T \mid |\langle T\xi_k, \eta_k\rangle| < \varepsilon, 1 \leq k \leq n\}$ with $\varepsilon > 0, \xi_k, \eta_k \in \mathcal{H}$.
- Strong operator topology (so): Induced by the seminorms $T \mapsto \|T\xi\|$ for $\xi \in \mathcal{H}$. So a basis of neighborhoods of 0 is given by $\{T \mid \|T\xi_k\| < \varepsilon, 1 \leq k \leq n\}$ with $\varepsilon > 0, \xi_k \in \mathcal{H}$.

Proposition 1.1. *Let $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ be linear. Then TFAE:*

(i) φ is wo-continuous.

(ii) φ is so-continuous.

(iii) $\varphi(T) = \sum_{j=1}^n \langle T\xi_j, \eta_j\rangle$ *for some* $\xi_j, \eta_j \in \mathcal{H}$.

Proof. The strong operator topology is stronger than the weak operator topology, so "$(i) \Rightarrow (ii)$". Also "$(iii) \Rightarrow (i)$" is clear because $\left|\sum_{j=1}^n \langle T\xi_j, \eta_j\rangle\right| \leq p_{\xi_1,\eta_1}(T) + \cdots + p_{\xi_n,\eta_n}(T)$. We are left to prove "$(ii) \Rightarrow (iii)$". Assume that (ii) holds. Then $|\varphi(T)| \leq \sum_{j=1}^n \|T\xi_j\|$ for all T and some fixed $\xi_j \in \mathcal{H}$. Consider $d : \mathcal{B}(\mathcal{H}) \to \mathcal{H}^n$ given by $d(T) = (T\xi_1, \ldots, T\xi_n)$. The map $d(\mathcal{B}(\mathcal{H})) \ni (T\xi_1, \ldots, T\xi_n) \mapsto \varphi(T)$ is well-defined and continuous. By Hahn-Banach it extends to a continuous linear functional $\mathcal{H}^n \to \mathbb{C}$, so by the Riesz representation theorem there are $\eta_1, \ldots, \eta_n$ such that $\varphi(T) = \langle (T\xi_1, \ldots, T\xi_n), (\eta_1, \ldots, \eta_n)\rangle = \sum_{j=1}^n \langle T\xi_j, \eta_j\rangle$. □

Corollary 1.2. *A convex subset of* $\mathcal{B}(\mathcal{H})$ *is so-closed if and only it is wo-closed.*

Proof. By Hahn-Banach separation, the closed convex subsets of a locally convex space are intersections of closed half spaces which are in turn defined by continuous functionals. But by the Proposition the continuous functionals for the two topologies are the same. □

2. Von Neumann's Theorem

Let $T \in \mathcal{B}(\mathcal{H})$. A closed subspace $\mathcal{X} \subseteq \mathcal{H}$ is invariant for T if $T\mathcal{X} \subseteq \mathcal{X}$. This is the case iff $TP_\mathcal{X} = P_\mathcal{X} T P_\mathcal{X}$ where $P_\mathcal{X}$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{X}$. Equivalently, $(I - P_\mathcal{X})TP_\mathcal{X} = 0$.

Now let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a subalgebra and let $\mathcal{X} \subseteq \mathcal{H}$ be invariant for all $T \in \mathcal{A}$. Then $\mathcal{H}$ is a module over $\mathcal{A}$ and $\mathcal{X}$ is a submodule. Again, the invariance is equivalent to $(I - P_\mathcal{X})\mathcal{A}P_\mathcal{X} = 0$.

What about T^*? If $\mathcal{X}$ is invariant for T, then $P_\mathcal{X} T^*(I - P_\mathcal{X}) = 0$. Suppose $\mathcal{A} = \mathcal{A}^*$. Then $\mathcal{X}$ is invariant for $\mathcal{A}$ if $(I - P_\mathcal{X})TP_\mathcal{X} = 0$ for all $T \in \mathcal{A}$ iff $P_\mathcal{X} T^*(I - P_\mathcal{X}) = 0$ for all $T \in \mathcal{A}$. By $\mathcal{A} = \mathcal{A}^*$ this holds iff $P_\mathcal{X} T(I - P_\mathcal{X}) = 0$ for all $T \in \mathcal{A}$. In this case we have $P_\mathcal{X} T - TP_\mathcal{X} = P_\mathcal{X} T(I - P_\mathcal{X}) - (I - P_\mathcal{X})TP_\mathcal{X} = 0$, i.e. $[T, P_\mathcal{X}] = 0$. The converse also holds: Suppose $[T, P_\mathcal{X}] = 0$. Then $(I - P_\mathcal{X})TP_\mathcal{X} = TP_\mathcal{X} - P_\mathcal{X} TP_\mathcal{X} = TP_\mathcal{X} - TP_\mathcal{X} P_\mathcal{X} = 0$.

Hence we have shown:

Proposition 2.1. *A closed subspace* $\mathcal{X} \subseteq \mathcal{H}$ *is invariant under both* T *and* T^* *if and only if* $[T, P_\mathcal{X}] = 0$. *If* $\mathcal{A}$ *is a* $*$*-closed subalgebra of* $\mathcal{B}(\mathcal{H})$*, then* $\mathcal{X}$ *is invariant under* $\mathcal{A}$ *if and only if* $[T, P_\mathcal{X}] = 0$ *for all* $T \in \mathcal{A}$.

Definition. *Let* $\Xi \subseteq \mathcal{B}(\mathcal{H})$ *be a subset. Define its* commutator *by:*

$$\Xi' = \{T \in \mathcal{B}(\mathcal{H}) \mid [T, \Xi] = 0\}.$$

Ξ' is an algebra and it is wo-closed. To see the latter note that for fixed $X \in \Xi$, we have $[T, X] = 0$ iff $\langle [T, X]\xi, \eta\rangle = 0$ for all $\xi, \eta \in \mathcal{H}$ iff $\langle TX\xi, \eta\rangle - \langle T\xi, X^*\eta\rangle = 0$ for all $\xi, \eta \in \mathcal{H}$. So we can write Ξ' as the intersection of zero sets of certain wo-continuous functionals. If $\Xi = \Xi^*$, then $(\Xi')^* = (\Xi^*)'$. So in this case Ξ' is a $*$-algebra.

Theorem 2.2 (von Neumann, Bicommutant Theorem, Double commutant theorem, von Neumann density theorem, ...). *Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a unital $*$-subalgebra. TFAE:*

(i) *$\mathcal{A}$ is wo closed.*

(ii) *$\mathcal{A}$ is so closed.*

(iii) *$\mathcal{A} = \mathcal{A}''$.*

Proof. $(iii) \Rightarrow (i) \iff (ii)$ are clear by the previous discussion. So we must show $(ii) \Rightarrow (iii)$. Assume $\mathcal{A}$ is so closed. Trivially, $\mathcal{A} \subseteq \mathcal{A}''$. We must show that if $T \in \mathcal{A}''$, then $T \in \mathcal{A}$. We show that T is in the so closure of $\mathcal{A}$. For this we have to show that for any $\xi_1, \ldots, \xi_n$, $\overline{d_n(\mathcal{A})\xi} \ni T\xi$ where $\xi = (\xi_1, \ldots, \xi_n)$ and $d_n : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}^n)$ is given by $d_n(S) = S \oplus \cdots \oplus S$. $\mathcal{X} := \overline{d_n(\mathcal{A})\xi}$ is invariant under the $*$-algebra $d_n(\mathcal{A})$, so $[P_\mathcal{X}, d_n(\mathcal{A})] = 0$. Let $A \in \mathcal{A}$. Write $P_\mathcal{X} = (P_{ij})_{i,j=1,\ldots,n}$ and $d_n(A)$ is a diagonal matrix with diagonal entries A. Then we must have that A commutes with every P_{ij}. Hence $P_{ij} \in \mathcal{A}'$, so $[T, P_{ij}] = 0$, and then also $[d_n(T), P_\mathcal{X}] = 0$. This implies that $\mathcal{X}$ is invariant under $d_n(T)$. Since $\xi \in \mathcal{X}$, then $T\xi \in \mathcal{X} = \overline{d_n(\mathcal{A})\xi}$. □

Definition. *$\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is a* von Neumann algebra *if it is a unital $*$-subalgebra satisfying $\mathcal{M} = \mathcal{M}''$.*

Let $\mathcal{M}$ be a von Neumann algebra. Then $\mathcal{M}'$ is a von Neumann algebra as $(\mathcal{M}')'' = (\mathcal{M}'')' = \mathcal{M}'$. The intersection of von Neumann algebras is again a von Neumann algebra.

$\mathcal{H}$ becomes a $(\mathcal{M}, \mathcal{M}'^{\mathrm{op}})$-bimodule.

Definition. *The* center *of a von Neumann algebra $\mathcal{M}$ is $Z(\mathcal{M}) := \mathcal{M} \cap \mathcal{M}'$. It is a commutative von Neumann algebra.*

We have $Z(\mathcal{M})' = (\mathcal{M} \cup \mathcal{M}')''$.

Definition. *A von Neumann algebra $\mathcal{M}$ is a* factor *if $Z(\mathcal{M}) = \mathbb{C}I$.*

Definition. *Let $\Xi \subseteq \mathcal{B}(\mathcal{H})$. The vN algebra generated by Ξ is $(\Xi \cup \Xi^*)''$.*

Example. Let $\mathcal{X}, \mathcal{Y}$ be Hilbert spaces. On $\mathcal{X} \otimes_{\mathrm{alg}} \mathcal{Y}$ put the inner product $\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle = \langle x_1, x_2\rangle\langle y_1, y_2\rangle$ and extend to a sesquilinear scalar product. Then $\mathcal{X} \otimes \mathcal{Y}$ is the completion of $\mathcal{X} \otimes_{\mathrm{alg}} \mathcal{Y}$. We have $(\mathcal{B}(\mathcal{X}) \otimes I_\mathcal{Y})' = I_\mathcal{X} \otimes \mathcal{B}(\mathcal{Y})$, so this gives a pair of von Neumann algebras algebras. Let $(e_i)_i, (f_j)_j$ be ONB of $\mathcal{X}, \mathcal{Y}$ resp. Then $\mathcal{X} \otimes \mathcal{Y} = \bigoplus_{j \in J} \mathcal{X} \otimes f_j$.

Example. Let G be a discrete group and take $\mathcal{H} = \ell^2(G)$ with ONB $(e_g)_{g \in G}$. Consider the left regular representation $g \mapsto \lambda(g)$, $ge_h = e_{gh}$. Similarly we have the right regular representation $g \mapsto \rho(g)$, $\rho(g)e_h = e_{hg^{-1}}$. We have $[\lambda(g_1), \rho(g_2)] = 0$. Put $L(G) = (\lambda(G))''$ and $R(G) = (\rho(G))''$. One can show that they are each other's commutant (later).

Let $\mathcal{A}$ be a von Neumann algebra. By the homework
$$\mathcal{A}' = \{V \in \mathcal{A}' \mid V \text{ unitary}\}''.$$
Then
$$\mathcal{A} = \{U \text{ unitary}, U \in \mathcal{A}'\}'.$$
Note that $[T, U] = 0$ iff $U^*TU = T$, hence
$$\mathcal{A} = \{T \in \mathcal{B}(\mathcal{H}) \mid UTU^* = T, \text{ for all } U \in \mathcal{A}', U \text{ unitary}\}.$$

Corollary 2.3. *Let $\mathcal{A}$ be a von Neumann algebra.*

(a) Let $N \in \mathcal{A}$ be normal, f a Borel function on $\sigma(N)$. Then $f(N) \in \mathcal{A}$.

(b) Let $T \in \mathcal{A}$, $T = V(T^*T)^{1/2}$ its polar decomposition. Then $V, (T^*T)^{1/2} \in \mathcal{A}$.

(c) Let $T \in \mathcal{A}$. Then $P_{\ker T}, P_{\overline{T\mathcal{H}}} \in \mathcal{A}$.

(d) Let $(P_i)_{i\in I} \subseteq \mathcal{A}$ be orthogonal projections. Let $\bigvee_{i\in I} P_i, \bigwedge_{i\in I} P_i$ be the orthogonal projections onto the closed linear span resp. intersection of the P_i. Then $\bigvee_{i\in I} P_i, \bigwedge_{i\in I} P_i \in \mathcal{A}$.

There are a couple other topologies on $\mathcal{B}(\mathcal{H})$ we are interested in. Consider $d_\infty(T) = T \oplus T \oplus T \oplus \cdots$ on $\mathcal{H} \oplus \mathcal{H} \oplus \cdots$. This can also be viewed as $T \mapsto T \otimes I_{\ell^2}$ on $\mathcal{H} \otimes \ell^2(\mathbb{N})$.

(i) The *ultraweak operator topology (uwo)*. The uwo topology is induced by the seminorms
$$p_{\xi_1,\xi_2,\ldots,\eta_1,\eta_2,\ldots}(T) = \left|\sum_j \langle T\xi_j, \eta_j\rangle\right|,$$
where $\xi_i, \eta_i \in \mathcal{H}$ with $\sum_i \|\xi_i\|^2, \sum_i \|\eta_i\|^2 < \infty$. . It is the subspace topology under $\mathcal{B}(\mathcal{H}) \hookrightarrow \mathcal{B}(\mathcal{H} \otimes \ell^2(\mathbb{N})), T \mapsto T \otimes I$ where $\mathcal{B}(\mathcal{H} \otimes \ell^2(\mathbb{N}))$ carries the wo topology. Then $T_i \to T$ in the uwo topology iff $d_\infty(T_i) \xrightarrow{\text{wo}} d_\infty(T)$.

(ii) The *ultrastrong operator topology (uso)*. The uwo topology is induced by the seminorms
$$p_{\xi_1,\xi_2,\ldots}(T) = \Big(\sum_j \|T\xi_j\|^2\Big)^{1/2},$$
where $\xi_i \in \mathcal{H}$ with $\sum_i \|\xi_i\|^2 < \infty$. Then $T_i \to T$ in the uso topology iff $d_\infty(T_i) \xrightarrow{\text{so}} d_\infty(T)$.

We have the following analogue of Proposition 1.1:

Proposition 2.4. *Let $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ be linear. TFAE:*

(1) φ is uwo-continuous.

(2) φ is uso-continuous.

(3) $\varphi(T) = \sum_j \langle T\xi_j, \eta_j\rangle = \langle d_\infty(T)(\xi_1, \ldots), (\eta_1, \ldots)\rangle$ for some $(\xi_i)_i, (\eta_j)_j \in \mathcal{H} \otimes \ell^2(\mathbb{N})$.

This can be deduced from Proposition 1.1 by applying it to $\mathcal{H} \otimes \ell^2(\mathbb{N})$.

Corollary 2.5. *Convex sets are uwo closed iff they are uso closed.*

Proposition 2.6. *On the unit ball $\mathcal{B}(\mathcal{H})_1$, the uwo and wo topologies coincide. Similarly the uso and so topologies coincide.*

Proposition 2.7. *Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a unital $*$-algebra. TFAE*

(i) $\mathcal{A}$ *is uwo closed.*

(ii) $\mathcal{A}$ *is uso closed.*

(iii) $\mathcal{A} = \mathcal{A}''$.

This can be deduced from the previous von Neumann bicommutant theorem by applying this to $d_\infty(\mathcal{A}) \subseteq \mathcal{B}(\mathcal{H} \otimes \ell^2(\mathbb{N}))$.

3. Commutative von Neumann algebras

Proposition 3.1. *Let $\mathcal{A}$ be a commutative von Neumann algebra with a cyclic vector $\xi \in \mathcal{H}$, i.e. $\overline{\mathcal{A}\xi} = \mathcal{H}$. Then $\mathcal{A} = \mathcal{A}'$, so $\mathcal{A}$ is a* maximal abelian *subalgebra.*

For the proof we need:

Lemma 3.2. *Let $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$ be a $*$-subalgebra. Then $\mathcal{B}$ is commutative if and only if $\|b\eta\| = \|b^*\eta\|$ for all $\eta \in \mathcal{H}$ and $b \in \mathcal{B}$.*

This is basically the operator algebra version of the familiar characterization of normal operators.

Proof. "$\Rightarrow$" $\|b\eta\|^2 = \langle b^*b\eta, \eta\rangle = \langle bb^*\eta, \eta\rangle = \|b^*\eta\|^2$. "$\Leftarrow$" We have $\langle b^*b\eta, \eta\rangle = \langle bb^*\eta, \eta\rangle$ for all b, η. By polarization, we get $b^*b = bb^*$. First let $x = x^*, y = y^* \in \mathcal{B}$ and take $b = x + iy$. Then $0 = [b, b^*] = [x+iy, x-iy] = 2i[y, x]$, so x, y commute. As every operator in $\mathcal{B}$ can be written as $x+iy$ with $x, y \in \mathcal{B}$ self-adjoint, $\mathcal{B}$ is commutative. $\square$

Proof of Proposition 3.1. We show using the lemma that $\mathcal{A}'$ is commutative. Let $y \in \mathcal{A}'$. Then $\|y\xi - x_n\xi\| \to 0$ as $n \to \infty$ for suitable $x_n \in \mathcal{A}$. The lemma implies that the map $\{x_n\xi\} \to \{x_n^*\xi\}, x_n\xi \mapsto x_n^*\xi$ is isometric. Since $x_n\xi$ is Cauchy, so is $x_n^*\xi$, so there is a $k \in \mathcal{H}$ such that $\|x_n^*\xi - k\| \to 0$ as $n \to \infty$. We prove that $k = y^*\xi$. For this it suffices to show that $\langle k, x\xi\rangle = \langle y^*\xi, x\xi\rangle$ for all $x \in \mathcal{A}$ as $\mathcal{A}\xi$ is dense in $\mathcal{H}$. We have

$$\begin{aligned}
\langle k, x\xi\rangle &= \lim_{n\to\infty} \langle x_n^*\xi, x\xi\rangle = \lim_{n\to\infty} \langle \xi, x_n x\xi\rangle \\
&= \lim_{n\to\infty} \langle \xi, x x_n\xi\rangle = \langle \xi, xy\xi\rangle \\
&\overset{y\in\mathcal{A}'}{=} \langle \xi, yx\xi\rangle \\
&= \langle y^*\xi, x\xi\rangle.
\end{aligned}$$

This proves $y^*\xi = k$. Next we prove $\|yx\xi\| = \|y^*x\xi\|$ for all $x \in \mathcal{A}$. Equivalently $\|xy\xi\| = \|xy^*\xi\|$. For this it suffices to prove $\|xx_n\xi\| = \|xx_n^*\xi\|$ for all n. This is the same as $\|x_nx\xi\| = \|x_n^*x\xi\|$ which is true by the lemma.

The lemma then implies that $\mathcal{A}'$ is commutative, so $\mathcal{A}' \subseteq \mathcal{A}'' = \mathcal{A}$. □

4. The Trace class

Let $\mathcal{H}$ be a Hilbert space and $(e_\alpha)_{\alpha\in I}$ an orthonormal basis. Let $T \in \mathcal{B}_+(\mathcal{H})$ be a positive operator. Then we define the *trace* of T to be

$$\operatorname{Tr} T = \sum_\alpha \langle Te_\alpha, e_\alpha\rangle \in [0,\infty].$$

Note that if $X \in \mathcal{B}(\mathcal{H})$, then

$$\begin{aligned}\operatorname{Tr}(X^*X) &= \sum_\alpha \langle X^*Xe_\alpha, e_\alpha\rangle = \sum_\alpha \|Xe_\alpha\|^2 = \sum_{\alpha,\beta} |\langle Xe_\alpha, e_\beta\rangle|^2 \\ &= \sum_{\alpha,\beta} |\langle X^*e_\beta, e_\alpha\rangle|^2 = \operatorname{Tr}(XX^*)\end{aligned}$$

So for $T \in \mathcal{B}_+(\mathcal{H})$ and U unitary, we have $\operatorname{Tr}(T) = \operatorname{Tr}(UTU^*)$ by taking $X = UT^{1/2}$.

Definition. *The space of* trace class *operators is*

$$\mathcal{C}_1(\mathcal{H}) = \{X \in \mathcal{B}(\mathcal{H}) \mid \operatorname{Tr}((X^*X)^{1/2}) < \infty\}.$$

It carries the norm $|T|_1 = \operatorname{Tr}((T^*T)^{1/2})$.

Proposition 4.1.

(i) $\mathcal{C}_1(\mathcal{H})$ *is a Banach space w.r.t. the norm* $|\cdot|_1$.

(ii) $\mathcal{C}_1(\mathcal{H})$ *is a two sided ideal in* $\mathcal{B}(\mathcal{H})$.

(iii) For $A, B \in \mathcal{B}(\mathcal{H})$ *and* $X \in \mathcal{C}_1(\mathcal{H})$, *we have* $|AXB|_1 \le \|A\|\,|X|_1\,\|B\|$.

(iv) The trace extends by linearity from $\mathcal{C}_1(\mathcal{H}) \cap \mathcal{B}_+(\mathcal{H})$ *to a linear map* $\operatorname{Tr} : \mathcal{C}_1(\mathcal{H}) \to \mathbb{C}$. *It satisfies* $|\operatorname{Tr}(X)| \le |X|_1$ *and* $\operatorname{Tr} X = \sum_\alpha \langle Te_\alpha, e_\alpha\rangle$ *and* $\sum_\alpha |\langle Te_\alpha, e_\alpha\rangle| \le |T|_1$.

(v) $\mathcal{C}_1(\mathcal{H}) \subseteq \mathcal{K}(\mathcal{H})$. *Here* $\mathcal{K}(\mathcal{H})$ *is the space of compact operators.*

4.1. Rank one operators

Let $\xi, \eta \in \mathcal{H}$. Let $E_{\xi,\eta} \in \mathcal{B}(\mathcal{H})$ denote the rank one operator $\langle\cdot,\eta\rangle\xi$. It satisfies $E_{\xi,\eta}^* = E_{\eta,\xi}$ and

$$E_{\xi,\eta}^*E_{\xi,\eta} = \langle\cdot,\eta\rangle \|\xi\|^2 \eta = \left\langle \cdot, \frac{\eta}{\|\eta\|}\right\rangle \frac{\eta}{\|\eta\|} \|\xi\|^2 \|\eta\|^2 = P_{\mathbb{C}\eta} \|\xi\|^2 \|\eta\|^2 .$$

So

$$\operatorname{Tr}((E_{\xi,\eta}^*E_{\xi,\eta})^{1/2}) = \|\xi\|\,\|\eta\|\,,$$

and so $|E_{\xi,\eta}|_1 = \|\xi\|\,\|\eta\|$.

Let $X \in \mathcal{C}_1(\mathcal{H}) \subseteq \mathcal{K}(\mathcal{H})$. Write $X = V(X^*X)^{1/2}$ (polar decomposition). Then

$$X = V(X^*X)^{1/2} = \sum_j s_j E_{Vh_j,h_j}$$

where h_j is an orthonormal system of eigenvectors of X^*X, and s_j the corresponding singular values, i.e. square roots of the eigenvalues of X^*X. Also $\sum_j s_j = \mathrm{Tr}((X^*X)^{1/2}) = |X|_1$. Letting $\xi_j = \sqrt{s_j}Vh_j, \eta_j = \sqrt{s_j}h_j$, we have $X = \sum_j E_{\xi_j,\eta_j}$ and $|X|_1 = \sum |E_{\xi_j,\eta_j}|_1$.

4.2. **Dual of $\mathcal{C}_1(\mathcal{H})$**

Theorem 4.2. *The map*

$$\mathcal{B}(\mathcal{H}) \ni T \longmapsto (X \longmapsto \mathrm{Tr}(XT)) \in \mathcal{C}_1(\mathcal{H})^*$$

is a well-defined isometric isomorphism.

Proof. We have $|\operatorname{Tr} XT| \leq |XT|_1 \leq |X|_1 \, \|T\|$, so $X \mapsto \operatorname{Tr} XT$ is a functional on $\mathcal{C}_1(\mathcal{H})$ of norm $\leq \|T\|$. Now take $\varphi \in \mathcal{C}_1(\mathcal{H})^*$. Then $(\xi,\eta) \mapsto \varphi(E_{\xi,\eta})$ is a sesquilinear form on $\mathcal{H}$ with $|\varphi(E_{\xi,\eta})| \leq \|\varphi\| \, |E_{\xi,\eta}|_1 = \|\varphi\| \, \|\xi\| \, \|\eta\|$. So there is $T \in \mathcal{B}(\mathcal{H})$ with $\|T\| \leq \|\varphi\|$ such that $\varphi(E_{\xi,\eta}) = \langle T\xi, \eta\rangle$. We have $TE_{\xi,\eta} = \langle \cdot, \eta\rangle T\xi = E_{T\xi,\eta}$ and $\operatorname{Tr} E_{\xi,\eta} = \langle \xi, \eta\rangle$, so $\varphi(E_{\xi,\eta}) = \mathrm{Tr}(TE_{\xi,\eta})$. We saw that we can write $X = \sum_j E_{\xi_j,\eta_j}$ with $\sum_j |E_{\xi_j,\eta_j}|_1 = |X|_1$. We then get $\mathrm{Tr}(TX) = \sum_j \varphi(E_{\xi_j,\eta_j}) = \varphi(X)$. So $\varphi = \mathrm{Tr}(\cdot T)$ and $\|T\| \leq \varphi$. We have already established the other inequality. □

A uwo continuous functional on $\mathcal{B}(\mathcal{H})$ is the same as a weak-$*$ continuous functional on the dual of $\mathcal{C}_1(\mathcal{H})$, i.e. a functional of the form $T \mapsto \mathrm{Tr}(TX)$ for fixed $X \in \mathcal{C}_1(\mathcal{H})$.

Consequence for von Neumann algebras: Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Then $\mathcal{M}$ is uwo closed, so

$$\mathcal{M} \cong (\mathcal{C}_1(\mathcal{H})/\mathcal{M}_\perp)^* = (\mathcal{C}_1(\mathcal{H})/\{X \in \mathcal{C}_1(\mathcal{H}) \mid \mathrm{Tr}(aX) = 0 \ \forall a \in \mathcal{M}\})^*.$$

Therefore $\mathcal{M}$ is the dual of a Banach space and the uwo continuous functionals on $\mathcal{M}$ are the weak-$*$ continuous functionals in this duality. $\mathcal{M}$ is the dual of the uwo continuous functionals on $\mathcal{M}$.

Theorem 4.3 (Sakai)**.** *Let $\mathcal{M}$ be a unital C^*-algebra. $\mathcal{M}$ is C^*-isomorphic to a von Neumann algebra if and only if $\mathcal{M}$ is the dual of a Banach space. The predual is isometrically unique.*

Corollary 4.4. *Let $\mathcal{M}_1, \mathcal{M}_2$ be von Neumann algebras and $\alpha : \mathcal{M}_1 \to \mathcal{M}_2$ an isomorphism of $*$-algebras. Then α is uwo continuous.*

Proof. First note that α is necessarily isometric since in a C^*-algebra the norm can be characterized purely algebraically. Then use uniqueness of the predual. □

5. The Kaplansky Density Theorem

Theorem 5.1. *Let $\mathcal{A} \subseteq \mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ where $\mathcal{A}$ is a (not necessarily unital) $*$-algebra, $\mathcal{M}$ a von Neumann algebra. Suppose $\mathcal{A}$ is wo dense in $\mathcal{M}$. Then*

- *$\mathcal{A}_1$ is wo dense in $\mathcal{M}_1$.*

- $\mathcal{A}_{h,1}$ *(the set of hermitian elements of norm* ≤ 1*) is wo dense in* $\mathcal{M}_{h,1}$.
- $(\mathcal{A}^+)_1$ *is wo dense in* $(\mathcal{M}^+)_1$.

Proof. We have $\overline{A}^{\mathrm{wo}} = \mathcal{M}$. We claim that $\overline{\mathcal{A}_h}^{\mathrm{wo}} = \mathcal{M}_h$. Indeed, if $a_i \xrightarrow{\mathrm{wo}} m = m^* \in \mathcal{M}_h$, then also $a_i^* \xrightarrow{\mathrm{wo}} m^* = m \in \mathcal{M}_h$, so $\frac{1}{2}(a_i + a_i^*) \xrightarrow{\mathrm{wo}} m$ and $\frac{1}{2}(a_i + a_i^*) \in \mathcal{A}_h$. Then also $\overline{\mathcal{A}_h}^{so} = \mathcal{M}_h$.

- $\overline{\mathcal{A}_{h1}}^{so} = \mathcal{M}_{h1}$: Consider the homeomorphism $[-1,1] \ni t \mapsto \frac{2t}{1+t^2} \in [-1,1]$. If $t = \tan\alpha$ with $\alpha \in [-\frac{\pi}{4}, \frac{\pi}{4}]$, then $\frac{2t}{1+t^2} = \sin 2\alpha$. So given $m \in \mathcal{M}_{h,1}$ there is $y \in \mathcal{M}_{h,1}$ such that $m = 2y(1+y^2)^{-1}$. Then there are $a_i \in \mathcal{A}_h$ such that $a_i \xrightarrow{so} y$. Then (noting $\frac{2t}{1+t^2} = (t+i)^{-1} + (t-i)^{-1}$)

$$\begin{aligned} 2a_i(1+a_i^2)^{-1} - 2y(1+y^2)^{-1} &= (a_i + i1)^{-1} + (a_i - i1)^{-1} - (y + i1)^{-1} - (y - i1)^{-1} \\ &= (a_i + i1)^{-1}(y - y_i)(y + i1)^{-1} + (a_i - i1)^{-1}(y - a_i)(y - i1)^{-1}. \end{aligned}$$

  Now $(y - a_i)(y + i1) \xrightarrow{so} 0$ and $(a_i + i1)^{-1}$ is bounded with norm ≤ 1, similarly for the other term, so the whole thing converges to 0 in the so topology. Next we claim that $2a_i(a_i^2+1)^{-1} \in \overline{\mathcal{A}_{h,1}}^{\|\cdot\|}$. For this we can approximate $\frac{2t}{1+t^2}$ as the uniform limit of polynomials on $[-\|a_i\|, \|a_i\|]$ with values in $[-1,1]$. To accommodate for the non-unital case, note that we may choose the polynomials to be without constant term because $\frac{2\cdot 0}{1+0^2} = 0$.

- Next $\overline{\mathcal{A}_1}^{wo} = \mathcal{M}_1$. Let $m \in \mathcal{M}_1$. For this apply the previous result to $M_2(\mathcal{A}) \subseteq M_2(\mathcal{M})$ acting on $\mathcal{H} \oplus \mathcal{H}$. Then

$$\begin{pmatrix} a_{11,i} & a_{12,i} \\ a_{21,i} & a_{22,i} \end{pmatrix} \xrightarrow{wo} \begin{pmatrix} 0 & m \\ m^* & 0 \end{pmatrix}$$

  with $a_{12,i} = a_{21,i}^*$. Then $\mathcal{A}_1 \ni a_{12,i} \xrightarrow{wo} m$.

- Finally $\overline{(\mathcal{A}^+)_1}^{wo} = (\mathcal{M}^+)_1$. Let $m \in \mathcal{M}_1^+$. Then $m^{1/2} \in \mathcal{M}_{h,1}$, so there are $a_i \in \mathcal{A}_{h,1}$ such that $a_i \xrightarrow{so} m^{1/2}$. Then $a_i^2 - m = a_i(a_i - m^{1/2}) - (m^{1/2} - a_i)m^{1/2}$. As a_i is bounded, this goes to 0 in the so topology.

$\square$

We introduce yet another topology on $\mathcal{B}(\mathcal{H})$: The $*$*-strong operator topology*. We say that $x_i \xrightarrow{*so} x$ if $x_i \xrightarrow{so} x$ and $x_i^* \xrightarrow{so} x^*$. The proof of the theorem shows that also $\overline{\mathcal{A}_1}^{*so} = \mathcal{M}_1$.

Corollary 5.2. *Let* $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ *be a unital* $*$*-subalgebra. Then* $\mathcal{A}$ *is von Neumann if and only if* $\mathcal{A}_1$ *is wo (or uwo, so, uso) closed.*

Proof. Apply the theorem to $\mathcal{M} = \overline{\mathcal{A}}^{wo}$. $\square$

Corollary 5.3. *Let* $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ *be a unital* $*$*-subalgebra. Then* $\mathcal{A}$ *is von Neumann if and only if* $\mathcal{A}_1$ *is wo (or uwo) compact.*

Let $\mathcal{A}$ be a C^*-algebra. Then $\mathcal{A}^{**}$ can again be made into a C^*-algebra, and it has a predual, so by Sakai's theorem, $\mathcal{A}^{**}$ can be realized as a von Neumann algebra.

6. Projections

Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $P \in \mathcal{B}(\mathcal{H})$ an orthogonal projection. Recall from earlier that $P \in \mathcal{M}$ iff $P\mathcal{H}$ is invariant under $\mathcal{M}'$.

Let $\Xi \subseteq \mathcal{H}$. Then $\mathcal{M}'\Xi$ is an $\mathcal{M}'$-invariant subspace, so $P_{\overline{\mathcal{M}'\Xi}} \in \mathcal{M}$ and similarly $P_{\overline{\mathcal{M}\Xi}} \in \mathcal{M}'$. In particular, if $\xi \in \mathcal{H}$, then $E_\xi := P_{\overline{\mathcal{M}\xi}} \in \mathcal{M}'$, $E'_\xi := P_{\overline{\mathcal{M}'\xi}} \in \mathcal{M}$.

Definition. *Let $\Xi \subseteq \mathcal{H}$.*

- Ξ *is* cyclic *for $\mathcal{M}$ if $\overline{\mathcal{M}\Xi} = \mathcal{H}$.*
- Ξ *is* separating *for $\mathcal{M}$ if $(\mathcal{M} \ni T \mapsto (\Xi \ni h \mapsto Th \in \mathcal{H}))$ is injective, i.e. when $T \in \mathcal{M}$ with $T\Xi = 0$, then $T = 0$.*

Proposition 6.1. *Let $\Xi \subseteq \mathcal{H}$, $\mathcal{M}$ a von Neumann algebra. Then Ξ is cyclic for $\mathcal{M}$ iff Ξ is separating for $\mathcal{M}'$.*

Proof. Assume $\overline{\mathcal{M}\Xi} = \mathcal{H}$ and $T\Xi = 0$ with $T \in \mathcal{M}'$. Then $0 = \mathcal{M}T\Xi = T\mathcal{M}\Xi$, so $T = 0$. Conversely, suppose Ξ is separating for $\mathcal{M}'$. Then $\overline{\mathcal{M}\Xi}$ is $\mathcal{M}$-invariant, so $P_{\overline{\mathcal{M}\Xi}} \in \mathcal{M}'$, so $T := 1 - P_{\overline{\mathcal{M}\Xi}} \in \mathcal{M}'$. Also $T\Xi = 0$, hence $T = 0$. So $1 = P_{\overline{\mathcal{M}\Xi}}$, so $\overline{\mathcal{M}\Xi} = \mathcal{H}$. □

Definition. *Let $\mathcal{M}$ be a von Neumann algebra and $P \in \mathcal{M}$ a projection. Then $P\mathcal{M}P|_{P\mathcal{H}} \subseteq \mathcal{B}(P\mathcal{H})$ is the* reduced algebra *by P, denoted $\mathcal{M}_P$. Similarly $\mathcal{M}'|_{P\mathcal{H}} \subseteq \mathcal{B}(P\mathcal{H})$ is the* induced algebra *by P, denoted $(\mathcal{M}')_P$.*

Proposition 6.2. *$\mathcal{M}_P, (\mathcal{M}')_P$ are von Neumann algebra and they are each other's commutant.*

Proof. We will prove that $(\mathcal{M}')_{P,1}$ is uwo compact. Consider the map $\Phi : \mathcal{M}' \to (\mathcal{M}')_P$, given by $\Phi(T) = T|_{T\mathcal{H}}$. It is a uwo continuous $*$-homomorphism. So $\Phi(\mathcal{M}_1)$ is uwo compact and clearly $\Phi(\mathcal{M}_1) \subseteq (\mathcal{M}')_{P,1}$, so it suffices to prove that we have equality in this inclusion. Let $T \in (\mathcal{M}')_{P,1}$. Then $T = S|_{P\mathcal{H}}$ for some $S \in \mathcal{M}'$. We have to show that we can find such S with $\|S\| \leq 1$. Write $S = V(S^*S)^{1/2}$. We have $(S^*S)^{1/2}|_{P\mathcal{H}} = \Phi((S^*S)^{1/2}) = (T^*T)^{1/2}$. Consider the function $f : \mathbb{R} \ni t \mapsto (t \vee 0) \wedge 1$. Then $f((T^*T)^{1/2}) = (T^*T)^{1/2}$ as $\|T\| \leq 1$. Also $\Phi(f((S^*S)^{1/2})) = f(\Phi((S^*S)^{1/2})) = (T^*T)^{1/2}$. So $\Phi(Vf((S^*S)^{1/2})) = T$ and $\left\|Vf(S^*S)^{1/2}\right\| \leq 1$. This shows that $(\mathcal{M}')_P$ is a von Neumann algebra.

For the second part note that clearly $[P\mathcal{M}P, P\mathcal{M}'P] = 0$, so $((\mathcal{M}')_P)' \supseteq \mathcal{M}_P$ and we must show the other inclusion. Take $X \in ((\mathcal{M}')_P)'$ and extend it to $\mathcal{H}$ by $\widetilde{X} = X \oplus 0_{(I-P)\mathcal{H}}$. Any $Y \in \mathcal{M}'$ has $P\mathcal{H}$ as an invariant subspace, so $y = Y_1 \oplus Y_1$ where $Y_1 = P\mathcal{H}, Y_2 = (I-P)\mathcal{H}$. Then $[Y, \widetilde{X}] = 0$, so $\widetilde{X} \in \mathcal{M}'' = \mathcal{M}$ and then $X = P\widetilde{X}P \in \mathcal{M}_P$. □

7. The non-unital case

Proposition 7.1. *Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a wo closed $*$-subalgebra, not necessarily with unit. Then*

(1) If $T \in \mathcal{A}$, then $P_{(\ker T)^\perp} \in \mathcal{A}$

(2) Let P, Q be self-adjoint projections in $\mathcal{A}$. Then $P \vee Q \in \mathcal{A}$.

(3) There is a largest projection $E \in \mathcal{A}$ and $ET = TE = T$ for all $T \in \mathcal{A}$.

(4) If $\mathcal{X} = \overline{\mathcal{A}\mathcal{H}}$, then $E = P_{\mathcal{X}}$ and $\mathcal{A}|_{\mathcal{X}}$ is a von Neumann algebra.

Proof. Homework. □

8. Ideals

Proposition 8.1. *Let $\mathcal{M}$ be a von Neumann algebra and $J \subseteq \mathcal{M}$ wo closed.*

(1) J is a left ideal of $\mathcal{M}$ if and only if $J = \mathcal{M}e$ for some projection $e \in \mathcal{M}$.

(2) J is a two-sided ideal of $\mathcal{M}$ if and only if $J = \mathcal{M}e$ for some projection $e \in Z(\mathcal{M})$.

Proof.

(1) Let $J \subseteq \mathcal{M}$ be a wo closed left ideal. Consider $J \cap J^*$. It is a wo closed $*$-subalgebra. Let E be the largest projection in $J \cap J^*$ as in Proposition 7.1. Let $m \in J$. Then $m^*m \in J \cap J^*$ and $m^*m = Em^*m = m^*m = Em^*mE$. Then $(m(I-E))^*(m(I-E)) = (I-E)m^*m(I-E) = m^*m - Em^*m - m^*mE + Em^*mE = 0$, so $m(I-E) = 0$, i.e. $m = mE$ and so $J = JE$. Then $J = \mathcal{M}E$.

(2) Let $J \subseteq \mathcal{M}$ be a wo closed two-sided ideal. Then by (1), $J = \mathcal{M}P = Q\mathcal{M}$ for some projections $P, Q \in \mathcal{M}$. Then $P = QP = Q$ and so $(I-P)\mathcal{M}P = 0 = P\mathcal{M}(I-P)$. Therefore $[P, \mathcal{M}] = 0$, so $P \in Z(\mathcal{M})$.

□

9. Murray-von Neumann equivalence

Let $\mathcal{M}$ be a von Neumann algebra and let $\mathcal{P}(\mathcal{M})$ denote its subset of projections.

Definition. *Given $e, f \in \mathcal{P}(\mathcal{M})$, write $e \sim f$ and say that e, f are Murray-von Neumann equivalent if $e = v^*v$, $f = vv^*$ for some partial isometry $v \in \mathcal{M}$*[a]

[a] A partial isometry is an element $v \in \mathcal{M}$ for which v^*v, vv^* are projections. Equivalently, it is unitary on some closed subspace of $\mathcal{H}$ and 0 on its complement. Equivalently $v = vv^*v$.

Spatially this means: $e\mathcal{H}, f\mathcal{H}$ are closed invariant subspaces under $\mathcal{U}(\mathcal{M}')$ and there is a unitary map $e\mathcal{H} \to f\mathcal{H}$ commuting with $\mathcal{U}(\mathcal{M}')$. In other words $e\mathcal{H}, f\mathcal{H}$ are unitarily equivalent as $\mathcal{U}(\mathcal{M}')$-representations.

We write $e \prec f$ if $e = v^*v$ and $f \geq vv^*$. In other words $e\mathcal{H}$ embeds isometrically into $f\mathcal{H}$ as $\mathcal{U}(\mathcal{M}')$-representations.

Proposition 9.1. *Some properties of* $\sim, \prec$.

(1) $\sim$ *is an equivalence relation.*

(2) $\prec$ *is transitive.*

(3) *If* $e, f \in \mathcal{P}(\mathcal{M})$, *then there is a projection* $P \in Z(\mathcal{M})$ *such that* $eP \prec fP$ *and* $e(I-P) \succ f(I-P)$.

(4) *(Schröder-Bernstein type property) If* $e \prec f$ *and* $e \succ f$, *then* $e \sim f$.

Proof.

(1) Spatially obvious.

(2) Spatially obvious.

(3) We will use a lemma and some definitions. Let $x \in \mathcal{M}$. Then $\overline{\mathcal{M}x}^{wo} = \mathcal{M}P_{(\ker x)^\perp}$. We let $r(x) := P_{(\ker x)^\perp}$ be the right support of x. Similarly $\overline{x\mathcal{M}}^{wo} = P_{(\ker x^*)^\perp}\mathcal{M}$ and $l(x) := P_{(\ker x^*)^\perp} = P_{\overline{x\mathcal{H}}}$ is the left support of x. Also $\overline{\mathcal{M}x\mathcal{M}}^{wo} = \mathcal{M}Z(x)$ where $Z(x)$ is the central support of x.

Lemma. *Let* $e, f \in \mathcal{P}(\mathcal{M})$. *TFAE:*

(i) *There is a partial isometry* $v \neq 0$ *such that* $v^*v \leq f, vv^* \leq e$.

(ii) $e\mathcal{M}f \neq 0$.

(iii) $Z(e)Z(f) \neq 0$.

Proof. "(i) $\Rightarrow$ (ii)" We have $0 \neq v = evf \in e\mathcal{M}f$. "(ii) $\Leftrightarrow$ (iii)" We have $0 \neq e\mathcal{M}f$ iff $\mathcal{M}e\mathcal{M}\mathcal{M}f\mathcal{M} \neq 0$ iff $0 \neq \overline{\mathcal{M}e\mathcal{M}}^{wo}\overline{\mathcal{M}f\mathcal{M}}^{wo}$ iff $0 \neq \mathcal{M}Z(e)\mathcal{M}Z(f)$ iff $0 \neq \mathcal{M}Z(e)Z(f)\mathcal{M}$ iff $0 \neq Z(e)Z(f)$. "(ii) $\Rightarrow$ (i)" Suppose $emf \neq 0$ for some $m \in \mathcal{M}$. Then let $emf = va$ be its polar decomposition. Then $vv^* \leq e$ and $v^*v = P_{(\ker emf)^\perp} \leq P_{(\ker f)^\perp} = f$. □

We now prove (3). Let $(e_i)_{i\in I}, (f_i)_{i \in I}$ be a maximal pair of families of pairwise orthogonal projections in $\mathcal{P}(\mathcal{M}) \setminus \{0\}$ such that $e_i \leq e, f_i \leq f$ and $e_i \sim f_i$. Then consider $\widetilde{e} = \sum_{i\in I} e_i, \widetilde{f} = \sum_{i \in I} f_i, e_0 = e - \widetilde{e}, f_0 = f - \widetilde{f}$. By maximality and the lemma we have $Z(e_0)Z(f_0) = 0$. Let $P = Z(f_0)$. Then $eP = \widetilde{e}P + e_0P = \widetilde{e}P$ and $fP = \widetilde{f}P + f_0$. Let $e_i \sim f_i$ via v_i. Then $\widetilde{e} \sim \widetilde{f}$ via $u = \sum v_i$. Then also $\widetilde{e}P \sim \widetilde{f}P$. Then $eP = \widetilde{e}P \sim \widetilde{f}P = fP - f_0 \leq fP$, so $eP \prec fP$. Similarly $e(I-P) = \widetilde{e}(I-P) + e_0(I-P)$ and $f(I-P) = \widetilde{f}(I-P) + f_0(I-P) = \widetilde{f}(I-P)$ and we get $e(I-P) \succ f(I-P)$.

(4) Let v, w be partial isometries such that $vv^* \leq e, v^*v = f$ and $ww^* \leq f, w^*w = e$. Let $e_0 = e - vv^*, f_0 = f - ww^*$. Inductively define $f_{n+1} = we_nw^*, e_{n+1} = vf_nv^*$. Then one proves $e_0 \perp e_j$ and $f_0 \perp f_j$ for all $j > 0$. Furthermore, $e_0e_1 \cdots e_n$ and e_j are pairwise orthogonal for $j > n$ and similarly for the f_i. Let $e_\infty = e - \sum_{n\geq 0} e_n$ and $f_\infty = f - \sum_{n \geq 0} f_n$. Note that

$e_n \sim f_{n+1}$ as $we_nw^* = f_{n+1}$ and also $e \sim f - f_0$ via w, so we get $e_\infty \sim f_\infty$ with $we_\infty w^* = f_\infty$. Now write $e = \sum_{k\geq 0}(e_{2k} + e_{2k+1}) + e_\infty \sim \sum_{k\geq 0}(e_{2k+1} + e_{2k+2}) + e_\infty = e - e_0 \sim f$.

□

10. Parallelogram rule

Theorem 10.1. *Let $e, f \in \mathcal{P}(M)$. Then:*

(a) $e \vee f - e \sim f - e \wedge f$.

(b) $e - e \wedge (1-f) \sim f - f \wedge (1-e)$.

Lemma 10.2.

(i) For any $x \in \mathcal{M}$, $l(x) \sim r(x)$ by the partial isometry in the polar decomposition.

(ii) $P_{\ker ef} = (1-f) + (1-e) \wedge f$ *and* $P_{(\ker ef)^\perp} = f - (1-e) \wedge f$.

Proof. Exercise. □

Proof of Theorem 10.1.

(a) $f - e \wedge f = f - (1-(1-e)) \wedge f \overset{(b)}{\sim} (1-e) - (1-f) \wedge (1-e) = (1-e) - (1 - f \vee e) = e \vee f - e$.

(b) $f - (1-e) \wedge f = r(ef) \sim l(ef) = r(fe) = e - (1-f) \wedge e$.

□

11. More stuff on projections...

Definition. $e \in \mathcal{P}(\mathcal{M})$ *is called* finite *if whenever* $e_1 \in \mathcal{P}(\mathcal{M})$ *is such that* $e_1 \leq e, e_1 \sim e$, *then* $e_1 = e$. *If e is not finite, then e is* infinite.

Note that if e is finite and $f \leq e$, then also f is finite. Indeed, if $f \sim f_1, f_1 \leq f$, then $e \sim (e-f) + f_1$ and $e - f + f_1 \leq e$, so $e - f + f_1 = e$ and then $f = f_1$.

We make some simplifying assumptions: Assume $\mathcal{H}$ is separable. We also assume that $\mathcal{M}$ is a factor.

Proposition 11.1. *Let $e \in \mathcal{P}(\mathcal{M})$ be infinite. Then $e = e_1 + e_2 + \ldots$ where the $e_1, e_2, \ldots$ are pairwise orthogonal and $e_1 \sim e_2 \sim e_3 \sim \ldots$.*

Proof. Omitted. □

Corollary 11.2. $e \neq 0$ *is infinite if and only if* $e = e' + e''$ *with* $e \sim e' \sim e''$.

Proof. "$\Leftarrow$" is clear, for the other implication take $e' = e_1 + e_3 + e_5 + \ldots$ and $e'' = e_2 + e_4 + e_6 + \ldots$ as in the proposition. □

Proposition 11.3. *If $e, f \in \mathcal{P}(\mathcal{M})$ are finite, then so is $e \vee f$.*

Proof.

Step 1. e, f can be assumed to be orthogonal. Indeed, $e \vee f = (e \vee f - e) + e \sim (f - e \wedge f)$ and $e \vee f - e$ is finite as it is $\sim f - e \wedge f \leq f$. Also $e \vee f - e \perp e$.

Step 2. Assume $e \vee f = e + f$ is infinite, $ef = 0$. Then by the corollary, $e + f = p + q$ where $p \sim q \sim p + q = e + f$. Note that automatically $pq = 0$. Since $\mathcal{M}$ is a factor, all projections are comparable, so we can assume $e \wedge p \prec q \wedge f$. We will prove $p \prec f$ which contradicts f finite and p infinite. We have

$$
\begin{aligned}
p &= e \wedge p + (p - e \wedge p) \\
&\sim e \wedge p + (e \vee p - e) \\
&\prec q \wedge f + (e \vee p - e) && e \vee p - e \perp q \wedge f \text{ as } q \perp p, f \perp e
\end{aligned}
$$

Now note that $e \vee p - e \leq e + f - e = f$, so both projections are $\leq f$ and we get $p \prec f$.

$\square$

For the following definition we don't make any assumptions on $\mathcal{H}, \mathcal{M}$.

Definition. *Let $e \in \mathcal{P}(\mathcal{M})$.*

- *e is* minimal *if $e\mathcal{M}e = \mathbb{C}e$.*
- *e is* abelian *if $e\mathcal{M}e$ is abelian.*
- *e is* semi-finite *if $e = \bigvee\{f \mid f \leq e,\ f \text{ finite}\}$.*

Examples.

- In $\mathcal{B}(\mathcal{H})$ the minimal projections are $P_{\mathbb{C}\xi}$ with $\xi \in \mathcal{H}$.
- Suppose $\mathcal{M} = \mathcal{B}(\mathcal{H}) \oplus \mathcal{B}(\mathcal{H})$. Then $P_{\mathbb{C}\xi} \oplus P_{\mathbb{C}\eta}$ is an abelian projection.
- Suppose $\mathcal{H}$ is a separable (not necessary) infinite dimensional Hilbert space. Then $I \in \mathcal{B}(\mathcal{H})$ is semifinite.

Definition. *Let $\mathcal{M}$ be a factor, $\mathcal{H}$ separable.*

- *$\mathcal{M}$ is of* type I *if $\mathcal{M}$ has a minimal projection.*
- *$\mathcal{M}$ is* finite *if I is a finite projection.*
- *$\mathcal{M}$ is* semi-finite *if there is a finite projection $0 \neq e \in \mathcal{M}$.*
- *$\mathcal{M}$ is of* type II_1 *if $\mathcal{M}$ is finite and has no minimal projection.*
- *$\mathcal{M}$ is of* type II_∞ *if I is not finite, but semi-finite, and $\mathcal{M}$ has no minimal projection.*
- *$\mathcal{M}$ is of* type III *if it has no finite projection.*

An example of a type II_∞ factor is $\mathcal{N} \otimes \mathcal{B}(\ell^2)$ where $\mathcal{N}$ is of type II_1.

Theorem 11.4. *Any type I algebra is isomorphic to $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

Proposition 11.5. *Let $\mathcal{M}$ be a type I factor and $e_0 \in \mathcal{M}$ a minimal projection. Let $(e_j)_{j\in J}$ be a maximal family of pairwise orthogonal projections such that $e_j \sim e_0$ for all $j \in J$. Then $\sum_{j\in J} e_j = I$.*

Proof. Proof. Let $f = I - \sum_{j\in J} e_j$. Assume $f \neq 0$. Since $\mathcal{M}$ is a factor, either $e_0 \prec f$ or $f \prec e_0$. In the second case we also get $e_0 \prec f$ as e_0 is minimal, so can assume $e_0 \prec f$. Then $v^*v = e_0, vv^* \leq f$ for some partial isometry v. Add the projection vv^* to $(e_j)_{j\in J}$. Then the new family is still pairwise orthogonal and $vv^* \sim e_0$, contradicting maximality. □

So we get $I = \sum_{j\in J} e_j$. Let v_j be a partial isometry from e_0 to e_j. Put $e_{ij} = v_i v_j^*$. This is a partial isometry from e_i to e_j. We have $e_{ij}e_{kl} = \delta_{jk}e_{il}$, $e_{ij}^* = e_{ji}$, $e_{ii} = e_i$ and $\sum e_{ii} = I$.

Now one can put all these e_j together to get an isomorphism $\mathcal{M} \cong \mathcal{B}(\mathcal{H}')$ for some Hilbert space $\mathcal{H}'$.

Pick $i_0 \in J$. Then $e_{i_0}\mathcal{H}$ is cyclic for $\mathcal{M}$. There is a partial isometry from $e_{i_0}\mathcal{H}$ to $e_j\mathcal{H}$. We have $\mathcal{M}e_{i_0}\mathcal{H} \supseteq e_j\mathcal{H}$ for all j. Then $e_{i_0}\mathcal{H}$ is separating for $\mathcal{M}'$. $\mathcal{M}'$ is isomorphic to $(\mathcal{M}')_{e_{i_0}}$ and the commutant is $\mathcal{M}_{e_{i_0}} = e_{i_0}\mathcal{M}e_{i_0}|_{e_{i_0}\mathcal{H}} = \mathbb{C}I_{e_{i_0}\mathcal{H}}$ and $(\mathcal{M}')_{e_{i_0}} = \mathcal{B}(e_{i_0}\mathcal{H})$. Then consider $\ell^2(J)\otimes e_{i_0}\mathcal{H} \xrightarrow{U} \mathcal{H}$ given by $\varepsilon_i \otimes h \mapsto v_i h$. Then one gets $(\mathcal{M}, \mathcal{M}') \sim (\mathcal{B}(\ell^2(J)) \otimes I_{e_{i_0}\mathcal{H}}, I_{\ell^2(J)} \otimes \mathcal{B}(e_{i_0}\mathcal{H}))$. □

Now suppose $\mathcal{M}$ is a type II_1 factor, so $\mathcal{M}$ has no minimal projection and I is finite. Hence every projection in $\mathcal{M}$ is finite. Note that equivalently, I is finite and $\mathcal{M}$ is infinite dimensional.

Recall that a state on a C^*-algebra $\mathcal{A}$ is a positive functional $\varphi : \mathcal{A} \to \mathbb{C}$ of norm 1. It is *tracial* (or a trace) if $\varphi(xy) = \varphi(yx)$. It is faithful if $\tau(x^*x) = 0$ implies $x = 0$.

Lemma 11.6. *Let $\mathcal{M}$ be a von Neumann algebra, $\tau : \mathcal{M} \to \mathbb{C}$ a faithful trace. Then any $0 \neq e \in \mathcal{P}(\mathcal{M})$ is finite. If $\mathcal{M}$ is a factor, then $e \prec f$ if and only if $\tau(e) \leq \tau(f)$, and $e \sim f$ if and only if $\tau(e) = \tau(f)$.*

Proof. Suppose $e \sim f$. Then $e = vv^*, f = v^*v$ for some partial isometry v and $\tau(vv^*) = \tau(v^*v)$, so $\tau(e) = \tau(f)$. Suppose $e \prec f$. Then $e \sim e_1 \leq f$. Then $\tau(e) = \tau(e_1)$ and $\tau(e_1) + \tau(f - e_1) = \tau(f)$. But note that $\tau(f - e_1) = \tau((f - e_1)^*(f - e_1)) > 0$ if $f - e_1 \neq 0$. So we get $\tau(e) \leq \tau(f)$ and if $\tau(e) = \tau(f)$ we get $e \sim f$. For the reverse implication use that $\mathcal{M}$ is a factor, so that all projections are comparable. □

12. Example: Groups

Let G be a group and consider $\ell^2(G)$ with ONB $(\varepsilon_g)_{g\in G}$. For $g \in G$, let $\lambda(g) \in \mathcal{B}(\ell^2(G))$ be defined by $\lambda(g)\varepsilon_h = \varepsilon_{gh}$. $\lambda(g)$ is unitary, $\lambda(g_1)\lambda(g_2) = \lambda(g_1g_2)$, $\lambda(g)^* = \lambda(g^{-1})$ and $\lambda(e) = I$. This is the left regular representation. Similarly, define the right regular representation ρ by $\rho(g)\varepsilon_h = \varepsilon_{hg^{-1}}$.

Proposition 12.1. *Let $\mathcal{M}$ be the weak operator closure of the linear span of $\lambda(G) \subseteq \mathcal{B}(\mathcal{H})$. Similarly let $\mathcal{N}$ be the wo closure of the span of $\rho(G)$. Then*

(i) *$\mathcal{M}, \mathcal{N}$ are von Neumann algebras and $[\mathcal{M}, \mathcal{N}] = 0$. ε_e is cyclic and separating for $\mathcal{M}$ and $\mathcal{N}$. $\tau(T) = \langle T\varepsilon_e, \varepsilon_e\rangle$ defines a faithful positive trace on $\mathcal{M}$.*

(ii) *$\mathcal{M}' = \mathcal{N}$ and $\mathcal{N}' = \mathcal{M}$.*

(iii) *$\mathcal{M}$ is a factor if and only if G has the infinite conjugacy class property (icc), meaning that every non-identity conjugacy class is infinite.*

Proof.

(i) Let $\mathcal{A}$ be the linear span $\lambda(G)$. This is clearly a $*$-subalgebra in $\mathcal{B}(\ell^2(G))$. Then of course $\overline{\mathcal{A}}^{wo} = \mathcal{A}''$ is von Neumann. We have $[\operatorname{span}\lambda(G), \operatorname{span}\lambda(H)] = 0$ because $\lambda(g)\rho(h)\varepsilon_k = \lambda(g)\varepsilon_{kh^{-1}} = \varepsilon_{gkh^{-1}} = \rho(h)\lambda(g)\varepsilon_k$. Then take wo closure (need to be a bit careful) and get $[\mathcal{M}, \mathcal{N}] = 0$. Let $\xi = \varepsilon_e$. Clearly ξ is cyclic for $\mathcal{M}, \mathcal{N}$. Therefore it is separating for $\mathcal{M}', \mathcal{N}'$. As $\mathcal{M} \subseteq \mathcal{N}', \mathcal{N} \subseteq \mathcal{M}'$, it is also separating for $\mathcal{M}, \mathcal{N}$. Take $x \in \mathcal{M}$. Then there is a net $(x_\iota)_{\iota\in\Lambda}$ in $\mathcal{A}$ such that $x_\iota \xrightarrow{so} x$. Recall from the Kaplansky density theorem, we can assume that also $x_\iota^* \xrightarrow{so} x^*$. Then we get $\|x_\iota\xi\|^2 \to \|x\xi\|^2$ and $\|x_\iota^*\xi\|^2 \to \|x^*\xi\|^2$, so $\tau(x_\iota^* x_\iota) \to \tau(x^*x)$ and $\tau(x_\iota x_\iota^*) \to \tau(xx^*)$. So if we show $\tau(y^*y) = \tau(yy^*)$ for $y \in \mathcal{A}$, then we get that this also holds for $y \in \mathcal{M}$. So let $y = \sum c_g\lambda(g)$ where only finitely many c_g are non-zero. Then $y^* = \sum \overline{c_g}\lambda(g^{-1}) = \sum \overline{c_{g^{-1}}}\lambda(g)$. Then $\tau(y^*y) = \|y\xi\|^2 = \sum |c_g|^2 = \|y^*\xi\|^2 = \tau(yy^*)$. Then for hermitian a, b we get $\tau(ab) = \tau(ba)$ by looking at $x = a + bi$, and therefore τ is a trace. τ is clearly positive. It is faithful because ξ is separating for $\mathcal{M}$.

(iii) Let $T \in \mathcal{M}$. Then $T \in Z(\mathcal{M})$ if and only if $[T, \lambda(g)] = 0$ for all $g \in G$. Also $[T, \lambda(g)] = 0$ if and only if $\lambda(g^{-1})T\lambda(g) = T$. Since ξ is separating, $T \in Z(\mathcal{M})$ if and only if $\lambda(g^{-1})T\lambda(g)\xi = T\xi$ for all $g \in G$. Write $T\xi = \sum c_h\varepsilon_h$ with $\sum |c_h|^2 < \infty$. Then

$$\begin{aligned}\lambda(g^{-1})T\lambda(g)\xi &= \lambda(g^{-1})T\varepsilon_g = \lambda(g^{-1})T\rho(g^{-1})\xi = \lambda(g^{-1})\rho(g^{-1})T\xi \\ &= \sum_h c_h\lambda(g^{-1})\rho(g^{-1})\varepsilon_h = \sum_h c_h\varepsilon_{g^{-1}hg} = \sum_h c_{ghg^{-1}}\varepsilon_h.\end{aligned}$$

So $\lambda(g^{-1})T\lambda(g)\xi = T\xi$ holds for all $g \in G$ if and only if $c_{ghg^{-1}} = c_h$ for all $g, h \in G$, i.e. $h \mapsto c_h$ is central (invariant under conjugation). If G is icc, then if $h \mapsto c_h$ is central and $(c_h)_{h\in G} \in \ell^2$, then $c_h = c\delta_{he}$, so $T = cI$. So if G is icc, then $Z(\mathcal{M}) = \mathbb{C}I$. Conversely, if G is not icc, some conjugacy class $X \subseteq G$ with $e \notin X$ is finite. Then $\sum_{g\in X}\lambda(g)$ is a non-trivial central operator in $\mathcal{M}$, the map $h \mapsto c_h$ is χ_X.

(ii) To prove $\mathcal{M} = \mathcal{N}', \mathcal{N} = \mathcal{M}'$ it suffices to show $[\mathcal{M}', \mathcal{N}'] = 0$. Indeed, then we get $\mathcal{N} \subseteq \mathcal{M}' \subseteq (\mathcal{N}')' = \mathcal{N}$ and $\mathcal{M} \subseteq \mathcal{N}' \subseteq (\mathcal{M}')' = \mathcal{M}$. How to describe $T \in \mathcal{N}'$ using $T\xi \in \ell^2(G)$? We know that $T \mapsto T\xi$ is injective because ξ is separating for $\mathcal{N}'$. Put $\eta = T\xi$. Then

$$T\varepsilon_h = T\rho(h^{-1})\xi = \rho(h^{-1})T\xi = \rho(h^{-1})\eta.$$

Depending on $\eta \in \ell^2(G)$, when does this define a bounded operator? We must have

$$\Big\|\sum_{h\in G} c_h\rho(h^{-1})\eta\Big\| \le C\Big(\sum |c_h|^2\Big)^{1/2}$$

for some constant $C > 0$. Not all η will satisfy this. If this holds, denote T by L_η. Similarly if $T \in \mathcal{M}'$ and $T\xi = \zeta$, then

$$T\varepsilon_h = T\lambda(h)\xi = \lambda(h)\zeta,$$

and the operator $\sum c_h \varepsilon_h \mapsto \sum c_h \lambda(h)\zeta$, denoted R_ζ, is bounded, i.e. we have

$$\left\| \sum_{h \in G} c_h \lambda(h)\zeta \right\| \le C\Big(\sum |c_h|^2\Big)^{1/2}.$$

We need to show $[L_\eta, R_\zeta] = 0$. Equivalently, $\langle R_\zeta \varepsilon_h, L_\eta^* \varepsilon_k\rangle = \langle L_\eta \varepsilon_h, R_\zeta^* \varepsilon_k\rangle$. So we need formulas for R_ζ^*, L_η^*[1]. Since $L_\eta \in \mathcal{N}'$, $L_\eta^* \in \mathcal{N}'$ and so $L_\eta^* = L_{\eta^*}$ for some $\eta^* \in \ell^2(G)$. We have $\langle \eta^*, \varepsilon_k\rangle = \langle L_\eta^* \xi, \varepsilon_k\rangle = \langle \xi, L_\eta \varepsilon_k\rangle = \langle \xi, \rho(k^{-1})\eta\rangle$. Write $\eta = \sum c_g \varepsilon_g$, so $\langle \eta^*, \varepsilon_k\rangle = \overline{c_{k^{-1}}}$ and then $\eta^* = \sum \overline{c}_{g^{-1}} \varepsilon_g$. Same type of formula for ζ^*, so that $R_\zeta^* = R_{\zeta^*}$. Write $\zeta = \sum b_h \varepsilon_h$, so that $\zeta^* = \sum \overline{b_{h^{-1}}} \varepsilon_h$. Then

$$\begin{aligned}
\langle R_\zeta \varepsilon_h, L_{\eta^*} \varepsilon_k\rangle &= \Big\langle \sum b_g \varepsilon_{hg}, \sum \overline{c_{g^{-1}}} \varepsilon_{gk}\Big\rangle \\
&= \sum_{hg=gk} b_g c_{g^{-1}} \\
&= \sum_{gh=kg} c_g b_{g^{-1}} \\
&= \Big\langle \sum c_g \varepsilon_{gh}, \sum \overline{b_{g^{-1}}} \varepsilon_{kg}\Big\rangle \\
&= \langle L_\eta \varepsilon_h, R_{\zeta^*} \varepsilon_k\rangle.
\end{aligned}$$

□

Example. Let S_∞ denote the group of permutations of $\mathbb{N}$ fixing all but finitely many elements. Then S_∞ has icc.

13. Type II_∞ factors

Proposition 13.1. *Let $\mathcal{M}$ be a factor, $p \in \mathcal{P}(\mathcal{M})$. Then $\mathcal{M}_p = p\mathcal{M}p|_{p\mathcal{H}}$ is a factor.*

Proof. Show that if $0 \neq x \in \mathcal{M}_p$, then $\overline{\mathcal{M}_p x \mathcal{M}_p}^{so} = \mathcal{M}_p$. We have $\overline{\mathcal{M}_p x \mathcal{M}_p}^{so} = \overline{p\mathcal{M}pxp\mathcal{M}p}^{so} = p\overline{\mathcal{M}x\mathcal{M}}^{so} p = pZ_{\mathcal{M}}(x)\mathcal{M}p = p\mathcal{M}p = \mathcal{M}_p$. □

Let $\mathcal{M}$ be a type II_∞ factor, so I is infinite, $\mathcal{M}$ has no nontrivial minimal projections and has a nontrivial finite projection.

Let $p \in \mathcal{P}(\mathcal{M})$ be a non-zero finite projection. Then $p\mathcal{M}p|_{p\mathcal{M}}$ is a factor and its identity is a finite projection. It has no minimal projection, so $\mathcal{M}_p$ is a type II_1 factor. There is a maximal family $(p_\iota)_{\iota \in J}$ of pairwise orthogonal projections with $p_\iota \sim p$, $\sum_{\iota \in J} p_\iota = I$.

Conclusion: $\mathcal{M}$ is isomorphic to $\mathcal{M}_p \otimes \mathcal{B}(\ell^2(J))$.

[1] L.T.: It seems to me that directly verifying $L_\eta R_\zeta \varepsilon_h = R_\zeta L_\eta \varepsilon_h$ is also not difficult, perhaps even quicker.

Remark. $(\mathcal{M}\otimes\mathcal{N})' = \mathcal{M}'\otimes\mathcal{N}'$ (non-trivial!). First solved as application of Tomita-Takesaki theory. Later simpler proofs were found.

14. Complements about GNS construction

Let $\mathcal{A}$ be a unital C^*-algebra. Let $\varphi : \mathcal{A} \to \mathbb{C}$ be a bounded positive linear functional. $\mathcal{A}$ is a left module over itself. Recall that in the GNS construction we define the semi inner product $\langle a, b\rangle_\varphi = \varphi(b^*a)$ with $a, b \in \mathcal{A}$. Let $\mathcal{H}_\varphi$ be the associated Hilbert space and $\xi_\varphi = 1_\mathcal{A} \in \mathcal{H}_\varphi$. Then let π_φ be the representation on $\mathcal{H}_\varphi$ of $\mathcal{A}$ given by $\pi_\varphi(a)a_1 = aa_1$. ξ_φ is a cyclic vector for π_φ.

If ρ is some other representation of $\mathcal{A}$ on a Hilbert space $\mathcal{K}$ with cyclic vector η such that $\langle\rho(a)\eta, \eta\rangle = \varphi(a)$, then these are unitarily equivalent, i.e. there is an unitary equivalence $U : \mathcal{H}_\varphi \to \mathcal{K}$ such that $U\xi_\varphi = \eta$ and $U\pi_\varphi(a) = \rho(a)U$.

Lemma 14.1. *Let $\psi, \varphi : \mathcal{A} \to \mathbb{C}$ with $\varphi \geq \psi \geq 0$. Then there is map $T : \mathcal{H}_\varphi \to \mathcal{H}_\psi$ with $\|T\| \leq 1$, $T\pi_\varphi(a) = \pi_\psi(a)T$ and $T\xi_\varphi = \xi_\psi$. Also $T^*\pi_\psi(a) = \pi_\varphi(a)T^*$.*

Lemma 14.2. *Let $\mathcal{A}$ be a unital C^*-subalgebra of $\mathcal{B}(\mathcal{H})$. Suppose $\xi, \eta \in \mathcal{H}$ are such that $\varphi \geq 0$ where $\varphi(a) = \langle a\xi, \eta\rangle$. Then there is a $\zeta \in \mathcal{H}$ so that $\varphi(a) = \langle a\zeta, \zeta\rangle$ and $(\mathcal{H}_\varphi, \pi_\varphi, \xi_\varphi) \cong (\overline{\mathcal{A}\zeta}, \mathcal{A}|_{\overline{\mathcal{A}\zeta}}, \zeta)$.*

Proof. Let $a \in \mathcal{A}$ with $a \geq 0$. Then $\varphi(a) = \frac{1}{4}(\langle a(\xi+\eta), (\xi+\eta)\rangle - \langle a(\xi-\eta), \xi-\eta\rangle)$ (the other terms in the polarization identity cancel because $\varphi(a) \in \mathbb{R}$). Then $0 \leq \varphi \leq \psi$ where $\psi(a) = \frac{1}{4}\langle a(\xi+\eta), (\xi+\eta)\rangle$. Then $(\mathcal{H}_\psi, \pi_\psi, \xi_\psi) \cong (\overline{\mathcal{A}\frac{\xi+\eta}{2}}, \mathcal{A}|_{\overline{\mathcal{A}\frac{\xi+\eta}{2}}}, \frac{\xi+\eta}{2})$. Denote the restriction $a|_{\overline{\mathcal{A}\frac{\xi+\eta}{2}}}$ by $\rho(a)$ for $a \in \mathcal{A}$ (i.e. the action of $\mathcal{A}$ on $\overline{\mathcal{A}\frac{\xi+\eta}{2}}$). Let $T : \overline{\mathcal{A}\frac{\xi+\eta}{2}} \to \mathcal{H}_\varphi$ be the map from the previous lemma. Then

$$\begin{aligned}\varphi(a) = \langle\pi_\varphi(a)\xi_\varphi, \xi_\varphi\rangle &= \langle\pi_\varphi(a)T\frac{\xi+\eta}{2}, T\frac{\xi+\eta}{2}\rangle \\ &= \langle T^*\pi_\varphi(a)T\frac{\xi+\eta}{2}, \frac{\xi+\eta}{2}\rangle \\ &= \langle\rho(a)T^*T\frac{\xi+\eta}{2}, \frac{\xi+\eta}{2}\rangle \\ &\overset{[T^*T,\rho(\mathcal{A})]=0}{=} \langle\rho(a)(T^*T)^{1/2}\frac{\xi+\eta}{2}, (T^*T)^{1/2}\frac{\xi+\eta}{2}\rangle.\end{aligned}$$

So take $\zeta = (T^*T)^{1/2}\frac{\xi+\eta}{2}$. □

Proposition 14.3. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Let $\varphi : \mathcal{M} \to \mathbb{C}$ be uwo continuous and $\varphi \geq 0$. Then there is a $\zeta = \sum_k \zeta_k \otimes e_k \in \mathcal{H} \otimes \ell^2(\mathbb{N})$ so that*

$$\varphi(m) = \langle(m \otimes I_{\ell^2})\zeta, \zeta\rangle = \sum_{k\in\mathbb{N}}\langle m\zeta_k, \zeta_k\rangle.$$

where $\zeta_k \in \mathcal{H}$, $\sum_k \|\zeta_k\|^2 < \infty$. Moreover, $\pi_\varphi : \mathcal{M} \to \mathcal{B}(\mathcal{H}_\varphi)$ is continuous if both spaces carry the uwo topology, and same for the uso topology.

Proof. Consider $\mathcal{M} \otimes I_{\ell^2(\mathbb{N})}$ acting on $\mathcal{H} \otimes \ell^2(\mathbb{N})$. Since φ is uwo continuous, there are $\xi, \eta \in \mathcal{H} \otimes \ell^2(\mathbb{N})$ such that $\varphi(m) = \langle (m \otimes I_{\ell^2(\mathbb{N})})\xi, \eta\rangle$ for all $m \in \mathcal{M}$. Now apply the lemma. So there is $\zeta \in \mathcal{H} \otimes \ell^2(\mathbb{N})$ such that $\varphi(m) = \langle (m \otimes I_{\ell^2(\mathbb{N})})\zeta, \zeta\rangle$.

For the last part: $(\mathcal{H}_\varphi, \pi_\varphi, \xi_\varphi)$ is unitarily equivalent to $(\overline{(\mathcal{M} \otimes I_{\ell^2(\mathbb{N})})\zeta}, \text{restriction to ...}, \zeta)$. The restriction to a subspace is clearly uwo continuous. □

Remark. If instead of uwo we have wo continuous, then we can use $\mathcal{H} \otimes \mathbb{C}^n$ instead of $\mathcal{H} \otimes \ell^2(\mathbb{N})$. Then π_φ is wo/so continuous.

15. Traces on Type II_1 Factors

Theorem 15.1. *Let $\mathcal{M}$ be a type II_1 factor. Then there is an uwo continuous linear functional $\tau : \mathcal{M} \to \mathbb{C}$ such that $\tau(1) = 1$, $\tau(x^*x) \geq 0$, $\tau([x, y]) = 0$. Furthermore, τ is faithful and unique and $\tau(\mathcal{P}(\mathcal{M})) = [0, 1]$.*

15.1. Idea of construction of τ

Idea: Construct a "dimension function" on $\mathcal{P}(\mathcal{M})$, i.e. a function $\delta : \mathcal{P}(\mathcal{M}) \to [0, 1]$ such that:

- If $P \sim Q$, then $\delta(P) = \delta(Q)$,
- If $P \perp Q$, then $\delta(P + Q) = \delta(P) + \delta(Q)$.

Recursively define projections: Take a projection P_n. Since $\mathcal{M}$ has no minimal projection, we can write $P_n = P_n' + P_n''$ with $P_n' \prec P_n''$ and $P_n', P_n'' \neq 0$ and then set $P_{n+1} = P_n'$.

Let $Q, P \in \mathcal{P}(\mathcal{M})$. Take $(T_\iota)_{\iota \in J}$ a maximal family of pairwise orthogonal projections such that $\sum_{\iota \in J} P_\iota \leq Q$ and $P_\iota \sim P$. Since Q is finite, we must have $|J| < \infty$. Put $[Q : P] = |J|$. This is well-defined (omitted). Now do something like

$$\lim_{n \to \infty} \frac{[P : P_n]}{[I : P_n]},$$

this is a candidate for $\delta(P)$.

Next observe that $\mathcal{P}(\mathcal{M})$ has linear span norm-dense in $\mathcal{M}$. Extend δ by linearity to get τ.

Proposition 15.2. *τ is faithful, i.e. if $\tau(x^*x) = 0$, then $x = 0$.*

Proof. Let $J = \{x \in \mathcal{M} \mid \sigma(x^*x) = 0\}$. Since τ is a trace, if $x \in J$, then $x^* \in J$. If $x \in J, y \in \mathcal{M}$, we have $0 \leq \tau((yx)^*yx) = \tau(x^*(y^*y)x) \leq \tau(x^* \|y\|^2 1x) = \|y\|^2 \tau(x^*x) = 0$. So J is a left-ideal, and then also a right ideal since it is a $*$-ideal. If $x \in J$, $\varepsilon > 0$, then $E([\varepsilon, \infty), x^*x) \in J$ (e.g. because this is $f(x^*x)x^*x$ for $f : \mathbb{R}_{>0} \to \mathbb{R}$ given by $f(t) = \frac{1}{t}$ if $t \geq \varepsilon$ and 0 otherwise). So if $J \neq 0$, there is a projection $0 \neq P \in J$. Then $I = P_1 + \cdots + P_n + P_{n+1}$ where $P_j \sim P$ for $j \leq n$ and $P \succ P_{n+1}$. Write $P_j = v_j P v_j^*$ where $P_j \sim P$ via v_j. Write $P_{n+1} = w^*w$ with $P \geq ww^*$, so $P_{n+1} = wPw^*$. Then we get $P_1, \ldots, P_{n+1} \in J$, and therefore $I \in J$, so $J = \mathcal{M}$. But this is impossible since $\tau \neq 0$. □

Note that this proof shows that in a II_1 factor any (algebraic) two-sided ideal is either 0 or $\mathcal{M}$.

Proposition 15.3. *Let $\tau : \mathcal{M} \to \mathbb{C}$ be as above. Then*

(a) *τ is unique.*

(b) *$\tau(\mathcal{P}(\mathcal{M})) = [0,1]$.*

Proof.

(a) Recall the construction of P_n. First show that $\tau|_{\mathcal{P}(\mathcal{M})}$ is unique. Fix n. If $Q \in \mathcal{P}(\mathcal{M})$, write $Q = \sum_{\iota \in J} P_\iota + \widetilde{P}$ with $P_\iota \sim P_n$ and $\widetilde{P} \prec P_n$. Then $\tau(P_n)|J| \leq \tau(Q) \leq \tau(P_n)(|J|+1)$. Apply this to $I = Q = \sum_{\iota \in \widetilde{J}} \widetilde{P}_\iota + \widetilde{\widetilde{P}}$ to get $\tau(P_n)|\widetilde{J}| \leq 1 \leq \tau(P_n)(|\widetilde{J}|+1)$. We get

$$\tau(Q) \in \Big[\frac{|J|}{|\widetilde{J}|+1}, \frac{|J|+1}{|\widetilde{J}|}\Big].$$

Also $|\widetilde{J}| \geq 2^{n+1}$, hence $\tau|_{\mathcal{P}(\mathcal{M})}$ is unique.

Next, for $x \geq 0$ consider

$$\frac{1}{n}\sum_{k=1}^{\infty} E([\frac{k}{n}, \infty), x) \leq x \leq \frac{1}{n} + \frac{1}{n}\sum_{k=1}^{\infty} E([\frac{k}{n}, \infty), x).$$

Note that the sums are actually finite. Therefore $\tau|_{\mathcal{P}(\mathcal{M})}$ determines τ on the set of positive elements, and hence everywhere.

(b) Let $\lambda \in (0,1)$. Construct recursively

$$P^{(1)} \leq P^{(2)} \leq \cdots \leq P^{(n)} \leq Q^{(n)} \leq Q^{(n-1)} \leq \cdots \leq Q^{(1)}$$

with $\tau(P^{(n)}) \leq \lambda \leq \tau(Q^{(n)})$ and $Q^{(n)} - P^{(n)} = E_1 + \cdots + E_m + E_{m+1}$, $E_j \sim P_n$ for $j = 1, \ldots, m$ and $E_{m+1} \prec P_n$. Take k such that $\tau(P^{(n)} + E_1 + \cdots + E_k) < \lambda < \tau(Q^{(n)} - E_{k+2} - \cdots - E_{m+1})$. Let $\bigvee P^{(n)} = P, \bigwedge Q^{(n)} = Q$. Then $\tau(P) \geq \tau(P^{(n)}) \geq \lambda - \tau(P_n)$ and $\tau(Q) \leq \lambda + \sigma(P_n)$.

□

15.2. Kadison-Fuglede positive determinant

Let $\mathcal{M}$ be a type II_1 factor. Take $x \in \mathrm{GL}_1(\mathcal{M})$. Set $\Delta(x) = \exp(\tau(\log(x^*x)^{1/2}))$. It is true that $\Delta(xy) = \Delta(x)\Delta(y)$.

16. The Standard Form of a von Neumann algebra with a faithful uwo-continuous tracial state

Theorem 16.1. *Let $\mathcal{M}$ be a von Neumann algebra with a faithful uwo continuous tracial state τ and consider the GNS construction $(\mathcal{H}_\tau, \pi_\tau, \xi_\tau)$. $\pi_\tau(\mathcal{M})$ is a von Neumann algebra isomorphic to $\mathcal{M}$ as a C^*-algebra with uwo-topology with remarkable spatial properties:*

- *ξ_τ is a cyclic and separating vector for $\pi_\tau(\mathcal{M})$.*
- *$(\pi_\tau(\mathcal{M}))'$ is isomorphic to $\mathcal{M}_{\mathrm{op}}$, the C^*-algebra with opposite multiplication and with the same predual as $\mathcal{M}$.*

- *The isomorphism with the opposite algebra has a spatial implementation by a conjugate-linear unitary operator J_τ with $J_\tau^2 = I$ and the antiisomorphism with the commutant is $\pi_\tau(\mathcal{M}) \ni a \mapsto Ja^*J \in \pi_\tau(\mathcal{M})'$.*

Definition. *If $\mathcal{N}$ is a $*$-algebra on $\mathcal{H}$, a vector $\xi \in \mathcal{H}$ is a* tracial *vector if $\|n\xi\| = \|n^*\xi\|$ for all $n \in \mathcal{N}$.*

The first step toward the proof of the theorem will be to show that via the GNS construction we can pass to the situation of a von Neumann algebra with a cyclic tracial vector.

Remarks.

- ξ is a tracial vector iff $\omega_\xi : \mathcal{N} \to \mathbb{C}$, $\omega_\xi(n) = \langle n\xi, \xi\rangle$ is a trace.
- If ξ is a cyclic tracial vector, then ξ is separating. Indeed, if $n\xi = 0$, then $n_1 n\xi = 0$ for all $n_1 \in \mathcal{N}$, so $n^* n_1^* \xi = 0$ as ξ is tracial, and then $n^*\mathcal{N}\xi = 0$, so $n^* = 0$, so $n = 0$.

Back to $\tau : \mathcal{M} \to \mathbb{C}$ uwo-continuous faithful tracial state. We first show that $\pi_\tau(\mathcal{M})$ is a von Neumann algebra. Here are the steps:

1. ξ_τ is a tracial vector for the C^*-algebra $\pi_\tau(\mathcal{M})$.
2. ξ_τ is cyclic by GNS, so also separating.
3. $\|m\| = \|\pi_\tau(m)\|$. Indeed, τ is faithful, so π_τ is injective, and an injective $*$-homomorphism of C^*-algebras is isometric.
4. $\pi_\tau(\mathcal{M})$ is a von Neumann algebra: π_τ is uwo continuous because τ is uwo continuous. By 3, $(\pi_\tau(\mathcal{M}))_1 = \pi_\tau(\mathcal{M}_1)$, hence $(\pi_\tau(\mathcal{M}))_1$ is uwo compact and hence Corollary 5.3, $\pi_\tau(\mathcal{M})$ is a von Neumann algebra.
5. Actually $\pi_\tau(\mathcal{M})$ and $\mathcal{M}$ are not only isomorphic as rings with involution via π_τ, but also ultraweakly (and still via π_τ). Indeed, π_τ being uwo continuous, the predual of $\pi_\tau(\mathcal{M})$ identi-fieswith a closed subspace (isometrically) of the predual of $\mathcal{M}$. But the predual is unique, so they must coincide.

So we reduced to the case of a von Neumann algebra with a tracial vector producing the tracial state, we are for the rest in this situation: $\mathcal{M}$ is a von Neumann algebra on $\mathcal{H}$, with cyclic tracial vector ξ. We define $J : \mathcal{H} \to \mathcal{H}$ by $Jm\xi = m^*\xi$ and extend the definition to $\mathcal{H}$ by continuity, since J is isometric. Since $J^2 = I$, we also have that J is invertible. So J is a conjugate-linear unitary operator. We call J an antiunitary operator.

Remark. Aside: How to handle conjugate-linear operators. Let $X : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded conjugate-linear operator. We can view X as a linear operator $X : \mathcal{H}_1 \to (\mathcal{H}_2)_c$ where $\mathcal{H}_c$ is the Hilbert space with the same underlying real vector space of $\mathcal{H}$, but $\lambda \cdot h_c = (\overline{\lambda} h)_c$ and $\langle h_c, h'_c\rangle = \overline{\langle h, h'\rangle}$. So then $X : \mathcal{H}_1 \to \mathcal{H}_{2,c}$ becomes unitary. For a conjugate-linear operator, the adjoint is defined as the adjoint of the operator $X : \mathcal{H}_1 \to \mathcal{H}_{2,c}$. Thus $\langle Xh_1, h_2\rangle_c = \langle h, X^*h_2\rangle$ or $\overline{\langle Xh_1, h_2\rangle} = \langle h, X^*h_2\rangle$.

For anti-unitaries when we polarize the isometricity relation, we get $\overline{\langle Xh, Xk\rangle} = \langle h, k\rangle$.

Introducing $\lambda(m), \rho(m)$. If $m \in \mathcal{M}$, we will also denote m by $\lambda(m)$. The reason is that $\lambda(m)m_1\xi = (mm_1)\xi$. We define $\rho(x) = J\lambda(x^*)J$. Since there are two J's, this is a linear operator and $\rho(cx) = c\rho(x)$. We have

$$\rho(m_1)m\xi = J\lambda(m_1^*)Jm\xi = J\lambda(m_1^*)m^*\xi = Jm_1^*m\xi = mm_1\xi.$$

So $\rho(m_1)$ is "right multiplication by m_1" on $\mathcal{M}\xi \sim \mathcal{M}$. [Actually often in this situation one identifies $\mathcal{H}$ with $L^2(\mathcal{M}, \tau)$ which we may view as another notation for the GNS Hilbert space $m\xi \to m$. Then $\lambda(m)$ and $\rho(m)$ are left and right multiplication and ρ is a representation of $\mathcal{M}^{\mathrm{op}}$.]

Theorem 16.2. *Let $\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$, ξ a cyclic tracial vector, J closure of $m\xi \mapsto m^*\xi$. Then*

$$J\mathcal{M}J = M'.$$

This will finish the proof of **??**.

Remark. $J\mathcal{M}J$ is actually by the definition of ρ just $\rho(\mathcal{M})$.

Proof.

1. ξ is cyclic and separating for $\mathcal{M}'$ (as it is separating and cyclic for $\mathcal{M}$).
2. $\rho(\mathcal{M}) \subseteq \mathcal{M}'$. Indeed, $\rho(m_1)\lambda(m_2)m\xi = \cdots = \lambda(m_2)\rho(m_1)m\xi$, hence $[\rho(m_1), \lambda(m_2)]\mathcal{M}\xi = 0$, so the commutator is 0.
3. The main idea of the proof will be to show $Jz\xi = z^*\xi$ if $z \in \mathcal{M}'$. Indeed, this would have the following consequences:
   (i) $\|z\xi\| = \|z^*\xi\|$ for $z \in \mathcal{M}'$ so that ξ is a tracial cyclic and separating vector $\mathcal{M}' \supseteq \rho(\mathcal{M})$.
   (ii) The map J closure of $z\xi \mapsto z^*\xi$ for $z \in \mathcal{M}'$ is the same as the map we got from $\mathcal{M}$. Then we get $J\mathcal{M}'J \subseteq (\mathcal{M}')' = \mathcal{M}$, hence $\mathcal{M} = J(J\mathcal{M}J)J = J\rho(\mathcal{M})J \subseteq J\mathcal{M}'J \subseteq \mathcal{M}$, so that $J\mathcal{M}'J = \mathcal{M}$ or equivalently $\mathcal{M}' = J\mathcal{M}J$.
4. Prove that $Jz\xi = z^*\xi$ for $z \in \mathcal{M}'$. Let $m \in \mathcal{M}, z \in \mathcal{M}'$. Then

$$\begin{aligned}\langle m\xi, z\xi\rangle = \langle z^*\xi, m^*\xi\rangle = \langle z^*\xi, Jm\xi\rangle &= \overline{\langle Jz^*\xi, J^2m\xi\rangle} \\ &= \overline{\langle Jz^*\xi, m\xi\rangle} = \langle m\xi, Jz^*\xi\rangle.\end{aligned}$$

   Since $\mathcal{M}\xi$ is dense in $\mathcal{H}$, we get $z\xi = Jz^*\xi$.

□

Corollary 16.3. *Let $\mathcal{M}$ be a von Neumann algebra with cyclic tracial vector. Then $\mathcal{M}^{\mathrm{op}} \sim \mathcal{M}'$.*

Note that moreover, $x \in Z(\mathcal{M}) \implies Jx^*J = x$. Indeed, $Jx^*Jm\xi = Jx^*m^*\xi = mx\xi = xm\xi$ and $\overline{\mathcal{M}\xi} = \mathcal{H}$.

Definition. *A von Neumann algebra $\mathcal{M}$ on $\mathcal{H}$ is in* standard form *if there is $J : \mathcal{H} \to \mathcal{H}$, anti-unitary, with $J^2 = I$ such that $J\mathcal{M}J = \mathcal{M}'$ and $Jx^*J = x$ if $x \in Z(\mathcal{M})$.*

Remarks.

- It is a fact: Every von Neumann algebra has a standard form [i.e. there is an uwo continuous representation π of $\mathcal{M}$ which is an isomorphism of C^*-algebras and so that $\pi(\mathcal{M})$ is in standard form]
- The von Neumann algebra $(\lambda(G))''$ of a group is isomorphic to its opposite, since $G \sim G^{\text{op}}$ (via $g \mapsto g^{-1}$).
- There is a II_1 factor $\mathcal{M}$ so that $\mathcal{M} \not\sim \mathcal{M}^{\text{op}}$ (due to Connes).

Example (to HW 6)**.** Suppose U, V are unitaries (on some Hilbert space) such that $UV = VUe^{2\pi i\theta}$ for some irrational θ. Let $\mathcal{A}_\theta$ be the C^*-algebra generated by them. E.g. on $L^2(S^1)$ consider $g \mapsto g(z)z$ and $g(\cdot) \mapsto g(\cdot e^{2\pi i\theta})$. Suppose $\tau : \mathcal{A}_\theta \to \mathbb{C}$ is a trace state. The closed linear span of the $U^m V^n$, $m, n \in \mathbb{Z}$, is $\mathcal{A}_\theta$. We have $\tau(U^m V^n) = \tau(V^n U^m) = e^{\pm 2\pi i mn\theta}\tau(U^m V^n)$. So if $mn \neq 0$, then $\tau(U^m V^n) = 0$. Also one shows that $\tau(U^n) = 0 = \tau(V^m)$ for $n, m \neq 0$. (e.g. $\theta(U) = \theta(V^*UV) = \dots$ similar argument as before).

17. Isomorphisms of von Neumann Algebras

There are two kinds of isomorphisms of von Neumann algebras. Let $\mathcal{M}_k$ be von Neumann algebras on Hilbert spaces $\mathcal{H}_k$, $k = 1, 2$.

(a) $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic as C^*-algebras and the isomorphism is a homeomorphism w.r.t. their uwo topologies. We already saw that the Sakai Theorem and the uniqueness of predual in its statement implies that $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic in this sense if $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphism as C^*-algebras. Actually, even more, it suffices that $\mathcal{M}_1, \mathcal{M}_2$ are isomorphic as $*$-algebras (purely algebraically). We will refer sometimes to such isomorphisms as *non-spatial.*

(b) $\mathcal{M}_1$ and $\mathcal{M}_2$ are spatially isomorphic if there is a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ so that $\mathcal{M}_1 = U^*\mathcal{M}_2 U$.

The following theorem says that abstract isomorphisms can turned into spatial isomorphisms in a suitable way.

Theorem 17.1. *If $\rho : \mathcal{M}_1 \to \mathcal{M}_2$ is an uwo continuous isomorphism, then there is a Hilbert space $\mathcal{X}$ and a projection $P \in (\mathcal{M}_1 \otimes I_{\mathcal{X}})' = \mathcal{M}_1' \otimes \mathcal{B}(\mathcal{X})$ (amplification) which is separating for $\mathcal{M}_1 \otimes I_{\mathcal{X}}$ (i.e. $P(\mathcal{H}_1 \otimes \mathcal{X})$ is separating) and so that there is a unitary operator $V : P(\mathcal{H}_1 \otimes \mathcal{X}) \to \mathcal{H}_2$ for which*

$$\rho(m) = V((m \otimes I_{\mathcal{X}})|_{P(\mathcal{H}_1 \otimes \mathcal{X})})V^*$$

for $m \in \mathcal{M}_1$. If $\mathcal{H}_2$ is separable, one may choose $\mathcal{X} = \ell^2(\mathbb{N})$.

Conversely, a homomorphism of $\mathcal{M}_1$ defined in this way (i.e. ampliation and restriction to a separating reducing subspace) has as range a von Neumann algebra and is an ultraweak isomorphism of the two.

Lemma 17.2. *Let $\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$, $\pi : \mathcal{M} \to \mathcal{B}(\mathcal{H}_2)$ a C^*-homomorphism,* $\ker \rho = 0$*, π uwo continuous. Then $\pi(\mathcal{M})$ is a von Neumann algebra and $\mathcal{M}$ and $\pi(\mathcal{M})$ are uwo isomorphic via π.*

Proof of Theorem 17.1. The converse is a direct application of the lemma, hence clear. For the other direction, the idea is to decompose $\mathcal{H}_2$ into $\mathcal{M}_2$-cyclic subspace, they correspond to GNS constructions, then pullback states to $\mathcal{M}_1$. See notes for rest. □

18. The hyperfinite II_1 factor

Here all Hilbert spaces will be assumed to be separable. For every $n \geq 1$, let $M_n = M_{n\times n}(\mathbb{C}) = \mathcal{B}(\mathbb{C}^n)$. For $n \mid m$, fix the embedding $M_n \hookrightarrow M_m$ given by $A \mapsto A \otimes I$ where we view $\mathbb{C}^m = \mathbb{C}^n \otimes \mathbb{C}^{m/n}$. If $n_1, n_2, \dots$ is a sequence with $n_j \mid n_{j+1}$ for all j, we can then consider $M_0 = \cup_j M_{n_j}$. On each M_n we normalize the trace such that $\tau(I) = 1$. Then we get an induced trace on M_0. Then denote by $\overline{\cup M_{n_j}}$ the corresponding GNS algebra. It turns out that these are all (for different sequences n_j) isomorphic type II_1 factors, and also isomorphic to $L(S_\infty)$. The key to this is the notion of hyperfiniteness.

Definition. *A II_1 factor $\mathcal{M}$ is* hyperfinite *if for any given $\varepsilon > 0$ and $x_1, \dots, x_n \in \mathcal{M}$, there is a finite-dimensional $*$-subalgebra $\mathcal{A} \subseteq \mathcal{M}$ and are $a_1, \dots, a_n \in \mathcal{A}$ such that $|a_j - x_j|_2 < \varepsilon$ where $|\cdot|_2$ is the Hilbert space norm coming from the GNS construction corresponding to τ (i.e. $|m|_2^2 = \tau(m^*m)$).*

The group S_∞ is the union of the finite groups $\bigcup_n S_n$ and similarly $\overline{\cup_{j\geq 1} M_{n_j}}$ contains the dense subalgebra $\cup_{j\geq 1} M_{n_j}$. So in both cases for the corresponding II_1 factors there are increasing sequences $1 \in \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \ldots \subseteq \mathcal{M}$ where $\mathcal{A}_j$ are $*$-subalgebras, $\dim \mathcal{A}_j < \infty$ and $\bigcup_{j\geq 1} \mathcal{A}_j$ is wo dense in $\mathcal{M}$. Then it is also so dense, so get density in $|\cdot|_2$-norm.

Theorem 18.1. *Let $\mathcal{M}_1, \mathcal{M}_2$ be separable II_1 factors. If $\mathcal{M}_1$ and $\mathcal{M}_2$ are hyperfinite, then they are isomorphic.*

Why 2-norm in the definition of hyperfiniteness and not the uniform norm? Suppose P, Q are orthogonal projections with $\|P - Q\| < 1$, then $P \sim Q$. Consider $G = PQ + (I - P)(I - Q)$. Then $GQ = PQ = PG$. Now $G - I = P(Q - P) + (I - P)((I - Q) - (I - P)) = (2P - I)(Q - P)$. Now $2P - I$ is unitary, so $\|G - I\| = \|Q - P\| < 1$, so G is invertible. So from $GQ = PQ$ we get $GQG^{-1} = P$, so P, Q have the same trace and are therefore equivalent. Suppose $P \in \mathcal{P}(\overline{\cup M_{n_j}}^{\|\cdot\|})$, there exists $Q \in \mathcal{P}(\cup M_{n_j})$ with $\|P - Q\| < \varepsilon$. Indeed, start with $X \in \cup M_{n_j}$ with $\|P - X\|$ small and X self-adjoint. Then $X - X^2$ is small, then take $Q = \varphi(X)$ for a suitable function φ ($\varphi = 0$ around 0 and $\varphi = 1$ around 1 (so that it is locally constant on $\sigma(X)$) and otherwise close to identity function). By what we have seen above, then $\tau(P) = \tau(Q)$. Therefore $\tau(\mathcal{P}(\overline{\cup M_{n_j}})) = \tau(\mathcal{P}(\overline{\cup M_{n_j}}))$. So $\bigcup_j \{\frac{k}{n_j} \mid 0 \leq k \leq n_j\}$ is an invariant of $\overline{\cup M_{n_j}}^{\|\cdot\|}$.

Now suppose $P, Q \in (\mathcal{M}, \tau)$ are such that $|P - Q|_2 < \varepsilon$. Then there are $P' \leq P, Q' \leq Q$ such that $\tau(P - P'), \tau(Q - Q')$ are small and $P' \sim Q'$.

Remark. One can define L^p spaces, e.g. $L^1(\mathcal{M},\tau)$ where $|m|_1 = \tau((m^*m)^{1/2})$. Then $L^1(\mathcal{M},\tau)^* = L^\infty(\mathcal{M},\tau)$. If $\mathcal{N} \subseteq \mathcal{M}$, then the dual of $L^1(\mathcal{N},\tau) \hookrightarrow L^1(\mathcal{M},\tau)$, i.e. a map $L^\infty(\mathcal{M},\tau) \to L^\infty(\mathcal{N},\tau)$, is the conditional expectation $E_{\mathcal{N}}^{\mathcal{M}}$.

19. Conditional Expectations in the case of a uwo continuous faithful tracial state

Let $\mathcal{M}$ be a von Neumann algebra and $\tau : \mathcal{M} \to \mathbb{C}$ a uwo continuous faithful tracial state. Via GNS $(\mathcal{H}_\tau, \pi_\tau, \xi_\tau)$ we can get a standard form, which makes this equivalent to $\mathcal{M}$ acting on $\mathcal{H}$ with cyclic and hence separating tracial vector ξ. There is also another equivalent notation for this which is quite convenient: $L^2(\mathcal{M},\tau) \supseteq L^\infty(\mathcal{M},\tau) \ni 1$.

$$\begin{tikzcd}
L^2(\mathcal{M},\tau) \arrow[d, dash] & \supseteq & L^\infty(\mathcal{M},\tau) \arrow[d, dash] & \ni & 1 \arrow[d, dash] \\
\mathcal{H}_\tau \arrow[d, dash] & & \mathcal{M}\xi \arrow[d, dash] & & \xi \\
\mathcal{H} & & \mathcal{M} & &
\end{tikzcd}$$

On $\mathcal{H} \sim \mathcal{H}_\tau \sim L^2(\mathcal{M},\tau)$ we then have $\lambda(m)$ and $\rho(m)$, $m \in \mathcal{M}$, acting, so that $\lambda(m)m_1\xi = mm_1\xi$ and $\rho(m) = Jm^*J$, $J(m)m_1\xi = m_1 m\xi$.

The standard form for $I \in \mathcal{N} \subseteq \mathcal{M}$, a unital von Neumann *sub*algebra, is then obtained from $\tau|_{\mathcal{N}}$, and amounts to restrictions of λ to $\mathcal{N}$ and $\overline{\mathcal{N}\xi}$, which identifies with $L^2(\mathcal{N},\tau|_{\mathcal{N}})$, and $J|_{\overline{\mathcal{N}\xi}}$ $(\simeq J|_{L^2(\mathcal{N},\tau|_{\mathcal{N}})})$ is the involutive antiunitary.

Let $E_{\mathcal{N}}^{\mathcal{M}}$ be the orthogonal projection from $\mathcal{H} = \overline{\mathcal{M}\xi}$ onto $\overline{\mathcal{N}\xi}$.

Note that $\mathcal{M} \sim L^\infty(\mathcal{M},\tau) \sim \mathcal{M}\xi$ is a dense subspace of $L^2(\mathcal{M},\tau)$ and similarly with $\mathcal{N}$.

Proposition 19.1. *Facts ($E = E_{\mathcal{N}}^{\mathcal{M}}$):*

- $EL^\infty(\mathcal{M},\tau) \subseteq L^\infty(\mathcal{N},\tau)$ *and* $\|Em\| \le \|m\|$, $E^2 = E$, $E(m^*) = E(m)^*$ *for* $m \in \mathcal{M}$.
- $E(n_1 m n_2) = n_1 E(m) n_2$ *for* $m \in \mathcal{M}, n_1, n_2 \in \mathcal{N}$.
- $E(m^*m) \ge E(m^*)E(m)$ *for* $m \in \mathcal{M}$.
- *If* $\mathcal{M} \ni m \ge 0$, *then* $E(m) \ge 0$.
- E *is uwo-continuous.*
- $\tau \circ E = \tau$.

Definition. *The* conditional expectation for $\mathcal{M}$ given $\mathcal{N}$ *is the induced map* $E : \mathcal{M} \to \mathcal{N}$.

Proof. $E^2 = E$, $E|_{\mathcal{N}} = \mathrm{id}_{\mathcal{N}}$ is obvious. $L^2(\mathcal{N}) = \overline{\mathcal{N}\xi} \subseteq \overline{\mathcal{M}\xi} = L^2(\mathcal{M},\tau)$ is an invariant subspace for $\lambda(\mathcal{N}), \rho(\mathcal{N}), J$. Since $\lambda(\mathcal{N})$,$\rho(\mathcal{N})$ are $*$-algebras, the orthogonal projection E onto an invariant subspace commutes with these: $[E, \lambda(\mathcal{N})] = [E, \rho(\mathcal{N})] = 0$. The relation $[E, J] = 0$ can be obtained similarly. If one wants to avoid discussing conjugate-linear operators, see notes.

To prove $EL^\infty(\mathcal{M}) \subseteq L^\infty(\mathcal{N})$, let $m \in \mathcal{M}$ and let $T = E\lambda(m)|_{\overline{\mathcal{N}\xi}}$ be the *compression* of T to $\overline{\mathcal{N}\xi}$. Clearly $\|T\| \leq \|m\|$. Also if $m \geq 0$, then $T \geq 0$, and $T^* = E\lambda(m^*)|_{\overline{\mathcal{N}\xi}}$. Since E and $\lambda(m)$ commute with $\rho(\mathcal{N})$, we get $[T, \rho(\mathcal{N})|_{L^2(\mathcal{N})}] = 0$. But $(L^2(\mathcal{N}), \lambda(\mathcal{N})|_{L^2(\mathcal{N})}, J|_{L^2(\mathcal{N})})$ is the standard form of $\mathcal{N}$ and $\rho(\mathcal{N})|_{L^2(\mathcal{N})}$ is just the corresponding $\rho(\mathcal{N})$ for $(\mathcal{N}, \tau|_{\mathcal{N}})$. Then $[T, \rho_{\mathcal{N}}(\mathcal{N})] = 0$ gives $T \in \lambda_{\mathcal{N}}(\mathcal{N})$, i.e. $T = \lambda(n)$ for some $n \in \mathcal{N}$. Then $Em\xi = E\lambda(m)E\xi = T\xi = \lambda_{\mathcal{N}}(n)\xi = n\xi$, so E takes $L^\infty(\mathcal{M})$ to $L^\infty(\mathcal{N})$.

The remaining properties follow easily. For example (see notes) □

Examples.

1. $\mathcal{M} = L^\infty(X, \Sigma, \mu)$, the L^∞-space of a probability measure space with σ-algebra Σ and $\mathcal{N} = L^\infty(X, \Sigma', \mu|_{\Sigma'})$ where $\Sigma' \subseteq \Sigma$ is a σ-subalgebra. $\mathcal{M}$ and $\mathcal{N}$ are von Neumann algebras of multiplication operators on $L^2(X, \Sigma, \mu)$. The tracial state is $\tau(f) = \int f d\mu = \langle M_f 1_X, 1_X \rangle_{L^2(X,\Sigma,\mu)}$ for $f \in L^\infty$ where 1_X is the constant function 1 and is the tracial vector. Then $E : L^\infty(X, \Sigma, \mu) \to L^\infty(X, \Sigma', \mu|_{\Sigma'})$ is the classical conditional expectation in probability theory.
2. G a discrete group, λ the left regular representation of G on $\ell^2(G)$, $H \subseteq G$ a subgroup. Let $\mathcal{M} = L(G) = \overline{\operatorname{span}\lambda(G)}^{uwo}$, $\mathcal{N} = \overline{\operatorname{span}\lambda(H)}^{uwo}$ and $L(H) = \overline{\operatorname{span}\lambda(H)}^{uwo}_{\ell^2(H)}$. Then $N \sim L(H)$ and
$$E \sum_{g\in G} c_g\lambda(g) = \sum_{g\in H} c_g\lambda(g).$$

Let $(\mathcal{M}, \tau)$ be a von Neumann algebra with faithful uwo continuous tracial state τ. Let G be a finite group acting on $(\mathcal{M}, \tau)$ (preserving τ) via α. Let
$$\mathcal{M}^{\alpha(G)} = \{m \in \mathcal{M} \mid \alpha(g)m = m \forall g \in G\}.$$
Then $E := E^{\mathcal{M}}_{\mathcal{M}^{\alpha(G)}}(x) = \frac{1}{|G|}\sum_{g\in G}\alpha(g)(x)$. For this verify that $E^2 = E$ and $\operatorname{Ran} E = \mathcal{M}^{\alpha(G)}$. Next $E^* = E$: $\alpha(g)$ is isometric in L^2: Indeed, $\tau(\alpha(g)(m)^*\alpha(g)(m)) = \tau(\alpha(g)(m^*m)) = \tau(m^*m)$. So α is unitary in L^2, i.e. $\alpha(g)^* = \alpha(g^{-1})$. Then
$$E = \frac{1}{|G|}\sum_{g\in G}\alpha(g) = \frac{1}{|G|}\frac{1}{2}\sum_{g\in G}\alpha(g) + \alpha(g^{-1}) = \frac{1}{2}(E + E^*),$$
so $E = E^*$.

Example. This can be applied to the second part of HW 7: Consider $G = \{\pm 1\}^n$, and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in G$ acts on $\mathcal{M}$ via $\alpha(\varepsilon)(m) = U_\varepsilon m U_\varepsilon^{-1}$ where $U_\varepsilon = \sum_{j=1}^n \varepsilon_j P_j$.

20. The Connes Theorem and Amenability

There is a C^*-algebraic notion related to the conditional expectation, that of *projection of norm one*:

Theorem 20.1. *Let $\mathcal{A}$ be a unital C^* algebra and $\mathcal{B}$ a unital C^*-subalgebra. Let $\pi : \mathcal{A} \to \mathcal{B}$ be a linear map so that $\pi|_{\mathcal{B}} = \mathrm{id}_{\mathcal{B}}$ and $\|\pi(a)\| \leq \|a\|$. Then:*

- *If $a \in \mathcal{A}, a \geq 0$, then $\pi(a) \geq 0$.*
- *$\pi(b_1 a b_2) = b_1 \pi(a) b_2$ for $b_1, b_2 \in \mathcal{B}$ and $a \in \mathcal{A}$.*
- *$\pi(a)^* \pi(a) \leq \pi(a^* a)$ for $a \in \mathcal{A}$.*

This also plays a role in the theory of von Neumann algebras.

Theorem 20.2 (Connes)**.** *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a type II_1 factor (with $\mathcal{H}$ separable). TFAE:*

1. *$\mathcal{M}$ is hyperfinite.*
2. *There is a projection E of norm one of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{M}$ as in the previous theorem.*
3. *Let $\mathcal{M}'$ be the commutant of $\mathcal{M}$ in the standard form of $\mathcal{M}$. Then the norm on the $*$-algebra generated by $\mathcal{M}$ and $\mathcal{M}'$ in $\mathcal{B}(L^2(\mathcal{M}, \tau))$ is the same as the norm on $\mathcal{M} \otimes \mathcal{M}'$ in $\mathcal{B}(L^2(\mathcal{M}, \tau) \otimes L^2(\mathcal{M}, \tau))$, i.e. $\|\sum x_k y_k\| = \|\sum x_k \otimes y_k\|$ for $x_k \in \mathcal{M}, y_k \in \mathcal{M}'$.*
4. *There are finite rank maps $\Phi_n : \mathcal{M} \to \mathcal{M}$ with $|\Phi_n(m) - m|_2 \xrightarrow{n \to \infty} 0$ for $m \in \mathcal{M}$, $\Phi_n(1) = 1$, $\tau \circ \Phi_n = \tau$, and that are completely positive, i.e. if $\sum m_{ij} \otimes e_{ij} \geq 0$, then $\sum \Phi_n(m_{ij}) \otimes e_{ij} \geq 0$ for $m_{ij} \in \mathcal{M}, e_{ij} \in M_N$.*

Note in 4. Φ is competeley positive if the induced map $M_N(\mathcal{M}) \to M_N(\mathcal{M})$ is completely positive for all N.

Corollary 20.3. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a hyperfinite II_1-factor. If $I \in \mathcal{N} \subseteq \mathcal{M}$ is a subfactor, which is not finite-dimensional, then $\mathcal{N}$ is also hyperfinite.*

Proof. Since $\mathcal{M}$ is hyperfinite, there is a projection of norm 1 of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{M}$. Then if $E^{\mathcal{M}}_{\mathcal{N}}$ is the conditional expectation of $\mathcal{M}$ onto N we have that $E^{\mathcal{M}}_{\mathcal{N}} \circ E$ is a projection of norm one of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{N}$. This implies by the Connes theorem that $\mathcal{N}$ is hyperfinite. $\square$

Definition. *A discrete group G is* amenable *if there is a state*

$$\Phi : \ell^\infty(G) \to \mathbb{C}$$

which is invariant under right shifts, i.e. $\Phi(f(\cdot h)) = \Phi(f)$ for all $h \in G$.

Lemma 20.4. *Assume G is amenable, $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ a von Neumann algebra and $u : G \to \mathcal{M}$ a homomorphism with $u(g)$ unitary and $(u(G))'' = \mathcal{M}$. Then there is a projection of norm one $E : \mathcal{B}(\mathcal{H}) \to \mathcal{M}'$.*

Proof. The idea is to define $E(T)$ where $T \in \mathcal{B}(\mathcal{H})$ as some kind of average of $u(g) T u(g)^*$, $g \in G$, where the averaging is done using the amenability of G. To do this we pass to the sesquilinear forms defined by operators. So if $T \in \mathcal{B}(\mathcal{H})$ and $\xi, \eta \in \mathcal{H}$, let

$$f_{\xi,\eta}(g) = \langle u(g)^* T u(g) \xi, \eta \rangle.$$

Then $f_{\xi,\eta} \in \ell^\infty(G)$. We define the sesquilinear form $B[T](\xi,\eta) = \Phi(f_{\xi,\eta})$, where Φ is the invariant state as in the definition. Since $|f_{\xi,\eta}(g)| \leq \|\xi\|\,\|\eta\|\,\|T\|$ and Φ is a state, which implies $|\Phi(f)| \leq \|f\|_\infty$, we get $|B[T](\xi,\eta)| \leq \|\xi\|\,\|\eta\|\,\|T\|$. It follows that there is a unique operator $E(T) \in \mathcal{B}(\mathcal{H})$ so that $B[T](\xi,\eta) = \langle E(T)\xi,\eta\rangle$ for all ξ,η and $\|E(T)\| \leq \|T\|$. Note that if $h \in G$,

$$\begin{aligned} f_{\xi,\eta}(gh) &= \langle u(gh)^* T u(gh)\xi,\eta\rangle \\ &= \langle u(g)^* T u(g) u(h)\xi, u(h)\eta\rangle \\ &= f_{u(h)\xi,u(h)\eta}(g) \end{aligned}$$

which implies $\Phi(f_{\xi,\eta}) = \Phi(f_{\xi,\eta}(\cdot h)) = \Phi(f_{u(h)\xi,u(h)\eta})$. Hence

$$B[T](\xi,\eta) = B[T](u(h)\xi,u(h)\eta)$$

so that

$$\langle E(T)\xi,\eta\rangle = \langle E(T)u(h)\xi,u(h)\eta\rangle = \langle u(h)^* E(T) u(h)\xi,\eta\rangle.$$

Since this holds for all $\xi,\eta \in \mathcal{H}$, we have $E(T) = u(h)^* E(T) u(h)$ which gives that $E(T) \in (u(G))' = \mathcal{M}'$. If $T \in \mathcal{M}' = (u(G))'$, we have $u(g)^* T u(g) = T$, so $f_{\xi,\eta}(g) = \langle T\xi,\eta\rangle$ (constant) and hence $\langle E(T)\xi,\eta\rangle = \Phi(f_{\xi,\eta}) = \langle T\xi,\eta\rangle$, so $E(T) = T$. □

Corollary 20.5. *If $\mathcal{M}$ is a II_1 factor so that $\mathcal{M} = (u(G))''$ for a unitary representation of an amenable group G, then $\mathcal{M}$ is hyperfinite.*

Proof. Use the standard form of $\mathcal{M}$, so that $\mathcal{M}' \sim \mathcal{M}^{\rm op}$. Then by the lemma there is a projection of norm one of $\mathcal{B}(L^2(\mathcal{M},\tau))$ onto $\mathcal{M}'$. By Connes' Theorem, $\mathcal{M}^{\rm op}$ is hyperfinite. Clearly $\mathcal{M}^{\rm op}$ is hyperfinite iff $\mathcal{M}$ is hyperfinite. □

As a converse of the lemma we have:

Proposition 20.6. *If G is a discrete group so that there is a projection of norm one $E : \mathcal{B}(\ell^2(G)) \to (\lambda(G))''$, then G is amenable.*

Proof. If $f \in \ell^\infty(G)$, let M_f be the multiplication operator in $\ell^2(G)$ and define $\Phi(f) = \tau(E(M_f))$ where τ is $\langle \cdot\,\varepsilon_e,\varepsilon_e\rangle$ on $(\lambda(G))''$. □

Conclusion:

Corollary 20.7. *If G is a countable icc group, then $(\lambda(G))''$ is hyperfinite iff G is amenable.*

Let F_n denote the free group on n generators $g_1,\dots,g_n$. For $n \geq 2$, F_n is icc and non-amenable. Here is how this can be seen when $n = 2$. Let $X(g_j^{\pm 1})$ be the subsets of $F_2 \setminus \{e\}$ which consist of reduced words which start on the right with a positive power of $g_j^{\pm 1}$. Then we have a disjoint decomposition

$$F_2 \setminus \{e\} = X(g_1) \sqcup X(g_1^{-1}) \sqcup X(g_2) \sqcup X(g_2^{-1}).$$

Then

$$F_2 = X(g_1) \cup X(g_1^{-1})g_1 = X(g_2) \cup X(g_2^{-1})g_2.$$

Let f_1, f_2, f_3, f_4 be the indicator functions of $X(g_1), X(g_1^{-1}), X(g_2), X(g_2^{-1})$. The decompositions above tell us that

$$\begin{aligned} 1 &\geq f_1 + f_2 + f_3 + f_4 \\ 1 &= f_1 + f_2(\cdot g_1) = f_3 + f_4(\cdot g_2) \end{aligned}$$

If $\Phi : \ell^\infty(G) \to \mathbb{C}$ is a right-invariant state, we would have $\Phi(f_j) \in [0,1]$ for $1 \leq j \leq 4$ and

$$\begin{aligned} 1 &\geq \Phi(f_1) + \Phi(f_2) + \Phi(f_3) + \Phi(f_4) \\ 1 &= \Phi(f_1) + \Phi(f_2) = \Phi(f_3) + \Phi(f_4) \end{aligned}$$

These imply $2 \leq 1$, so F_2 is non-amenable. (These decompositions are examples of "paradoxical decompositions")

F_2 is icc. If $g \in F_2 \setminus \{e\}$, then $g \in F(g_j^{\pm 1})$ for some j and with a $+$ or $-$ sign. If $k \in \{1,2\} \setminus \{j\}$, then the length of $g_k^n g g_k^{-n}$ goes to ∞ as $n \to \infty$. Therefore F_2 is icc.

21. Property Γ

Since $L(F_n)$ for $n \geq 2$ is not amenable, it cannot be isomorphic to the hyperfinite II_1 factor. This can also be proven in a different way. Murray and von Neumann defined a so-called *property* Γ, a property of asymptotic center. The hyperfinite II_1 factor has Γ, but $L(F_n)$ does not.

Let $\mathcal{M}$ by a type II_1 factor. The orginial definition of Γ involved unitary operators:

> **Definition.** *$\mathcal{M}$ has* property Γ *if for every $x_1, \ldots, x_n \in \mathcal{M}$ and $\varepsilon > 0$ there is $u \in \mathcal{M}$, unitary, $\tau(u) = 0$, so that $|[u, x_j]|_2 < \varepsilon$.*

This is equivalent to the following: Given $x_1, \ldots, x_n \in \mathcal{M}$ there is a sequence $y_k \in \mathcal{M}$ so that $\sup_{k \in \mathbb{N}} \|y_k\| < \infty$ and $\lim_{k \to \infty} |[y_k, x_j]|_2 = 0$ for $j = 1, \ldots, n$, but $\liminf_{k \to \infty} |y_k - \tau(y_k)1|_2 > 0$. Such a sequence $(y_k)_{1 \leq k}$ is called a *central sequence* and the last condition is that it is non-trivial. If we don't restrict, like here, to the separable setting, then one has to consider central nets.

The condition about the bound on the uniform norms is still unpleasant. In the big paper of Connes on injective II_1 factors, he also shows that Γ is equivalent to: Given $x_1, \ldots, x_n \in \mathcal{M}$ and $\varepsilon > 0$ there is $\xi \in L^2(\mathcal{M}, \tau)$ so that $|\xi|_2 = 1$, $|\lambda(x_j)\xi - \rho(x_j)\xi|_2 < \varepsilon$, but $\langle \xi, 1 \rangle = 0$.

We record this here:

> **Theorem 21.1.** *The following are equivalent:*
>
> *(1) $\mathcal{M}$ has property Γ,*
>
> *(2) Given $x_1, \ldots, x_n \in \mathcal{M}$ there is a non-trivial central sequence, i.e. a sequence $y_k \in \mathcal{M}$ so that $\sup_{k \in \mathbb{N}} \|y_k\| < \infty$ and $\lim_{k \to \infty} |[y_k, x_j]|_2 = 0$ for $j = 1, \ldots, n$, but $\liminf_{k \to \infty} |y_k - \tau(y_k)1|_2 > 0$,*
>
> *(3) Given $x_1, \ldots, x_n \in \mathcal{M}$ and $\varepsilon > 0$ there is $\xi \in L^2(\mathcal{M}, \tau)$ so that $|\xi|_2 = 1$, $|\lambda(x_j)\xi - \rho(x_j)\xi|_2 < \varepsilon$, but $\langle \xi, 1 \rangle = 0$.*

> **Proposition 21.2.** *$L(F_2)$ does not have property Γ.*

Proof. We will use the last condition. Let $x_1 = \lambda(g_1)$, $x_2 = \lambda(g_2)$ where g_1, g_2 are the two generators of F_2. Assume there are $\xi_k \in \ell^2(F_2)$ so that $|\lambda(x_j)\xi_k - \rho(x_j)\xi_k|_2 \to 0$ (beware of the two distinct λ's! we have $\lambda(x_j) = \lambda(g_j)$, but $\rho(x_j) = \rho(g_j^{-1})$). So $|\lambda(g_j)\rho(g_j)\xi_k - \xi_k|_2 \to 0$. Moreover $\langle \xi_k, 1\rangle = 0$ means $\xi_k(e) = 0$. Let S be the set of words in F_2 ending on a non-zero power of g_1. Then $S \cup g_1 S g_1^{-1} = F_2 \setminus \{e\}$. But $g_2^k S g_2^{-k}$ are pairwise disjoint in F_2 for $k \in \mathbb{Z}$. Let P be the projection of $\ell^2(F_2)$ onto $\ell^2(S)$. The projection onto $\ell^2(gSg^{-1})$ is just $\lambda(g)\rho(g)P\lambda(g^{-1})\rho(g^{-1})$. Note that since $g \mapsto \lambda(g)\rho(g)$ is a representation of $G = F_2$, we get that $\lim_{k\to\infty} |\xi_k - \lambda(g)\rho(g)\xi_k|_2 = 0$ implies $||P\xi_k|_2 - |\lambda(g)\rho(g)P\lambda(g^{-1})\rho(g^{-1})\xi_k|_2| \le ||P\xi_k|_2 - |P\lambda(g^{-1})\rho(g^{-1})\xi_k|_2| \to 0$. Since $S \cup g_1 S g_1^{-1} = F_2 \setminus \{e\}$ and $\xi_k(e) = 0$, $||P\xi_k|_2 - |\lambda(g)\rho(g)P\lambda(g^{-1})\rho(g^{-1})\xi_k|_2| \to 0$, we get that

$$1 = |\xi_k|_2^2 \le |P\xi_k|_2^2 + |\lambda(g_1)\rho(g_1)P\lambda(g_1^{-1})\rho(g_1^{-1})\xi_k|_2^2$$

implies that if k is sufficiently large, $|P\xi_k|_2 > \frac{1}{\sqrt{2}} - \varepsilon$ for a given $\varepsilon > 0$, for instance $|P\xi_k|_2 > \frac{2}{3}$. If k is sufficiently large, $|\lambda(g^p)\rho(g^p)P\lambda(g^{-p})\rho(g^{-p})\xi_k|_2 > \frac{2}{3}$ for $p = 0, 1, 2$. But

$$\sum_{p\in\mathbb{Z}} |\lambda(g_2^p)\rho(g_2^p)P\lambda(g_2^{-p})\rho(g_2^{-p})\xi_k|_2^2 \le 1,$$

giving a contradiction. □

Property Γ is related to other important properties of II_1 factors. Other names: non-Γ, spectral gap, full.

Fact:

Theorem 21.3. *If G has property Γ, then the action of G on G by conjugation is amenable.*[a] *The converse is not true.*

[a] I believe an action of a group on a set X is *amenable* if there is an invariant mean on X.

Proposition 21.4. *If $\mathcal{M}$ is the hyperfinite II_1 factor, then $\mathcal{M}$ has property Γ.*

Proof. We can construct $\mathcal{M}$ via GNS from $\overline{\cup_{k\ge1} M_{2^k}}^{\text{norm}}$ w.r.t. its unique trace state. Let π be the GNS representations. Then $\mathcal{M} = \overline{\pi(\cup_{k\ge1} M_{2^k}^{\text{norm}})}^{wo}$. Then $y_k = I_2 \otimes \dots I_2 \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in M_{2^k}$ gives a non-trivial central sequence $(\pi(y_k))_{k\ge1}$. If $x \in \mathcal{M}$, there is a $a \in M_{2^n}$ for some n so that $|x - \pi(a)|_2 < \varepsilon$ (use Kaplansky applied to the dense subalgebra and $\frac{x}{\|x\|} \in \mathcal{M}_1$). Then if $k > n$,

$$|[x, \pi(y_k)]|_2 \le |[a, \pi(y_k)]|_2 + |[x - a, \pi(y_k)]|_2 = |[x - a, \pi(y_k)]|_2 \le 2|x - \pi(a)|_2 \cdot \|\pi(y_k)\| \le 2\varepsilon.$$

Also $\tau(\pi(y_k)) = 0$ while $|\pi(y_k)|_2 = 1$, so that $\liminf_{k\to\infty} |\pi(y_k) - \tau(\pi(y_k))1|_2 > 0$. □

Let $\mathrm{Aut}(\mathcal{M})$ be the automorphism group of $\mathcal{M}$. By uniqueness of the trace we always have $\tau \circ \alpha = \tau$ if $\alpha \in \mathrm{Aut}(\mathcal{M})$. In particular, α gives rise to a unitary operator on $L^2(\mathcal{M}, \tau)$:

$$|\alpha(x)|_2 = \tau(\alpha(x)^*\alpha(x))^{1/2} = ((\tau \circ \alpha)(x^*x))^{1/2} = \tau(x^*x)^{1/2} = |x|_2.$$

Let U_α denote the unitary operators defined by α on $L^2(\mathcal{M}, \tau)$. In this way one can embed $\mathrm{Aut}(\mathcal{M})$ into $U(L^2(\mathcal{M}, \tau))$. In $\mathrm{Aut}(\mathcal{M})$ there is the subgroup $\mathrm{Int}(\mathcal{M})$ of inner automorphisms $\mathcal{M} \ni m \mapsto umu^{-1} \in \mathcal{M}$. This is a normal subgroup.

Theorem 21.5. *Let $\mathcal{M}$ be a separable II_1-factor. Then TFAE*

(1) *$\mathcal{M}$ does not have property Γ.*

(2) *$\mathrm{Int}(\mathcal{M})$ is closed in $\mathrm{Aut}(\mathcal{M})$.*

(3) *$C^*(\lambda(\mathcal{M}) \cup \rho(\mathcal{M})) \supseteq \mathcal{K}(L^2(\mathcal{M}, \tau))$ where $\mathcal{K}(\mathcal{H})$ is the set of compact operators on a Hilbert space $\mathcal{H}$. ($\lambda(\mathcal{M})$, $\rho(\mathcal{M})$ in the standard form)*

22. Haagerup property

Let $\mathcal{M}$ be a separable II_1 factor.

Definition. *$\mathcal{M}$ has the* Haagerup property *if there are completeley positive maps $\Phi_j : \mathcal{M} \to \mathcal{M}$ so that $\tau \circ \Phi_j \leq \tau$, $\Phi_j(I) \leq I$ and $\lim_{j\to\infty} |\Phi_j(x) - x|_2 = 0$ if $x \in \mathcal{M}$ and Φ_j are compact in $L^2(\mathcal{M})$.*

On the free groups F_n let $l(g)$ be the length of the reduced word g. Then if $t > 0$, consider the multiplication operator on $\ell^2(F_n)$:

$$(M_t\xi)(g) = e^{-tl(g)}\xi(g)$$

for $\xi \in \ell^2(F_n)$. One can show that $M_t L(F_n)\varepsilon_e \subseteq L(F_n)\varepsilon_e$. Then let $\Phi_t : \mathcal{M} \to \mathcal{M}$ be defined by $\Phi_t(x)\varepsilon_e = M_t(x\varepsilon_e)$. One can show that these are completely positive, have the other properties that establish that $L(F_n)$ has the Haagerup property.

Later we will use another construction based on free probability to prove that $L(F_n)$ has the Haagerup property, see Corollary 27.3.

23. Affiliated operators

Let $\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$.

Definition. *An unbounded operator T, defined on $\mathcal{D}(T) \subseteq \mathcal{H}$, is affiliated to $\mathcal{M}$ if for any $u \in \mathcal{U}(\mathcal{M}')$, $u\mathcal{D}(T) = \mathcal{D}(T)$ and $Tuh = uTh$ for $h \in \mathcal{D}(T)$.*

More concisely: $uTu^* = T$ (as unbounded operators).

Examples.

- The bounded operators affiliated to $\mathcal{M}$ are precisely the operators in $\mathcal{M}$.
- Let $\mathcal{H} = L^2(X, \Sigma, \mu)$ where (X, Σ, μ) is a probability space. Let $\mathcal{M}$ be the von Neumann algebra $L^\infty(X, \Sigma, \mu)$ acting as multiplication operators on $L^2(X, \Sigma, \mu)$ and note that $\mathcal{M} = \mathcal{M}'$. Let $f : X \to \mathbb{C}$ be measurable. For each $\lambda \geq 0$, let $X_\lambda = f^{-1}(\{|z| \leq \lambda\}) \subseteq X$ and $\mathcal{D} = \{h \in L^2(X, \Sigma, \mu) \mid h^{-1}(\mathbb{C} \setminus \{0\}) \subseteq X_\lambda$ for some $\lambda \geq 0\}$. Then T defined as $Th = fh$, $h \in \mathcal{D}$ is an unbounded operator affiliated to $\mathcal{M}$. Remark that unitary operators in $\mathcal{M} = \mathcal{M}'$ are multiplication operators by $g \in L^\infty(X, \Sigma, \mu)$ where $|g| = 1$.

- Let $\mathcal{M}$ be a II_1 factor on $\mathcal{H}$ and let $X \in \mathcal{M}$ be an operator so that $\ker X = 0$. If $\mathcal{D} = \operatorname{Ran} X$. We define T on $\mathcal{D}$ to be the inverse of X, i.e. $TXh = h$ for $h \in \mathcal{H}$. If $u \in \mathcal{M}'$ is unitary, then $uXu^* = X$ implies $Xuh = uXh$ so that $u\mathcal{D} = \mathcal{D}$. Then $TuXh = TXuh = uh = uTXh$, so that $uTu^* = T$.

To deal with the graph of affiliated operators to $\mathcal{M}$ we introduce $M_2(\mathcal{M}) \sim M_2 \otimes \mathcal{M}$, the von Neumann algebra of 2×2 matrices with entries in $\mathcal{M}$ acting on $\mathcal{H} \oplus \mathcal{H}$. Then

$$(M_2(\mathcal{M}))' = I_2 \otimes \mathcal{M}' = \left\{ \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}) \mid X \in \mathcal{M}' \right\}.$$

If T with domain $\mathcal{D}(T)$ is affiliated with $\mathcal{M}$, then the graph of T

$$G_T = \{(h, Th) \in \mathcal{H} \oplus \mathcal{H} \mid h \in \mathcal{D}(T)\}.$$

is invariant under the unitary operators in $(M_2(\mathcal{M}))'$, which are $\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}$ with $u \in \mathcal{U}(\mathcal{M}')$. If G_T is closed, then $P_{G_T} \in M_2(\mathcal{M})$. The densely defiend closed affiliated operators to $\mathcal{M}$ can be described using the projection onto the graphs which satisfy

$$P \wedge \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0,$$

i.e. $G_T \cap (0 \oplus \mathcal{H}) = 0$, and with right support $P \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ (densely defined)

If $\mathcal{M}$ is finite, the affiliated operators which are closed and densely defined are particularly nice. Let $\operatorname{Aff}(\mathcal{M})$ be the densely defined closed operators affiliated to $\mathcal{M}$.

Proposition 23.1.

- *If* $T_1, T_2 \in \operatorname{Aff}(\mathcal{M})$ *and* $T_1 \subseteq T_2$, *then* $T_1 = T_2$.
- *If* $T_1, T_2 \in \operatorname{Aff}(\mathcal{M})$, *then the closures* $\overline{(T_1 + T_2)}, \overline{T_1 T_2} \in \operatorname{Aff}(\mathcal{M})$, *and if* $T \in \operatorname{Aff}(\mathcal{M})$, *then* $T^* \in \operatorname{Aff}(\mathcal{M})$.
- $\operatorname{Aff}(\mathcal{M})$ *with these operations is a* $*$*-algebra.*
- *If* T *is a densely defined operator on* $\mathcal{H}$ *which is self-adjoint, then* $T \in \operatorname{Aff}(\mathcal{M})$ *iff its spectral measure is in* $\mathcal{M}$
- *etc.*

Part 2. Free Probability

Reference: [VDN92].

24. Noncommutative Probability

In classical probability theory a probability space (Ω, Σ, μ) corresponds roughly to an algebra of numerical random variables $f : \Omega \to \mathbb{C}$ and the expectation functional $Ef = \int f d\mu$.

So, an algebraic caricature of this is a *noncommutative probability space*:

Definition. *A* noncommutative probability space *is* $(\mathcal{A}, \varphi)$*, where* $\mathcal{A}$ *is an unital algebra over* $\mathbb{C}$*, and* $\varphi : \mathcal{A} \to \mathbb{C}$ *a linear map satisfying* $\varphi(1) = 1$*, called the* expectation functional*. The elements* $a \in \mathcal{A}$ *will be called* noncommutative random variables*.*

This is like in quantum mechanics: $\mathcal{A}$ is an algebra of operators, observables, and $\varphi(T) = \langle T\xi, \xi \rangle$, where $\xi \in \mathcal{H}$ is the state vector.

To add to the purely algebraic framework positivity we need to use C^*-algebras (this more or less generalizes the quantum mechanics framework):

Definition. *A* C^**-probability space is* $(\mathcal{A}, \varphi)$*,* $\mathcal{A}$ *a unital* C^**-algebra and* $\varphi : \mathcal{A} \to \mathbb{C}$ *a state.*

To introduce measurability we must pass to von Neumann algebras.

Definition. *A* W^*-probability space *is a von Neumann algebra* $\mathcal{A}$ *with an uwo continuous state* $\varphi : \mathcal{A} \to \mathbb{C}$*.*

In the classical setting, the W^*-probability spaces $(\mathcal{A}, \varphi)$ are $\mathcal{A} = L^\infty(X, \Sigma, \mu)$ and $\varphi(f) = \int f d\mu$. To deal with unbounded random variables, there are affiliated operators.

Denote by $\mathbb{C}\langle X_\iota \mid \iota \in I \rangle$ the noncommutative polynomial ring in the variables X_ι.

Definition. *If* $a \in \mathcal{A}$*, the* distribution *of* a *is the map*

$$\begin{aligned} \mu_a : \mathbb{C}\langle X \rangle &\longrightarrow \mathbb{C}, \\ p &\longmapsto \varphi(p(a)). \end{aligned}$$

The space of distributions is the set Σ *of linear functionals* $\mu : \mathbb{C}[X] \to \mathbb{C}$ *with* $\mu(1) = 1$*.*

Classically, $(\Omega, \Sigma, d\omega)$, the distribution $\mu_{(f_\iota)_{\iota \in I}}$ of a family $f_\iota : \Omega \to \mathbb{C}, \iota \in I$, of random variables, is the pushforward $((f_\iota)_{\iota \in I})_*(d\omega)$ on $\mathbb{C}^I$. Roughly as a functional over functions on $\mathbb{C}^I$ this corresponds to

$$\text{Functions}(\mathbb{C}^I) \xrightarrow{\circ (f_\iota)_{\iota \in I}} (\text{random var. over } \Omega) \xrightarrow{E} \mathbb{C}.$$

So the definition in noncommutative algebraic caricature is: If $(a_\iota)_{\iota \in I} \subseteq (\mathcal{A}, \varphi)$ noncommutative random variables, their joint distribution is

$$\begin{aligned} \mu_{(a_\iota)_{\iota \in I}} &: \mathbb{C}\langle X_\iota \mid \iota \in I \rangle \to \mathbb{C} \\ \mu_{(a_\iota)_{\iota \in I}}(P(X_\iota \mid \iota \in I)) &= \varphi(P(a_\iota \mid \iota \in I)). \end{aligned}$$

This amounts to giving the noncommutative moments of the $(a_\iota)_{\iota \in I}$: $\varphi(a_{\iota_1} \cdots a_{\iota_n})$, $n \in \mathbb{N}, \iota_1, \ldots, \iota_n \in I$. Of course there are many variants with additional structure, instead of $\mathbb{C}\langle X_\iota \mid \iota \in I \rangle$ take certain universal C^*-algebras etc.

In case $(\mathcal{A}, \varphi)$ is a C^*-probability space and $a = a^* \in \mathcal{A}$, is a hermitian random variable, then

$$\begin{aligned} &\mathbb{C}[X] \to C(\sigma(a)) \to \mathcal{A} \\ &P(X) \mapsto \text{polynomial function on } \sigma(a) \mapsto \text{continuous functional calculus} \end{aligned}$$

Thus actually $C(\sigma(a)) \ni f \mapsto \varphi(f(a))$ is a genuine probability measure on $\sigma(a)$.

So μ_a can be identified with the measure on $\mathbb{R}$ so that $\mu_a(f) = \int f d\mu_a$.

If we are in the W^*-probability context with $(\mathcal{A}, \varphi)$, then actually if $\Delta \subseteq \mathbb{R}$ is a Borel set, then $\mu_a(\Delta) = \varphi(E(a; \Delta))$ where $E(a; -)$ is the spectral measure of a. In the classical context, the spectral projection for Δ in $L^\infty(\Omega, \Sigma, d\omega)$ is $\chi_{a^{-1}(\Delta)}$ and $E(\chi_{a^{-1}(\Delta)}) = \omega(a^{-1}(\Delta))$ which gives that $\mu_a(\Delta) = \omega(a^{-1}(\Delta))$.

25. Classical Independence

There is a classical noncommutative probability theory corresponding to the quantum mechanics prescriptions. There independence of observables corresponds to tensor products. That is T and S are independent, if roughly T and S have distribution like $X \otimes I, I \otimes Y$ w.r.t. $\varphi = \varphi_1 \otimes \varphi_2$.

In the algebraic caricature one could define this as follows:

Definition. *A family $(\mathcal{A}_\iota)_{\iota \in I}$ of unital subalgebras in $(\mathcal{A}, \varphi)$ is* classically independent *if $[\mathcal{A}_i, \mathcal{A}_j] = 0$ for all $i \neq j, i, j \in I$ and $\varphi(a_1 \cdots a_n) = \varphi(a_1) \cdots \varphi(a_n)$ for $a_j \in \mathcal{A}_{\iota(j)}$ and $\iota(1), \ldots, \iota(n)$ are pairwise distinct.*

This generalizes the classical probability independence. Subsets of $(\mathcal{A}, \varphi)$, $\omega_j \subseteq \mathcal{A}, \iota \in I$ are independent if the unital subalgebras they generate are independent.

26. Free Independence

What distinguishes free probability from other varieties of noncommutative probability theory, is the definition of independence.

Definition. *In $(\mathcal{A}, \varphi)$ a family $(\mathcal{A}_\iota)_{\iota \in I}$ of unital subalgebras is* freely independent *(or* free*) if $\varphi(a_1 \cdots a_n) = 0$ whenever $a_j \in \mathcal{A}_{\iota(j)}$, $\varphi(a_j) = 0$ for $1 \leq j \leq n$, and consecutive indices $\iota(j), \iota(j+1)$ are distinct for $1 \leq j \leq n-1$.*

This is very noncommutative. But a very large part of basic probability theory has free analogues.

Again, subsets are freely independent if the unital subalgebras they generate are freely independent.

Examples.

- Let $G = *_{\iota \in I} G_\iota$ be a free product of a family of groups. Let $\mathbb{C}[G_\iota] \subseteq \mathbb{C}[G]$ be the corresponding group rings. Then in $(\mathbb{C}[G], \tau)$ (where τ is the von Neumann trace $\tau(x) = \langle x\varepsilon_e, \varepsilon_e \rangle$) the $\mathbb{C}[G_\iota]$ are freely independent. Indeed, $G = *_{\iota \in I} G_\iota$ means the G_ι's generate G and every reduced word $g_1 \cdots g_n$ where $g_j \in G_{\iota(j)} \setminus \{e\}$ with $\iota(j) \neq \iota(j+1)$, $1 \leq j < n$ is non-trivial, i.e. $\neq e$. If $a_j \in \mathbb{C}[G_{\iota(j)}]$, $1 \leq j \leq n$ and $(\varepsilon_g)_{g \in G}$ is the basis in $\mathbb{C}[G]$ and $\iota(j) \neq \iota(j+1)$, then a_j is a linear combination of $\varepsilon_g, g \in G_{\iota(j)}$ and $\tau(a_j) = 0$ means no constant term, so a_j is a linear combination of $\varepsilon_g \in G_{\iota(j)} \setminus \{e\}$. Expanding $a_1 \cdots a_n$ we get a sum of terms of the form $\tau(\varepsilon_{g_1} \cdots \varepsilon_{g_n})$ where $g_j \in G_{\iota(j)} \setminus \{e\}$, $\iota(j) \neq \iota(j+1)$. Then $\varepsilon_{g_1} \cdots \varepsilon_{g_n} = \varepsilon_{g_1 \cdots g_n}$ which has trace 0.
- *Creation and destruction* (or annihilation) operators on the *full Fock space*. Let $\mathcal{H}$ be a complex Hilbert space. Let $\mathcal{TH} = \bigoplus_{n \geq 0} \mathcal{H}^{\otimes n}$ where $\mathcal{H}^{\otimes 0} = \mathbb{C}1$ ($1 =$ *vacuum vector*).

The *creation operator* $\ell(h)$ on $\mathcal{TH}$, where $h \in \mathcal{H}$, is $\ell(h) \in \mathcal{B}(\mathcal{TH})$ given by $\ell(h)\xi = h \otimes \xi$. The adjoint $\ell^*(h) := \ell(h)^*$ is given by $\ell^*(h)1 = 0$ and $\ell^*(h)h_1 \otimes \cdots \otimes h_n = \langle h_1, h\rangle h_2 \otimes \cdots \otimes h_n$.

If $(e_\iota)_{\iota \in I}$ is an ONB in $\mathcal{H}$ we let $\ell_\iota = \ell(e_\iota)$, and $\ell_\iota^* = \ell^*(e_\iota) = (\ell_\iota)^*$.

Proposition 26.1. *$\{\ell_\iota, \ell_\iota^*\}_{\iota \in I}$ is freely independent in $(\mathcal{B}(\mathcal{TH}), \langle \cdot 1, 1\rangle)$.*

Proof. We have to check that the algebras these elements generate are freely independent. The algebra generated by ℓ_ι, ℓ_ι^* consists of finite sums $\sum_{p,q\geq 0} c_{p,q}\ell_\iota^p(\ell_\iota^*)^p$ (use that $\ell_\iota^*\ell_\iota = \mathrm{id}$ and $\ell_\iota\ell_\iota^* = ...$). After expanding sums, the free independence boils down to

$$\langle \ell_{\iota_1}^p(\ell_{\iota_1}^*)^{q_1}\ell_{\iota_2}^p(\ell_{\iota_2}^*)^{q_2}\cdots\ell_{\iota_n}^p(\ell_{\iota_n}^*)^{q_n}1, 1\rangle = 0$$

when $p_j + q_j > 0, p_j \geq 0, q_j \geq 0, \iota_j \neq \iota_{j+1}$. (Note that the expectation of $\ell^p(\ell^*)^q$ is 1 if $p = q = 0$ and 0 otherwise). Assume the expectation is not 0. Then we must have $p_1 = 0$, otherwise $(\ell_{\iota_1}^*)^{p_1}1 = 0$. Then $q_1 > 0$. But then $(\ell_{\iota_1}^*)^{q_1}\ell_{\iota_2}^{p_2}$ is 0 if $p_2 > 0$, so $p_2 = 0$, and then $q_2 > 0$ etc. Then also $q_n > 0$, but then $(\ell_{\iota_n}^*)^{q_n}1 = 0$. ☐

If $(\mathcal{A}_\iota)_{\iota \in I}$ are subalgebras of $\mathcal{A}$, denote by $\bigvee_{\iota \in I} A_\iota$ the subalgebra they generate.

Some properties of free independence:

Proposition 26.2.

(1) If $(\mathcal{A}_\iota)_{\iota \in I}$ are freely independent unital subalgebras in $(\mathcal{A}, \varphi)$, then $\varphi|_{\bigvee_{\iota \in I}\mathcal{A}_\iota}$ is completely determined by the $\varphi|_{\mathcal{A}_\iota}$.

(2) If $(\mathcal{A}_\iota)_\iota$ are freely independent unital subalgebras in $(\mathcal{A}, \varphi)$, and $I = \bigsqcup_{j \in J} I_j$ is a partition and $\mathcal{B}_j = \bigvee_{\iota \in I_j}\mathcal{A}_\iota$, then the $(B_j)_{j \in J}$ are freely independent in $(\mathcal{A}, \varphi)$.

(3) If $(\mathcal{A}_\iota)_\iota$ are freely independent unital subalgebras in $(\mathcal{A}, \varphi)$, and $(\mathcal{C}_{\iota,k})_{k \in K_\iota}$ are freely independent subalgebras in $(\mathcal{A}_\iota, \varphi|_{\mathcal{A}_\iota})$, then $(\mathcal{C}_{\iota,k})_{(i,k)\in\bigsqcup_{i \in I}\{\iota\}\times K_\iota}$ is freely independent in $(\mathcal{A}, \varphi)$

(4) If $\mathcal{A} = \bigvee_{\iota \in I}\mathcal{A}_\iota$ and the $(\mathcal{A}_\iota)_\iota$ are freely independent in $(\mathcal{A}, \varphi)$, then if all $\varphi|_{\mathcal{A}_\iota}$ are traces, then also φ is a trace.

Proof.

(1) The elements of $\mathcal{A}$ are linear combinations of products $a_1, \ldots, a_n$ where $a_j \in \mathcal{A}_{\iota(j)}$, so must show that free independences gives us what $\varphi(a_1 \cdots a_n)$ should be if we know the $\varphi|_{\mathcal{A}_\iota}$. We shall use induction over n. For $n = 1$ this is clear, assume known for $\varphi(a_1 \cdots a_n)$ if $n < N$. Then consider $\varphi(a_1 \cdots a_N)$ if $\iota(j) = \iota(j+1)$ for some j reduces to a product of $N - 1$, ... If $\iota(j) \neq \iota(j+1)$ for all j, then consider $a_j^\circ = a_j - \varphi(a_j)1$, so that $\varphi(a_j^\circ) = 0$. Then expand $\varphi(a_1 \cdots a_N) = \varphi((a_1^\circ + \varphi(a_1)) \cdots (a_N^\circ + \varphi(a_N)1))$ and use induction.

(2) Exercise or see [VDN92].

(3) Exercise or see [VDN92].

(4) Given an algebra $\mathcal{A}$ with expectation φ, denote $\mathcal{A}^\circ = \ker\varphi$. In this case $\mathcal{A} = \bigvee_{\iota\in I}\mathcal{A}_\iota$ gives

$$\mathcal{A} = \mathbb{C}1 + \sum_{n\geq 1}\sum_{\iota_1,\ldots,\iota_n\in I^n, \iota_j\neq\iota_{j+1}} \mathcal{A}_{\iota_1}^\circ\cdots\mathcal{A}_{\iota_n}^\circ$$

So we must show $\varphi(xy) = \varphi(yx)$ when $x = a_1\cdots a_n, y = b_m\cdots b_1$ with $a_k\in\mathcal{A}_{\iota(k)}, b_k\in\mathcal{A}_{j(k)}$ where $\varphi(a_k) = 0, \varphi(b_k) = 0$ and $\iota(k)\neq\iota(k+1), j(k)\neq j(k+1)$. Then

$$\begin{aligned}\varphi(a_1\cdots a_n b_m\cdots b_1) &= \delta_{\iota(n),j(m)}(\varphi(a_n b_m)\varphi(a_1\cdots a_{n-1}b_{m-1}\cdots b_1) + \varphi(a_1\cdots a_{n-1}(a_n b_m)^\circ b_{m-1}\cdots b_1))\\ &= \delta_{\iota(n),j(m)}\varphi(a_n b_m)\varphi(a_1\cdots a_{n-1}b_{m-1}\cdots b_1)\end{aligned}$$

Repeat this (say $m\leq n$), so that

$$\varphi(a_1\cdots a_n b_m\cdots b_1) = \delta_{m,n}\varphi(a_n b_m)\varphi(a_{n-1}b_{m-1})\cdots\varphi(a_{n-m+1}b_1)\delta_{\iota(n),j(m)}\cdots\delta_{\iota(n-m+1),j(1)}.$$

Now do the same with yx and we see that we get the same if $\varphi|_{\mathcal{A}_\iota}$ are traces.

□

Corollary 26.3. *If $\mathcal{A} = \bigvee_{\iota\in I}\mathcal{A}_\iota$ and the $\mathcal{A}_\iota$ are freely independent and commutative, then φ is a trace.*

27. The Semicircular function (aka free Gaussian functor)

Lemma 27.1. *Let ℓ_1 be the creation operator corresponding to some norm 1 vector in $\mathcal{H}$. Then*

$$\mu_{\ell_1+\ell_1^*} = \frac{1}{2\pi}\chi_{[-2,2]}\sqrt{4-t^2}\mathrm{d}t$$

in $(\mathcal{B}(\mathcal{TH}), \langle\cdot 1, 1\rangle)$, where $\mathrm{d}t$ is the Lebesgue measure.

This can be proven directly, but we will deduce it later.

Theorem 27.2. *Let $\mathcal{H}$ be a real Hilbert space, $\mathcal{H}_\mathbb{C}$ its complexification and $\mathcal{TH}_\mathbb{C}$ the full Fock space over $\mathcal{H}_\mathbb{C}$. For $h\in\mathcal{H}\subseteq\mathcal{H}_\mathbb{C}$, let $s(h) = \frac{1}{2}(\ell(h)+\ell(h)^*)$ and let $\Phi(\mathcal{H}) = (s(\mathcal{H}))''\subseteq\mathcal{B}(\mathcal{TH}_\mathbb{C})$, and let $\tau_\mathcal{H}$ be the restriction of $\langle\cdot 1, 1\rangle$ to $\Phi(\mathcal{H})$. Then*

(i) $\Phi(\mathcal{H})\simeq L(F_{\dim\mathcal{H}})$. (For $\dim\mathcal{H} = 1$ this is just $L(\mathbb{Z})$).

(ii) 1 is a cyclic and separating trace vector for $\Phi(\mathcal{H})$.

(iii) If $(h_\iota)_{\iota\in I}$ are orthogonal vectors in $\mathcal{H}$, then the $(s(h_\iota))_{\iota\in I}$ are freely independent and $\mu_{s(h)} = \frac{2}{\pi\|h\|^2}\chi_{(-\|h\|,\|h\|)}\sqrt{\|h\|^2-t^2}d\lambda(t)$.

(iv) If $T:\mathcal{H}_1\to\mathcal{H}_2$ is a linear map with $\|T\|\leq 1$ between real Hilbert spaces, let $\mathcal{T}(T_\mathbb{C}) = \bigoplus_{n\geq 1}T_\mathbb{C}^{\otimes n}:\mathcal{T}((\mathcal{H}_1)_\mathbb{C})\to\mathcal{T}((\mathcal{H}_2)_\mathbb{C})$ and there is a a unique map $\Phi(T):\Phi(\mathcal{H}_1)\to\Phi(\mathcal{H}_2)$ such that $(\Phi(T)X)1 = \mathcal{T}(T_\mathbb{C})(X1)$ for all $X\in\Phi(\mathcal{H}_1)$. $\Phi(T)$ is linear, bounded, completely positive, unital, trace-preserving. If T is isometric, then $\Phi(T)$ is a faithful homomorphism and if T is the orthogonal projection onto a subspace $\mathcal{H}_2\subseteq\mathcal{H}_1$, then $\Phi(T)$ is the conditional expectation onto $\Phi(\mathcal{H}_2)\simeq(s(Th_1), h_1\in\mathcal{H}_1)''\subseteq\Phi(\mathcal{H}_1)$.

(v) If $(\mathcal{H}_\iota)_{\iota\in I}$ is a family of pairwise orthogonal subspaces in $\mathcal{H}$ and $V_\iota : H_\iota \to \mathcal{H}$ are the inclusions, then the family of subalgebras $(\Phi(V_\iota))(\Phi(\mathcal{H}_\iota))$ are freely independent in $(\Phi(\mathcal{H}), \tau_{\mathcal{H}})$.

(vi) $\Phi(\mathcal{H})$ is in standard form on $\mathcal{T}(\mathcal{H}_\mathbb{C})$ and $Jh_1 \otimes \cdots \otimes h_n = h_n \otimes \cdots \otimes h_1$, if $h_j \in \mathcal{H}$, and $\Phi(\mathcal{H})' = (d(\mathcal{H}))''$ where $d(h) = \frac{1}{2}(r(h) + r(h)^*)$ and $r(h)\xi = \xi \otimes h$ is the right creation operator.

Corollary 27.3. *$L(F_n)$ has the Haagerup compact approximation property.*

Proof. $\Phi(\mathbb{R}^n) = L(F_n)$ and $\mathcal{T}((rI_{\mathbb{R}^n})_\mathbb{C}) = \bigoplus_{m\geq 0} r^m I_{\mathcal{H}_\mathbb{C}^{\otimes m}}$, $\mathcal{H} = \mathbb{R}^n$, is compact if $0 \leq r < 1$. But $\Phi(rI_{\mathbb{R}^n})$ are completely positive and as $r \nearrow 1$ they converge strongly to the identity and note that $\Phi(\mathcal{H}) \ni X \mapsto X1 \in \mathcal{T}(\mathcal{H}_\mathbb{C})$ is just the map $\Phi(\mathcal{H}) \hookrightarrow L^2(\Phi(\mathcal{H}), \tau_\mathcal{H})$, 1 being a separating and cyclic tracial vector. $\square$

Proof of Theorem 27.2. (iii) follows immediately by dilation from Lemma 27.1. The free independence statement is due to the fact that the $(\{\ell(h_\iota), \ell^*(h_\iota)\})_{\iota\in I}$ are freely independent in $(\mathcal{B}(\mathcal{T}\mathcal{H}_\mathbb{C}), \langle\cdot 1, 1\rangle)$. Next (ii).

(a) We prove by induction over n that $h_1 \otimes \cdots \otimes h_n \in \Phi(\mathcal{H})1$. If $n = 0$, $1 = I1$. Also $s(h_1)\cdots s(h_n)1 \in 2^{-n}h_1 \otimes \cdots \otimes h_n + \bigoplus_{0\leq k<n} \mathcal{H}^{\otimes k}$. So 1 is cyclic.

(b) Separating. Let $(e_\iota)_{\iota\in I}$ be an ONB in $\mathcal{H}$ (hence in $\mathcal{H}_\mathbb{C}$) and $\ell_\iota = \ell(e_\iota), r_\iota = r(e_\iota)$. Then

$$[\ell_\iota, r_j] = [\ell_\iota^*, r_j^*] = 0$$
$$[\ell_\iota^*, r_j] = \delta_{\iota j} P_{\mathbb{C}1}$$
$$[r_\iota^*, \ell_j] = \delta_{\iota j} P_{\mathbb{C}1}$$

All the commutators are 0 on $\bigoplus_{n>0} \mathcal{H}^{\otimes n}$, so only need to check the equalities on 1. We infer that $[\ell_\iota + \ell_\iota^*, r_j + r_j^*] = \delta_{\iota j}(-P_{\mathbb{C}1} + P_{\mathbb{C}1}) = 0$. Since $\Phi(\mathcal{H}) = ((s(e_\iota))_{\iota\in I})''$, we have $\Phi(\mathcal{H})' \supseteq ((d(e_\iota))_{\iota\in I})''$ and by the argument in (a) applied to the $d(e_\iota))_{\iota\in I})$ we get that 1 is cyclic for $(\Phi(\mathcal{H}))'$, hence separating for $\Phi(\mathcal{H})$.

(c) Since the $(s(e_\iota))_{\iota\in I}$ are freely independent, we infer that on the algebra generated by the $(s(e_\iota))_{\iota\in I}$, $\tau_\mathcal{H}$ is a trace (as this is generated by abelian freely independent subalgebras). Then pass to the closure (maybe use Kaplansky)

Next (i). $\Phi(\mathcal{H})$ is generated by the commutative freely independent subalgebras $((s(e_\iota))'')_{\iota\in I}$. Using the lemma one finds that $((s(e_\iota))'', \tau_\mathcal{H}|_{(s(e_\iota))''}) \simeq (L^\infty([-1,1], \text{semicircle measure}, \int)$. The function $g(x) = \int_{-1}^x \frac{2}{\pi}\sqrt{1-t^2}dt$ is continuous, strictly increasing, $g(-1) = 0, g(1) = 1$, a homeomorphism of $[-1,1]$ and $[0,1]$. Note that μ_g = Lebesgue measure on $[0,1]$. Hence $e^{2\pi i g}$ is an isomorphism from $[-1,1)$ with the semicircle measure and S^1 with the Haar measure. Then

$$L^\infty([-1,1], \text{semicircle}) \simeq \ldots L^\infty(S^1, \text{haar}) \simeq (L(\mathbb{Z}), \tau_\mathbb{Z}) \simeq (\lambda_\mathbb{Z}(1))'' \simeq (\lambda_{F_{\dim \mathcal{H}}}(g_\iota))'.$$

The isomorphism $\Phi(\mathcal{H}) \simeq L(F_{\dim \mathcal{H}})$ follows from the fact that the sets $(e^{\pm 2\pi i g(s(e_\iota))})_{\iota\in I}$ and $(\lambda(g_\iota), \lambda(g_\iota)^{-1})_{\iota\in I}$ are generators for the two algebras, have the same distribution etc.

(iv)

(a) Suppose first that T is an isometric injection. Clearly $\mathcal{T}(T_{\mathbb{C}}) : \mathcal{T}(\mathcal{H}_{1,\mathbb{C}}) \to \mathcal{T}(\mathcal{H}_{2,\mathbb{C}})$ isometric injection and the map intertwines $s_{\mathcal{H}_1}(h)$ and $s_{\mathcal{H}_2}(Th)$, $h \in \mathcal{H}_1$. Since $\mathcal{T}(T_{\mathbb{C}})\mathcal{T}(\mathcal{H}_{1,\mathbb{C}})$ contains the cyclic tracial vector 1 for $\Phi(\mathcal{H}_2)$, the subspace $\mathcal{T}(T\mathcal{H}_1)_{\mathbb{C}}$ is separating for $(s_{\mathcal{H}_2}(T\mathcal{H}_1))''$. We then get a map $\Phi(\mathcal{H}_1) \to \Phi(\mathcal{H}_2)$, given by $X \mapsto X|_{\mathcal{T}(T\mathcal{H}_{1,\mathbb{C}})}$. This is $\Phi(T)$.

Why completely positive? If $\mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism, then $M_n(\mathcal{A}) \to M_n(\mathcal{B})$ is still a $*$-homomorphism, so completely positive.

(b) Next assume that T is a projection. Then $\mathcal{T}(T_{\mathbb{C}})$ is just the conditional expectation on L^2 spaces. Define then $\Phi(T)$ to be the conditional expectation defined earlier.

(c) General case. Write T as an isometry followed by a projection, e.g.

$$\mathcal{H}_1 \longrightarrow \mathcal{H}_1 \oplus \mathcal{H}_2 \longrightarrow \mathcal{H}_2$$
$$h \longmapsto ((I - T^*T)^{1/2}h, Th) \longmapsto Th$$

Note that $\left\|((I - T^*T)^{1/2}h, Th)\right\|^2 = \|Th\|^2 + \left\|(I - T^*T)^{1/2}h\right\|^2 = \langle T^*Th, h\rangle + \langle (I-T^*T)^{1/2}h, (I-T^*T)^{1/2}h\rangle = \|h\|^2$, so the first map is indeed an isometry. Then define $\Phi(T)$ as the composition of the corresponding Φ's. Note that the characterization of Φ is compatible with composition.

Trace is preserved because Φ doesn't change the degree 0 (vacuum) component. $\square$

28. The free product construction

Definition. *The* algebraic free product $\mathcal{A} = *_{\iota \in I}\mathcal{A}_\iota$ *of algebras* $\mathcal{A}_\iota$ *is the free product with amalgamation over* $\mathbb{C}1$*, defined by the universal property: There are maps* $\mathcal{A}_\iota \xrightarrow{\psi_\iota} \mathcal{A}$ *such that if there are maps* $\mathcal{A}_\iota \xrightarrow{f_\iota} \mathcal{B}$*, then there is a unique morphism* $\mathcal{A} \to \mathcal{B}$ *making the obvious triangle commute.*

As a vector space, if $\mathcal{A}_\iota = \mathbb{C}1 \oplus V_\iota$, one could identify $\mathcal{A}$ as a vector space with

$$\mathbb{C}1 \oplus \bigoplus_{n\geq 1} \bigoplus_{\iota_1 \neq \iota_2 \neq \dots \iota_n} V_{\iota_1} \otimes \dots \otimes V_{\iota_n}$$

and ψ_ι the inclusion of $\mathbb{C}1 \oplus V_\iota$ (to $n = 1$).

Similarly one can define the free product of C^*-algebras. This is obtained by taking the algebraic free product, and completing it with respect to a suitable norm (taking the universal enveloping C^*-algebra).

Now if we actually have noncommutative probability spaces, i.e. if the algebras are equipped with states φ_ι, we want to give the free product the structure of such a space, i.e. extend the state. In the algebraic this is clear. In the C^*-algebra case less so.

If we have a family of Hilbert spaces $\mathcal{H}_\iota$ with distinguished norm 1 vectors ξ_ι, then we can form their free product $(\mathcal{H}, \xi)$ with respect to the ξ_ι. Denote $\mathcal{H}_\iota^\circ = \mathcal{H}_\iota \ominus \mathbb{C}\xi_\iota$, then

$$\mathbb{C}\xi \oplus \bigoplus_{n\geq 1} \bigoplus_{\iota_1 \neq \iota_2 \neq \dots \iota_n} \mathcal{H}_{\iota_1}^\circ \otimes \dots \otimes \mathcal{H}_{\iota_n}^\circ.$$

Let $(\mathcal{H}(\iota), \xi)$ be the free product over all the $(\mathcal{H}_j, \xi_j)$ except for $j = \iota$. Then we have unitary identification operators $V_\iota : \mathcal{H}_\iota \otimes \mathcal{H}(\iota) \to \mathcal{H}$ given by

$$V_\iota(\xi_\iota \otimes \eta) = \eta,$$

$$V_\iota(h_\iota \otimes (h_{\iota_1} \otimes \cdots \otimes h_{\iota_n})) = h_\iota \otimes h_{\iota_1} \otimes \cdots \otimes h_{\iota_n},$$

where $\eta \in \mathcal{H}(\iota)$ and $h_\iota \in \mathcal{H}_\iota$, $h_{\iota_j} \in \mathcal{H}_{\iota_j}$, $\iota \neq \iota_1$ and $\iota_j \neq \iota_{j+1}$. Then via V_ι we may embed $\mathcal{B}(\mathcal{H}_\iota)$ into $\mathcal{B}(\mathcal{H})$.

Now going back to the setting with algebras. Suppose $\mathcal{A}_\iota$ are C^*-algebras and for each ι we have a representation ρ_ι on $(\mathcal{H}_\iota, \xi_\iota)$. Then we get an induced representation ρ of $\mathcal{A} = *_\iota A_\iota$ on $\mathcal{H}$. Then if $i_\iota : \mathcal{A}_\iota \to \mathcal{A}$ is the natural map, then $\langle \rho(i_\iota(a))\xi, \xi\rangle = \langle \rho_\iota(a)\xi, \xi\rangle$ for $a \in \mathcal{A}_\iota$. Hence if each $\mathcal{A}_\iota$ is equipped with a state φ_ι, then we can take the ρ_ι to be the associated GNS representations, and in this way get an induced state $\varphi = *_\iota \varphi_\iota$ on $\mathcal{A}$.

Proposition 28.1. *The $\mathcal{A}_\iota$ are freely independent in $\mathcal{A}$, i.e. if $a_j \in \mathcal{A}_{\iota_j}$ satisfy $\varphi_{\iota_j}(a_j) = 0$ for all $j = 1, \ldots, n$ and $\iota_j \neq \iota_{j+1}$, then $\varphi(a_1 \cdots a_n) = 0$.*

Proof. [VDN92, Proposition 1.5.5] □

Finally if the $\mathcal{A}_\iota$ are W^* algebras, and the φ_j are uwo continuous states, then one does the same thing. The W^* free product is the uwo closure of the GNS representation of $*_\iota \mathcal{A}_\iota$ w.r.t. $*_\iota \varphi_\iota$.

Examples.

- If G_ι are groups, then
$$*_\iota(\ell^2(G_\iota), \varepsilon_e) = (\ell(*_\iota G_\iota), \varepsilon_e).$$
- $$*_\iota(\mathcal{T}(\mathcal{H}_\iota), 1) = (\mathcal{T}(\bigoplus_\iota \mathcal{H}_\iota), 1)$$

There was a guest lecture by Takahiro Hasebe: Regrettably I didn't take notes for this.

29. Free Brownian Motion

Take $X_t = s(\chi_{[0,t)})$ in $\mathcal{H} = L^2([0,\infty))$. Then $\chi_{[0,t_1)}, \chi_{[t_1,t_2)}, \chi_{[t_2,t_3)}, \ldots$ are orthogonal in $\mathcal{H}$, so $s(\chi_{[0,t_1)}), s(\chi_{[t_2,t_3)}), s(\chi_{[t_3,t_4)}), \ldots$ are freely independent and they follow a semicircular law.

30. Free Convolution

Suppose $(\mathcal{A}, \varphi)$ is a noncommutative probability space. If $a, b \in \mathcal{A}$, how do we find the distribution of $a + b$ in terms of those of a, b? In general not really possible without more information, but if a, b are freely independent, we can do it. Indeed, then by Proposition 26.2, the expectation φ on the subalgebra generated by a, b is determined by the distributions of a, b, hence the distribution of $a + b$ is determined.

Definition. *The* additive free convolution *of distributions $\mu : \mathbb{C}[X] \to \mathbb{C}, \nu : \mathbb{C}[Y] \to \mathbb{C}$, is the distribution denoted $\mu \boxplus \nu$ and obtained through the composition $\mathbb{C}[Z] \to \mathbb{C}[X] *_{\mathbb{C}1} \mathbb{C}[Y] \to \mathbb{C}$ where the first map maps $Z \mapsto X + Y$ and the second is $\mu * \nu$.*

Proposition 30.1. *If $a, b \in \mathcal{A}$ are freely independent, then $\mu_{a+b} = \mu_a \boxplus \mu_b$.*

If $(\mathcal{A}, \varphi)$ is a C^*-noncommutative probability space, and a, b are self-adjoint, we can do the same thing, and then μ_a, μ_b, μ_{a+b} are compactly supported measures on $\mathbb{R}$.

Similarly we can define:

Definition. *The* multiplicative free convolution *of distributions $\mu : \mathbb{C}[X] \to \mathbb{C}, \nu : \mathbb{C}[Y] \to \mathbb{C}$, is the distribution denoted $\mu \boxtimes \nu$ and obtained through the composition $\mathbb{C}[Z] \to \mathbb{C}[X] *_{\mathbb{C}1} \mathbb{C}[Y] \to \mathbb{C}$ where the first map maps $Z \mapsto XY$ and the second is $\mu * \nu$.*

Proposition 30.2. *If $a, b \in \mathcal{A}$ are freely independent, then $\mu_{ab} = \mu_a \boxtimes \mu_b$.*

Proposition 30.3. *If $a, b \in \mathcal{A}$ are freely independent, then $\mu_{ab} = \mu_{ba}$, i.e. multiplicative free convolution is commutative.*

Proof. The subalgebras generated by a and b respectively are commutative., hence by Proposition 26.2 (4), φ is a trace on the algebra generated by a, b. Therefore $\varphi((ab)^n) = \varphi((ba)^n)$ for all n which gives the claim. $\square$

31. The R-Transform

How do we compute $\mu \boxplus \nu$? In the classical setting we have the Fourier transform $\mathcal{F}$ which satisfies $\mathcal{F}(\mu * \nu) = (\mathcal{F}\mu)(\mathcal{F}\nu)$, and we can further take log to linearize. In our setting we have the *R-transform*.

Theorem 31.1. *For a distribution $\mu : \mathbb{C}[X] \to \mathbb{C}$ let*

$$\begin{aligned} G_\mu(z) &= \sum_{n \geq 0} z^{-n-1} \mu(X^n), \\ K_\mu(z) &= G_\mu^{-1}(z) \quad \textit{(compositional inverse)}, \\ R_\mu(z) &= K_\mu(z) - z^{-1}. \end{aligned}$$

Then for any distributions $\mu, \nu : \mathbb{C}[X] \to \mathbb{C}$ we have

$$R_{\mu \boxplus \nu} = R_\mu + R_\nu.$$

R_μ is the *R-transform* of μ. We won't prove it, but the main idea is:

Lemma 31.2. *Let e_1, e_2 be orthonormal vectors in a Hilbert space and ℓ_1, ℓ_2 the corresponding creation operators on $\mathcal{TH}$. Let*

$$\begin{aligned} T_1 &= \ell_1^* + \sum_{k \geq 0} \alpha_{k+1} \ell_1^k, \\ T_2 &= \ell_2^* + \sum_{k \geq 0} \beta_{k+1} \ell_2^k, \end{aligned}$$

$$T_3 = \ell_1^* + \sum_{k\geq 0}(\alpha_{k+1}+\beta_{k+1})\ell_1^k.$$

Then

$$\mu_{T_3} = \mu_{T_1+T_2} = \mu_{T_1} \boxplus \mu_{T_2}.$$

Here of course the expectation functional is $\omega = \langle \cdot 1, 1\rangle$ as in the example before Proposition 26.2.

Proof. One can do this relatively directly by comparing $\omega(T_3^k)$ and $\omega((T_1+T_2)^k)$. □

The idea is that every distribution comes from a unique T as in the lemma, then can use the linearization in the lemma to prove the theorem.

Theorem 31.3. *Let μ be a distribution. Then there are unique $\alpha_1, \alpha_2, \dots$ such that $T = \ell_1^* + \sum_{k\geq 0}\alpha_{k+1}\ell_1^k$ has $\mu_T = \mu$, and moreover $R_\mu = \sum_{k=0}^{\infty}\alpha_{k+1}z^k$.*

Proof. [VDN92, 3.2.2] and [VDN92, Theorem 3.3.1]. □

Theorem 31.1 follows at once from this and the lemma.

Lastly consider the case of a C^*-algebra where our distributions are genuine measures. Then

$$G_\mu(z) = \sum_{n\geq 0} z^{-n-1}\int t^n \mathrm{d}\mu(t) = \int (z-t)^{-1}\mathrm{d}\mu(t)$$

is called the *Cauchy transform*, or *Stieltjes transform*, of μ. $G_\mu(z)$ is holomorphic in $(\mathbb{C}\cup\{\infty\})\setminus \operatorname{supp}\mu$.

We can recover μ from $G_\mu(z)$? We have

$$G_\mu(x+i\varepsilon) = \int \frac{\mathrm{d}\mu(t)}{x+i\varepsilon-t},$$

and

$$-\frac{1}{\pi}\operatorname{Im}\frac{1}{x+i\varepsilon-t} = \frac{1}{\pi}\frac{\varepsilon}{(x-t)^2+\varepsilon^2}.$$

Then, if $d\lambda$ denotes the Lebesgue measure on $\mathbb{R}$, we get

$$\begin{aligned}
-\frac{1}{\pi}G_\mu(x+i\varepsilon)\mathrm{d}\lambda(x) &= \left(\int \frac{1}{\pi}\frac{\varepsilon}{(x-t)^2+\varepsilon^2}\mathrm{d}\mu(t)\right)\mathrm{d}\lambda(x) \\
&= \mu * \left(\frac{1}{\pi}\frac{\varepsilon}{(x-t)^2+\varepsilon^2}\mathrm{d}\lambda(x)\right)
\end{aligned}$$

Now note that

$$\frac{1}{\pi}\frac{\varepsilon}{(x-t)^2+\varepsilon^2}\mathrm{d}\lambda(x) \longrightarrow \delta_0$$

weakly as $\varepsilon \searrow 0$, hence

$$-\frac{1}{\pi}\operatorname{Im} G_\mu(x+i\varepsilon)\mathrm{d}\lambda(x) \to \mu$$

weakly as $\varepsilon \searrow 0$.

Example. What is $\mu_{\ell_1^*+\ell_1}$ in $(\mathcal{B}(\mathcal{TH}), \langle \cdot 1, 1\rangle)$? We have $R_\mu = z$, so $K_\mu = z^{-1} + z$. We can thus find G_μ as the compositional inverse: Solving $\frac{1}{G} + G = z$ gives

$$G_\mu = \frac{z \pm \sqrt{z^2 - 4}}{2} = \frac{z \pm z\sqrt{1 - \frac{4}{z^2}}}{2}.$$

To figure out the sign, or equivalently what square root to take, note that we must have $G_\mu(\infty) = 0$, so we take the square root for which $\sqrt{1} = 1$, and then

$$G_\mu = \frac{z - z\sqrt{1 - \frac{4}{z^2}}}{2}.$$

We see that

$$-\frac{1}{\pi} \operatorname{Im} G_\mu(x + i\varepsilon)$$

converges uniformly to

$$\frac{1}{2\pi} \chi_{[-2,2]} \sqrt{4 - x^2}$$

as $\varepsilon \searrow 0$. Hence,

$$\mu_{\ell_1^*+\ell_1} = \frac{1}{2\pi} \chi_{[-2,2]} \sqrt{4 - t^2} \mathrm{d}\lambda(t),$$

finally proving Lemma 27.1.

32. The Central Limit Theorem

Theorem 32.1. *Let $a_1, a_2, \dots \in (\mathcal{A}, \varphi)$ be freely independent. Assume*

- *the a_i are centered, i.e. $\varphi(a_i) = 0$,*
- $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^n \varphi(a_j^2) = \frac{\alpha^2}{2}$,
- $\sup_{j\geq 1} \left|\varphi(a_j^k)\right| < \infty$.

Then

$$\mu_{\frac{1}{\sqrt{n}}(a_1 + \cdots + a_n)} \longrightarrow \frac{2}{\pi\alpha^2} \chi_{[-\alpha,\alpha]} \sqrt{\alpha^2 - t^2} \mathrm{d}\lambda(t)$$

in moments as $n \to \infty$.

Sketch of proof. Similar idea as in the classical case, but with the R-transform in place of the Fourier transform. See notes for details. $\square$

Is there similarly a Poisson limit law? In the classical setting:

$$\lim_{n\to\infty} \left((1 - \frac{\lambda}{n})\delta_0 + \frac{\lambda}{n}\delta_1 \right)^{*n} = \sum_{k\geq 0} \frac{\lambda^k}{k!} e^{-\lambda} \delta_k.$$

The free analogue would be

$$\lim_{n\to\infty} \left((1 - \frac{\lambda}{n})\delta_0 + \frac{\lambda}{n}\delta_1 \right)^{\boxplus n} = ?$$

Let $\mu_n = (1 - \frac{\lambda}{n})\delta_0 + \frac{\lambda}{n}\delta_1$. Then

$$G_{\mu_n}(z) = \int (z-t)^{-1} \mathrm{d}\mu_n(t) = (1 - \frac{\lambda}{n})z^{-1} + \frac{\lambda}{n}(z-1)^{-1} = \frac{z - (1 - \frac{\lambda}{n})}{z(z-1)}.$$

We can then solve for K_{μ_n} and R_{μ_n} and obtain

$$R_{\mu_n} = \frac{z-1}{2z}\left(1 - \sqrt{1 + \frac{4\lambda}{n}\frac{z}{(z-1)^2}}\right),$$

where we take the branch of the square root with $\sqrt{1} = 1$. We have

$$R_{\mu_n} = \frac{z-1}{2z}\left(-\frac{2\lambda}{n}\frac{z}{(n-1)^2} + O(n^{-2})\right),$$

so we see that

$$\lim_{n\to\infty} R_{\mu_n^{\boxplus n}} = \lim_{n\to\infty} nR_{\mu_n}(z) = \frac{\lambda}{1-z}.$$

So this should be the R-transform of the free Poisson measure ν. We then get $K_\nu(z) = z^{-1} + \frac{\lambda}{1-z}$, and

$$G_\nu = \frac{1 - \lambda + z + \sqrt{(z-1-\lambda)^2 - 4\lambda}}{2z}.$$

From this one can get

$$\nu = \begin{cases} (1-\lambda)\delta_0 + \Xi & \text{if } 0 \le \lambda \le 1, \\ \Xi & \text{if } \lambda > 1, \end{cases}$$

where

$$\Xi = \frac{1}{2\pi t}\chi_{[(1-\sqrt{\lambda})^2,(1+\sqrt{\lambda})^2]}\sqrt{4\lambda - (t-(1+\lambda))^2}\mathrm{d}\lambda(t).$$

This corresponds to the Marchenko–Pastur law from random matrix theory.

33. S-Transform

We used the R-transform to compute the additive free convolution. To compute the multiplicative free convolution we use the S-transform.

Theorem 33.1. *Let μ be a distribution with $\mu(X) \neq 0$. Let*

$$\begin{aligned} \psi_\mu(z) &= \sum_{k\ge1}\mu(X^k)z^k \\ \chi_\mu(z) &= \psi_\mu^{-1}(z) \quad \textit{(compositional inverse)}, \\ S_\mu(z) &= \chi_\mu(z)\frac{1+z}{z}. \end{aligned}$$

Then for any two such distributions μ, ν:

$$S_{\mu\boxtimes\nu} = S_\mu S_\nu.$$

34. Free Convolution of Measures with unbounded support

One can extend the operations $\boxplus, \boxtimes$ to general probability measures on $\mathbb{R}$ or $(0, \infty)$ (not necessarily compactly supported!), I didn't really take notes on this. For example the R-transform of the Cauchy distribution $\mu = \frac{1}{\pi} \frac{1}{1+x^2} \mathrm{d}\lambda(x)$ is $r_\mu(z) = -i$.

35. Free Independence with Amalgamation

The idea is to define free independence conditioned on a subalgebra.

Definition. *Let $\mathcal{B}$ be a unital algebra. A* noncommutative $\mathcal{B}$-probability space *is $(\mathcal{A}, \Phi)$ where $\mathcal{A}$ is a unital algebra with $\mathcal{B}$ as a subalgebra, and $\Phi : \mathcal{A} \to \mathcal{B}$ a $B - B$-bimodule map so that $\Phi|_B = \mathrm{id}_B$.*

Definition. *If $(\mathcal{A}, \Phi)$ is a noncommutative $\mathcal{B}$-probability space, and $(\mathcal{A}_\iota)_\iota$ subalgebras with $\mathcal{B} \subseteq \mathcal{A}_\iota \subseteq \mathcal{A}$, then the $(\mathcal{A}_\iota)_\iota$ are* B-free *if*

$$\Phi(a_1 \cdots a_n) = 0,$$

whenever $a_j \in \mathcal{A}_{\iota_j}$, $\Phi(a_j) = 0$ for $j = 1, \ldots, n$ and $\iota_1 \neq \iota_2 \neq \ldots \neq \iota_n$.

There are also C^*-, W^*-analgoues of this.

36. Multivariate Normal Form

37. Free analogue of Wick's Theorem

38. Random Matrices

References

[VDN92] D. Voiculescu, K. J. Dykema, and A. Nica. *Free Random Variables.* 1st ed. Vol. 1. Providence: American Mathematical Society, 1992.