Algebraic Geometry

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Taught by Paul Vojta
Notes taken by Leonard Tomczak

Contents

1 (Co)homological Algebra

Throughout, $\mathcal{A}$ is an abelian category.

Definition.

(a) A complex $A^\bullet$ *in* $\mathcal{A}$ *is a collection* $(A^i)_{i\in\mathbb{Z}}$ *of objects in* $\mathcal{A}$ *together morphisms* $d^i : A^i \to A^{i+1}$ *for all* $i \in \mathbb{Z}$ *such that* $d^{i+1} \circ d^i = 0$ *for all* i*. Often we will give the* A^i *only for* i *in a proper subset of* $\mathbb{Z}$*. If so, then by convention,* $A^i = 0$ *for all other* i*.*

(b) Let $A^\bullet, B^\bullet$ *be complexes in* $\mathcal{A}$*. Then a* morphism $f^\bullet : A^\bullet \to B^\bullet$ of complexes *is a collection* $(f^i)_{i\in\mathbb{Z}}$ *of morphisms* $f^i : A^i \to B^i$ *for all* i *such that the diagram*

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A^{i-1} & \longrightarrow & A^i & \longrightarrow & A^{i+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f^{i-1}} & & \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle f^{i+1}} & & \\ \cdots & \longrightarrow & B^{i-1} & \longrightarrow & B^i & \longrightarrow & B^{i+1} & \longrightarrow & \cdots \end{array}$$

commutes. This gives a category of complexes in $\mathcal{A}$*. It is an abelian category.*

Definition. *Let* $A^\bullet$ *be a complex in* $\mathcal{A}$*. Then for all* i *the* i-th cohomology $h^i(A^\bullet)$ *is the quotient*

$$h^i(A^\bullet) = \ker d^i / \operatorname{im} d^{i-1} := \textit{codomain of } \operatorname{coker}(\ker(\operatorname{coker} d^{i-1}) \to \ker d^i)$$

for all i*.*

Also for all morphisms $f^\bullet : A^\bullet \to B^\bullet$ *of complexes, we have a morphism* $h^i(f^\bullet) : h^i(A^\bullet) \to h^i(B^\bullet)$ *for all* i *defined uniquely by the condition that*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{im}(d_A^{i-1}) & \longrightarrow & \ker(d_A^i) & \longrightarrow & h^i(A^\bullet) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f^{i-1}} & & \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle h^i(f^\bullet)} & & \\ 0 & \longrightarrow & \operatorname{im}(d_B^{i-1}) & \longrightarrow & \ker(d_B^i) & \longrightarrow & h^i(B^\bullet) & \longrightarrow & 0 \end{array}$$

commutes. So h^i *isa covariant functor* $\{$*complexes in* $\mathcal{A}\} \to \mathcal{A}$*.*

Proposition 1.1.

(a) Let $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ *be a short exact sequence of complexes in* $\mathcal{A}$*. Then there are natural maps* $\delta^i : h^i(C^\bullet) \to h^{i+1}(A^\bullet)$ *such that the sequence*

$$\cdots \to h^{i-1}(C^\bullet) \xrightarrow{\delta^{i-1}} h^i(A^\bullet) \to h^i(B^\bullet) \to h^i(C^\bullet) \xrightarrow{\delta^i} h^{i+1}(A^\bullet) \to \dots$$

is exact.

(b) Furthermore, δ^i is functorial, in the following sense. Let

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1^\bullet & \longrightarrow & B_1^\bullet & \longrightarrow & C_1^\bullet & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle f_A^\bullet} & & \big\downarrow{\scriptstyle f_B^\bullet} & & \big\downarrow{\scriptstyle f_C^\bullet} & & \\
0 & \longrightarrow & A_2^\bullet & \longrightarrow & B_2^\bullet & \longrightarrow & C_2^\bullet & \longrightarrow & 0
\end{array}$$

be a commutative diagram of complexes in $\mathcal{A}$ with exact rows. Then the diagram

$$\begin{array}{ccc}
h^i(C_1^\bullet) & \xrightarrow{\delta_1^i} & h^{i+1}(A_1^\bullet) \\
\big\downarrow{\scriptstyle h^i(f_C^\bullet)} & & \big\downarrow{\scriptstyle h^i(f_A^\bullet)} \\
h^i(C_i^\bullet) & \xrightarrow{\delta_2^i} & h^{i+1}(A_2^\bullet)
\end{array}$$

commutes.

Proof. Use Freyd's theorem and the Snake Lemma for abelian groups (see Lang's *Algebra* or the movie *It's My Turn*). □

Definition. *Let $f^\bullet, g^\bullet : A^\bullet \to B^\bullet$ be morphisms of complexes in $\mathcal{A}$. We say that $f^\bullet$ and $g^\bullet$ are* homotopic *and write $f^\bullet \sim g^\bullet$, if there exist morphisms $k^i : A^i \to B^i$ in $\mathcal{A}$ for all i such that*

$$f^\bullet - g^\bullet = kd + dk.$$

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A^{i-1} & \longrightarrow & A^i & \longrightarrow & A^{i+1} & \longrightarrow & \cdots \\
 & \swarrow{\scriptstyle k^{i-1}} & \big\downarrow{\scriptstyle f,g} & \swarrow{\scriptstyle k^i} & \big\downarrow{\scriptstyle f,g} & \swarrow{\scriptstyle k^{i+1}} & \big\downarrow{\scriptstyle f,g} & & \\
\cdots & \longrightarrow & B^{i-1} & \longrightarrow & B^i & \longrightarrow & B^{i+1} & \longrightarrow & \cdots
\end{array}$$

Such a collection is called a homotopy operator*.*

Fact. If $f^\bullet \sim g^\bullet$, then $h^i(f^\bullet) = h^i(g^\bullet)$ for all i.

Proof. Use Freyd and chase definitions. □

Fact. $\sim$ is an equivalence relation.

Recall. Let $A \in \mathcal{A}$. Then $\operatorname{Hom}(A, \cdot) : B \mapsto \operatorname{Hom}(A, B)$ and $\operatorname{Hom}(\cdot, A) : B \mapsto \operatorname{Hom}(B, A)$ are covariant and contravariant functors respectively, from $\mathcal{A}$ to **Ab**. They are also additive and left exact.

Definition. *An object $I \in \mathcal{A}$ is* injective *if the functor* $\operatorname{Hom}(\cdot, I)$ *is (right) exact.*

In conrete terms, this means for all short exact sequences $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$, the map $\operatorname{Hom}(A, I) \to \operatorname{Hom}(A', I)$ is surjective. So all maps $A' \to I$ can be extended to give maps $A \to I$.

Dually, an objective $P \in \mathcal{A}$ is projective if $\operatorname{Hom}(P, \cdot)$ is right exact, i.e. $\operatorname{Hom}(P, A) \to \operatorname{Hom}(P, A'')$ surjective.

Definition. *A* coresolution*, or* right resolution*, of an object $A \in \mathcal{A}$ is an exact sequence $0 \to A \xrightarrow{\varepsilon} E^0 \to E^1 \to \cdots$ in $\mathcal{A}$ (equivalently, a complex $E^\bullet$ in $\mathcal{A}$, zero in all degrees < 0, together with an augmentaion map $\varepsilon : A \to E^0$ such that the above sequence is exact). Dually, a* (left) resolution *is an exact sequence $\cdots \to E_2 \to E_1 \to E_0 \to A \to 0$ in $\mathcal{A}$.*

Definition. *Let P be a property of objects of $\mathcal{A}$. Then a P resolution or coresolution of an object A in $\mathcal{A}$ is a resolution or coresolution $E_\bullet$ or $E^\bullet$ of A in which E_i or E^i has P for all i.*

Definition. *An abelian category $\mathcal{A}$ has* enough injectives *if for all $A \in \mathcal{A}$ there is a monomorphism from A to an injective object of $\mathcal{A}$.*

Proposition 1.2. *If $\mathcal{A}$ has enough injectives, then every object of $\mathcal{A}$ has an injective resolution.*

Proof. Let $A \in \mathcal{A}$. We construct $E^\bullet$ inductively. Let $A \to E^0$ be a monomorphism such that E^0 is injective. Given $0 \to A \to E^0 \to \cdots E^n$, let $\operatorname{coker}(E^{n-1} \to E^n) \to E^{n+1}$ be a monomorphism with E^{n+1} injective for $n > 0$ and for $n = 0$ take $\operatorname{coker}(A \to E^0) \to E^1$. Then $0 \to A \to E^0 \to E^1 \to \cdots$ is an injective resolution of A. □

Lemma 1.3. *Let $\varphi : A \to B$ be a morphism in $\mathcal{A}$, and let $I^\bullet, J^\bullet$ resp. be right resolutions of A, B with $J^\bullet$ injective. Then there is a map $f^\bullet : I^\bullet \to J^\bullet$ such that the diagram*

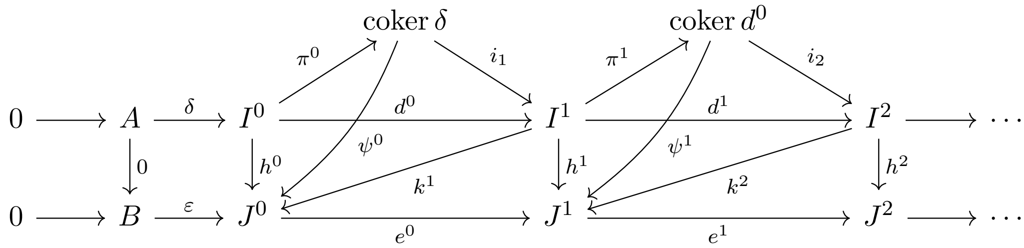
$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \\ 0 & \longrightarrow & B & \longrightarrow & J^0 & \longrightarrow & J^1 & \longrightarrow & \cdots \end{array}$$

commutes.

Proof. Use induction and the definition of injective. Exercise. □

Lemma 1.4. *With notation as in the previous lemma, any two such morphisms $f^\bullet, g^\bullet : I^\bullet \to J^\bullet$ are homotopic.*

Proof. Let $h^\bullet = f^\bullet - g^\bullet$. Then we have the following diagram in which the rectangles commute:



Since $h^0 \circ \delta = 0$, h^0 factors through $\operatorname{coker} \delta$, giving $\psi^0 : \operatorname{coker} \delta \to J^0$. Since $\operatorname{coker} \delta \to I^1$ is a monomorphism, and J^0 is injective, this extends to a map $k^1 : I^1 \to J^0$. So $k^1 i_1 = \psi^0$ and $\psi^0 \pi^0 = h^0$. Also let $k^0 : I^0 \to J^{-1} = 0$ be the zero map. Then $e^{-1}k^0 + k^1 d^0 = k^1 d^0 = k^1 i_1 \pi^0 = \psi^0 \pi^0 = h^0$. Now $h^1 - e^0 k^1$ vanishes on $\operatorname{im} d^0$ because $(h^1 - e^0 k^1) d^0 = e^0 h^0 - e^0 h^0 = 0$. So $h^1 - e^0 k^1$ factors through $\operatorname{coker} d^0$, i.e. there is $\psi^1 : \operatorname{coker} d^0 \to J^1$ such that $h^1 - e^0 k^1 = \psi^1 \circ \pi^1$. Since $\operatorname{coker} d^0 \to I^2$ is a monomorphism and J^1 is injective, ψ^1 extends to $k^2 : I^2 \to J^1$ (so $k^2 \circ i_2 = \psi^1$) and $e^0 k^1 + k^2 d^1 = h^1 - \psi^1 \pi^1 + k^2 d^1 = h^1 - k^2 i_2 \pi^1 + k^2 d^1 = h^1$. Then proceed by induction. □

1.1 Right-derived functors

For this section, $\mathcal{A}$ is an abelian category with enough injectives, $\mathcal{B}$ is an abelian category, and $F : \mathcal{A} \to \mathcal{B}$ is a covariant left exact functor.

Definition.

(a) *For each object $A \in \mathcal{A}$, choose an injective resolution $0 \to A \to I^\bullet$. Then we define*

$$R^i F(A, I^\bullet) = h^i(F(I^\bullet)) \quad \forall i \in \mathbb{N}.$$

(b) *For each morphism $\varphi : A \to B$ in $\mathcal{A}$, choose injective resolutions $0 \to A \to I^\bullet, 0 \to B \to J^\bullet$, and choose $f^\bullet : I^\bullet \to J^\bullet$ as in Lemma 1.3. Then we define*

$$R^i F(\varphi, f^\bullet) : R^i F(A, I^\bullet) \to R^i F(B, J^\bullet)$$

to be $h^i(F(f^\bullet))$ for all $i \in \mathbb{N}$.

Lemma 1.5. *Let $\varphi : A \to B, I^\bullet, J^\bullet$ as in (b) of the above definition. Then, for any two morphisms $f^\bullet, g^\bullet : I^\bullet \to J^\bullet$ as in Lemma 1.3, we have $R^i F(\varphi, f^\bullet) = R^i F(\varphi, g^\bullet)$ for all i.*

Proof. By Lemma 1.4, $f^\bullet$ and $g^\bullet$ are homotopic. Let $k^\bullet$ be a homotopy operator between them. Then $F(k^\bullet)$ is a homotopy operator between $F(f^\bullet)$ and $F(g^\bullet)$, so $R^i F(\varphi, f^\bullet) = h^i(F(f^\bullet)) = h^i(F(g^\bullet)) = R^i F(\varphi, g^\bullet)$. □

Definition. *Continuation of the above definition. We define $R^i F(\varphi, I^\bullet, J^\bullet) := R^i F(\varphi, f^\bullet)$ for any $f^\bullet : I^\bullet \to J^\bullet$ as in Lemma 1.3. By the lemma this is well defined.*

Now let $\varphi : A \to B$ and $\psi : B \to C$ be morphisms in $\mathcal{A}$, and let

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \longrightarrow & I^\bullet \\ & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle f^\bullet} \\ 0 & \longrightarrow & B & \longrightarrow & J^\bullet \\ & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle g^\bullet} \\ 0 & \longrightarrow & C & \longrightarrow & K^\bullet \end{array}$$

be a commutative diagram in which the rows are injective resolutions. Then

$$\begin{array}{ccc} R^iF(A, I^\bullet) & \xrightarrow{R^iF(\varphi, I^\bullet, J^\bullet)} & R^iF(B, J^\bullet) \\ & \searrow{\scriptstyle R^iF(\psi\circ\varphi, I^\bullet, K^\bullet)} & \downarrow{\scriptstyle R^iF(\psi, J^\bullet, K^\bullet)} \\ & & R^iF(C, K^\bullet) \end{array}$$

commutes for all i. So

$$R^iF(\mathrm{id}_A, I^\bullet, J^\bullet) : R^iF(A, I^\bullet) \to R^iF(A, J^\bullet)$$

is an isomorphism for all injective resolutions $0 \to A \to I^\bullet$ and $0 \to A \to J^\bullet$ of A with inverse $R^iF(\mathrm{id}_A, J^\bullet, I^\bullet)$. Therefore $R^iF(A)$ is well defined up to isomorphism for all i, and so is $R^iF(\varphi)$ for all $\varphi : A \to B$ (exercise).

Also R^iF is an additive functor for all i (because all steps in the above construction are additive).

This proves part (a) of Theorem 1.1A in [Har77]. Part (b): $F \cong R^0F$, canonically.

Proof. Let $A \in \mathcal{A}$, and let $0 \to A \to I^\bullet$ be an injective resolution. Since $0 \to A \to I^0 \to I^1$ is exact, so is $0 \to F(A) \to F(I^0) \to F(I^1)$. So $R^0F(A) = h^0(F(I^\bullet)) = \ker(F(I^0) \to F(I^1)) = F(A)$. For a morphism $\varphi : A \to B$ in $\mathcal{A}$, we get a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & I^0 & \longrightarrow & I^1 \\ & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} \\ 0 & \longrightarrow & B & \longrightarrow & J^0 & \longrightarrow & J^1 \end{array}$$

Apply F to it, and use commutativity and exactness to get $F(\varphi) = h^0(f^\bullet) = R^0F(\varphi)$. $\square$

Theorem 1.6 (1.1A(c))**.** *For each short exact sequence*

$$0 \to A' \to A \to A'' \to 0$$

in $\mathcal{A}$, there are morphisms $\delta^i : R^iF(A'') \to R^{i+1}F(A^i)$ for all $i \geq 0$, giving a long exact sequence

$$\cdots \to R^{i-1}F(A'') \xrightarrow{\delta^{i-1}} R^iF(A') \to R'F(A) \to R^iF(A'') \xrightarrow{\delta^i} R^{i+1}F(A') \to \cdots$$

in R^iF.

Proof. This is "almost" the snake lemma. "Almost" because we need a SES of injective resolutions. To do this, choose injective resolutions $0 \to A' \to I^\bullet$ and $0 \to A'' \to J^\bullet$; then we will construct a commutative diagram with exact rows (given by the obvious maps $I^i \to I^i \oplus J^i$ and $I^i \oplus J^i \to J^i$) and the middle column is an injective resolution of A.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varepsilon'} & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon''} & & \\
0 & \longrightarrow & I^0 & \longrightarrow & I^0 \oplus J^0 & \longrightarrow & J^0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d^{0'}} & & \downarrow{\scriptstyle d^0} & & \downarrow{\scriptstyle d^{0''}} & & \\
0 & \longrightarrow & I^1 & \longrightarrow & I^1 \oplus J^1 & \longrightarrow & J^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

The key step is: Given a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K' & \longrightarrow & K & \longrightarrow & K'' & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle f'} & & & & \big\downarrow{\scriptstyle f''} & & \\
 & & I & & & & J & &
\end{array}$$

in which the top row is exact and I is injective, it can be filled in to give a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K' & \longrightarrow & K & \longrightarrow & K'' & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f''} & & \\
0 & \longrightarrow & I & \longrightarrow & I \oplus J & \longrightarrow & J & \longrightarrow & 0
\end{array}$$

in which the maps $0 \to I \to I \oplus J \to J \to 0$ are the obvious ones, and f is injective. We really just need maps $g' : K \to I, g'' : K \to J$ such that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K' & \longrightarrow & K & \longrightarrow & K'' & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle f'} & \swarrow{\scriptstyle g'} & & \searrow{\scriptstyle g''} & \big\downarrow{\scriptstyle f''} & & \\
 & & I & & & & J & &
\end{array}$$

commutes. g'' is clear, for g' use that I is injective. Then set $f = (g', g'')$.

Remark. In an abelian category any SES $0 \to B' \to B \to B'' \to 0$, in which B' is injective, splits.

Use this with K', K, K'', I, J equal to A', A, A'', I^0, J^0 to get $\varepsilon : A \to I^0 \oplus J^0$ and then with $\operatorname{coker} \varepsilon', \operatorname{coker} \varepsilon, \operatorname{coker} \varepsilon'', I^1, J^1$ to get $d^0 : I^0 \oplus J^0 \to I^1 \oplus J^1$, etc. □

Theorem 1.7 (1.1A(d))**.** *δ^i is natural with respect to morphisms of SES's in $\mathcal{A}$. In other words, for all i, δ^i is a natural transformation from*

$$(0 \to A' \to A \to A'' \to 0) \longmapsto R^iF(A')$$

to

$$(0 \to A' \to A \to A'' \to 0) \longmapsto R^{i+1}F(A'').$$

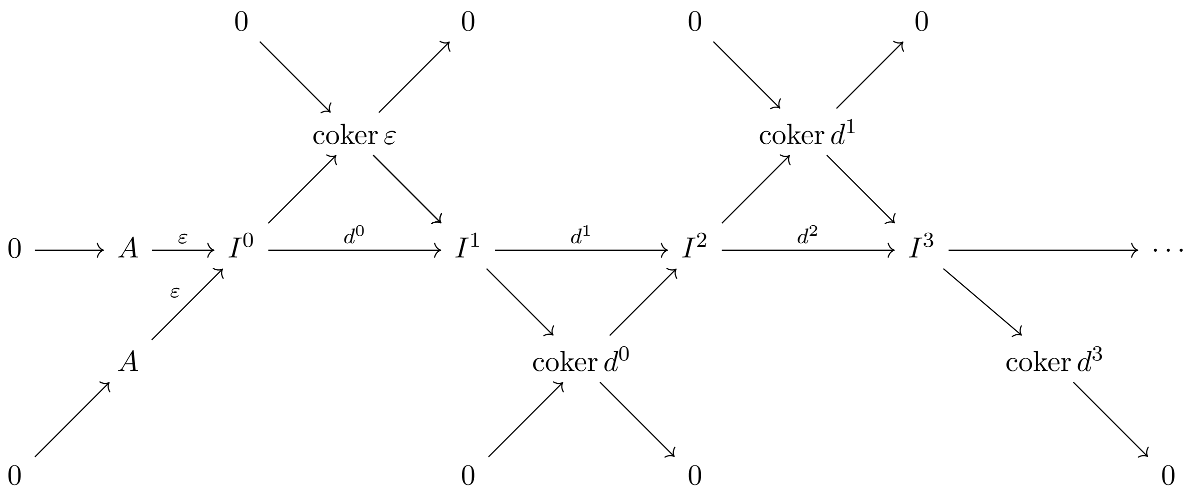
Theorem 1.8 (1.1A(e))**.** *If I is an injective object of $\mathcal{A}$, then $R^iF(I) = 0$ for all $i > 0$.*

Proof. Use the injective resolution $0 \to I \to I \to 0 \to \cdots$ of I. □

Definition. *An object $A \in \mathcal{A}$ is* acyclic for F, *or* F-acyclic, *if $R^iF(A) = 0$ for $i > 0$.*

So all injective objects of $\mathcal{A}$ are F-acyclic for all left-exact convariant functors $F : \mathcal{A} \to \mathcal{B}$.

Fact. In the diagram below,



the horizontal sequence is exact iff all of the diagonal sequences are.

Proposition 1.9 (1.2A)**.** *Let $A \in \mathcal{A}$ and let $0 \to A \xrightarrow{\varepsilon} J^\bullet$ be an F-acyclic resolution of A. Then there are canonical isomorphisms*

$$R^iF(A) \cong h^i(F(J^\bullet))$$

for all i.

Proof. Long exact sequences and induction on i (exercise). □

Definition. *A (covariant)* δ-functor *from* $\mathcal{A}$ *to* $\mathcal{B}$ *is a collection* $(T^i)_{i\in\mathbb{N}}$ *of additive functors* T^i*, together with morphisms* $\delta^i : T^i(A'') \to T^{i+1}(A')$ *for all short exact sequences* $0 \to A' \to A \to A'' \to 0$ *in* $\mathcal{A}$ *and for all* $i \geq 0$ *such that*

(i) There is a long exact sequence as in Theorem 1.6, and

(ii) δ^i *is natural as in Theorem 1.7.*

Example. $(R^iF)_{i\geq 0}$ is a δ-functor (provided F is left exact and $\mathcal{A}$ has enough injectives).

Definition. *A* δ*-functor* $T = (T^i : \mathcal{A} \to \mathcal{B})_{i\geq 0}$ *is* initial *if, for any other* δ*-functor* $U = (U^i : \mathcal{A} \to \mathcal{B})_{i\geq 0}$ *and for any morphism* $\varphi : T^0 \to U^0$*, there is a unique sequence* $(f^i : T^i \to U^i)_{i\geq 0}$ *of morphisms of functors with* $f^0 = \varphi$*, that commute with the* δ^i *for all short exact sequences and for all* i*.*

If an initial δ-functor exists, it is unique up to unique isomorphism.

Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories.

Definition. *An additive functor* $F : \mathcal{A} \to \mathcal{B}$ *is* effaceable *if for all* $A \in \mathcal{A}$ *there is a monomorphism* $u : A \to M$ *for some* M *such that* $F(u) = 0$*.*

Theorem 1.10 (1.3A)**.** *Let* $T = (T^i)_{i\geq 0}$ *be a* δ*-functor from* $\mathcal{A}$ *to* $\mathcal{B}$*. If* T^i *is effaceable for all* $i > 0$*, then* T *is initial.*

Corollary 1.11. *Assume that* $\mathcal{A}$ *has enough injectives. Then*

(a) For any left-exact, covariant functor $F : \mathcal{A} \to \mathcal{B}$*, the right-derived functors* $(R^iF)_{i\geq 0}$ *form an initial* δ*-functor with* $R^0F \cong F$*.*

(b) If $T = (T^i)_{\geq 0}$ *is any initial* δ*-functor, then* T^0 *is left-exact, and* $T^i \cong R^iT^0$ *for all* $i \geq 0$*.*

Proof.

(a) We already know that $(R^iF)_{i\geq 0}$ is a δ-functor with $R^0F \cong F$. It remains only to show that δ is initial. By Theorem 1.10, it is enough to show that T^i is effaceable for all $i > 0$. Let $i > 0$ and $A \in \mathcal{A}$. Pick a monomorphism $u : A \to M$ with M injective. Then, for all $i > 0$, $R^iF(u) = 0$ because $R^iF(M) = 0$.

(b) Let T be as given. Then for all short exact sequences $0 \to A' \to A \to A'' \to 0$, the long exact sequence begins $0 \to T^0A' \to T^0A \to T^0A'' \to \ldots$, so T^0 is left-exact. Since both $(T^i)_{i\geq 0}$ and $(R^iT^0)_{i\geq 0}$ are initial δ-functors and $T^0 \cong R^0T^0$, we have $T^i \cong R^iT^0$ for all i (uniquely).

□

1.2 Categories with enough Injectives

To use this theory, we need to show that certain categories have enough injectives. We start with **Ab**.

Lemma 1.12. *An abelian group A is injective if and only if it is divisible.*

Proof. "$\Rightarrow$" Given $a \in A$ and $n \in \mathbb{Z}$, $n \neq 0$, let $\varphi : \mathbb{Z} \to A$ be defined by $1 \mapsto a$. Then there is $\psi : \mathbb{Z} \to A$ such that

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot n} & \mathbb{Z} \\ & & \downarrow{\scriptstyle \varphi} & \swarrow{\scriptstyle \exists \psi} & \\ & & A & & \end{array}$$

commutes. Then $na' = a$ for $a' = \psi(1)$.

"$\Leftarrow$" Let A be a divisible abelian group, and let

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \xrightarrow{\cdot n} & M' \\ & & \downarrow{\scriptstyle \varphi} & & \\ & & A & & \end{array}$$

be a diagram in **Ab**, with M a subgroup of M'. For any M'' with $M \subseteq M'' \subsetneq M'$, any $\psi : M'' \to A$ extending φ, any $x \in M'$, we can extend ψ to a map $\rho : \langle M'', x\rangle \to A$ as follows. Let $H = \{n \in \mathbb{Z} : nx \in M''\}$; it is a subgroup of $\mathbb{Z}$. If $H = 0$, then $\langle M'', x\rangle \cong M'' \oplus \mathbb{Z}$, so we can extend ψ by letting $\rho(x) = 0$. Otherwise, $H = n\mathbb{Z}$ with $n \neq 0$. Choose $a_0 \in A$ such that $na_0 = \psi(nx)$. Then define ρ by $\rho(m + kx) = \psi(m) + ka_0$ where $m \in M'', k \in \mathbb{Z}$. This is well defined and works. Then conclude by Zorn's lemma. □

Proposition 1.13. *The category* **Ab** *has enough injectives.*

Proof. Recall the Pontryagin dual of an abelian group A is $\widehat{A} = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$. Then $A \mapsto \widehat{A}$ is a contravariant left-exact functor which is in fact exact as $\mathbb{Q}/\mathbb{Z}$ is divisible, hence injective.

Step 1. The natural map $A \to \widehat{\widehat{A}}$ is injective. Suppose $0 \neq a \in A$. Then define $\varphi : \langle a\rangle \to \mathbb{Q}/\mathbb{Z}$ by $\varphi(a) = \frac{1}{2}$ if the order of a is infinite and $\frac{1}{m}$ if the order of a is $m < \infty$. Since $\mathbb{Q}/\mathbb{Z}$ is injective, φ extends to $\alpha : A \to \mathbb{Q}/\mathbb{Z}$. Then $\alpha(a) = \varphi(a) \neq 0$, so $a \notin \ker(A \to \widehat{\widehat{A}})$.

Note. $\widehat{\widehat{A}}$ is not always injective, for example if $A = \mathbb{Z}/2\mathbb{Z}$, then $\widehat{\widehat{A}} \cong \widehat{A} \cong \mathbb{Z}/2\mathbb{Z}$.

Step 2. Construct an injection from $\widehat{\widehat{A}}$ to some injective object. Choose a surjection $\bigoplus_{i\in I} \mathbb{Z} \twoheadrightarrow \widehat{A}$. This gives an injection

$$\widehat{\widehat{A}} \to \big(\bigoplus_{i\in I} \mathbb{Z}\big)^{\widehat{}} = \mathrm{Hom}\big(\bigoplus_{i\in I} \mathbb{Z}, \mathbb{Q}/\mathbb{Z}\big) = \prod_{i\in I} \mathrm{Hom}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \prod_{i\in I} \mathbb{Q}/\mathbb{Z},$$

which is divisible, and hence injective.

□

Example. Let $A = \mathbb{Z}$. Then $\widehat{A} = \mathrm{Hom}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$. The easiest surjection is $\bigoplus_{i=1}^{\infty} \mathbb{Z} \to \mathbb{Q}/\mathbb{Z}, e_i \mapsto \frac{1}{i}$ for all i. So $\mathbb{Z} \hookrightarrow \prod_{i=1}^{\infty} \mathbb{Q}/\mathbb{Z}, n \mapsto (n/1, n/2, n/3, \dots)$.

So we have proved:

Theorem 1.14. **Ab** *has enough injectives.*

Theorem 1.15. *Let A be a commutative ring. Then* $\mathbf{Mod}(A)$ *has enough injectives.*

Proof. See Homework 2. □

Theorem 1.16. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then* $\mathbf{Mod}(X)$ *has enough injectives.*

Proof. Let $\mathcal{F} \in \mathbf{Mod}(X)$. For all $x \in X$ choose an embedding of the stalk $\mathcal{F}_x$ into an injective $\mathcal{O}_{X,x}$-module I_x. Let $\mathcal{I} = \prod_{x \in X} j_{x*} I_x$, where $j_x : \{x\} \to X$ is the inclusion (and therefore $j_{x*} I_x$ is a skyscraper sheaf). So for all $U \subseteq X$ open, $\mathcal{I}(U) = \prod_{x \in U} I_x$. We have a naturally defined map $\mathcal{F} \to \mathcal{I}$, given by

$$\underset{\in \mathcal{F}(U)}{f} \longmapsto \underset{\in \prod_{x\in U}}{(f_x)_{x \in U}} \longmapsto \underset{\in \prod_{x \in U} I_x = \mathcal{I}(U)}{(f_x)_{x \in U}}$$

which is injective as it is injective on stalks. We claim that $\mathcal{I}$ is injective in $\mathbf{Mod}(X)$. Let $\mathcal{G} \in \mathbf{Mod}(X)$. Then

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{I}) = \prod_{x \in X} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, j_{x*} I_x) = \prod_{x \in X} \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x).$$

Let $\mathcal{G} \to \mathcal{H}$ be an injection of sheaves: Then $\mathcal{G}_x \to \mathcal{H}_x$ is injective for all x, so $\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{H}_x, I_x) \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x)$ is surjective for all x, therefore $\prod_{x \in X} \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{H}_x, I_x) \to \prod_{x \in X} \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x)$ is surjective, so $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{I}) \to \mathrm{Hom}_{\mathcal{O}_x}(\mathcal{G}, \mathcal{I})$ is surjective, and then $\mathcal{I}$ is injective. □

Corollary 1.17. *Let X be a topological space. Then* $\mathbf{Ab}(X)$ *as enough injectives.*

Proof. Take $\mathcal{O}_X = \underline{\mathbb{Z}}$ be the constant sheaf of rings $\mathbb{Z}$ on X. Then $(X, \mathcal{O}_X)$ is a ringed space and $\mathbf{Mod}(X) \cong \mathbf{Ab}(X)$. □

So, we have proved that $\mathbf{Ab}, \mathbf{Mod}(A), \mathbf{Mod}(X)$ and $\mathbf{Ab}(X)$ have enough injectives.

Definition. *Let X be a topological space. Since* $\mathbf{Ab}(X)$ *has enough injectives and* $\Gamma(X, \cdot) : \mathbf{Ab}(X) \to \mathbf{Ab}$ *is covariant and left-exact, there are right-derived functors* $R^i\Gamma(X, \cdot) : \mathbf{Ab}(X) \to \mathbf{Ab}$ *for all i. We define the* cohomology functors $H^i(X, \cdot) : \mathbf{Ab}(X) \to \mathbf{Ab}$ *to be* $R^i\Gamma(X, \cdot)$ *for all i.*

Next: We will show that, for any ringed space $(X, \mathcal{O}_X)$, $H^i(X, \cdot)$ can be computed as the right-derived functors of $\Gamma(X, \cdot) : \mathbf{Mod}(X) \to \mathbf{Ab}$ for all $\mathcal{F} \in \mathbf{Mod}(X)$.

Recall: A sheaf $\mathcal{F} \in \mathbf{Ab}(X)$ is *flasque* if the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective for all opens $V \subseteq U$ in X.

Example. The injective sheaf $\mathcal{I}$ from the earlier proof is flasque.

Lemma 1.18. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then any injective module in $\mathbf{Mod}(X)$ is flasque.*

Proof. Let $\mathcal{I} \in \mathbf{Mod}(X)$ be an injective object. Let $U \subseteq X$ be open and let $j : U \to X$ be the inclusion map, and let $\mathcal{O}_U = j_!(\mathcal{O}_X|_U)$. This is an $\mathcal{O}_X$-module. Recall that if $\mathcal{F}$ is a sheaf on U, then $j_!\mathcal{F}$ is defined to be the sheaf associated to the presheaf

$$W \mapsto \begin{cases} \mathcal{F}(W) & \text{if } W \subseteq U, \\ 0 & \text{otherwise} \end{cases}$$

the stalks of which are

$$(j_!\mathcal{F})_x = \begin{cases} \mathcal{F}_x & \text{if } x \in U, \\ 0 & \text{otherwise.} \end{cases}$$

Also $(j_!\mathcal{F})|_U \cong \mathcal{F}$. For all open $V \subseteq U \subseteq X$, we have an injection $\mathcal{O}_V \hookrightarrow \mathcal{O}_U$ of sheaves on X. Since $\mathcal{I}$ is injective,

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_U, \mathcal{I}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{I})$$

is surjective. Note that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_U, \mathcal{I}) = \mathcal{I}(U)$ and under this identification the above surjective map corresponds to the restriction $\mathcal{I}(U) \to \mathcal{I}(V)$, so $\mathcal{I}$ is flasque. To see that equality, let $\mathcal{O}'_U$ be the presheaf $W \mapsto \begin{cases} \mathcal{F}(W) & \text{if } W \subseteq U, \\ 0 & \text{otherwise.} \end{cases}$ Then

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_U, \mathcal{I}) &= \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}'_U, \mathcal{I}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{O}'_U|_U, \mathcal{I}|_U) \\ &= \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{O}_X|_U, \mathcal{I}|_U) = \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(U), \mathcal{I}(U)) = \mathcal{I}(U). \end{aligned}$$

□

Lemma 1.19. *Let X be a topological space and let $\mathcal{F} \in \mathbf{Ab}(X)$. If $\mathcal{F}$ is flasque, then it is acyclic.*

Proof. Assume that $\mathcal{F}$ is flasque. Embed it into an injective sheaf $\mathcal{I}$, and let $\mathcal{G} = \mathcal{I}/\mathcal{F}$. Then $0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0$ is exact. Since $\mathcal{F}$ and $\mathcal{I}$ are flasque, so is $\mathcal{G}$ ([Har77, Ex. 1.16c]), so by [Har77, Ex. 1.16b],

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{I}) \to \Gamma(X, \mathcal{G}) \to 0$$

is exact. Since $\mathcal{I}$ is acyclic, we get $H^1(X, \mathcal{F}) = 0$ from the long exact sequence, and $H^i(X, \mathcal{F}) \cong H^{i-1}(X, \mathcal{G})$ for all $i > 1$ from the LES, so $H^i(X, \mathcal{F}) = 0$ by induction. □

Proposition 1.20. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then $H^i(X, \cdot)$ can also be computed as the i-th right derived functor of* $\Gamma(X, \cdot) : \mathbf{Mod}(X) \to \mathbf{Ab}$ *for all $i \geq 0$.*

Proof. Use the previous two results together with the fact that the right derived functors can be computed from an acyclic resolution. □

Corollary 1.21. *If X is a scheme over* $\operatorname{Spec} A$ *for some ring A, then $H^i(X, \mathcal{F})$ has a naturally defined A-module structure for all i, for all $\mathcal{F} \in \mathbf{Mod}(X)$.*

Proof. For such X, all sheaves in $\mathbf{Mod}(X)$ are also sheaves of A-modules. So we can let $\Gamma_A(X, \cdot)$ be the global section functor $\mathbf{Mod}(X) \to \mathbf{Mod}(A)$. Then

$$\begin{array}{ccc} \mathbf{Mod}(X) & \xrightarrow{\Gamma_A(X,\cdot)} & \mathbf{Mod}(A) \\ & {\scriptstyle \Gamma(X,\cdot)} \searrow \quad \swarrow {\scriptstyle \text{forgetful}} & \\ & \mathbf{Ab} & \end{array}$$

commutes. Therefore the diagram

$$\begin{array}{ccc} \mathbf{Mod}(X) & \xrightarrow{R^i\Gamma_A(X,\cdot)} & \mathbf{Mod}(A) \\ & {\scriptstyle H^i(X,\cdot)} \searrow \quad \swarrow {\scriptstyle \text{forgetful}} & \\ & \mathbf{Ab} & \end{array}$$

commutes, because the forgetful functor is exact. So we can identify $H^i(X, \mathcal{F})$ with $R^i\Gamma_A(X, \cdot)$, thus giving $H^i(X, \mathcal{F})$ the structure of an A-module for all $i, \mathcal{F}$. □

2 Cohomology of Coherent Sheaves on Noetherian Schemes

2.1 Grothendieck's Vanishing Theorem

Lemma 2.1. *Let X be a noetherian topological space, and let $(\mathcal{F}_\alpha)_{\alpha\in A}$ be a directed system of flasque sheaves on X. Then $\varinjlim \mathcal{F}_\alpha$ is flasque.*

Proof. Let $V \subseteq U$ be open subsets of X. Then $\mathcal{F}_\alpha(U) \to \mathcal{F}_\alpha(V)$ is onto for all α. Then $\varinjlim \mathcal{F}_\alpha(U) \to \varinjlim \mathcal{F}_\alpha(V)$ is surjective as $\varinjlim$ is an exact functor in $\mathbf{Ab}$. Also $(\varinjlim \mathcal{F}_\alpha)(U) = \varinjlim \mathcal{F}_\alpha(U)$ since X is noetherian ([Har77, Exercise II, 1.11]). □

Proposition 2.2. *Let X be a noetherian topological space, and let $(\mathcal{F}_\alpha)_{\alpha\in A}$ be a direct system in $\mathbf{Ab}(X)$. Then for all i there are natural isomorphisms*

$$\varinjlim H^i(X, \mathcal{F}_\alpha) \xrightarrow{\simeq} H^i(X, \varinjlim \mathcal{F}_\alpha)$$

compatible with the restriction maps.

Proof. Let $\mathcal{C}$ be the category of A-directed systems in $\mathbf{Ab}(X)$. We need to show that $\varinjlim H^i(X,\cdot)$ and $H^i(X, \varinjlim \cdot)$ are isomorphic functors $\mathcal{C} \to \mathbf{Ab}$. For each α, inject $\mathcal{F}_\alpha$ into its sheaf $\mathcal{G}^0_\alpha$ $(= U \mapsto \prod_{x\in U} (\mathcal{F}_\alpha)_x)$ of discontinuous sections, which is flasque. Map $\operatorname{coker}(\mathcal{F}_\alpha \to \mathcal{G}^0_\alpha)$ into its sheaf $\mathcal{G}^1_\alpha$ of discontinuous sections, etc. Then for all α we get a flasque resolution

$$0 \to \mathcal{F}_\alpha \to \mathcal{G}^\bullet_\alpha$$

of $\mathcal{F}_\alpha$. This process is functorial, so $(\mathcal{G}^i_\alpha)_{\alpha\in A}$ is a directed system for all i. Then we have

$$\begin{aligned}\varinjlim H^i(X,\mathcal{F}_\alpha) = \varinjlim h^i(\Gamma(X,\mathcal{G}^\bullet_\alpha)) &\underset{\varinjlim \text{ exact in } \mathbf{Ab}}{=} h^i(\varinjlim \Gamma(X,\mathcal{G}^\bullet_\alpha)) \\ &\underset{\text{[Har77, Exercise II, 1.11]}}{=} h^i(\Gamma(X, \varinjlim \mathcal{G}^\bullet_\alpha)).\end{aligned}$$

Claim: $\varinjlim$ is an exact functor on $\mathbf{Ab}(X)$.

Proof. Because $\varinjlim$ commutes with $\varinjlim$ (over two different directed systems), $(\varinjlim \mathcal{G}^i_\alpha)_P = \varinjlim((\mathcal{G}^i_\alpha)_P)$ for all i, P and $\varinjlim$ is exact in $\mathbf{Ab}$. □

Then $0 \to \varinjlim \mathcal{F}_\alpha \to \varinjlim \mathcal{G}^\bullet_\alpha$ is a flasque resolution, so

$$h^i(\Gamma(X, \varinjlim \mathcal{G}^i_\alpha)) = H^i(X, \varinjlim \mathcal{F}_\alpha).$$

□

Corollary 2.3. *On a noetherian space cohomology commutes with arbitrary direct sums.*

Proof. Any direct sum is the $\varinjlim$ of its finite subsums. □

Lemma 2.4. *Let Y be a closed subset of a topological space X, let $j : Y \to X$ be the inclusion map, and let $\mathcal{F} \in \mathbf{Ab}(Y)$. Then $H^i(X, j_*\mathcal{F}) \cong H^i(Y, \mathcal{F})$ for all i.*

Proof. Let $\mathcal{I}^\bullet$ be a flasque resolution of $\mathcal{F}$ on Y. Then $j_*(\mathcal{I}^\bullet)$ is a flasque resolution of $j_*\mathcal{F}$ on X. Indeed, since j is a closed embedding, j_* is exact. Also $\Gamma(X, j_*\mathcal{F}) = \Gamma(Y, \mathcal{F})$, so the cohomology is the same. □

Note. By abuse of notation, we will often write $H^i(X, \mathcal{F})$ instead of $H^i(X, j_*\mathcal{F})$.

Definition. *Let X be a topological space, $Z \subseteq X$ a closed subset, $U = X \setminus Z$ and let $i : Z \to X, j : U \to X$ be the inclusion maps. For $\mathcal{F} \in \mathbf{Ab}(X)$ let $\mathcal{F}_Z$ denote $i_*(\mathcal{F}|_Z) := i_*(j^*\mathcal{F})$ and let $\mathcal{F}_U := j_!(\mathcal{F}|_U)$.*

Proposition 2.5 ([Har77, Exercise II 1.19])**.** *We have an exact sequence*

$$0 \to \mathcal{F}_U \to \mathcal{F} \to \mathcal{F}_Z \to 0.$$

Theorem 2.6 (Grothendieck's Vanishing Theorem)**.** *Let X be a noetherian topological space of dimension n, and let $\mathcal{F} \in \mathbf{Ab}(X)$. Then $H^i(X, \mathcal{F}) = 0$ for all $i > n$.*

Proof. We induct on (n, m) where $n = \dim X$ and m is the number of irreducible components of X, and the tuples are ordered lexicographically.

Step 1. Reduce from X arbitrary of dimension $n \geq 0$ to X irreducible of dimension $\leq n$.

Proof. We induct on m as above. Since $X \neq \emptyset$, $m > 0$. If $m = 1$, then X is irreducible and we are done, so assume $m > 1$. Choose an irreducible component Z of X and let $U = X \setminus Z$. Then

$$0 \to \mathcal{F}_U \to \mathcal{F} \to \mathcal{F}_Z \to 0 \qquad (*)$$

is exact.

Claim 1. $H^i(X, \mathcal{F}_Z) = 0$ for all $i > n$.

Proof. Since Z is irreducible, $H^i(Z, \mathcal{F}|_Z) = 0$ for all $i > n$ (by the case we are reducing to). Conclude by Lemma 2.4. □

Claim 2. $H^i(X, \mathcal{F}_U) = 0$ for all $i > n$.

Proof. Subclaim: There is a sheaf $\mathcal{G}$ on $\overline{U}$ such that $\mathcal{F}_U \cong j_*\mathcal{G}$, where $j : \overline{U} \hookrightarrow X$ is the inclusion map. So $\mathcal{F}_U$ can be regarded as a sheaf on $\overline{U}$. To see this note that

$$0 \to (\mathcal{F}_U)_{X\setminus\overline{U}} \to \mathcal{F}_U \to (\mathcal{F}_U)_{\overline{U}} \to 0$$

is exact. But also $(\mathcal{F}_U)_{X\setminus\overline{U}} = 0$ because all of its stalks are 0. Therefore $\mathcal{F}_U \cong (\mathcal{F}_U)_{\overline{U}} = j_*(\mathcal{F}_U|_{\overline{U}})$, so we can let $\mathcal{G} = \mathcal{F}_U|_{\overline{U}}$.

Next note that $H^i(X, \mathcal{F}_U) \cong H^i(\overline{U}, \mathcal{F}_U|_{\overline{U}}) = 0$ for all $i > n$ by inductive hypothesis as $\overline{U}$ has $m - 1$ irreducible components. □

Claims 1 and 2 then show what we want using the long exact sequence of $(*)$ as we have $0 = H^i(X, \mathcal{F}_U) \to H^i(X, \mathcal{F}) \to H^i(X, \mathcal{F}_Z) = 0$ for all $i > n$. □

Step 2. Prove the base cases $X = \emptyset$ and X irreducible of dimension 0.

Proof. The case $X = \emptyset$ is trivial, so let $\dim X = 0$ and X irreducible. Note that then the only closed subsets of X are $\emptyset$ and X itself, so the only open subsets are $\emptyset$ and X, so $\mathcal{F}$ is flasque, hence acyclic. □

Step 3. X irreducible of dimension $n > 0$, assuming everything holds already for all X of dimension $< n$.

Step 3a Reduce to the case where $\mathcal{F}$ is finitely generated.

Proof. Let $B = \bigsqcup_{U \subseteq X \text{ open}} \mathcal{F}(U)$, and let A be the collection of finite subsets of B. Then A is a directed set. For all $\alpha \in A$, let $\mathcal{F}_\alpha$ be the subsheaf of $\mathcal{F}$ generated by α. Then $\mathcal{F} = \varinjlim_\alpha \mathcal{F}_\alpha$, so $H^i(X, \mathcal{F}) = \varinjlim_\alpha H^i(X, \mathcal{F}_\alpha)$ and it suffices to prove $H^i(X, \mathcal{F}_\alpha) = 0$.

Step 3b Reduce to the case in which $\mathcal{F}$ is generated by one element. Use a long exact sequence argument involving short exact sequences

$$0 \to \mathcal{F}_{\alpha'} \to \mathcal{F}_\alpha \to \mathcal{G} \to 0$$

where $\emptyset \neq \alpha' \subsetneq \alpha \in A$ and $\mathcal{G} := \mathcal{F}_\alpha / \mathcal{F}_{\alpha'}$ is generated by the images of the elements of $\alpha \setminus \alpha'$. □

Step 3c. Now assume $\mathcal{F}$ is generated by one section $s \in \mathcal{F}(U)$ for some open $U \subseteq X$. Reduce to $\mathbb{Z}_U$ and subsheaves of $\mathbb{Z}_U$ where $\mathbb{Z}_U := j_!(\underline{\mathbb{Z}}|_U)$ and $\underline{\mathbb{Z}}$ is the constant sheaf $\mathbb{Z}$ on X.

Proof. We may assume $U \neq \emptyset$. Then the map $\mathbb{Z}_U \to \mathcal{F}$ taking $1 \in \mathbb{Z}_U(U) = \mathbb{Z}$ to $s \in \mathcal{F}(U)$ (this map exists on the presheaf used to define $j_!(\underline{\mathbb{Z}}|_U)$, so extends to a map on $\mathbb{Z}_U$). Let $\mathcal{R}$ be the kernel of this map, so that

$$0 \to \mathcal{R} \to \mathbb{Z}_U \to \mathcal{F} \to 0$$

is exact. Thus it suffices to prove the theorem for $\mathbb{Z}_U$ and for $\mathcal{R}$, since $H^i(X, \mathbb{Z}_U) \to H^i(X, \mathcal{F}) \to H^{i+1}(X, \mathcal{R})$ is exact. □

Step 3d. Reduce to showing it for $\mathbb{Z}_U$.

Proof. Let $\mathcal{R}$ be a subsheaf of $\mathbb{Z}_U$. We can assume $\mathcal{R} \neq 0$. Let d be the smallest positive integer such that $d \in \mathcal{R}_x \subseteq \mathbb{Z}_x = \mathbb{Z}$ as $x \in U$ varies, and let $V = \{x \in$

$U : \mathcal{R}_x \ni d\}$. Then $v \neq \emptyset$ and V is open (as in Problem 1 on HW1). Then, by minimality of d, $\mathcal{R}_x = d\mathbb{Z}$ for all $x \in V$, so $R_V \cong \mathbb{Z}_V$ as sheaves on X. Then

$$0 \to \mathbb{Z}_V \xrightarrow{d} \to \mathcal{R} \to \mathcal{R}/\mathbb{Z}_V \to 0$$

is an exact sequence of sheaves on X, and $\operatorname{Supp}(\mathcal{R}/\mathbb{Z}_V) \subseteq \overline{U \setminus V}$ has dimension $< n$. By inductive hypothesis, on dimension, it suffices to show the theorem for $\mathbb{Z}_V$. □

Step 3e. Prove it for $\mathbb{Z}_U$.

Proof. $0 \to \mathbb{Z}_U \to \underline{\mathbb{Z}} \to \mathbb{Z}_{X \setminus U} \to 0$ is exact, so for all $i > n$,

$$H^{i-1}(X, \mathbb{Z}_{X\setminus U}) \to H^i(X, \mathbb{Z}_U) \to H^i(X, \underline{\mathbb{Z}})$$

is exact. But $H^{i-1}(X, \mathbb{Z}_{X\setminus U}) = H^{i-1}(X \setminus U, \mathbb{Z}|_{X\setminus U}) = 0$ by inductive hypothesis since $\dim(X \setminus U) \leq n - 1 < i - 1$. Also $H^i(X, \mathbb{Z}) = 0$ for all $i > 0$ since by [Har77, Exercise II 1.16a], a constant sheaf on an irreducible space is flasque. □

□

Example ([Har77, Exercise III 2.1a])**.** Let k be an infinite field and let $X = \mathbb{A}^1_k$. Let P and Q be distinct closed points of X and let $U = X \setminus \{P, Q\}$. Then $H^1(X, \mathbb{Z}_U) \neq 0$.

Proof. Let $j : U \hookrightarrow X$ be the inclusion map. Note

$$\mathcal{F}_U = j_!(\text{const. sheaf } \mathcal{F} \text{ on } U) = \left(V \mapsto \begin{cases} \mathbb{Z} & \text{if } V \subseteq U \text{ and } V \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \right)$$

is already sheaf because U is irreducible. Also $\mathbb{Z}|_{\{P,Q\}}$ is the direct sum $(\mathbb{Z} \text{ at } P) \oplus (\mathbb{Z} \text{ at } A)$ of two skyscraper sheaves. $\mathbb{Z}_{\{P,Q\}}$ has the same description on X. Now look at the short exact sequence

$$0 \to \mathbb{Z}_U \to \mathbb{Z} \to \mathbb{Z}_{\{P,Q\}} \to 0.$$

From the long exact sequence we get:

$$\underbrace{H^0(X, \mathbb{Z})}_{=\mathbb{Z}} \to \underbrace{H^0(X, \mathbb{Z}_{\{P,Q\}})}_{=\mathbb{Z}\oplus\mathbb{Z}} \to H^i(X, \mathbb{Z}_U)$$

The first map cannot be surjective, so $H^1(X, \mathbb{Z}_U) \neq 0$. □

2.2 Serre's Criterion for Affineness

Proposition 2.7. *Let I be an injective A-module where A is a noetherian ring. Then the sheaf $\widetilde{I}$ on* Spec A *is flasque.*

Proof. Omitted. □

Theorem 2.8. *Let $X = \operatorname{Spec} A$ with A noetherian. Then all quasi-coherent sheaves on X are acyclic.*

Proof. Let M be an A-module and let $\mathcal{F} = \widetilde{M}$. Let $0 \to M \to I^\bullet$ be an injective resolution of M in $\mathbf{Mod}(A)$. Then $0 \to \widetilde{M} \to \widetilde{I}^\bullet$ is a flasque, hence acyclic, resolution in $\mathbf{Mod}(X)$ by Proposition 2.7, so $H^i(X, \widetilde{M}) = h^i(\Gamma(X, \widetilde{I}^\bullet)) = h^i(I^\bullet) = 0$ for all $i > 0$. □

Theorem 2.9 (Serre)**.** *Let X be a noetherian scheme. Then TFAE:*

(i) *X is affine.*

(ii) *$H^i(X, \mathcal{F}) = 0$ for all quasi-coherent sheaves $\mathcal{F}$ on X, for all $i > 0$.*

(iii) *$H^1(X, \mathcal{I}) = 0$ for all quasi-coherent sheaves $\mathcal{I}$ of ideals on X.*

Proof. "$(i) \Rightarrow (ii)$" has been proved in Theorem 2.8 and "$(ii) \Rightarrow (iii)$" is immediate. So we prove "$(iii) \Rightarrow (i)$". We will use the criterion of [Har77, Exercise II 2.17]: A scheme X is affine if and only if there is a finite subset $\{f_1, \dots, f_r\}$ of $A := \Gamma(X, \mathcal{O}_X)$ such that the open set X_{f_i} is affine for all i and $(f_1, \dots, f_r) = (1)$ in A.

Claim 1. For all closed points $P \in X$ there is an $f \in A$ such that $P \in X_f$ and X_f is affine.

Proof. Let $P \in X$ be a closed point, let U be an open affine neighborhood of P in X, and let $Y = X \setminus U$. Then we have a short exact sequence

$$0 \to \mathcal{I}_{Y \cup \{P\}} \to \mathcal{I}_Y \to k(P) \to 0$$

of sheaves on X, where $\mathcal{I}_{Y \cup \{P\}}$ and $\mathcal{I}_Y$ are the (coherent) ideal sheaves on X corresponding to $Y \cup \{P\}$ and Y respectively, and $k(P)$ is the skyscraper sheaf corresponding to the residue field at P. [Proof that it is exact: On $X \setminus \{P\}$, the first map is an isomorphism and $k(P) = 0$; on $U = \operatorname{Spec} B$ it is $0 \to \widetilde{\mathfrak{m}}_P \to \widetilde{B} \to \widetilde{k(P)} \to 0$ where $\mathfrak{m}_P$ is the maximal ideal of B corresponding to P.]

Then

$$H^0(X, \mathcal{I}_Y) \to H^0(X, k(p)) \to H^1(X, \mathcal{I}_{Y \cup \{P\}})$$

is exact and $H^1(X, \mathcal{I}_{Y \cup \{P\}}) = 0$ by assumption (iii), therefore there exists $f \in H^0(X, \mathcal{I}_Y) \subseteq A$ such that $f \notin \mathfrak{m}_P$. Moreover, $X_f \subseteq U_f$ (because $f \in \mathcal{I}_Y$, so $Y \cap X_f = \emptyset$), so $X_f = \operatorname{Spec} B_f$ is affine. □

We now use [Vak22, Exercise 5.1E]: Every non-empty quasi-compact scheme has a closed point. [Let X be such a scheme. By quasi-compactness, X is covered by $U_i = \operatorname{Spec} A_i$, $i = 1, \dots, n$ with $n > 0$. Suppose X no closed points. Pick a point $z_0 \in U_1$ corresponding to a maximal ideal of A_1. Then $\{z_0\}$ is closed in U_1, but not in X, so its closure contains points $\neq z_0$. These points necessarily lie outside of U_1. Let y be such a point. Pick i_1 such that $y \in U_{i_1} \setminus U_1$, so $\overline{\{y\}} \subseteq X \setminus U_1$. Pick $z_1 \in U_{i_1}$ corresponding to a maximal ideal in A_{i_1} containing the ideal of $\overline{\{y\}} \cap U_{i_1}$. Repeating (and letting $i_0 = 1$), we get a sequence of points $z_0 \rightsquigarrow z_1 \rightsquigarrow z_2 \rightsquigarrow \cdots \rightsquigarrow z_n$, such that each z_j corresponds to a maximal ideal in A_{i_j} and $z_j \notin \overline{\{z_{j+1}\}}$ for all $0 \le j < n$. By the pigeonhole principle, there are j, l with $0 \le j < l \le n$ such that $i_j = i_l$. But we have $z_j \notin \overline{\{z_l\}}$, so $z_j \ne z_l$, yet $z_j, z_l \in U_{i_j} = U_{i_l}$ and $z_l \in \overline{\{z_{j+1}\}} \not\ni z_j$, contradicting the fact that z_j corresponds to a maximal ideal of A_{i_j}, so $\overline{\{z_j\}} \cap U_{i_j} = \{z_j\}$.]

Claim 2. There exist $f_1, \ldots, f_n \in A$ such that $\bigcup X_{f_i} = X$ and X_{f_i} is affine for all i.

Proof. Let X_{cl} denote the set of closed points of X. By Claim 1, for all points $P \in X_{\mathrm{cl}}$ there is $f_P \in A$ such that $P \in X_{f_P}$ and X_{f_P} is affine. Let $Z = X \setminus \bigcup_{P \in X_{\mathrm{cl}}} X_{f_P}$. Then Z is closed. Regard it as a closed subscheme of X with the reduced induced subscheme structure. It is quasi-compact. If $Z \ne \emptyset$, then it has a closed point P which is also closed in X. But then $P \in \bigcup_{P' \in X_{\mathrm{cl}}} X_{f_{P'}}$, a contradiction as $P \in X_{\mathrm{cl}} \subseteq X_{f_P}$. Therefore $\{X_{f_P} \mid P \in X_{\mathrm{cl}}\}$ is an open covering of X. Let $f_1, \ldots, f_n \in A$ correspond to a finite subcovering. Then $\bigcup_{i=1}^n X_{f_i} = X$. □

Finally we show that $(f_1, \ldots, f_n) = (1)$ in $A = \Gamma(X, \mathcal{O}_X)$, finishing the proof. Define $\alpha : \mathcal{O}_X^n \to \mathcal{O}_X$ by $\mathcal{O}_X^n(U) \ni (a_1, \ldots, a_n) \mapsto \sum a_i f_i \in \mathcal{O}_X(U)$. This is surjective on stalks, because $\bigcup X_{f_i} = X$, so for all $x \in X$ there is some i such that $x \in X_{f_i}$ and then f_i is not in the maximal ideal of $\mathcal{O}_{X,x}$, and then $\mathcal{O}_{X,x} \xrightarrow{\cdot f_i} \mathcal{O}_{X,x}$ is surjective. Let $\mathcal{F} = \ker \alpha$. Then

$$0 \to \mathcal{F} \to \mathcal{O}_X^n \to \mathcal{O}_X \to 0$$

is exact and we get the exact sequence

$$\Gamma(X, \mathcal{O}_X^n) \to \Gamma(X, \mathcal{O}_X) \to H^1(X, \mathcal{F}).$$

We want to show $H^1(X, \mathcal{F}) = 0$ which gives what we want. We prove by induction that $H^1(X, \mathcal{F} \cap \mathcal{O}_X^i) = 0$ for all $i = 0, \ldots, n$. The case $i = 0$ is trivial. Let $i > 0$ and assume $H^1(X, \mathcal{F} \cap \mathcal{O}_X^{i-1}) = 0$. Consider the exact sequence

$$0 \to \mathcal{F} \cap \mathcal{O}_X^{i-1} \to \mathcal{F} \cap \mathcal{O}_X^i \to \mathcal{G}_i \to 0.$$

Then $\mathcal{G}_i$ is a subsheaf of $\mathcal{O}_X$. It is a coherent ideal sheaf, so by assumption $H^1(X, \mathcal{G}_i) = 0$ and the long exact sequence gives $H^1(\mathcal{F} \cap \mathcal{O}_X^i) = 0$. □

Corollary 2.10 (of Proposition 2.7)**.** *Let X be a noetherian scheme. Then every quasi-coherent sheaf on X can be embedded in a flasque quasi-coherent sheaf on X.*

Proof. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Cover X with finitely many open affines $U_i = \operatorname{Spec} A_i$, and for all i let M_i be an A_i-module such that $\mathcal{F}|_{U_i} \cong \widetilde{M}_i$. Embed M_i into an injective A_i-module I_i so that $\widetilde{I}_i$ is flasque on U_i by Proposition 2.7. Then for all i we have an injective map $\mathcal{F}|_{U_i} \to \widetilde{I}_i$, which gives a map $\mathcal{F} \to (f_i)_* \widetilde{I}_i$ where $f_i : U_i \to X$ is the inclusion map. These combine to give a map $\mathcal{F} \to \bigoplus_{i=1}^n (f_i)_* \widetilde{I}_i$. This map is injective on stalks, hence injective. For each i, $\widetilde{I}_i$ is flasque on U_i, hence $(f_i)_*\widetilde{I}_i$ is flasque on X and therefore $\bigoplus_{i=1}^n (f_i)_*\widetilde{I}_i$ is flasque. It is also quasi-coherent each U_i is noetherian. □

3 Čech Cohomology

Let X be a topological space, $\mathcal{F} \in \mathbf{Ab}(X)$ and $\mathcal{U} = (U_i)_{i\in I}$ an open covering of X. We fix a well-ordering of I.

Notation. $U_{i_0\cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$ for all $i_0, \ldots, i_p \in I$.

Definition. *For all $p \in \mathbb{N}$, $C^p(\mathcal{U}, \mathcal{F}) := \prod_{i_0<\cdots<i_p} \mathcal{F}(U_{i_0\ldots i_p})$. The components of $\alpha \in C^p(\mathcal{U}, \mathcal{F})$ will be denoted $\alpha_{i_0\ldots i_p} \in \mathcal{F}(U_{i_0\ldots i_p})$.*

Convention. Let $\alpha \in C^p(\mathcal{U}, \mathcal{F})$. For arbitrary $i_0, \ldots, i_p \in I$, we define

$$\alpha_{i_0\ldots i_p} := \begin{cases} 0 & \text{if some index is repeated} \\ (\operatorname{sign}\sigma)\alpha_{i_{\sigma(0)}\cdots i_{\sigma(p)}} & \text{otherwise, where } \sigma \in S_{p+1} \text{ is the unique permutation such that } i_{\sigma(0)} < \ldots i_{\sigma(p)} \end{cases}$$

This notation is compatible with our earlier notation. In fact, for all $i_0, \ldots, i_p \in I$ and $\sigma \in S_{p+1}$, $\alpha_{i_0\ldots i_p} = \operatorname{sign}\sigma\alpha_{i_{\sigma(0)}\cdots i_{\sigma(p)}}$.

Definition. *Define $d : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ by*

$$(d\alpha)_{i_0\ldots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \alpha_{i_0\ldots\widehat{i_j}\ldots i_{p+1}} \tag{1}$$

for all $i_0 < \cdots < i_{p+1}$ in I.

Lemma 3.1. *The formula (1) holds for all $i_0, \ldots, i_{p+1} \in I$.*

Proof. First, it is true for arbitrary $i_0, \ldots, i_{p+1}$ if and only if it is true for $i_{\sigma(0)}, \ldots, i_{\sigma(p+1)}$ for all $\sigma \in S_{p+1}$.

Proof. We can assume $\sigma = (l\ l+1)$. So we need to show that both sides are multiplied by -1 in this case. This is true for the LHS by definition and individually for all terms on the RHS where $j \neq l, l+1$. Then the $j = l$ term on the original RHS is the $j = l+1$ term on the new RHS because the subscripts on α are the same and $(-1) = -(-1)^{l+1}$. Similarly $j = l+1$ term on the original RHS and the $j = l$ term on the new RHS. $\square$

If there are no repeated indices, then we are done. Otherwise, we may assume $i_0 = i_1$. Then the LHS is 0, all terms for $j > 1$ on the RHS are 0, and the $j = 0, 1$ terms cancel. $\square$

Lemma 3.2. $d^2 = 0$.

Proof. Exercise. $\square$

Therefore, $C^\bullet(\mathcal{U}, \mathcal{F})$ is a complex of abelian groups.

Definition. *The* Čech Cohomology *groups for $\mathcal{F}$ with respect to $\mathcal{U}$ are defined by $\check{H}^p(\mathcal{U},\mathcal{F}) = h^p(C^\bullet(\mathcal{U},\mathcal{F}))$ for all $p \in \mathbb{N}$.*

Example. If $U = \{X\}$, then

$$C^p(\mathcal{U},\mathcal{F}) = \begin{cases} H^0(X,\mathcal{F}) & \text{if } p = 0, \\ 0 & \text{otherwise} \end{cases}$$

and $\check{H}^p(\mathcal{U},\mathcal{F})$ is the same.

Remark. Here we usually don't get a long exact sequence in cohomology.

Remark. For all $X, \mathcal{F}$ and $\mathcal{U}$ we have $\check{H}^0(\mathcal{U},\mathcal{F}) = \Gamma(X,\mathcal{F})$.

Proof. By the sheaf axioms,

$$0 \to \Gamma(X,\mathcal{F}) \to \prod_i \mathcal{F}(\mathcal{U}_i) \xrightarrow{d} \prod_{i<j} \mathcal{F}(U_i \cap U_j)$$

is exact and the claim follows. □

Example. Let k be a field, let $X = \mathbb{P}^1_k = \operatorname{Proj} k[x,y]$, and let $\mathcal{U} = \{U, V\}$ where $U = D_+(x), V = D_+(y)$ and $U < V$. Then

$$\begin{aligned} C^0(\mathcal{U},\mathcal{O}(1)) &= \Gamma(U,\mathcal{O}(1)) \times \Gamma(V,\mathcal{O}(1)) \\ &= xk[t^{-1}] \times yk[t] && t = \frac{x}{y} \\ &= xk[y/x] \times yk[x/y] \\ C^1(\mathcal{U},\mathcal{O}(1)) &= \Gamma(U \cap V, \mathcal{O}(1)) = xk[t,t^{-1}] \\ C^p(\mathcal{U},\mathcal{O}(1)) &= 0 && p > 1 \end{aligned}$$

Let $S = k[x,y]$. Then $\mathcal{O}(1) = \widetilde{S(1)}$, and

	Global Sections	Restriction map
$\mathcal{O}(1)\vert_{D_+(x)} = \widetilde{S(1)_{(x)}} = x\widetilde{S_{(x)}}$	$xk[t^{-1}]$	x
$\mathcal{O}(1)\vert_{D_+(y)} = \widetilde{S(1)_{(y)}} = y\widetilde{S_{(y)}}$	$yk[t]$	y
$\mathcal{O}(1)\vert_{D_+(xy)} = \widetilde{S(1)_{(xy)}} = x\widetilde{S_{(xy)}} = y\widetilde{S_{(xy)}}$	$xk[t,t^{-1}]$	x $\quad \frac{x}{t} = y$

Then

$$\check{H}^0(\mathcal{U},\mathcal{O}(1)) = xk[t^{-1}] \cap yk[t] = xk[t^{-1}] \cap xt^{-1}k[t^{-1}] = kx \oplus kxt^{-1} = kx \oplus ky = S_1$$

and

$$\check{H}^1(\mathcal{U},\mathcal{O}(1)) = \operatorname{coker}(C^0 \to C^1) = 0$$

because $C^0 \to C^1$ is onto: All basis elements xt^n ($n \in \mathbb{Z}$) of $C^1 = xk[t,t^{-1}]$ lie in the image of $C^0 \to C^1$: For $n \le 0$ they are in the image of $\mathcal{F}(U) \to C^1$ and for $n \ge 1$ they are in the image of $\mathcal{F}(V) \to C^1$. Also $\check{H}^p(\mathcal{U},\mathcal{O}(1)) = 0$ for all $p > 1$.

3.1 Comparison of Čech cohomology with sheaf cohomology

Definition. *The* Čech complex $\mathcal{C}^\bullet(\mathcal{U},\mathcal{F})$ of sheaves on X, *is defined by*

$$\mathcal{C}^p(\mathcal{U},\mathcal{F}) = \prod_{i_0<\dots<i_p} j_*(\mathcal{F}|_{U_{i_0\dots i_p}}),$$

where $j : U_{i_0\dots i_p} \hookrightarrow X$ *is the inclusion map. Also define* $d : \mathcal{C}^p(\mathcal{U},\mathcal{F}) \to \mathcal{C}^{p+1}(\mathcal{U},\mathcal{F})$ *as before.*

So for all open $V \subseteq X$,

$$\mathcal{C}^p(\mathcal{U},\mathcal{F})(V) = \prod_{i_0<\dots<i_p} j_*(\mathcal{F}|_{U_{i_0\dots i_p}})(V) = \prod_{i_0<\dots<i_p} \mathcal{F}(U_{i_0\dots i_p}\cap V) = C^p((U_i\cap V)_{i\in I}, \mathcal{F}|_V).$$

In particular, $\Gamma(X,\mathcal{C}^p(\mathcal{U},\mathcal{F})) = C^p(\mathcal{U},\mathcal{F})$ for all p.

Lemma 3.3. *Define* $\varepsilon : \mathcal{F} \to \mathcal{C}^0(\mathcal{U},\mathcal{F})$ *by*

$$\mathcal{F}(V) \ni s \longmapsto (\dots, s|_{U_i\cap V}, \dots) \in \mathcal{C}^0(\mathcal{U},\mathcal{F})(V)$$

for all open $V \subseteq X$*. Then*

$$0 \to \mathcal{F} \xrightarrow{\varepsilon} \mathcal{C}^0(\mathcal{U},\mathcal{F}) \to \mathcal{C}^1(\mathcal{U},\mathcal{F}) \to \dots \qquad (*)$$

is exact as a sequence of sheaves on X*.*

Proof (sketch). The sequence

$$0 \to \mathcal{F}(V) \to \prod_{i\in I} \mathcal{F}(U_i\cap V) \to \prod_{i<j} \mathcal{F}(U_i\cap U_j\cap V)$$

is exact by the sheaf axioms, so $(*)$ is exact at $\mathcal{F}$ and at $\mathcal{C}^0(\mathcal{U},\mathcal{F})$. To prove exactness at $\mathcal{C}^p(\mathcal{U},\mathcal{F})$ for all $p > 0$, it suffices to prove that it is exact at stalks at x for all $x \in X$. This is done by constructing the appropiate homotopy for all $x \in X$, details omitted. □

So $(*)$ is a (right) resolution of $\mathcal{F}$.

Proposition 3.4. *If* $\mathcal{F}$ *is flasque, then* $\check{H}^p(\mathcal{U},\mathcal{F}) = 0$ *for all* $p > 0$.

Proof. Since $\mathcal{F}$ is flasque, $\mathcal{F}|_{U_{i_0\dots i_p}}$ is flasque for all $p \in \mathbb{N}, i_0, \dots, i_p$, so $j_*(\mathcal{F}|_{U_{i_0\dots i_p}})$ is flasque for all $p, i_0, \dots, i_p$. Then $\mathcal{C}^p(\mathcal{U},\mathcal{F})$ is flasque for all $p \geq 0$ and $\mathcal{C}^\bullet(\mathcal{U},\mathcal{F})$ is a flasque resolution of $\mathcal{F}$. Therefore, we can use it to compute $H^p(X,\mathcal{F})$ for all $p \in \mathbb{N}$. We get

$$0 = H^p(X,\mathcal{F}) = h^p(\Gamma(X,\mathcal{C}^\bullet(\mathcal{U},\mathcal{F}))) = h^p(C^\bullet(\mathcal{U},\mathcal{F})) = \check{H}(\mathcal{U},\mathcal{F})$$

for $p > 0$. □

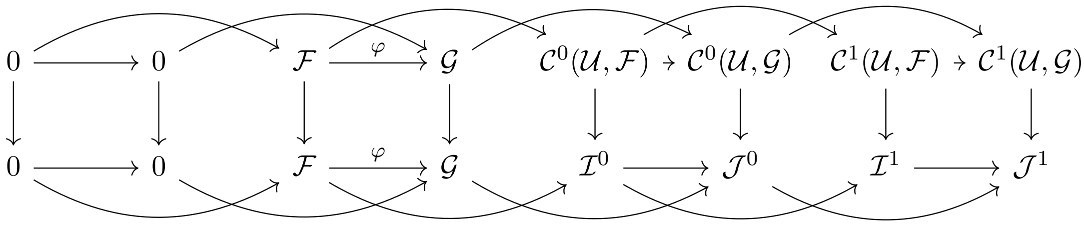
Lemma 3.5. *For all $p \geq 0$, there is a canonical map $\check{H}^p(\mathcal{U}, \mathcal{F}) \to H^p(X, \mathcal{F})$, functorial in $\mathcal{F}$.*

Proof. Let $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$. We also have a resolution $0 \to \mathcal{F} \to \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$. Since $\mathcal{I}^\bullet$ is injective, there is a morphism $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}) \to \mathcal{I}^\bullet$ of complexes, such that the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}) \\ & & \| & & \downarrow \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{I}^\bullet \end{array}$$

commutes. Now apply $h^p(\Gamma(X, \cdot))$ to the diagram to get a well-defined map $\check{H}^p(\mathcal{U}, \mathcal{F}) \to H^p(\mathcal{U}, \mathcal{F})$. It is canonical, because there are no choices for computing $\check{H}^p(\mathcal{U}, \mathcal{F})$, and the map is independent of the choice of $\mathcal{I}^\bullet$ by earlier results.

For functoriality in $\mathcal{F}$, let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism in $\mathbf{Ab}(X)$, and let $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ and $0 \to \mathcal{G} \to \mathcal{J}^\bullet$ be injective resolutions. Then we have a diagram



leading to a diagram

$$\begin{array}{ccc} \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}) & \longrightarrow & \mathcal{C}^\bullet(\mathcal{U}, \mathcal{G}) \\ \downarrow & & \downarrow \\ \mathcal{I}^\bullet & \longrightarrow & \mathcal{J}^\bullet \end{array}$$

of complexes and morphisms that commutes up to homotopy. The top map is $\mathcal{C}^\bullet(\mathcal{U}, \varphi)$, and the others are constructed using injectivity of $\mathcal{I}^\bullet$ and $\mathcal{J}^\bullet$. Then the corresponding diagram of $\check{H}^p(\mathcal{U}, \mathcal{F})$ and $H^p(X, \mathcal{F})$ commutes for all p. □

Theorem 3.6 (Comparison Theorem)**.** *Let X be a noetherian separated scheme, let $\mathcal{U}$ be an open* affine *covering of X, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then for all $p \in \mathbb{N}$, the maps $\check{H}^p(\mathcal{U}, \mathcal{F}) \to H^p(X, \mathcal{F})$ are isomorphism.*

Proof. By induction on p. For $p = 0$, they are both $\Gamma(X, \mathcal{F})$ and the map is the identity map. Inductive step: Assume $p > 0$ and that $\check{H}^{p-1}(\mathcal{U}, \mathcal{F}) \to H^{p-1}(X, \mathcal{F})$ is an isomorphism. By Corollary 2.10 we can embed $\mathcal{F}$ into a flasque quasi-coherent sheaf $\mathcal{G}$. Let $\mathcal{R} = \mathcal{G}/\mathcal{F}$, so

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{R} \to 0$$

is exact. For all $q > 0$, $i_0 < \dots i_q$ and all open affines V, the set $U_{i_0 \dots i_q} \cap V$ is affine (recall that a finite intersection of affines in a separated scheme is affine). Since $\mathcal{F}$ is quasi-coherent, $H^1(U_{i_0 \dots i_q} \cap V, \mathcal{F}) = 0$ by Theorem 2.8, so

$$0 \to \mathcal{F}(U_{i_0 \dots i_q} \cap V) \to \mathcal{G}(U_{i_0 \dots i_q} \cap V) \to \mathcal{R}(U_{i_0 \dots i_q} \cap V) \to 0$$

is exact. Then

$$0 \to \mathcal{C}^q(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^q(\mathcal{U}, \mathcal{G}) \to \mathcal{C}^q(\mathcal{U}, \mathcal{R}) \to 0$$

is exact as it is exact on global sections over open affines V. So we get a long exact sequence of Čech cohomology groups by the Snake lemma. Also $H^q(X, \mathcal{G}) = \check{H}^q(\mathcal{U}, \mathcal{G}) = 0$ for all $q > 0$ because $\mathcal{G}$ is flasque.

Now if $p > 1$, then we have a commutative diagram

$$\begin{array}{ccccccc} 0 = \check{H}^{p-1}(\mathcal{U}, \mathcal{G}) & \longrightarrow & \check{H}^{p-1}(\mathcal{U}, \mathcal{R}) & \xrightarrow[\simeq]{\delta} & \check{H}^p(\mathcal{U}, \mathcal{F}) & \longrightarrow & \check{H}^p(\mathcal{U}, \mathcal{G}) = 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 = H^{p-1}(X, \mathcal{G}) & \longrightarrow & H^{p-1}(X, \mathcal{R}) & \xrightarrow[\simeq]{\delta} & H^p(X, \mathcal{F}) & \longrightarrow & H^p(X, \mathcal{G}) = 0 \end{array}$$

By induction the map $\check{H}^{p-1}(\mathcal{U}, \mathcal{R}) \to H^{p-1}(X, \mathcal{R})$ is an isomorphism, therefore also $\check{H}^p(\mathcal{U}, \mathcal{F}) \to H^p(X, \mathcal{F})$ is an isomorphism.

If $p = 1$, our diagram is

$$\begin{array}{ccccccc} \Gamma(X, \mathcal{G}) & \longrightarrow & \Gamma(X, \mathcal{R}) & \xrightarrow{\delta} & \check{H}^1(\mathcal{U}, \mathcal{F}) & \longrightarrow & \check{H}^1(\mathcal{U}, \mathcal{G}) = 0 \\ \| & & \| & & \downarrow & & \downarrow \\ \Gamma(X, \mathcal{G}) & \longrightarrow & \Gamma(X, \mathcal{R}) & \xrightarrow{\delta} & H^1(X, \mathcal{F}) & \longrightarrow & H^1(X, \mathcal{G}) = 0 \end{array}$$

This time the map $\check{H}^1(\mathcal{U}, \mathcal{F}) \to H^1(X, \mathcal{F})$ is an isomorphism by the five lemma. □

Remarks.

(a) If X is a scheme over $\operatorname{Spec} A$, for some ring A, and if $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, then $C^\bullet(\mathcal{U}, \mathcal{Z})$ is a complex in $\mathbf{Mod}(A)$, so $\check{H}^p(\mathcal{U}, \mathcal{F})$ has a canonical A-module structure for all p.

(b) Likewise, the complexes $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$ of sheaves have a natural structure of sheaves of A-modules on X, so the map (resp. isomorphism) $\check{H}^p(\mathcal{U}, \mathcal{F}) \to H^p(X, \mathcal{F})$ of Lemma 3.5 (resp. Theorem 3.6) is a map of A-modules.

There is another comparison theorem. It works for arbitrary topological spaces X and $\mathcal{F} \in \mathbf{Ab}(X)$, but only for H^1 and only if you take limits of coverings.

Let X be an arbitrary topological space and let $\mathcal{Z} \in \mathbf{Ab}(X)$.

Definition. *A* refinement *of an open covering* $\mathcal{U} = (U_i)_{i\in I}$ *of* X *is an open covering* $\mathcal{V} = (V_j)_{j\in J}$ *of* X *such that there is a map* $\lambda : J \to I$ *such that* $V_j \subseteq U_{\lambda(j)}$ *for all* $j \in J$. *This gives natural maps* $\lambda^p : \check{H}^p(\mathcal{U}, \mathcal{F}) \to \check{H}^p(\mathcal{V}, \mathcal{F})$ *for all* p.

The natural map is defined as follows: Define $\varphi^p : C^p(\mathcal{U}, \mathcal{F}) \to C^p(\mathcal{V}, \mathcal{F})$ by $(\varphi^p(\alpha))_{j_0\cdots j_p} = \alpha_{\lambda(j_0)\dots\lambda(j_p)}|_{V_{j_0\cdots j_p}}$ for all p, $j_0, \dots, j_p \in J$. This is compatible with the boundary maps, so we have a map $C^\bullet(\mathcal{U}, \mathcal{F}) \to C^\bullet(\mathcal{V}, \mathcal{F})$ of complexes, hence maps $\lambda^p : \check{H}^p(\mathcal{U}, \mathcal{F}) \to \check{H}^p(\mathcal{V}, \mathcal{F})$ for all p. This is functorial in $\mathcal{F}$ (obvious), and functorial in refinements, as follows. Let $\mathcal{W} = (W_k)_{k\in K}$ be refinement of V, and let $\mu : K \to J$ be a map such that $W_k \subseteq V_{\mu(k)}$ for all k. Then $\mathcal{W}$ is also a refinement of $\mathcal{U}$, via $\lambda \circ \mu : K \to I$, and the diagram

$$\begin{array}{ccc} \check{H}^p(\mathcal{U}, \mathcal{F}) & \xrightarrow{(\lambda\circ\mu)^p} & \check{H}^p(\mathcal{W}, \mathcal{F}) \\ & {\scriptstyle\lambda^p}\searrow \quad \nearrow{\scriptstyle\mu^p} & \\ & \check{H}^p(\mathcal{V}, \mathcal{F}) & \end{array}$$

commutes for all p.

Lemma 3.7. *If* $\mathcal{U}, \mathcal{V}$ *are as above, then the maps* $\lambda^p : \check{H}^p(\mathcal{U}, \mathcal{F}) \to \check{H}^p(\mathcal{V}, \mathcal{F})$ *are independent of the choice of* λ.

Proof. See Stacks, 09UY. □

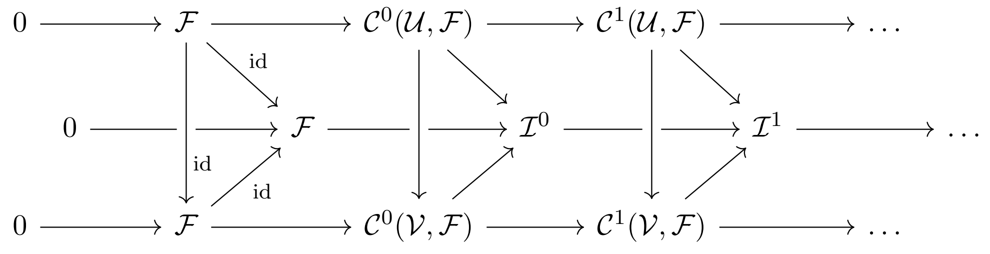
Furthermore, $(\check{H}^p(\mathcal{U}, \mathcal{F}))_{\mathcal{U}}$ is a direct system, indexed by coverings $\mathcal{U}$ of X, partially ordered by refinement, for all p. So we can take $\varinjlim_{\mathcal{U}} \check{H}^p(\mathcal{U}, \mathcal{F})$. This is often denoted $\check{H}^p(X, \mathcal{F})$.

Lemma 3.8. *For all* $\mathcal{F}$ *and all* p*, the natural maps of Lemma 3.5 are compatible with the refinement maps* λ^p*, i.e. the diagram*

$$\begin{array}{ccc} \check{H}^p(\mathcal{U}, \mathcal{F}) & \longrightarrow & H^p(X, \mathcal{F}) \\ & {\scriptstyle\lambda^p}\searrow \quad \nearrow & \\ & \check{H}^p(\mathcal{V}, \mathcal{F}) & \end{array}$$

commutes.

Proof. Let $\mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$. We have a commutative diagram



with exact rows. Now take gloal sections and cohomology to get that the earlier diagram commutes for all p. (Recall that $\check{H}^p(\mathcal{U},\mathcal{F}) = h^p(\Gamma(X, \mathcal{C}^p(\mathcal{U},\mathcal{F}))).$) □

We now have well-defined maps $\varinjlim \check{H}^p(\mathcal{U},\mathcal{F}) \to H^p(X,\mathcal{F})$.

Theorem 3.9 ([Har77, Exercise III 4.4])**.** *This is an isomorphism for $p \le 1$.*

Proof. If $p = 0$, then both sides are $\Gamma(X,\mathcal{F})$. This leaves $p = 1$. Embed $\mathcal{F}$ into a flasque sheaf $\mathcal{G}$ and let $\mathcal{R} = \mathcal{G}/\mathcal{F}$, so we get a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{R} \to 0.$$

We also have injections $C^p(\mathcal{U},\mathcal{F}) \to C^p(\mathcal{U},\mathcal{G})$ for all $\mathcal{U}$ and p, so let $D^p(\mathcal{U}) = C^p(\mathcal{U},\mathcal{G})/C^p(\mathcal{U},\mathcal{F})$ for all p so that we have a short exact sequence

$$0 \to C^\bullet(\mathcal{U},\mathcal{F}) \to C^\bullet(\mathcal{U},\mathcal{G}) \to D^\bullet(\mathcal{U}) \to 0$$

of complexes. If $\mathcal{V}$ is a refinement of $\mathcal{U}$, then we get a commutative diagram with exact rows

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \check{H}^0(\mathcal{U},\mathcal{F}) & \longrightarrow & \check{H}^0(\mathcal{U},\mathcal{G}) & \longrightarrow & h^0(D^\bullet(\mathcal{U})) & \longrightarrow & \check{H}^1(\mathcal{U},\mathcal{F}) & \to & \check{H}^1(\mathcal{U},\mathcal{G}) = 0 \\
 & & \| & & \| & & \vdots & & \downarrow \lambda^1 & & \downarrow \lambda^1 \\
0 & \longrightarrow & \check{H}^0(\mathcal{V},\mathcal{F}) & \longrightarrow & \check{H}^0(\mathcal{V},\mathcal{G}) & \longrightarrow & h^0(D^\bullet(\mathcal{V})) & \longrightarrow & \check{H}^1(\mathcal{V},\mathcal{F}) & \to & \check{H}^1(\mathcal{V},\mathcal{G}) = 0
\end{array} \qquad (*)$$

Here the middle map is induced by the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^\bullet(\mathcal{U},\mathcal{F}) & \longrightarrow & C^\bullet(\mathcal{U},\mathcal{G}) & \longrightarrow & D^\bullet(\mathcal{U}) & \longrightarrow & 0 \\
 & & \downarrow \varphi^\bullet & & \downarrow \varphi^\bullet & & \vdots & & \\
0 & \longrightarrow & C^\bullet(\mathcal{V},\mathcal{F}) & \longrightarrow & C^\bullet(\mathcal{V},\mathcal{G}) & \longrightarrow & D^\bullet(\mathcal{V}) & \longrightarrow & 0
\end{array}$$

We also have a commutative diagram with exact rows

$$\begin{array}{ccccc}
0 & \longrightarrow & h^0(D^\bullet(\mathcal{U})) & \longrightarrow & h^0(C^\bullet(\mathcal{U},\mathcal{R})) \\
 & & \downarrow & & \downarrow \psi \\
0 & \longrightarrow & h^0(D^\bullet(\mathcal{V})) & \longrightarrow & h^0(C^\bullet(\mathcal{V},\mathcal{R}))
\end{array}$$

Now $h^0(C^\bullet(\mathcal{U},\mathcal{R})) = h^0(C^\bullet(\mathcal{V},\mathcal{R})) = \Gamma(X,\mathcal{R})$, $h^0(C^\bullet(\mathcal{U},\mathcal{R})) = \check{H}^0(\mathcal{U},\mathcal{R})$, $h^0(C^\bullet(\mathcal{V},\mathcal{R})) = \check{H}^0(\mathcal{V},\mathcal{R})$ and ψ is the identity map (via these isomorphisms). So

$$\begin{array}{ccc}
h^0(D^\bullet(\mathcal{U})) & & \\
\downarrow & \searrow & \\
 & & \Gamma(X,\mathcal{R}) \\
\downarrow & \nearrow & \\
h^0(D^\bullet(\mathcal{V})) & &
\end{array}$$

commutes, so $h^0(D^\bullet(\mathcal{U})) \to h^0(D^\bullet(\mathcal{V}))$ is injective, so applying the five lemma to the diagram $(*)$ gives that $\lambda^1 : \check{H}^1(\mathcal{U}, \mathcal{F}) \to \check{H}^1(\mathcal{V}, \mathcal{F})$ is injective. Now consider the following diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \Gamma(X,\mathcal{F}) & \longrightarrow & \Gamma(X,\mathcal{G}) & \longrightarrow & \varinjlim h^0(D^\bullet(\mathcal{U})) & \to & \varinjlim \check{H}^1(\mathcal{U},\mathcal{F}) & \to & \overset{=0}{\varinjlim \check{H}^1(\mathcal{U},\mathcal{G})} \\ & & \| & & \| & (A) & \downarrow f & (B) & \downarrow g & & \downarrow \\ 0 & \longrightarrow & \Gamma(X,\mathcal{F}) & \longrightarrow & \Gamma(X,\mathcal{G}) & \longrightarrow & \Gamma(X,\mathcal{R}) & \longrightarrow & H^1(X,\mathcal{F}) & \longrightarrow & \underset{=0}{H^1(X,\mathcal{G})} \end{array} \qquad (**)$$

The first and last squares commute, for obvious reasons. **Claim.** The top row is exact.

Proof. The top row in $(*)$ was

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G}) \to h^0(D^\bullet(\mathcal{R})) \to \check{H}^1(\mathcal{U}, \mathcal{F}) \to \check{H}^1(\mathcal{U}, \mathcal{G}) = 0$$

and is exact. The bottom row of $(*)$ was the same, but with $\mathcal{U}$ replaced by a refinement $\mathcal{V}$. The vertical maps in $(*)$ were all injections or equalities, so by arrow chasing and noting that all of the direct limits are unions, we get that the top row of $(**)$ is exact. □

The second row in $(**)$ is exact because it is part of the long exact sequence of sheaf cohomology.

Square (A) commutes because $\varinjlim h^0(D^\bullet(\mathcal{U}))$ is a union of subgroups of $\Gamma(X, \mathcal{R})$, so it obviously commutes.

Claim. (B) commutes.

Proof. Let $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ and $0 \to \mathcal{R} \to \mathcal{J}^\bullet$ be injective resolutions. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{C}^\bullet(\mathcal{U},\mathcal{F}) & \longrightarrow & \mathcal{C}^\bullet(\mathcal{U},\mathcal{G}) & \longrightarrow & \mathcal{C}^\bullet(\mathcal{U},\mathcal{R}) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}^\bullet & \longrightarrow & \mathcal{I}^\bullet \oplus \mathcal{J}^\bullet & \longrightarrow & \mathcal{J}^\bullet & \longrightarrow & 0 \end{array}$$

with exact rows by the handout. Now take global sections and restrict to $D^\bullet(\mathcal{U})$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^\bullet(\mathcal{U},\mathcal{F}) & \longrightarrow & C^\bullet(\mathcal{U},\mathcal{G}) & \longrightarrow & D^\bullet(\mathcal{U}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Gamma(X,\mathcal{I}^\bullet) & \longrightarrow & \Gamma(X,\mathcal{I}^\bullet \oplus \mathcal{J}^\bullet) & \longrightarrow & \Gamma(X,\mathcal{J}^\bullet) & \longrightarrow & 0 \end{array}$$

This is commutative with exact rows. The naturatlity of the long exact sequence in cohomology then implies commutativity. □

Hence $(**)$ is a commutative diagram with exact rows. We also know that f is injective since $h^0(D^\bullet(\mathcal{U})) \to \Gamma(X, \mathcal{R})$ is injective for all $\mathcal{U}$. We want to show that g is an isomorphism. By the five lemma, it suffices to show that f is an isomorphism, so it suffices to show: **Claim.** f is surjective.

Proof. Let $\psi : \mathcal{G} \to \mathcal{R}$ be the quotient map, and fix $s \in \Gamma(X, \mathcal{R})$. For all x, the map $\psi_x : \mathcal{G}_x \to \mathcal{R}_x$ is surjective, so there is an open neighborhood U_x of x and $t^{(x)} \in \mathcal{G}(U_x)$ such that $\psi_x(t_x^{(x)}) = s_x \in \mathcal{R}_x$. After shrinking U_x, we may assume that ψ takes $t^{(x)}$ to $s|_{U_x}$. Let $\mathcal{U} = (U_x)_{x \in X}$ and choose a well-ordering on the set of points in X. Then $t := (t^{(x)})_{x \in X}$ is an element of $C^0(\mathcal{U}, \mathcal{G})$ and $C^0(\mathcal{U}, \psi)$ takes it to $(s|_{U_x})_{x \in X} \in C^0(\mathcal{U}, \mathcal{R})$. So $s \in h^0(D^\bullet(\mathcal{U}))$ because it is in the image of $C^0(\mathcal{U}, \psi)$. So f is surjective. □

□

3.2 Cohomology of Sheaves on Projective Space

Let A be a noetherian ring, $S = A[x_0, x_1, \ldots, x_r]$ with $r \geq 1$ and $X = \operatorname{Proj} S = \mathbb{P}_A^r$. Recall $\mathcal{O}_X(n) = \widetilde{S(n)}$ and that that for any $\mathcal{O}_X$-module $\mathcal{F}$, $\Gamma_*(\mathcal{F})$ is defined as $\bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n))$. It is a $\mathbb{Z}$-graded S-module.

Theorem 3.10. *With notation as above*

(a) The natural map $S \to \Gamma_*(\mathcal{O}_X) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{O}_X(n))$ is an isomorphism of $\mathbb{Z}$-graded S-modules.

(b) $H^i(X, \mathcal{O}_X(n)) = 0$ for all $0 < i < r$ and $n \in \mathbb{Z}$. (It is also true for all $i > r$ by Theorem 2.6, if $\dim A = 0$)

(c) $H^r(X, \mathcal{O}_X(-r-1)) \cong A$.

(d) For all $n \in \mathbb{Z}$, the natural map

$$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-r-1-n)) \to H^r(X, \mathcal{O}_X(-r-1)) \cong A$$

is a perfect pairing of free A-modules of finite rank.

Proof. We will use Čech cohomology and Theorem 3.6. Let $\mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(n)$. This is a quasi-coherent sheaf of graded S-modules on X. Since cohomolgoy commutes with $\bigoplus_{n \in \mathbb{Z}}$ on a noetherian topological space, $H^i(X, \mathcal{F}) \cong \bigoplus_{n \in \mathbb{Z}} H^i(X, \mathcal{O}_X(n))$ for all i and all of our computations will respect the grading, so we don't need a graded revision of Theorem 3.6. Also, as noted earlier, all $H^i(X, \mathcal{O}_X(n))$ have natural A-module structures and we have an A-module version of Theorem 3.6.

Our computations will use Čech cohomology, with $\mathcal{U} = (D_+(x_j))_{j=0,\ldots,r}$. These sets are affine, so Theorem 3.6 applies: $\check{H}^p(\mathcal{U}, \mathcal{O}_X(n)) \cong H^p(X, \mathcal{O}_X(n))$ for all n and p, as A-modules. Therefore in particular $H^p(X, \mathcal{O}_X(n)) = 0$ for all $p > r$ and for all n.

Also for all $i_0, \dots, i_p$, $U_{i_0 \dots i_p} = D_+(x_{i_0} \cdots x_{i_p})$ and $\mathcal{F}(U_{i_0 \dots i_p}) = S_{x_{i_0} \cdots x_{i_p}}$ with $\mathbb{Z}$-graded S-module structure. So $C^\bullet(\mathcal{U}, \mathcal{F})$ is

$$0 \to \prod_{i_0} S_{x_{i_0}} \xrightarrow{d^0} \prod_{i_0 < i_1} S_{x_{i_0} x_{i_1}} \to \cdots \to S_{x_0 \dots x_r} \to 0,$$

with grading (from $\mathcal{F}$ on $C^\bullet$, from S on the above).

Proof of (a). We take the intersections in $S_{x_0 \dots x_r}$, which works because all localization maps involved are injections. See [Har77, II 5.13] to get $\check{H}^0(\mathcal{U}, \mathcal{F}) = \ker d^0 = \bigcap_i S_{x_i} = S$. In fact, $S_{x_0} \cap S_{x_1} = S$ and $r \geq 1$. □

Proof of (c). **Claim.** $\check{H}^r(\mathcal{U}, \mathcal{F})$ is the free graded A-module with basis $\{x_0^{l_0} \cdots x_r^{l_r} : l_i < 0 \,\forall i\}$.

Note that in degree $-r-1$, this has rank 1 because $l_0 + \cdots + l_r = -r-1$ iff $l_i = -1$ for all i, so the claim implies (c).

We have

$$\check{H}^r(\mathcal{U}, \mathcal{F}) = \operatorname{coker}\Big(\prod_k S_{x_0 \cdots \widehat{x_k} \cdots x_r} \xrightarrow{d^{r-1}} S_{x_0 \dots x_r}\Big).$$

Here $S_{x_0 \dots x_r}$ is the free A-module with basis $\{x_0^{l_0} \cdots x_r^{l_r} \mid l_0, \dots, l_r \in \mathbb{Z}\}$ and the image of d^{r-1} is the A-submodule spanned by $\{x_0^{l_0} \cdots x_r^{l_r} \mid l_0, \dots, l_r \in \mathbb{Z}, l_k \geq 0 \text{ for at least one } k\}$. So the cokernel is isomorphic with grading to the free submodule spanned by $\{x_0^{l_0} \cdots x_r^{l_r} : l_0, \dots, l_r < 0\}$. □

Note. The isomorphism $\check{H}^r(\mathcal{U}, \mathcal{O}_K(-r-1)) \cong A$ is not canonical.

Proof of (d). The definition of the pairing is as follows: Let $s \in H^0(X, \mathcal{O}_X(n))$. Then tensoring with s gives an $\mathcal{O}_X$-module homomorphism $\mathcal{O}_X(-r-1-n) \to \mathcal{O}_X(-r-1-n) \otimes O_X(n) \cong \mathcal{O}_X(-r-1)$, so we get a map

$$H^r(X, - \otimes s) : H^r(X, \mathcal{O}_X(-r-1-n)) \to H^r(X, \mathcal{O}_X(-r-1)),$$

which is A-linear, so we get a map

$$H^0(X, \mathcal{O}_X(n)) \to \operatorname{Hom}_A(H^r(X, \mathcal{O}_X(-r-1-n)), H^r(X, \mathcal{O}_X(-r-1))).$$

This map is also A-linear, so we get a bilinear pairing of A-modules.

Next: If $n < 0$, then both factors in the pairing are 0: $H^0(X, \mathcal{O}_X(n)) = 0$ by (a), and $H^r(X, \mathcal{O}_X(-r-1-n))$ by the claim in the proof of (c) because $\sum l_i = -r-1-n > -r-1$, so some l_i is ≥ 0. So we may assume $n \geq 0$. The pairing takes pairs of cocycles represented by $(x_0^{m_0} \cdots x_r^{m_r}, x_0^{l_0} \cdots x_r^{l_r})$ to a cocycle represented by $x_0^{m_0 + l_0} \cdots x_r^{m_r + l_r}$ (proof later), or to 0 if $m_i + l_i \geq 0$ for some i. Here $\sum m_i = n$ and $\sum l_i = -r-1-n$.

Claim. The above description determines a unique, well-defined map $H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-r-1-n)) \to H^r(X, \mathcal{O}_X(-r-1))$.

Proof of claim. We need a map from $(\ker d^0)_n \times (C^r(\mathcal{U},\mathcal{F})_{-r-1-n}/\operatorname{im} d^{r-1}_{-r-1-n})$ to $C^r(\mathcal{U},\mathcal{F})_{-r-1}/\operatorname{im} d^{r-1}_{-r-1}$. We have $(\ker d^0)_n = S_n = \bigoplus_{\sum m_i = n, m_i \geq 0 \forall i} A\cdot(x_0^{m_0}\cdots x_r^{m_r})$, $C^r(\mathcal{U},\mathcal{F})_{-r-1-n} = \bigoplus_{\sum l_i = -r-1-n, l_i\in\mathbb{Z}}$, $\operatorname{im} d^{r-1}_{-r-1-n} = \bigoplus_{\sum l_i=-r-1-n, \exists i: l_i\geq 0}$, $C^r(\mathcal{U},\mathcal{F})_{-r-1} = \bigoplus_{\sum l_i = -r-1, l_i\in\mathbb{Z}}$ and $\operatorname{im}^{r-1}_{-r-1} = \bigoplus_{\sum l_i=-r-1,\exists i: l_i\geq 0}$. The pairing is given by multiplication. To show that it is well-defined, we need to show that elements of $S_n \times (\operatorname{im} d^{r-1})_{-r-1-n}$ map to elements of $(\operatorname{im} d^{r-1})_{-r-1}$. This is true because $(m_i \geq 0 \forall i)\wedge(\exists j : l_j \geq 0) \implies \exists i (l_i + m_i \geq 0)$. □

Claim: The two pairings coincide.

Proof. Each $s \in H^0(X, \mathcal{O}_X(n))$ is an element of S_n canonically, by (a). Tensoring with s gives a map $\mathcal{O}_X(-r-1-n) \xrightarrow{\otimes s} \mathcal{O}_X(-r-1)$, giving $H^r(X,\mathcal{O}_X(-r-1-n)) \to H^r(X,\mathcal{O}_X(-r-1))$ which comes from $\mathcal{C}^\bullet(\mathcal{U},\mathcal{O}_X(-r-1-n)) \to \mathcal{C}^\bullet(\mathcal{U},\mathcal{O}_X(-r-1))$ which corresponds to multiplication by s on $C^\bullet(\mathcal{U},\mathcal{F})_{-r-1-n} = C^\bullet(\mathcal{U},\mathcal{O}_X(-r-1-n)) \to C^\bullet(\mathcal{U},\mathcal{O}_X(-r-1)) = C^\bullet(\mathcal{U},\mathcal{F})_{-r-1}$. Recall $\mathcal{F} = \widetilde{T}$ where T is bi-graded: $T = \bigoplus S(n)$. Consequently, our map in $\check{H}^r$ is given by multiplication by s. □

Claim: This pairing is perfect.

Proof. $H^0(X,\mathcal{O}_X(n))$ has a basis $\{x_0^{m_0}\cdots x_r^{m_r} : m_i \in \mathbb{N}\,\forall i, \sum m_i = n\}$ and $H^r(X,\mathcal{O}_X(-r-1-n))$ has basis $\{[x_0^{-m_0-1}\cdots x_r^{-m_r-1}] \in \check{H}^r(\mathcal{U},\mathcal{O}_X(-r-1-n)) : -m_1-1 < 0\,\forall i, \sum(-m_i-1) = -r-1-n\}$. These are dual bases with basis elements corresponding in the obvious way because the sets of eligible $(m_0,\ldots,m_r)$ are the same, and the pairing takes $(x_0^{m_0},\ldots,x_r^{m_r},[x_0^{-m_0'-1},\ldots,x_r^{-m_r'-1}])$ to 1 if $m_i = m_i'$ for all i and 0 otherwise. □

So the pairing is perfect and we have seen that the cohomology modules are free modules of finite rank. □

Proof of (b). We induct on r. Recall $r \geq 1$. If $r = 1$, there is nothing to prove. So suppose $r > 1$ and that (b) holds for $r-1$.

Claim. For all $i > 0$, every element of $H^i(X,\mathcal{F})$ is annihilated by some power of x_r.

Proof. Localizing $C^\bullet(\mathcal{U},\mathcal{F})$ by x_r gives $C^\bullet(\mathcal{U}\cap U_r, \mathcal{F}|_{U_r})$, whose cohomology in degree $i > 0$ is 0 because it is $H^i(U_r,\mathcal{F}|_{U_r})$ by Theorem 2.8 and Theorem 3.6, since U_r is affine and noetherian, and $\mathcal{F}_{U_r}$ is quasi-coherent. Since localization is an exact functor, it preserves cohomology, so $\check{H}^i(\mathcal{U},\mathcal{F})_{x_r} = H^i(X,\mathcal{F})_{x_r} = 0$ for all $i > 0$. □

Claim. For all $0 < i < r$, multiplication by x_r is injective on $H^i(X,\mathcal{F})$.

Proof. We have an exact sequence

$$0 \to S(-1) \xrightarrow{\cdot x_r} S \to S/(x_r) \to 0$$

of graded S-modules, hence an exact sequence

$$0 \to S(n-1) \xrightarrow{x_r} S(n) \to S(n)/(x_r) \to 0$$

of graded A-modules for all $n \in \mathbb{Z}$. Therefore

$$0 \to \mathcal{F}(-1) \xrightarrow{\cdot x_r} \mathcal{F} \to \mathcal{F}_H \to 0$$

is exact, where $\mathcal{F}_H = \bigoplus_{n\in\mathbb{Z}} \mathcal{O}_H(n)$ and H is the hyperplane $\{x_r = 0\}$ in $\mathbb{P}^r_A$. Note $H \cong \mathbb{P}^{r-1}_A$. This gives a long exact sequence in cohomology, which gives

$$H^{i-1}(X, \mathcal{F}_H) \to H^i(X, \mathcal{F}(-1)) \xrightarrow{\cdot x_r} H^i(X, \mathcal{F}).$$

For $1 < i < r$ we have $H^{i-1}(X, \mathcal{F}_H) = H^{i-1}(H, \mathcal{F}_H) = 0$ by Lemma 2.4 and the inductive hypothesis. This proves the claim for $1 < i < r$. For $i = 1$ we have that

$$S = H^0(X, \mathcal{F}) \xrightarrow{\varphi} S/(x_r) = H^0(X, \mathcal{F}_H) \xrightarrow{\psi} H^1(X, \mathcal{F}(-1)) \xrightarrow{\cdot x_r} H^1(X, \mathcal{F})$$

is exact. Now φ is the quotient map $S \to S/(x_r)$. It is surjective, so $\psi = 0$ and therefore the last map $\cdot x_r$ is injective. □

The two claims together now imply that $H^i(X, \mathcal{F}) = 0$ for all $0 < i < r$. □

□

Theorem 3.11. *Let X be a projective scheme over a noetherian ring A, let $\mathcal{O}_X(1)$ be a very ample line bundle on X over A, and let $\mathcal{F}$ be a coherent sheaf on X. Then:*

(a) $H^i(X, \mathcal{F})$ is a finitely generated A-module for all $i \geq 0$,

(b) There is an integer n_0, depending only on $\mathcal{F}$ and $\mathcal{O}_X(1)$, such that $H^i(X, \mathcal{F}(n)) = 0$ for all $n \geq n_0$ and all $i > 0$.

Proof.

Step 1. Reduce to the case $X = \mathbb{P}^r_A$ and $\mathcal{O}_X(1) = \mathcal{O}(1)$ with $r > 0$.

Proof. Pick a closed embedding $j : X \to \mathbb{P}^r_A$ over A with $r > 0$ such that $\mathcal{O}_X(1) = j^*\mathcal{O}(1)$. Then Step 1 is immediate, because $j_*\mathcal{F}$ is coherent ([Har77, Exercise II 5.5]) and $H^i(X, \mathcal{F}) = H^i(\mathbb{P}^r_A, j_*\mathcal{F})$. Next we have $(j_*\mathcal{F})(n) \cong j_*(\mathcal{F}(n))$ for all n because this is $(j_*\mathcal{F}) \otimes \mathcal{O}_{\mathbb{P}^r_A}(n) \cong j_*(\mathcal{F} \otimes j^*\mathcal{O}_{\mathbb{P}^r_A}(n))$ which is true by the projection formula, [Har77, Exercise II 5.1d] which also gives (b). □

Step 2. The claims are true for $\mathcal{F} \cong \mathcal{O}(q)$, $q \in \mathbb{Z}$.

Proof. This is immediate from Theorem 3.10. □

Step 3. It is true if $\mathcal{F}$ is isomorphic to a finite direct sum $\mathcal{O}_X(q_1) \oplus \cdots \oplus \mathcal{O}_X(q_k)$.

Proof. This is immediate. ☐

Step 4. The general case: $\mathcal{F}$ is coherent on $X = \mathbb{P}^r_A$, $r > 0$.

Proof. We will use descending induction on i. Base case: If $i > r$, then $H^i(X, \mathcal{F}) = 0$ for all coherent $\mathcal{F}$ (e.g. a suitable Čech complex ends at $i = r$). For the inductive step let $0 \le i \le r$, and assume the theorem is true for $i + 1$. There is a surjection $\mathcal{E} \to \mathcal{F}$, where $\mathcal{E} = \mathcal{O}_X(q_1) \oplus \cdots \oplus \mathcal{O}_X(q_k)$, by [Har77, Corollary II 5.18]. So we get a short exact sequence

$$0 \to \mathcal{R} \to \mathcal{E} \to \mathcal{F} \to 0$$

of coherent sheaves on X.

(a) From the long exact sequence we get that

$$H^i(X, \mathcal{E}) \to H^i(X, \mathcal{F}) \to H^{i+1}(X, \mathcal{R})$$

is exact for all $i \ge 0$, with $H^i(X, \mathcal{E})$ and $H^{i+1}(X, \mathcal{R})$ finitely generated over A by Step 3 and by the inductive hypothesis. Therefore $H^i(X, \mathcal{F})$ is finitely generated.

(b) For all $n \in \mathbb{Z}$,

$$0 \to \mathcal{R}(n) \to \mathcal{E}(n) \to \mathcal{F}(n) \to 0$$

is exact, so the long exact sequence gives

$$H^i(X, \mathcal{E}(n)) \to H^i(X, \mathcal{F}(n)) \to H^{i+1}(X, \mathcal{R}(n)).$$

By Step 3 and the inductive hypothesis, $H^i(X, \mathcal{E}(n))$ and $H^{i+1}(X, \mathcal{R}(n))$ are 0 for all n sufficiently large. So $H^i(X, \mathcal{F}(n)) = 0$ for all $n \ge n_i$, then we are done by induction with $n_0 = \max\{n_1, \ldots, n_r\}$.

☐

☐

Corollary 3.12. $\Gamma(X, \mathcal{F})$ *is a finitely generated A-module.*

Remark. This generalizes [Har77, II 5.19]

Corollary 3.13. *Let A be a noetherian ring, and let X be a closed subscheme of $\mathbb{P}^r_A$ for some $r \ge 0$. Then the restriction map*

$$\rho : \Gamma(\mathbb{P}^r_A, \mathcal{O}(n)) \to \Gamma(X, \mathcal{O}_X(n))$$

is surjective for all $n \gg 0$.

Proof. Let $\mathcal{I}$ be the ideal sheaf in $\mathcal{O}_{\mathbb{P}^r_A}$ corresponding to X. Let $i : X \to \mathbb{P}^r_A$ be the closed embedding. Since $\mathcal{I}$ is coherent, $H^i(\mathbb{P}^r_A, \mathcal{I}(n)) = 0$ for all $n \gg 0$. Taking the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^r_A} \to i_*\mathcal{O}_X \to 0,$$

tensoring it with $\mathcal{O}(n)$ and taking the long exact sequence in cohomology gives that

$$\Gamma(\mathbb{P}^r_A, \mathcal{O}_{\mathbb{P}^r_A}(n)) \to \Gamma(\mathbb{P}^r_A, (i_*\mathcal{O}_X)(n)) \to H^1(\mathbb{P}^r_A, \mathcal{I}(n))$$

Now note that $(i_*\mathcal{O}_X)(n) = i_*\mathcal{O}_X(n)$ and so $\Gamma(\mathbb{P}^r_A, (i_*\mathcal{O}_X)(n)) = \Gamma(X, \mathcal{O}_X(n))$. □

3.3 Ample Line Bundles

Theorem 3.14 ([Har77, II 5.17], Serre). *Let X be a projective scheme over a noetherian ring A, let $\mathcal{O}_X(1)$ be a very ample line bundle on X (over A), and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then there is an $n_0 \in \mathbb{Z}$ (depending on $\mathcal{F}$) such that the sheaf $\mathcal{F}(n)$ can be generated by finitely many global sections.*

Proof.

Step 1. Reduce to $X = \mathbb{P}^r_A$ for some r and $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^r_A}(1)$.

Proof. Choose an embedding $i : X \hookrightarrow \mathbb{P}^r_A$ for some r such that $\mathcal{O}_X(1) \cong i^*\mathcal{O}_{\mathbb{P}^r_A}(1)$. Then i is a closed embedding, so i is finite and therefore $i_*\mathcal{F}$ is coherent on $\mathbb{P}^r_A$ by [Har77, Exercise II 5.5]. By definition of i_*, $H^0(X, \mathcal{F}) = H^0(\mathbb{P}^r_A, i_*\mathcal{F})$. For all open affines $U = \operatorname{Spec} B$ in $\mathbb{P}^r_A$, $i^{-1}(U) =: V$ is an open affine $\operatorname{Spec} B/I$ in X and $\mathcal{F}|_V \cong \widetilde{M}$ for some finitely generated B/I-module M. Also $(i_*\mathcal{F})|_U = \widetilde{{}_BM}$, and if $s_1, \dots, s_n \in H^0(\mathbb{P}^r_A, i_*\mathcal{F})$ generate $i_*\mathcal{F}$, then they correspond to $m_1, \dots, m_n \in M$ which generate ${}_BM$ as a B-module, hence they generate the (B/I)-module M. □

Step 2. Prove the case $X = \mathbb{P}^r_A$, $\mathcal{O}_X(1) = \mathcal{O}(1)$.

Proof. Cover $X = \mathbb{P}^r_A$ with open sets $D_+(x_i)$, $i = 0, 1, \dots, r$. For each i, $\mathcal{F}|_{D_+(x_i)} \cong \widetilde{M}_i$ for some finitely generate module M_i over $B_i := A[x_0/x_i, \dots, x_r/x_i]$. For all i, let $\{s_{ij} : j = 1, \dots, m_i\}$ be a finite generating set for M_i. Then by [Har77, Lemma II 5.14] for all i, j there is $n_{ij} \in \mathbb{N}$ such that $x_i^{n_{ij}} s_{ij}$ extends to a global section of $\mathcal{F}(n_{ij})$. We may take all $n_{ij} = n$ for some fixed n, so $x_i^n s_{ij}$ extends to a global section of $\mathcal{F}(n)$ for all i, j. Now $\mathcal{F}(n)|_{D_+(x_i)} \cong \widetilde{M'_i}$ for some $D_+(x_i)$-module M'_i for all i, and $\cdot x_i^n : \mathcal{F} \to \mathcal{F}(n)$ induces an isomorphism $M_i \xrightarrow{\cong} M'_i$, therefore $x_i^n s_{ij}$ $(j = 1, \dots, m_i)$ generate M'_i for all i, so $\{x_i^n s_{ij}\}_{ij}$ generate $\mathcal{F}(n)$. □

□

Recall the following definition:

Definition. *A line bundle $\mathcal{L}$ on a noetherian scheme X is* ample *if for every coherent sheaf $\mathcal{F}$ on X there exists $n_0 \in \mathbb{N}$ depending on $\mathcal{F}$ such that $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections (gbgs) for all $n \geq n_0$.*

Examples.

- If X is affine (and noetherian), then every line bundle on X is ample, because every coherent sheaf on X is gbgs.
- If X is projective over a noetherian ring A, then any very ample line bundle on X over A is ample by Theorem 3.14.

Proposition 3.15. *Let $\mathcal{L}$ be a line bundle on a noetherian scheme X. Then TFAE:*

(i) *$\mathcal{L}$ is ample;*

(ii) *$\mathcal{L}^m$ is ample for all $m > 0$;*

(iii) *$\mathcal{L}^m$ is ample for some $m > 0$.*

Proof. "$(i) \implies (ii) \implies (iii)$" are easy. "$(iii) \implies (i)$" Assume $\mathcal{L}^m$ is ample for some $m > 0$, and let $\mathcal{F}$ be a coherent sheaf on X. Then for all $i = 0, 1, \ldots, m-1$, $\mathcal{F} \otimes \mathcal{L}^i$ is coherent, so there are $n_i \in \mathbb{Z}$ such that $(\mathcal{F} \otimes \mathcal{L}^i) \otimes (\mathcal{L}^m)^j$ is gbgs for all $j \geq n_i$. Let $N := \max\{i + mn_i : 0 \leq i < m\}$. Then for $n \geq N$, write $n = i + mj$ with $0 \leq i < m$. Then, since $n \geq N \geq i + mn_i$, $j \geq n_i$, so $\mathcal{F} \otimes \mathcal{L}^n = (\mathcal{F} \otimes \mathcal{L}^i) \otimes (\mathcal{L}^m)^j$ is gbgs. □

Lemma 3.16. *Let $\mathcal{L}$ be an ample line bundle on a noetherian scheme X, and let U be an open subscheme of X. Then $\mathcal{L}|_U$ is ample on U.*

Proof. Let $\mathcal{F}$ be a coherent sheaf on U. By [Har77, Exercise II 5.15] there is a coherent sheaf $\mathcal{F}'$ on X such that $\mathcal{F}'|_U \cong \mathcal{F}$. Choose $n_0 \in \mathbb{Z}$ such that $\mathcal{F}' \otimes \mathcal{L}^n$ is gbgs for all $n \geq n_0$. Then $(\mathcal{F}' \otimes \mathcal{L}^n)|_U \cong \mathcal{F} \otimes (\mathcal{L}|_U)^n$ is gbgs for all $n \geq n_0$, so $\mathcal{L}|_U$ is ample. □

Theorem 3.17. *Let X be a scheme of finite type over a noetherian ring A, and let $\mathcal{L}$ be a line sheaf on X. Then $\mathcal{L}$ is ample iff $\mathcal{L}^m$ is very ample over* $\operatorname{Spec} A$ *for some $m > 0$.*

Proof. "$\Leftarrow$" Suppose $\mathcal{L}^m$ is very ample on X over A for some $m > 0$. Let $i : X \to \mathbb{P}^r_A$ be an embedding such that $\mathcal{L}^m \cong i^*\mathcal{O}(1)$. Then i factors as $i = i_2 \circ i_1$ where i_1 is an open embedding $i_1 : X \to \overline{X}$ and i_2 is a closed embedding $i_2 : \overline{X} \to \mathbb{P}^r_A$ over A. Then $\mathcal{O}_{\overline{X}}(1) = i_2^*\mathcal{O}(1)$ is very ample over A, hence ample. So by Lemma 3.16, $\mathcal{L}^m = i_1^*\mathcal{O}_{\overline{X}}(1)$ is ample, and then $\mathcal{L}^m$ is ample.

"$\Rightarrow$" **Claim.** For any $P \in X$ there is $n > 0$ and a section $s \in \Gamma(X, \mathcal{L}^n)$ such that X_s contains P and is affine.

Proof. Let $P \in X$ and let U be an open affine neighborhood of P such that $\mathcal{L}|_U \cong \mathcal{O}_U$. Let $Y = X \setminus U$ be the closed subscheme with the reduced subscheme structure. Then $\mathcal{I}_Y$ is a coherent sheaf on X, so there is $n > 0$ such that $\mathcal{I}_Y \otimes \mathcal{L}^n$ is generated by global sections. So there is $s \in \Gamma(X, \mathcal{I}_Y \otimes \mathcal{L}^n)$ such that $s_P \notin \mathfrak{m}_P(\mathcal{I}_Y \otimes \mathcal{L}^n)_P$. Since $\mathcal{I}_Y$ is a subsheaf of $\mathcal{O}_X$ and $\mathcal{L}^n$ is locally free, $\mathcal{I}_Y \otimes \mathcal{L}^n$ is a subsheaf of $\mathcal{L}^n$, so we can regard s as a global section of $\mathcal{L}^n$. Since $P \notin Y$, $(i_*\mathcal{O}_Y)_P = 0$ (where $i : Y \to X$ is the inclusion) and so $(\mathcal{I}_Y)_P = \mathcal{O}_{X,P}$. Therefore $s_p \notin \mathfrak{m}_P(\mathcal{I}_Y \otimes \mathcal{L}^n)_P = \mathfrak{m}_P\mathcal{L}^n_P$. Also $s_Q \in \mathfrak{m}_Q\mathcal{L}^n$ for all $Q \in Y$

because $(\mathcal{I}_Y)_Q \subseteq \mathfrak{m}_Q$. So $P \in X_s$ and $X_s \subseteq U$. But also $\mathcal{L}^n|_U \cong \mathcal{O}_U$, so s corresponds to an element $f \in \Gamma(U, \mathcal{O}_U)$ and so $X_s = U_f$ is affine. □

By quasi-compactness, X can be covered by finitely many such X_s, say by $X_{s_i}, i = 1, \ldots, k$ with $s_i \in \Gamma(X, \mathcal{L}^{n_i})$ for all i. Letting $n = \operatorname{lcm}(n_1, \ldots, n_k)$ and replacing s_i with s_i^{n/n_i} for all i, X_{s_i} remains unchanged and we may assume $n_i = n$ for all i. Finally, since $\mathcal{L}^n$ is ample, we can replace $\mathcal{L}$ with $\mathcal{L}^n$, so we may assume $n = 1$. So there are $s_1, \ldots, s_k \in \Gamma(X, \mathcal{L})$ such that the X_{s_i} are all affine and they cover X. Now, for all i let $X_i = X_{s_i}$ and let $B_i = \Gamma(X_i, \mathcal{O}_{X_i})$, so $X_i = \operatorname{Spec} B_i$ for all i. Since X is of finite type over A, each B_i is finitely generated over A, say $B_i = A[b_{i1}, \ldots, b_{ik_i}]$. As noted earlier, for all i, j there exists $n_{ij} > 0$ such that $s_i^{n_{ij}} b_{ij}$ extends to a global section $c_{ij} \in \Gamma(X, \mathcal{L}^{n_{ij}})$. We can assume that $n_{ij} = n$ for all i, j. So now we have $\{s_i^n\} \cup \{c_{ij}\} \subseteq \Gamma(X, \mathcal{L}^n)$, a finite subset which generates $\mathcal{L}^n$ (because the s_i^n do). So there is a unique morphism $\varphi : X \to \mathbb{P}_A^N$ over A, where $N = k + \sum_{i=1}^k k_i - 1$, such that $\varphi^*\mathcal{O}(1) \cong \mathcal{L}^n$ in such a way that $\varphi^* x_i = s_i^n$ and $\varphi^* x_{ij} = c_{ij}$ for all i, j. Moreover, for all $i_0 = 1, \ldots, k$, letting $U_{i_0} = D_+(x_{i_0})$ (standard open affines in $\mathbb{P}_A^N$) with affine rings $A[\{x_i/x_{i_0}\}_i, \{x_{ij}/x_{i_0}\}_{ij}]$, we have that $\varphi(X_{i_0}) \subseteq U_{i_0}$ for all i_0, and $\varphi^{-1}(U_{i_0}) = X_{i_0}$ is affine, $\cong \operatorname{Spec} B_{i_0}$, where the map of affine rings is

$$\begin{aligned} A[\{x_i/x_{i_0}\}_i, \{x_{ij}/x_{i_0}\}_{ij}] &\longrightarrow B_{i_0} \\ x_i/x_{i_0} &\longmapsto s_i^n/s_{i_0}^n, \\ x_{ij}/x_{i_0} &\longmapsto c_{ij}/s_{i_0}^n. \end{aligned}$$

This map is surjective, because $x_{i_0 j}/x_{i_0} \mapsto c_{i_0 j}/s_{i_0}^n = b_{i_0 j}$ for all j, and these generate B_{i_0} over A. Therefore φ is a closed immersion into $\bigcup U_i$ (an open subscheme of $\mathbb{P}_A^N$), so φ is an immersion. □

More properties:

1. If a line bundle $\mathcal{O}(1)$ on X is very ample over Y, for some morphism $X \to Y$, then so is $\mathcal{O}(n)$ for all $n > 0$ (n-uple embedding which is a closed embedding). If line bundles $\mathcal{L}$ and $\mathcal{M}$ on X are very ample over Y, then so is $\mathcal{L} \otimes \mathcal{M}$.
2. Ampleness. Let X be a scheme.
    - If $\mathcal{L}$ and $\mathcal{M}$ are ample, then so is $\mathcal{L} \otimes \mathcal{M}$.
    - If $\mathcal{L}$ is ample, and $i : X' \to X$ is an embedding, then $i^*\mathcal{L}$ is ample.
    - For all $r > 0$, $n \leq 0$, $\mathcal{O}(n)$ is not ample on $\mathbb{P}_A^r$ (A noetherian).

    So the set of ample (or very ample) line bundles on X forms a cone in $\operatorname{Pic}(X)$.

Proposition 3.18. *Let X be a proper scheme over a noetherian ring A. Let $\mathcal{L}$ be a line sheaf on X. TFAE:*

(i) $\mathcal{L}$ is ample.

(ii) For each coherent sheaf $\mathcal{F}$ on X there is an integer n_0 such that $H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0$ for all $i > 0$, $n \geq n_0$.

Proof. "$\Rightarrow$" Assume that $\mathcal{L}$ is ample. Then $\mathcal{L}^m$ is very ample over A for some $m > 0$ and X is projective over A. Apply Theorem 3.11 to $\mathcal{F} \otimes \mathcal{L}^j$ for $j = 0, 1, \ldots, m-1$ to obtain that $\mathcal{F} \otimes \mathcal{L}^n = \mathcal{F} \otimes \mathcal{L}^j \otimes (\mathcal{L}^m)^{n_1}$ if $n = mn_1 + j$, $0 \leq j < n$, is acyclic for all $n \gg 0$.

"$\Leftarrow$" We will show that $\mathcal{L}$ is ample by verifying the condition in its definition, i.e. that for all coherent sheaves $\mathcal{F}$, $\mathcal{F} \otimes \mathcal{L}^n$ is gbgs for all $n \gg 0$.

Claim 1. For all coherent sheaves $\mathcal{F}$ on X and for all closed points P on X there is an integer $n_0 = n_0(\mathcal{F}, P)$ such that for each $n \geq n_0$, there exists an open neighborhood U of P in X such that the global sections in $\Gamma(X, \mathcal{F} \otimes \mathcal{L}^n)$ generate the stalks of $\mathcal{F} \otimes \mathcal{L}^n$ at every point in U (we say that $\mathcal{F} \otimes \mathcal{L}^n$ is gbgs over U).

Proof. Let $\mathcal{F}$ and P be as above, and let $\mathcal{I}_P$ be the ideal sheaf of $\{P\}$ in X. Then there is an exact sequence

$$0 \to \mathcal{I}_P\mathcal{F} \to \mathcal{F} \to \mathcal{F} \otimes \kappa(P) \to 0,$$

where $\kappa(P)$ is the skyscraper sheaf $\mathcal{O}_X/\mathcal{I}_P$ at P.[1] Then

$$0 \to \mathcal{I}_P\mathcal{F} \otimes \mathcal{L}^n \to \mathcal{F} \otimes \mathcal{L}^n \to \mathcal{F} \otimes \kappa(P) \otimes \mathcal{L}^n \to 0$$

is exact for all $n \in \mathbb{Z}$. By (ii) there is $n_0 = n_0(\mathcal{F}, P)$ such that $H^1(X, \mathcal{I}_P\mathcal{F} \otimes \mathcal{L}^n) = 0$ for all $n \geq n_0$. Therefore,

$$\Gamma(X, \mathcal{F} \otimes \mathcal{L}^n) \to \Gamma(X, \mathcal{F} \otimes \mathcal{L}^n \otimes \kappa(P))$$

is surjective for all $n \geq n_0$. For all such n, by Nakayama's lemma, the germs at P of elements of $\Gamma(X, \mathcal{F} \otimes \mathcal{L}^n)$ generate the stalk $(\mathcal{F} \otimes \mathcal{L}^n)_P$ as an $\mathcal{O}_{X,P}$-module. Then there is an open set U such that $P \in U$ and $\mathcal{F} \otimes \mathcal{L}^n$ is gbgs over U, take e.g. U to be the complement of the support of the coherent sheaf $\operatorname{coker}(\mathcal{O}_X^N \to \mathcal{F} \otimes \mathcal{L}^n)$, where $(\mathcal{F} \otimes \mathcal{L}^n)_P$ can be generated by N elements of $\Gamma(X, \mathcal{F} \otimes \mathcal{L}^n)$. □

Claim 2. In Claim 1, we can take U to be independent of n.

Proof. Fix P. By Claim 1, with $\mathcal{F} = \mathcal{O}_X$, there exists $m > 0$ and open $V \subseteq X$ depending on P such that $P \in V$ and $\mathcal{L}^m$ is gbgs over V. By Claim 1, applied to $\mathcal{F}$, there is $n_0 \in \mathbb{Z}$ such that for all $r = 0, 1, \ldots, m-1$, there is an open $U_r \subseteq X$ such that $P \in U_r$ and $\mathcal{F} \otimes \mathcal{L}^{n_0+r}$ is gbgs over U_r. Let $U_P = V \cap U_0 \cap \cdots \cap U_{m-1}$. Then we are done. Given $n \geq n_0$, write $n = n_0 + n_1 m + r$ with $n_1 \in \mathbb{N}$ and $0 \leq r < m$. Then $\mathcal{F} \otimes \mathcal{L}^n \cong (\mathcal{F} \otimes \mathcal{L}^{n_0+r}) \otimes (\mathcal{L}^m)^{n_1}$, and $\mathcal{F} \otimes \mathcal{L}^{n_0}$ and $\mathcal{L}^n$ are gbgs over U_r and V resp., hence over U_P, therefore so is $\mathcal{F} \otimes \mathcal{L}^n$. □

[1] Note this comes from tensoring $0 \to \mathcal{I}_P \to \mathcal{O}_X \to \kappa(P) \to 0$ with $\mathcal{I}_P$ and then replacing $\mathcal{I}_P \otimes \mathcal{F}$ by its image $\mathcal{I}_P\mathcal{F}$ in $\mathcal{F}$.

Finishing the proof: We need $n_0 = n_0(\mathcal{F})$ such that $\mathcal{F} \otimes \mathcal{L}^n$ is gbgs over X for all $n \geq n_0$. Since X is quasi-compact, every non-empty closed subset contains a closed point. Therefore $\bigcup_P U_P = X$ where the U_P are as in Claim 2. By quasi-compactness, choose a finite subcover $U_{P_1}, \dots, U_{P_m}$. Then $\mathcal{F} \otimes \mathcal{L}^n$ is gbgs over $\bigcup U_{P_i} = X$ for all $n \geq \max\{n(\mathcal{F}, P_i) : 1 \leq i \leq m\}$. □

3.4 Euler Characteristic

Definition. *If X is a projective variety over a field k, and let $\mathcal{F}$ be a coherent sheaf on X. The* Euler characteristic $\chi(\mathcal{F})$ *is defined by*

$$\chi(\mathcal{F}) := \sum_{i=0}^{\dim(X)} (-1)^i \dim_k H^i(X, \mathcal{F}).$$

Proposition 3.19. *Let X be a projective scheme over a field k, and let $0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_n \to 0$ be an exact sequence of coherent sheaves on X. Then*

$$\sum_{i=0}^{n} (-1)^i \chi(\mathcal{F}_i) = 0.$$

Proof. By [Har77, Exercise III 5.1], this is true for $n = 2$. The general case follows by induction on n and using the same method as was used in the exercise. □

Proposition 3.20 ([Har77, Exercise III 5.2a])**.** *Let X be a projective scheme over a field k, let $\mathcal{O}_X(1)$ be a very ample line bundle on X over k, and let $\mathcal{F}$ be a coherent sheaf on X. Then there is a polynomial $P \in \mathbb{Q}[z]$ such that $\chi(\mathcal{F}(n)) = P(n)$ for all $n \in \mathbb{Z}$. We call P the* Hilbert polynomial *of $\mathcal{F}$ with respect to $\mathcal{O}_X(1)$.*

Proof. We may assume that X is a closed subscheme of $\mathbb{P}^r_k$ for some $r > 0$, and that $\mathcal{O}_X(1) = i^*\mathcal{O}(1)$, where $i : X \to \mathbb{P}^r_k$ is a fixed closed embedding. We use noetherian induction on $\operatorname{Supp} \mathcal{F}$. We may assume that $X = \mathbb{P}^r_k$ and i is the identity map, since $\chi(\mathcal{F}) = \chi(i_*\mathcal{F})$ by Lemma 2.4. For the base case suppose $\operatorname{Supp} \mathcal{F} = \emptyset$. Then $\mathcal{F} = 0$, so $\chi(\mathcal{F}(n)) = 0$ for all n, so it is true with $P = 0$. Next the inductive step. Since $\cap_{i=1}^r \{x_r = 0\} = \emptyset$ and $\operatorname{Supp} \mathcal{F} \neq \emptyset$, we may choose j such that $\operatorname{Supp} \mathcal{F} \nsubseteq \{x_j = 0\}$. Let $\mathcal{R}$ and $\mathcal{Q}$ be the kernel and cokernel of $\mathcal{F}(-1) \xrightarrow{\cdot x_j} \mathcal{F}$, respectively (which is an isomorphism outside of $\{x_j = 0\}$). Then

$$0 \to \mathcal{R} \to \mathcal{F}(-1) \xrightarrow{\cdot x_j} \mathcal{F} \to \mathcal{Q} \to 0$$

is exact. Twisting by $\mathcal{O}(n)$ gives that

$$0 \to \mathcal{R}(n) \to \mathcal{F}(n-1) \to \mathcal{F}(n) \to \mathcal{Q}(n) \to 0$$

is exact for all n. So

$$\chi(\mathcal{F}(n)) - \chi(\mathcal{F}(n-1)) = \chi(\mathcal{Q}(n)) - \chi(\mathcal{R}(n))$$

for all $n \in \mathbb{Z}$. Since $\cdot x_j$ is an isomorphism on $D_+(x_j)$, the supports of $\mathcal{Q}$ and $\mathcal{R}$ are contained in $\operatorname{Supp}\mathcal{F} \cap \{x_j = 0\} \subsetneq \operatorname{Supp}\mathcal{F}$. Therefore by the inductive hypothesis there are polynomials $R, Q \in \mathbb{Q}[z]$ such that $\chi(\mathcal{Q}(n)) = Q(n)$ and $\chi(\mathcal{R}(n)) = R(n)$ for all $n \in \mathbb{Z}$. Then by [Har77, Proposition I 7.3b] (which is still true if we replace "$n \gg 0$" with "$n \in \mathbb{Z}$" throughout), there is a polynomial $P_0 \in \mathbb{Q}[z]$ such that $\chi(\mathcal{F}(n+1)) = P_0(n)$ for all $n \in \mathbb{Z}$. $\square$

Now let $X = \mathbb{P}^r_k$ (with $r > 0$), and let $M = \Gamma_*(\mathcal{F}) = \bigoplus_{n\in\mathbb{Z}} H^0(X, \mathcal{F}(n))$. Then the Hilbert polynomial P of $\mathcal{F}$ just defined is the same as the Hilbert polynomial of M defined in [Har77, I §7], i.e. $P_M \in \mathbb{Q}[z]$ such that $\dim_k M_n = P_M(n)$ for all $n \gg 0$. Indeed, since $\dim_k M_n = \dim_k H^0(X, \mathcal{F}(n))$ for all n, this amounts to showing that $\chi(\mathcal{F}(n)) = \dim_k H^0(X, \mathcal{F}(n))$ for all $n \gg 0$. This is true by definition of χ and Theorem 3.11.

4 Divisors and Curves

Didn't really take notes for this section, it was mostly [Har77, II §6].

5 Duality

5.1 Ext Groups and Sheaves

For the moment, let $(X, \mathcal{O}_X)$ be a ringed space (not necessarily locally ringed). We will need two cases in particular:

- X is a scheme.
- X is a point, and $\mathcal{O}_X$ is a ring.

All sheaves considered will be $\mathcal{O}_X$-modules, so we will work in $\mathbf{Mod}(X)$. Recall that $\mathbf{Mod}(X)$ has enough injectives, and cohomology of an $\mathcal{O}_X$-module can be computed using an injective resolution on $\mathbf{Mod}(X)$. For $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, recall that $\mathrm{Hom}_X(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ is an abelian group. Also recall that $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is the sheaf $U \mapsto \mathrm{Hom}_U(\mathcal{F}|_U, \mathcal{G}|_U)$. This is a sheaf of $\mathcal{O}_X$-modules. For all $\mathcal{F}$, $\mathrm{Hom}(\mathcal{F}, -)$ and $\mathcal{H}om(\mathcal{F}, -)$ are left exact.

Definition. *The functors* $\mathrm{Ext}^i(\mathcal{F}, -)$ *and* $\mathcal{E}xt^i(\mathcal{F}, -)$ *are the right derived functors of* $\mathrm{Hom}(\mathcal{F}, -)$ *and* $\mathcal{H}om(\mathcal{F}, -)$ *respectively. They are covariant functors* $\mathbf{Mod}(X) \to \mathbf{Ab}$ *and* $\mathbf{Mod}(X) \to \mathbf{Mod}(X)$ *respectively.*

Lemma 5.1. *If* $\mathcal{I} \in \mathbf{Mod}(X)$ *is injective and* $U \subseteq X$ *is an open subset, then* $\mathcal{I}|_U$ *is injective in* $\mathbf{Mod}(U)$.

Proof. Given a diagram
$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G} \\ & & \downarrow & & \\ & & \mathcal{I}|_U & & \end{array}$$
in $\mathbf{Mod}(U)$ with exact top row, we get a diagram
$$\begin{array}{ccccc} 0 & \longrightarrow & j_!\mathcal{F} & \longrightarrow & j_!\mathcal{G} \\ & & \downarrow & \swarrow_{f} & \\ & & j_!(\mathcal{I}|_U) & & \\ & & \downarrow & & \\ & & \mathcal{I} & & \end{array}$$
where $j : U \hookrightarrow X$ is the inclusion map and $j_!$ is the functor extending a sheaf by 0. The top row of the second diagram is exact because it is exact on stalks. So by injectivity of $\mathcal{I}$ there exists $f : j_!\mathcal{G} \to \mathcal{I}$ such that the diagram commutes. Then $f|_U$ maps $\mathcal{G} = (j_!\mathcal{G})|_U$ to $\mathcal{I}_U$ and extends the given map $\mathcal{F} \to \mathcal{I}|_U$. □

Proposition 5.2. $\mathcal{E}xt^i(\mathcal{F}|_U, \mathcal{G}|_U) \cong \mathcal{E}xt^i(\mathcal{F}, \mathcal{G})|_U$ *for all* $\mathcal{F}, \mathcal{G} \in \mathbf{Mod}(X)$, *open* $U \subseteq X$ *and for all* i.

Proof. Let $0 \to \mathcal{G} \to \mathcal{I}^\bullet$ be an injective resolution in $\mathbf{Mod}(X)$. Then $\mathcal{I}^\bullet|_U$ is an injective resolution of $\mathcal{G}|_U$ in $\mathbf{Mod}(U)$. By definition we have $\mathcal{H}om_X(\mathcal{F}, \mathcal{G})|_U = \mathcal{H}om_U(\mathcal{F}|_U, \mathcal{G}|_U)$.

We get

$$\begin{aligned}\mathcal{E}xt^i_U(\mathcal{F}|_U, \mathcal{G}|_U) &= h^i(\mathcal{H}om_U(\mathcal{F}|_U, \mathcal{I}^\bullet|_U)) \\ &= h^i(\mathcal{H}om_X(\mathcal{F}, \mathcal{I}^\bullet)|_U) = h^i(\mathcal{H}om_X(\mathcal{F}, \mathcal{I}^\bullet))|_U \\ &= \mathcal{E}xt^i(\mathcal{F}, \mathcal{G})|_U\end{aligned}$$

□

Proposition 5.3. $\mathcal{E}xt^i(\mathcal{O}_X, \mathcal{G}) = \begin{cases}\mathcal{F} & \text{if } i = 0, \\ 0 & \text{if } i > 0.\end{cases}$ *and* $\operatorname{Ext}^i(\mathcal{O}_X, \mathcal{G}) = H^i(X, \mathcal{G})$ *for all* i.

Proof. By computation. Let $\mathcal{I}^\bullet$ be an injective resolution of $\mathcal{G}$. Then

$$\mathcal{E}xt^i(\mathcal{O}_X, \mathcal{G}) = h^i(\mathcal{H}om(\mathcal{O}_X, \mathcal{I}^\bullet)) = h^i(\mathcal{I}^\bullet) = \begin{cases}\mathcal{G} & \text{if } i = 0, \\ 0 & \text{if } i > 0.\end{cases}$$

and

$$\operatorname{Ext}^i(\mathcal{O}_X, \mathcal{G}) = h^i(\operatorname{Hom}(\mathcal{O}_X, \mathcal{I}^\bullet)) = h^i(\Gamma(X, \mathcal{I}^\bullet)) = H^i(X, \mathcal{G}).$$

□

In particular we see that $\Gamma(X, \mathcal{E}xt^i(\mathcal{F}, \mathcal{G})) \neq \operatorname{Ext}^i(\mathcal{F}, \mathcal{G})$ in general (unless $i = 0$).

Recall: If $\mathcal{E}, \mathcal{F}, \mathcal{G} \in \mathbf{Mod}(X)$ with $\mathcal{E}$ locally free of finite rank, then $\mathcal{H}om(\mathcal{E}, \mathcal{F}) \cong \mathcal{E}^\vee \otimes \mathcal{F}$, more generally

$$\mathcal{H}om(\mathcal{F} \otimes \mathcal{E}, \mathcal{G}) \cong \mathcal{H}om(\mathcal{F}, \mathcal{E}^\vee \otimes \mathcal{G}) \cong \mathcal{H}om(\mathcal{F}, \mathcal{G}) \otimes \mathcal{E}^\vee,$$

and

$$\operatorname{Hom}(\mathcal{F} \otimes \mathcal{E}, \mathcal{G}) \cong \operatorname{Hom}(\mathcal{F}, \mathcal{H}om(\mathcal{E}, \mathcal{G})) \cong \operatorname{Hom}(\mathcal{F}, \mathcal{E}^\vee \otimes \mathcal{G}).$$

Lemma 5.4. *Let* $\mathcal{E} \in \mathbf{Mod}(X)$ *be locally free of finite rank. If* $\mathcal{I} \in \mathbf{Mod}(X)$ *is injective, then so is* $\mathcal{E} \otimes \mathcal{I}$.

Proof. $- \otimes \mathcal{E}^\vee$ and $\operatorname{Hom}(-, \mathcal{I})$ are exact functors, thus so is their composite $\operatorname{Hom}(- \otimes \mathcal{E}^\vee, \mathcal{I}) \cong \operatorname{Hom}(-, \mathcal{E} \otimes \mathcal{I})$. □

Proposition 5.5. *Let* $\mathcal{E}$ *be a locally free sheaf of finite rank. Then for all* $\mathcal{F}, \mathcal{G}$*:*

(a) $\operatorname{Ext}^i(\mathcal{F} \otimes \mathcal{E}, \mathcal{G}) \cong \operatorname{Ext}^i(\mathcal{F}, \mathcal{E}^\vee \otimes \mathcal{G})$*;*

(b) $\mathcal{E}xt^i(\mathcal{F} \otimes \mathcal{E}, \mathcal{G}) \cong \mathcal{E}xt^i(\mathcal{F}, \mathcal{E}^\vee \otimes \mathcal{G}) \cong \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \otimes \mathcal{E}^\vee$ *for all* i.

Proof. The $i = 0$ case follows from [Har77, Exercise II 5.1]. For general $i \geq 0$: Let $0 \to \mathcal{G} \to \mathcal{I}^\bullet$ be an injective resolution. Then $0 \to \mathcal{E}^\vee \otimes \mathcal{G} \to \mathcal{E}^\vee \otimes \mathcal{I}^\bullet$ is also an injective resolution by the previous lemma. Therefore

$$\mathrm{Ext}^i(\mathcal{F} \otimes \mathcal{E}, \mathcal{G}) = h^i(\mathrm{Hom}(\mathcal{F} \otimes \mathcal{E}, \mathcal{I}^\bullet)) \cong h^i(\mathrm{Hom}(\mathcal{F}, \mathcal{E}^\vee \otimes \mathcal{I}^\bullet)) = \mathrm{Ext}^i(\mathcal{F}, \mathcal{E}^\vee \otimes \mathcal{G}).$$

Likewise for $\mathcal{E}xt$. Also

$$\begin{aligned}\mathcal{E}xt(\mathcal{F}, \mathcal{E}^\vee \otimes \mathcal{G}) &= h^i(\mathcal{H}om(\mathcal{F}, \mathcal{E}^\vee \otimes \mathcal{I}^\bullet)) = h^i(\mathcal{H}om(\mathcal{F}, \mathcal{I}^\bullet) \otimes \mathcal{E}^\vee) \\ &= h^i(\mathcal{H}om(\mathcal{F}, \mathcal{I}^\bullet)) \otimes \mathcal{E}^\vee = \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \otimes \mathcal{E}^\vee.\end{aligned}$$

□

Proposition 5.6. *Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules. Then for all $\mathcal{G}$ there is a long exact sequence*

$$0 \to \mathcal{H}om(\mathcal{F}'', \mathcal{G}) \to \mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om(\mathcal{F}', \mathcal{G}) \to \mathcal{E}xt^1(\mathcal{F}'', \mathcal{G}) \to \mathcal{E}xt^1(\mathcal{F}', \mathcal{G}) \to \dots$$

and likewise for Hom *and* Ext.

Proof. Let $0 \to \mathcal{G} \to \mathcal{I}^\bullet$ be an injective resolution. Then for all i and all open $U \subseteq X$ we get a short exact sequence

$$0 \to \mathrm{Hom}(\mathcal{F}''|_U, \mathcal{I}^i|_U) \to \mathrm{Hom}(\mathcal{F}|_U, \mathcal{I}^i|_U) \to \mathrm{Hom}(\mathcal{F}'|_U, \mathcal{I}^i|_U) \to 0$$

since $\mathrm{Hom}_U(-, \mathcal{I}^i|_U)$ is exact for all i, U. Therefore

$$0 \to \mathcal{H}om(\mathcal{F}'', \mathcal{I}^i) \to \mathcal{H}om(\mathcal{F}, \mathcal{I}^i) \to \mathcal{H}om(\mathcal{F}', \mathcal{I}^i) \to 0$$

is exact for all i. So we get a short exact sequence of complexes in $\mathbf{Mod}(X)$. We conclude by applying the Snake lemma. The proof for Ext is similar. □

Proposition 5.7. *Let $\mathcal{E}_\bullet \to \mathcal{F} \to 0$ be a left resolution of $\mathcal{F}$ in which all $\mathcal{E}_i$ are locally free of finite rank. Then for all $\mathcal{G}$,*

$$\mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) = h^i(\mathcal{H}om(\mathcal{E}_\bullet, \mathcal{G})) \qquad \forall\, i \geq 0$$

Proof. Postponed until we do spectral sequences. □

Note. This is not true for regular Ext (compare Proposition 5.3).

Proposition 5.8. *Let X be a noetherian scheme, let $\mathcal{F}$ be a coherent sheaf on X and $\mathcal{G}$ any sheaf of $\mathcal{O}_X$-modules. Let $x \in X$. Then:*

$$\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})_x \cong \mathrm{Ext}^i_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x) \quad \forall\, i \geq 0.$$

Here $\mathrm{Ext}^i_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$ is Ext of modules over the ring $\mathcal{O}_{X,x}$ which is the same as Ext of $\mathcal{O}_Y$-modules where Y is the ringed space $(Y = \{x\}, \mathcal{O}_Y(Y) = \mathcal{O}_{X,x})$.

Proof. Since $\mathcal{E}xt$ commutes with restricting to an open subset $U \subseteq X$, we may assume that X is affine, equal to $\operatorname{Spec} A$. Then A is a noetherian ring and $\mathcal{F} \cong \widetilde{M}$ where M is a finitely generated A-module. Then there is a left resolution $E_\bullet \to M \to 0$ with E_i free of finite rank for all i. Then $\widetilde{E}_\bullet \to \mathcal{F} \to 0$ is a free coherent left resolution of $\mathcal{F}$. Now we compute

$$\begin{aligned}\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})_x &= h^i(\mathcal{H}om(\widetilde{E}_\bullet, \mathcal{G}))_x = h^i(\mathcal{H}om(\widetilde{E}_\bullet, \mathcal{G})_x) \\ &= h^i((\widetilde{E}_\bullet^\vee \otimes \mathcal{G})_x)\end{aligned}$$

and

$$\mathrm{Ext}^i_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x) = h^i(\mathrm{Hom}_{\mathcal{O}_{X,x}}((E_\bullet)_x, \mathcal{G}_x)) = h^i((E_\bullet)_x^\vee \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x)$$

These are the same ($(E_\bullet)_x^\vee = (E_\bullet^\vee)_x$ as $E_\bullet$ is free). □

5.2 $\mathcal{E}xt$ and $\mathcal{O}(1)$

Proposition 5.9. *Let X be a projective scheme over a noetherian ring A, let $\mathcal{O}(1)$ be a very ample line bundle on X over* $\operatorname{Spec} A$*, and let $\mathcal{F}, \mathcal{G}$ be coherent sheaves on X. Then there exists $n_0 \in \mathbb{Z}$, depending only on $\mathcal{F}, \mathcal{G}$, and $\mathcal{O}(1)$ such that*

$$\Gamma(X, \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}(n))) \cong \mathrm{Ext}^i(\mathcal{F}, \mathcal{G}(n)) \quad \forall\, i, \forall\, n \geq n_0.$$

Proof. We induct on i.

Step 0. For $i = 0$ it is true for all $n \in \mathbb{Z}$ by definition of $\mathcal{H}om$ and Hom. So we may assume $i > 0$.

Step 1. If $\mathcal{F}$ is locally free, then the result is true for all i, because

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}(n)) \cong \mathrm{Ext}^i(\mathcal{O}_X, \mathcal{F}^\vee \otimes \mathcal{G}(n)) \cong H^i(X, (\mathcal{F}^\vee \otimes \mathcal{G})(n)) = 0 \quad \forall\, n \gg 0$$

and $\Gamma(X, \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}(n))) = \Gamma(X, \mathcal{E}xt^i(\mathcal{O}_X, \mathcal{F}^\vee \otimes \mathcal{G}(n))) = \Gamma(X, 0) = 0$ for all $n \in \mathbb{Z}$.

Step 2. General case. Use induction on $i \geq 1$. Given $\mathcal{F}$, there is a locally free coherent sheaf $\mathcal{E}$ on X that maps onto $\mathcal{F}$, so there is a short exact sequence

$$0 \to \mathcal{R} \to \mathcal{E} \to \mathcal{F} \to 0$$

with $\mathcal{R}$ coherent. Since $\mathcal{E}$ is locally free, $\mathrm{Ext}^i(\mathcal{E}, \mathcal{G}(n)) = 0$ for all $n \gg 0$ and $\mathcal{E}xt^i(\mathcal{E}, \mathcal{G}(n)) = 0$ for all $n \in \mathbb{Z}$. So from the long exact sequences in $\mathrm{Ext}(-, \mathcal{G}(n))$ and $\mathcal{E}xt(-, \mathcal{G}(n))$, if $i > 1$, we have isomorphisms

$$\mathrm{Ext}^i(\mathcal{R}, \mathcal{G}(n)) \xrightarrow{\simeq} \mathrm{Ext}^{i+1}(\mathcal{F}, \mathcal{G}(n))$$

$$\mathcal{E}xt^i(\mathcal{R},\mathcal{G}(n)) \xrightarrow{\simeq} \mathcal{E}xt^{i+1}(\mathcal{F},\mathcal{G}(n))$$

for all n, and the latter gives $\Gamma(X,\mathcal{E}xt^i(\mathcal{R},\mathcal{G}(n))) \xrightarrow{\simeq} \Gamma(X,\mathcal{E}xt^{i+1}(\mathcal{F},\mathcal{G}(n)))$ for all n. So we get the inductive step for $i > 1$ as indicated. It remains to consider $i = 1$. We have exact sequences

$$0 \to \operatorname{Hom}(\mathcal{F},\mathcal{G}(n)) \to \operatorname{Hom}(\mathcal{E},\mathcal{G}(n)) \xrightarrow{\beta} \operatorname{Hom}(\mathcal{R},\mathcal{G}(n)) \to \operatorname{Ext}^1(\mathcal{F},\mathcal{G}(n)) \to \operatorname{Ext}^1(\mathcal{E},\mathcal{G}(n)) = 0$$

and

$$0 \to \mathcal{H}om(\mathcal{F},\mathcal{G}(n)) \to \mathcal{H}om(\mathcal{E},\mathcal{G}(n)) \xrightarrow{\alpha} \mathcal{H}om(\mathcal{R},\mathcal{G}(n)) \to \mathcal{E}xt^1(\mathcal{F},\mathcal{G}(n)) \to \mathcal{E}xt^1(\mathcal{E},\mathcal{G}(n)) = 0$$

The last terms are 0 by ... and ... for $n \gg 0$. Taking global sections of the second sequences gives a sequence of global sections:

$$\begin{aligned} 0 &\to \Gamma(X,\mathcal{H}om(\mathcal{F},\mathcal{G}(n))) \to \Gamma(X,\mathcal{H}om(\mathcal{E},\mathcal{G}(n))) \to \Gamma(X,\mathcal{H}om(\mathcal{R},\mathcal{G}(n))) \\ &\to \Gamma(X,\mathcal{E}xt^1(\mathcal{F},\mathcal{G}(n))) \to \Gamma(X,0) = 0. \end{aligned}$$

and this sequence is exact for all $n \gg 0$ by 6.7 (can pull out (n)) and Exercise 5.10. Therefore we get isomorphisms

$$\operatorname{Ext}^1(\mathcal{F},\mathcal{G}(n)) \cong \operatorname{coker}\beta \cong \operatorname{coker}\Gamma(X,\alpha) \cong \Gamma(X,\mathcal{E}xt^1(\mathcal{F},\mathcal{G}(n)))$$

for all $n \gg 0$.

To see that we can choose n_0 independent of i, e.g. note that if $X \hookrightarrow \mathbb{P}^r_A$ corresponds to $\mathcal{O}(1)$, then

$$\operatorname{Ext}^i(\mathcal{E},\mathcal{G}(n)) \cong \operatorname{Ext}^i(\mathcal{O}_X,\widehat{\mathcal{E}} \otimes \mathcal{G}(n)) \cong H^i(X,(\widetilde{\mathcal{E}} \otimes G)(n)) = 0$$

for all $i > r$ and all $n,\mathcal{E},\mathcal{G}$, and we always have $\mathcal{E}xt^i(\mathcal{E},\mathcal{G}(n)) = 0$ for all $i > 0, n, \mathcal{E}, \mathcal{G}$. So for all $i > r$ the isomorphism holds for all n. □

5.3 Serre Duality on $\mathbb{P}^n_k$

Let k be a field and let $n \in \mathbb{N}_0$. Recall that the *canonical sheaf* of a nonsingular variety X over k of dimension n is $\omega_X = \wedge^n \Omega_{X/k}$. Here $\Omega_{X/k} = T_X^\vee$ is the cotangent bundle, which is locally free of rank n. We know that when $X = \mathbb{P}^n_A$, then $\omega_X \cong \mathcal{O}_X(-n-1)$, see [Har77, Example II 8.20.1].

Theorem 5.10 (Duality for $\mathbb{P}^n_k$)**.** *Let k be a field and let $X = \mathbb{P}^n_k$ with $n > 0$. Then*

(a) $H^n(X,\omega_X) \cong k$ (non-canonically). Fix one such isomorphism $t : H^n(X,\omega_X) \xrightarrow{\simeq} k$ over k.

(b) For any coherent sheaf $\mathcal{F}$ on X, the natural k-bilinear map

$$\begin{aligned}\mathrm{Hom}(\mathcal{F},\omega_X)\times H^n(X,\mathcal{F}) &\longrightarrow H^n(X,\omega_X),\\ (\varphi,c) &\longmapsto H^n(X,\varphi)(c)\end{aligned}$$

when composed with t, is a perfect pairing of finite dimensional vector spaces.

(c) For every $i \geq 0$ there is a natural functorial isomorphism

$$\mathrm{Ext}^i(\mathcal{F},\omega_X) \xrightarrow{\cong} \mathrm{Hom}(H^{n-i}(X,\mathcal{F}), H^n(X,\omega_X)),$$

which, when composed with t, gives a non-canonical isomorphism

$$\mathrm{Ext}^i(\mathcal{F},\omega_X) \xrightarrow{\cong} H^{n-i}(X,\mathcal{F})^\vee.$$

When $i = 0$, these maps are the ones induced by the pairing of (b).

Proof.

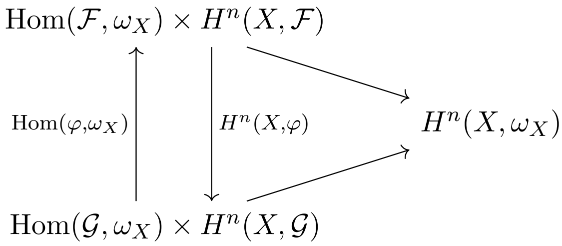
(a) This is immediate from Theorem 3.10 c since $\omega_X \cong \mathcal{O}_X(-n-1)$.

(b) The finite dimensionality of $H^n(X,\mathcal{F})$ follow from Theorem 3.10 a. We prove the rest of the statement in steps:

Case 1. $\mathcal{F} = \mathcal{O}(q)$ for some $q \in \mathbb{Z}$. Then $\mathrm{Hom}(\mathcal{F},\omega_X) \cong H^0(X,\mathcal{O}_X(-q-n-1))$ and $H^n(X,\mathcal{F}) = H^n(X,\mathcal{O}_X(q))$, so this is Theorem 3.10 d.

Case 2. $\mathcal{F} = \oplus\mathcal{O}(q_i)$ (finite $\oplus$). This is immediate.

Interlude. What does "natural" mean for the pairing? Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of schemes. Then "naturality" means that the following diagram commutes



by which we mean that for $g \in \mathrm{Hom}(\mathcal{G},\omega_X)$ and $c \in H^n(X,\mathcal{F})$ we have

$$\langle g\circ\varphi, c\rangle_{\mathcal{F}} = \langle g, H^n(X,\varphi)(c)\rangle_{\mathcal{G}}.$$

They are equal because the top pairing takes $(g\circ\varphi, c)$ to $H^n(X, g\circ\varphi)(c)$ and value of the bottom pairing on $(g, H^n(X,\varphi)(c))$ is $H^n(X,g)(H^n(X,\varphi)(c))$ and they are equal because $H^n(X,-)$ is a functor.

Case 3. The general case. By [Har77, Corollary II 5.18] there is an exact sequence $\mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$ with $\mathcal{E}_0 \cong \oplus\mathcal{O}(q_i)$ and $\mathcal{E}_1 \cong \mathcal{O}(r_j)$ (both $\oplus$) being finite. Then we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Hom}(\mathcal{F},\omega) & \longrightarrow & \mathrm{Hom}(\mathcal{E}_0,\omega) & \longrightarrow & \mathrm{Hom}(\mathcal{E}_1,\omega) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & H^n(X,\mathcal{F})^\vee & \longrightarrow & H^n(X,\mathcal{E}_0)^\vee & \longrightarrow & H^n(X,\mathcal{E}_1)^\vee \end{array}$$

Exactness of the top row is clear. For the bottom row this follows since for this X, $H^n(X,-)$ is a right exact functor (because this is the end of the LES), so $H^n(X,-)^\vee$ is a left exact contravariant functor. The vertical arrows all come from the pairing. The squares commute because the pairing is functorial. Then by the previous steps the center and right vertical arrows are isomorphism, so the first one also is.

(c) $i = 0$ is proved in (b). The rest is an exercise.

□

Remark. The isomorphism $t : H^n(X,\omega) \to k$ is in fact invariant under automorphisms of $\mathbb{P}^n_k$ $(= \mathrm{PGL}_n(k))$, i.e. $\mathrm{PGL}_n(k) \to \mathrm{Aut}(H^n(\mathbb{P}^n_k, \mathcal{O}(-n-1)))$ is the trivial map, see [Har77, III 7.1.1].

5.4 Dualizing Sheaves

Definition (First version)**.** *Let X be a proper scheme over k of dimension n. Then a* dualizing sheaf *for X over k is an ordered pair (ω_X°, t) consisting of a coherent sheaf ω_X° on X and a* trace morphism *$t : H^n(X, \omega_X^\circ) \to k$ such that*

$$\mathrm{Hom}(\mathcal{F},\omega_X^\circ) \times H^n(X,\mathcal{F}) \to H^n(X,\omega_X^\circ) \xrightarrow{t} k$$

induces an isomorphism $\mathrm{Hom}(\mathcal{F},\omega_X^\circ) \to H^n(X,\mathcal{F})^\vee$ *for all coherent sheaves $\mathcal{F}$ on X.*

Definition (Second version)**.** *Let X be a proper scheme over k, and let $n \geq \dim X$. Then an n-*dualizing sheaf *for X is a coherent sheaf ω_X° on X that represents the functor $F : \mathbf{Coh}(X) \to \mathbf{Mod}(k)$ given by $\mathcal{F} \mapsto H^n(X,\mathcal{F})^\vee$.*

As usual, if ω_X° exists, it is unique up to unique isomorphism.

Proposition 5.11. *If $n > \dim X$, then F is the zero functor and $\omega_X^\circ = 0$.*

Proof. Easy exercise. □

Proposition 5.12. *Let X be as above and let $n = \dim X$. Then an n-dualizing sheaf exists iff there is a pair (ω_X°, t) as in the first version of the definition. If so, then ω_X° is the same in both definitions and t is likewise uniquely determined.*

Proof. Clear from Yoneda lemma. □

Proposition 5.13 (Revised)**.** *Let $n = \dim X$. Let $\mathcal{C}_1$ be the category whose objects are (ordered) pairs (ω, t) where $\omega \in \mathbf{Coh}(X)$ and $t : H^n(X, \omega) \to k$ is a k-linear map, and whose morphisms $(\omega_1, t_1) \to (\omega_2, t_2)$ are maps $\varphi : \omega_1 \to \omega_2$ in $\mathbf{Coh}(X)$ such that the obvious triangle commutes. Let $\mathcal{C}_2$ be the category whose objects are ordered pairs (ω, ψ), where $\omega \in \mathbf{Coh}(X)$ and ψ is a natural transformation $\mathrm{Hom}(-, \omega) \to H^n(X, -)^\vee$ (of contravariant functors $\mathbf{Coh}(X) \to \mathbf{Mod}(k)$), and whose morphisms $(\omega_1, \psi_1) \to (\omega_2, \psi_2)$ are morphisms $\varphi : \omega_1 \to \omega_2$ in $\mathbf{Coh}(X)$ such that*

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{F}, \omega_1) & & \\ & \searrow^{\psi_1(\mathcal{F})} & \\ \Big\downarrow {\scriptstyle -\circ\varphi} & & H^n(X, \mathcal{F})^\vee \\ & \nearrow_{\psi_2(\mathcal{F})} & \\ \mathrm{Hom}(\mathcal{F}, \omega_2) & & \end{array}$$

commutes for all $\mathcal{F}$. Then:

(a) Define $F_1 : \mathcal{C}_1 \to \mathcal{C}_2$ on objects by $(\omega, t) \mapsto (\omega, \psi)$ where $\psi(\mathcal{F})$ is the map $\mathrm{Hom}(\mathcal{F}, \omega) \to H^n(X, \mathcal{F})^\vee$ given by the pairing $\mathrm{Hom}(\mathcal{F}, \omega) \times H^n(X, \mathcal{F}) \to H^n(X, \omega) \xrightarrow{\omega} k$. On morphisms, define F_1 to take $\varphi(\omega_1, t_1) \to (\omega_2, t_2)$ in $\mathcal{C}_1$ to the morphism $\Phi : (\omega_1, \psi_1) \to (\omega_2, \psi_2)$ in $\mathcal{C}_2$ also given by φ. Then F is a functor $\mathcal{C}_1 \to \mathcal{C}_2$.

(b) Define $F_2 : \mathcal{C}_2 \to \mathcal{C}_1$ as follows. On objects, it takes $(\omega, \psi) \in \mathcal{C}_2$ to $(\omega, t) \in \mathcal{C}_1$, where $t : H^n(X, \omega) \to k$ is defined as follows: $\psi(\omega)$ takes $\mathrm{Hom}(\omega, \omega)$ to $H^n(X, \omega)^\vee$, then t is the image of id_ω. On morphisms F_2 takes $(\omega_1, \psi_1) \to (\omega_2, \psi_2)$ in $\mathcal{C}_2$ corresponding to $\varphi : \omega_1 \to \omega_2$ to a map $(\omega_1, t_1) \to (\omega_2, t_2)$ in $\mathcal{C}_1$, also given by φ. Then F_2 is a functor $\mathcal{C}_2 \to \mathcal{C}_1$.

(c) These functors are mutually inverse, so they give isomorphisms of categories.

(d) Let $(\omega, t) \in \mathcal{C}_1$ and let $(\omega, \psi) = F_1(\omega, t)$. Then (ω, t) is a dualizing sheaf as in the first definition if and only if (ω, ψ) is an n-dualizing sheaf.

Proof. Exercise. □

Corollary 5.14. *An n-dualizing sheaf is unique up to unique isomorphism.*

In the case of $X = \mathbb{P}^n_k$, $\omega \cong \mathcal{O}(-n-1)$, and $h^n(X, \mathcal{O}(-n-1)) \cong k$ (non-canonically), but the pair $(\mathcal{O}(-n-1), t)$ gives a canonical element $\alpha \in H^n(X, \mathcal{O}(-n-1))$, defined by $t(\alpha) = 1$.

5.5 Duality for more general X

Lemma 5.15. *Let $P = \mathbb{P}^N_k$ with $N > 0$, let X be a closed subscheme of P, and $r = \operatorname{codim}_P X$. Then*

$$\mathcal{E}xt^i_P(\mathcal{O}_X, \omega_P) = 0 \quad \forall\, i < r.$$

Proof. Let $\mathcal{F}^i = \mathcal{E}xt^i_P(\mathcal{O}_X, \omega_P)$. By [Har77, Exercise III 6.3], $\mathcal{F}^i$ is coherent, so $\mathcal{F}^i(q)$ is gbgs for all $q \gg 0$. So it will suffice to show that $\Gamma(P, \mathcal{F}^i(q)) = 0$ for all $q \gg 0$. We have

$$\begin{aligned}\Gamma(P, \mathcal{F}^i(q)) &= \Gamma(P, \mathcal{E}xt^i_P(\mathcal{O}_X, \omega_P(q))) \\ &= \operatorname{Ext}^i_P(\mathcal{O}_X, \omega_P(q)) \\ &= \operatorname{Ext}^i_P(\mathcal{O}_X(-q), \omega_P) \cong H^{N-i}(X, \mathcal{O}_X(-q))^\vee = 0.\end{aligned}$$

□

More generally, this is true for all P which are equidimensional and Cohen-Macaulay (later).

Note: In the following we will be suppressing pushforwards when considering sheaves on X or P.

Lemma 5.16. *Let k, N, P, X and r as above. Let $\omega^\circ_X = \mathcal{E}xt^r(\mathcal{O}_X, \omega_P)$. Then $\operatorname{Hom}_X(\mathcal{F}, \omega^\circ_X) \cong \operatorname{Ext}^r_P(\mathcal{F}, \omega_P)$ for all $\mathcal{O}_X$-modules $\mathcal{F}$, functorially in $\mathcal{F}$.*

Proof. Let $0 \to \omega_P \to \mathcal{I}^\bullet$ be an injective resolution of ω_P (in $\mathbf{Mod}(P)$), so that $\mathcal{E}xt^i_P(\mathcal{F}, \omega_P) = h^i(\mathcal{H}om_P(\mathcal{F}, \mathcal{I}^\bullet))$ and $\operatorname{Ext}^i_P(\mathcal{F}, \omega_P) = h^i(\operatorname{Hom}_P(\mathcal{F}, \mathcal{I}^\bullet))$.

Claim. Let A be a commutative ring, I an ideal in A, M an A/I-module, and N an A-module. Then

$$\operatorname{Hom}_A(M, N) \cong \operatorname{Hom}_{A/I}(M, \operatorname{Hom}_A(A/I, N)).$$

Proof. Clear from $\operatorname{Hom}_A(A/I, N) \cong \{n \in N \mid In = 0\}$. □

Corollary. *If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules and $\mathcal{I}$ is a sheaf of $\mathcal{O}_P$-modules, then $\operatorname{Hom}_P(\mathcal{F}, \mathcal{I}) \cong \operatorname{Hom}_X(\mathcal{F}, \mathcal{H}om_P(\mathcal{O}_X, \mathcal{I}))$.*

Proof. First, we have

$$\operatorname{Hom}_X(\mathcal{F}, \mathcal{H}om_P(\mathcal{O}_X, \mathcal{I})) = \operatorname{Hom}_P(\mathcal{F}, \mathcal{H}om_P(\mathcal{O}_X, \mathcal{I})) \hookrightarrow \operatorname{Hom}_P(\mathcal{F}, \mathcal{I})$$

as follows: $\mathcal{H}om_P(\mathcal{O}_X, \mathcal{I})$ is killed by the ideal sheaf of X in P, so it can be regarded as a sheaf of $\mathcal{O}_X$-modules. That is the first map. For the second map, the surjection $\mathcal{O}_P \to \mathcal{O}_X$ gives an injection $\mathcal{H}om_P(\mathcal{O}_X, \mathcal{I}) \to \mathcal{H}om_P(\mathcal{O}_P, \mathcal{I}) = \mathcal{I}$, which implies that the map $\operatorname{Hom}_P(\mathcal{F}, \mathcal{H}om_P(\mathcal{O}_X, \mathcal{I})) \to \operatorname{Hom}_P(\mathcal{F}, \mathcal{I})$ is injective. To show surjectivity, it suffices to show that, for all $\varphi \in \operatorname{Hom}_P(\mathcal{F}, \mathcal{I})$, we have $\operatorname{im}\varphi \subseteq \mathcal{H}om_P(\mathcal{O}_X, \mathcal{I})$. Let $p \in P$. Since $\mathcal{O}_{X,p}$ is a finitely generated $\mathcal{O}_{P,p}$-module, $\mathcal{H}om_P(\mathcal{O}_X, \mathcal{I})_p = \operatorname{Hom}_{\mathcal{O}_{P,p}}(\mathcal{O}_{X,p}, \mathcal{I}_p)$. So it suffices to show that $\operatorname{im}\varphi_p \subseteq \operatorname{Hom}_{\mathcal{O}_{P,p}}(\mathcal{O}_{X,p}, \mathcal{I}_p)$. Note that $\operatorname{Hom}_{\mathcal{O}_{P,p}}(\mathcal{O}_{X,p}, \mathcal{I}_p) = \{t_p \in \mathcal{I}_p : \operatorname{Ann}(t_p) \supseteq \mathcal{J}_p$ where $\mathcal{J}$ is the ideal sheaf of X in $P\}$. So $\operatorname{im}\varphi_p \subseteq \operatorname{Hom}_{\mathcal{O}_{P,p}}(\mathcal{O}_{X,p}, \mathcal{I}_p)$ because $\operatorname{Ann}(s_p) \supseteq \mathcal{J}_p$ for all $s_p \in \mathcal{F}_p$. □

Claim. If $\mathcal{I}$ is an injective $\mathcal{O}_P$-module, then $\mathcal{J} := \operatorname{Hom}_P(\mathcal{O}_X, \mathcal{I})$ is an injective $\mathcal{O}_X$-module.

Proof. By the corollary, $\operatorname{Hom}_X(\mathcal{F}, \mathcal{J}) = \operatorname{Hom}_P(\mathcal{F}, \mathcal{I})$ and $\operatorname{Hom}_P(-, \mathcal{I})$ is an exact functor ($j : X \hookrightarrow P$ is a closed immersion, so j_* is exact in this case). □

Recall that $0 \to \omega_P \to \mathcal{I}^\bullet$ is an injective resolution of ω_P in $\mathbf{Mod}(P)$.

Claim. Let $\mathcal{J}^i = \mathcal{H}om_P(\mathcal{O}_X, \mathcal{I}^i)$ for all i. Then $0 \to \mathcal{J}^0 \to \mathcal{J}^1 \to \cdots \to \mathcal{J}^r$ is exact.

Proof. By Lemma 5.15 and the corollary we have

$$0 = \mathcal{E}xt^i_P(\mathcal{O}_X, \omega_P) = h^i(\mathcal{H}om_P(\mathcal{O}_X, \mathcal{I}^\bullet)) = h^i(\mathcal{H}om_X(\mathcal{O}_X, \mathcal{J}^\bullet)) = h^i(\mathcal{J}^\bullet)$$

for $i < r$. □

Now let $\mathcal{J}_1^r = \operatorname{im}(d^{r-1} : \mathcal{J}^{r-1} \to \mathcal{J}^r)$. Then $\mathcal{J}_1^r \subseteq \mathcal{J}^r$ and $0 \to \mathcal{J}^0 \to \mathcal{J}^1 \to \cdots \to \mathcal{J}^{r-1} \to \mathcal{J}_1^r \to 0$ is exact.

Fact. Let $0 \to \mathcal{J}' \to \mathcal{J} \to \mathcal{J}'' \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules.

(a) If $\mathcal{J}'$ is injective, then this sequence splits.

(b) If $\mathcal{J}'$ and $\mathcal{J}$ are injective, then so is $\mathcal{J}''$.

By induction and the claim, $\mathcal{J}^i \cong \operatorname{im} d^{i-1} \oplus \operatorname{im} d^i$, and both factors are injective for all $i < r$. Therefore $\mathcal{J}_1^r = \operatorname{im} d^{r-1}$ is injective, and so is $\mathcal{J}_2^r := \mathcal{J}^r/\mathcal{J}_1^r$ and we have $\mathcal{J}^r \cong \mathcal{J}_1^r \oplus \mathcal{J}_2^r$. Now let

$$\mathcal{J}_1^i = \begin{cases} \mathcal{J}^i & \text{if } i < r, \\ \mathcal{J}_1^r & \text{if } i = r, \\ 0 & \text{if } i > r, \end{cases} \quad \text{and} \quad \mathcal{J}_2^i = \begin{cases} 0 & \text{if } i < r, \\ \mathcal{J}_2^r & \text{if } i = r, \\ \mathcal{J}^i & \text{if } i > r. \end{cases}$$

Then we have complexes $\mathcal{J}_1^\bullet, \mathcal{J}_2^\bullet$ with $\mathcal{J}_1^\bullet$ exact, all $\mathcal{J}_1^i$ and $\mathcal{J}_2^i$ are injective, $\mathcal{J}_1^\bullet$ is in degrees $\leq r$, $\mathcal{J}_2^\bullet$ is in degrees $\geq r$, and $\mathcal{J}^\bullet = \mathcal{J}_1^\bullet \oplus \mathcal{J}_2^\bullet$.

Moreover, $\omega_X^\circ = \mathcal{E}xt^r_P(\mathcal{O}_X, \omega_P) = h^r(\mathcal{J}^\bullet) = h^r(\mathcal{J}_2^\bullet) = \ker(\mathcal{J}_2^r \to \mathcal{J}_2^{r+1})$. Also, for any $\mathcal{O}_X$-module $\mathcal{F}$,

$$\begin{aligned} \operatorname{Ext}^r(\mathcal{F}, \omega_P) &= h^r(\operatorname{Hom}_P(\mathcal{F}, \mathcal{I}^\bullet)) = h^r(\operatorname{Hom}_X(\mathcal{F}, \mathcal{J}^\bullet)) \\ &= h^r(\operatorname{Hom}_X(\mathcal{F}, \mathcal{J}_1^\bullet)) \oplus h^r(\operatorname{Hom}_X(\mathcal{F}, \mathcal{J}_2^\bullet)) \end{aligned}$$

Recall that $\mathcal{J}_1^r$ is a direct summand of $\mathcal{J}^{r-1} = \mathcal{J}_1^{r-1}$, so $\operatorname{Hom}_X(\mathcal{F}, \mathcal{J}_1^{r-1}) \to \operatorname{Hom}_X(\mathcal{F}, \mathcal{J}_1^r)$ is surjective and so the first term in the direct sum is 0. For the second term $0 \to \omega_X^\circ \to$

$\mathcal{J}_2^r \to \mathcal{J}_2^{r+1}$ is exact and $\operatorname{Hom}_X(\mathcal{F}, -)$ is left exact, so $\operatorname{Hom}_X(\mathcal{F}, \omega_X^\circ) \cong \ker(\operatorname{Hom}_X(\mathcal{F}, \mathcal{J}_2^r) \to \operatorname{Hom}_X(\mathcal{F}, \mathcal{J}_2^{r+1})) \cong h^r(\operatorname{Hom}_X(\mathcal{F}, \mathcal{J}_2^\bullet))$.

This is all functorial in $\mathcal{F}$. □

Proposition 5.17. *Let X be a nonempty projective scheme over a field k, and let $n = \dim X$. Then X has an n-dualizing sheaf (and therefore a dualizing sheaf).*

Proof. Embed X into $\mathbb{P}_k^N =: P$ with $N > 0$, let $r = N - n = \operatorname{codim}_P X$, and let $\omega_X^\circ = \mathcal{E}xt_P^r(\mathcal{O}_X, \omega_P)$. Then for all coherent $\mathcal{O}_X$-modules $\mathcal{F}$, by the lemma and duality on P we have

$$\operatorname{Hom}_X(\mathcal{F}, \omega_X^\circ) \cong \operatorname{Ext}_P^r(\mathcal{F}, \omega_P) \cong H^{N-r}(P, \mathcal{F})^\vee = H^n(X, \mathcal{F})^\vee$$

contravariantly functorially in $\mathcal{F}$. Therefore ω_X° represents $H^n(X, -)^\vee$. □

Theorem 5.18 (Duality)**.** *Let X, k and n be as above, let ω_X° be an $(n-)$-dualizing sheaf for X, and let $\mathcal{O}(1)$ be a very ample line bundle on X. Then*

(a) For all $i \geq 0, \mathcal{F} \in \mathbf{Coh}(X)$ there are natural functorial maps

$$\theta^i : \operatorname{Ext}_X^i(\mathcal{F}, \omega_X^\circ) \to H^{n-i}(X, \mathcal{F})^\vee$$

such that when $i = 0$, θ^0 is the map in the definition of the n-dualizing sheaf.

(b) TFAE:

(i) X is Cohen-Macaulay and equidimensional;

(ii) For any locally free $\mathcal{F}$ on X, $H^i(X, \mathcal{F}(-q)) = 0$ for all $i < 0$ and $q \gg 0$ (depending on $\mathcal{F}$);

(ii') $H^i(X, \mathcal{O}_X(-q)) = 0$ for all $i < n$ and $q \gg 0$;

(iii) The maps θ^i are isomorphisms for all i and $\mathcal{F}$.

Proof.

(a) Write $\omega = \omega_X^\circ$. Let $\mathcal{O}_X(1)$ be a very ample line bundle on X over k. Given a coherent sheaf $\mathcal{F}$ on X, there is a surjection $\mathcal{E} \to \mathcal{F}$, with $\mathcal{E}$ of the form $\mathcal{O}_X(-q_l)$ where $q_l \gg 0$ for all l. Then

$$\operatorname{Ext}_X^i(\mathcal{E}, \omega) = \bigoplus_l \operatorname{Ext}_X^i(\mathcal{O}_X, \omega(q_l)) = \bigoplus_l H^i(X, \omega(q_l)) = 0$$

for all $i > 0$ and $q_l \gg 0$. Therefore $\operatorname{Ext}_X^\bullet(-, \omega)$ is a coeffaceable contravariant δ-functor $\mathbf{Coh}(X) \to \mathbf{Mod}(k)$, so it is universal. Since $H^{n-\bullet}(X, -)^\vee$ is also a contravariant δ-functor, we get unique morphisms θ^i as above, including the condition on θ^0.

(b) "$(ii) \Rightarrow (ii')$" special case. "$(ii') \Rightarrow (iii)$" Given $\mathcal{F}$ and $\mathcal{E} \to \mathcal{F}$ as above, we have $H^{n-i}(X,\mathcal{E})^\vee \cong \oplus H^{n-i}(X,\mathcal{O}_X(-q_l)) = 0$ for all $i > 0, q_l \gg 0$. So $H^{n-i}(X,-)^\vee$ is also coeffaceable, so in this case the θ^i are all isomorphisms.

"$(iii) \Rightarrow (ii)$" Let $\mathcal{F}$ be locally free. Then

$$\begin{aligned} H^i(X,\mathcal{F}(-q)) &\cong \operatorname{Ext}_X^{n-i}(\mathcal{F}(-q),\omega)^\vee = \operatorname{Ext}_X^{n-i}(\mathcal{O}_X,\omega\otimes\mathcal{F}^\vee(q))^\vee \\ &\cong H^{n-i}(X,\omega\otimes\mathcal{F}^\vee(q))^\vee. \end{aligned}$$

This is 0 for all $q \gg 0$ and $i < n$ as shown earlier.

We will not prove "$(i) \Leftrightarrow (ii)$".

□

Note. if X is regular, then it is Cohen-Macaulay.

Remark. The requires that k be algebraically closed. This isn't necessary - see the statement in parentheses in the proof of "$(i) \Rightarrow (ii)$", and also [Eis95, Exercise 19.3] (which basically says: Assume A is noetherian. Then A is regular iff $A[x_1,\dots,x_n]$ is regular, so $\operatorname{Spec} A$ is regular iff $\mathbb{P}^N_A$ is regular). Also in Bourbaki or Stacks.

Corollary 5.19. *Let X, n and ω_X° be as above, and assume that X satisfies the conditions of part (b) (e.g. X is regular and equidimensional). Then for any locally free coherent sheaf $\mathcal{F}$ on X, there are natural isomorphisms*

$$H^i(X,\mathcal{F}) \xrightarrow{\simeq} H^{n-i}(X,\mathcal{F}^\vee\otimes\omega_X^\circ)^\vee$$

for all $i \geq 0$.

Proof. $H^i(X,\mathcal{F})^\vee \cong \operatorname{Ext}_X^{n-i}(\mathcal{F},\omega) \cong \operatorname{Ext}_X^{n-i}(\mathcal{O}_X,\mathcal{F}^\vee\otimes\omega) \cong H^{n-i}(X,\mathcal{F}^\vee\otimes\omega)$. □

5.6 Computing the Dualizing Sheaf

Definition. *Let A be a ring, and let $f_1,\dots,f_r \in A$, $r \in \mathbb{N}$. Then the* Koszul complex *$K_\bullet(f_1,\dots,f_r)$ of A is defined as follows. Let $K_p(f_1,\dots,f_r) = \bigwedge^p(A^r)$ for all $p \in \mathbb{N}_0$, so $K_p = 0$ for all $p > r$, $K_1 = A^r$ and $K_0 = A$. Let $e_1,\dots,e_r$ be the standard basis of $K_1 = A^r$, so $K_p = \bigoplus_{0<i_1<\dots<i_p\leq r}(e_{i_1}\wedge\dots e_{i_p})A$ for all $p \geq 0$. Also define $d = d_p : K_p \to K_{p-1}$ by*

$$d(e_{i_1}\wedge\cdots\wedge e_{i_p}) = \sum_{j=1}^{p}(-1)^{j-1}f_{i_j}e_{i_1}\wedge\cdots\wedge\widehat{e_{i_j}}\wedge\cdots\wedge e_{i_p}.$$

This is a complex of free A-modules. If M is an A-module, then we define $K_\bullet(f_1,\dots,f_r;M) := K_\bullet(f_1,\dots,f_r)\otimes_A M$.

Definition. *Let A and M be as above. Then a* regular sequence *for M is a sequence $f_1,\dots,f_r \in A$ such that f_i is a nonzerodivisor on $M/(f_1,\dots,f_{i-1})M$ for all i.*

Proposition 5.20. *Let A and M be as above, and let $f_1, \dots, f_r$ be a regular sequence for M. Then*

$$h_i(K_\bullet(f_1, \dots, f_r; M)) = \begin{cases} 0 & i > 0, \\ M/(f_1, \dots, f_r)M & i = 0. \end{cases}.$$

Proof. Omitted. See [Eis95, 17.5]. □

Remark. In the above situation, $K_0 = M$ and $K_1 \to K_0$ has image $(f_1, \dots, f_r)M$, so

$$K_\bullet(f_1, \dots, f_r; M) \to M/(f_1, \dots, f_r)M \to 0$$

is a free resolution of $M/(f_1, \dots, f_r)M$.

Theorem 5.21. *Let k be a field, let $P = \mathbb{P}^N_k$, and let X be a nonempty closed subscheme of P with ideal sheaf $\mathcal{I}$. Assume that X is a locally complete intersection in P of codimension r. Then $\omega_X^\circ \cong \omega_P \otimes \wedge^r(\mathcal{I}/\mathcal{I}^2)^\vee$. In particular, ω_X° is a line bundle on X.*

Proof (sketch). We calculate $\omega_X^\circ = \mathcal{E}xt^r_P(\mathcal{O}_X, \omega_P)$.

Step 1. Compute it locally. Fix a closed point $x \in X$, and let $U = \operatorname{Spec} A$ be an open affine neighborhood of x in P. After shrinking U we may assume $X \cap U = V(f_1, \dots, f_r)$ with $f_1, \dots, f_r \in A$. Let $\mathfrak{m} \subseteq A$ be the maximal ideal corresponding to x. Since $A_\mathfrak{m}$ is a regular local ring (since P is regular), $A_\mathfrak{m}$ is Cohen-Macaulay, so $f_1, \dots, f_r$ is a regular sequence in $A_\mathfrak{m}$ (as a module over itself), see [Har77, Theorem 8.21A (c)]. Therefore $K_\bullet(f_1, \dots, f_r; A_\mathfrak{m})$ is a free resolution of $A_\mathfrak{m}/(f_1, \dots, f_r)A_\mathfrak{m} = \mathcal{O}_{X,x}$. Then, after shrinking U if necessary we may assume that $K_\bullet(f_1, \dots, f_r)$ is a free resolution of $A/(f_1, \dots, f_r)$ over A as A-modules, so it is a free resolution of $\mathcal{O}_X|_U$ over $\mathcal{O}_U$ after applying $\tilde{\ }$. So

$$\mathcal{E}xt^r_P(\mathcal{O}_X, \omega_P)|_U \cong h^r(K_\bullet(f_1, \dots, f_r; \mathcal{O}_P(U))^\sim, \omega_P|_U)$$

$$= \frac{\overbrace{\mathcal{H}om_U(K_\bullet(f_1, \dots, f_r; \mathcal{O}_P(U))^\sim, \omega_P|_U)}^{=\ker d^r \text{ (end of } K_\bullet)}}{\operatorname{im}\left(\mathcal{H}om_U(K_{r-1}(f_1, \dots, f_r; \mathcal{O}_P(U))^\sim, \omega_P|_U) \to \mathcal{H}om_U(K_r(\dots)^\sim, \omega_P(U))\right)}$$

Now $K_{r-1}(f_1, \dots, f_r; \mathcal{O}_P(U))^\sim$ and $K_r(\dots)^\sim$ are free of rank r and 1 respectively. Then $\mathcal{H}om_U(K_{r-1}(f_1, \dots, f_r; \mathcal{O}_P(U))^\sim, \omega_P|_U) = (\omega_P|_U)^r$ and $\mathcal{H}om_U(K_r(\dots)^\sim, \omega_P(U))) = \omega_P|_U$ and the map is given by $\begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix}$. So

$$\mathcal{E}xt^r_P(\mathcal{O}_X, \omega_P) \cong \omega_P|_U/(f_1, \dots, f_r)\omega_P|_U \cong (\omega_P \otimes \mathcal{O}_X)|_U.$$

Step 2. Glue. As the point x and the neighborhood U vary, this isomorphism changes in such a way that if you recast it as

$$\mathcal{E}xt^r_P(\mathcal{O}_X, \omega_P)|_U \cong (\omega_P \otimes \mathcal{O}_X \otimes \wedge^r(\mathcal{I}/\mathcal{I}^2)^\vee)|_U,$$

the isomorphisms (for various U) are compatible on intersections and you get the theorem. This amounts to noting that $K_r(f_1, \dots, f_r; \mathcal{O}_P(U))^{\sim} \otimes \mathcal{O}_X \cong \wedge^r(\mathcal{I}/\mathcal{I}^2)$ canonically.

□

Corollary 5.22. *If X is regular, projective over a field k and nonempty, then $\omega_X^\circ \cong \omega_X$, where $\omega_X = \wedge^{\dim X}\Omega_{X/k}$.*

Proof. Embed X into $\mathbb{P}_k^N =: P$ for some N. Then $\omega_X^\circ \cong \omega_P \otimes \wedge^r(\mathcal{I}/\mathcal{I}^2)^\vee \cong \omega_X$ by the adjunction formula, [Har77, II 8.20]. □

Corollary 5.23. *If X is as above and itnegral, and $n = \dim X$, then $H^n(X, \omega_X) \cong H^0(X, \mathcal{O}_X)^\vee \cong k$.*

Corollary 5.24. *If X is a projective, regular scheme of dimension 1 over k, then $H^1(X, \mathcal{O}_X) \cong H^0(X, \omega_X)^\vee$. In particular,*

$$p_a(X) = h^1(X, \mathcal{O}_X) = h^0(X, \omega_X) = p_g(X).^{1}$$

Now assume X is an integral, projective scheme of dimension 1 over k, possibly singular. Then p_a is still defined as before, and $p_g = p_g(\widetilde{X})$ where $\pi : \widetilde{X} \to X$ is the normalization. If X is singular, then $p_a(X) > p_g(X)$. Moreover, it is known that

$$p_a(X) - p_g(X) = \sum_{x \in X_{\text{sing}}} [k(x) : k]\delta_x,$$

where

$$\delta_x = \dim_{k(x)}(\pi_*\mathcal{O}_{\widetilde{X}}/\mathcal{O}_X)_x = \begin{cases} 0 & \text{if } x \text{ is a regular point,} \\ 1 & \text{if } x \text{ is a node or a simple cusp,} \\ > 1 & \text{otherwise.} \end{cases}$$

Theorem 5.25 (Kodaira Vanishing). *Let X be a nonsingular projective variety over $\mathbb{C}$, let $n = \dim X$, and let $\mathcal{L}$ be an ample line bundle on X. Then*

(a) $H^i(X, \mathcal{L} \otimes \omega_X) = 0$ for all $i > 0$; equivalently

(b) $H^i(X, \mathcal{L}^\vee) = 0$ for all $i < n$.

Proof. Omitted (uses analytic methods). □

Corollary 5.26. *The same is true over any field k of characteristic 0.*

It is false in positive characteristic.

---

[1] p_a, p_g are the arithmetic and geometric genus resp., the first equality holds by [Har77, Exercise III 5.3 (a)] and the last equality is the definition of p_g.

Proof. We will prove (a) using the Lefschetz principle. Let X be a smooth projective variety over an arbitrary field k of characteristic 0, and let $n, \mathcal{L}$ be as above. Let $i : X \hookrightarrow \mathbb{P}^N_k$ be a closed embedding over k for some N. Then there exist:

- a field k_0, finitely generated over $\mathbb{Q}$,
- a scheme X_0 over k,
- a closed embedding $i_0 : X_0 \hookrightarrow \mathbb{P}^N_{k_0}$ over k_0, and
- a line bundle $\mathcal{L}_0$ over X_0,

such that

- $i_0 \times_{k_0} k : X_0 \times_{k_0} X \hookrightarrow \mathbb{P}^N_{k_0} \times_{k_0} k \cong \mathbb{P}^N_k$ is isomorphic over $\mathbb{P}^N_k$ to $i : X \hookrightarrow \mathbb{P}^N_k$ and
- the pull-back of $\mathcal{L}_0$ to X via $X \cong X_0 \times_{k_0} X \to X_0$ is isomorphic to $\mathcal{L}$.

This is because the ideal sheaf of X in $\mathbb{P}^N_k$ and the description of $\mathcal{L}$ via finitely many trivializing open subsets and cocycle conditions involve only finitely many elements of k, so they are all contained in such a field k_0. Also we may assume that X_0 is smooth over k_0.

There also exists an embedding $k_0 \hookrightarrow \mathbb{C}$. Then the Kodaira Vanishing for $\mathcal{L}_\mathbb{C}$ on $X_\mathbb{C} = X_0 \times_{k_0} \mathbb{C}$ implies Kodaira Vanishing for $\mathcal{L}_0$ on X_0. We then get Kodaira Vanishing for $\mathcal{L}$ on X. This holds because for all field extensions k'/k, separated finite type schemes X/k, and quasi-coherent sheaves $\mathcal{F}$ on X, we have

$$H^i(X', \mathcal{F}') \cong H^i(X, \mathcal{F}) \otimes_k k'$$

for all $i \geq 0$, where $X' = X \times_k k'$, $p : X' \to X$ is the projection, and $\mathcal{F}' = p^*\mathcal{F}$. This is true by Flat base change, [Har77, III 9.3], or by computation using Čech cohomology. $\square$

6 Spectral Sequences

We will mostly follow Lang [Lan02], but also Vakil [Vak22]. Note that Vakil interchanges the roles of p and q.

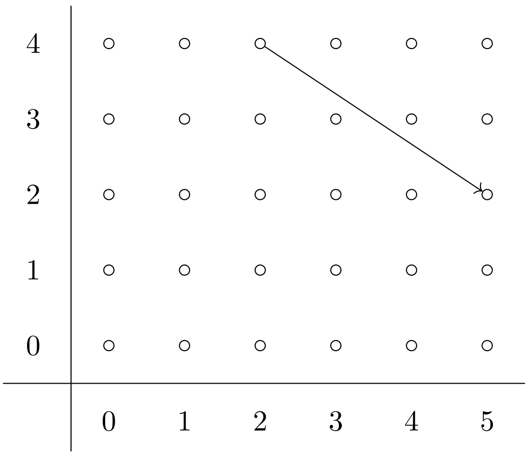
For concreteness, we will work over the category **Ab** of abelian groups, but this theory works over any abelian category. Recall that here $\mathbb{N} = \mathbb{Z}_{\geq 0}$.

Definition. *A* spectral sequence *is a sequence* $\{E_r, d_r\}_{r\geq 0}$ *of bigraded objects* $E_r = \bigoplus_{p,q\in\mathbb{N}} E_r^{p,q}$ *together with homomorphisms, called* differentials*,* $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ *of* bidegree $(r, 1-r)$ *such that*

1. $d_r^2 = 0$*, and*
2. $H(E_r) = E_{r+1}$*, i.e.* $E_r^{p,q} = \frac{\ker\left(d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}\right)}{\operatorname{im}\left(d_r : E_r^{p-r,q+r-1} \to E_r^{p,q}\right)}$ *for all* p, q, r.

In the above, and generally we let $E_r^{p,q} = 0$ for all $r \in \mathbb{N}$ and $p, q \in (\mathbb{Z} \times \mathbb{Z}) \setminus (\mathbb{N} \times \mathbb{N})$.

Here is a typical picture. The arrow drawn is $d_r : E_3^{2,4} \to E_3^{5,2}$.



Note. Antidiagonals play a key role in this theory: If $n = p + q$, then d_r is of degree 1 in n for all r. We will sometimes write $E_r^{n,p}$ to mean $E_r^{p,q}$ with $p + q = n$.

Note. If $r > n + 1$, then $q - r + 1 < 0$ and $p - r < 0$ for all $(p, q) \in \mathbb{N} \times \mathbb{N}$ such that $p + q = n$. Therefore $d_r^{p,q} = d_r^{p-r,q+r-1} = 0$, so $E_{r+1}^{p,q} = E_r^{p,q}$ and so $E_{n+1}^{p,q} = E_{n+2}^{p,q} = \dots$ for all $p, q \in \mathbb{N}$ where $n = p + q$. We call this limiting value $E_\infty^{p,q}$.

Note. d_0 goes this way: $\uparrow$, and d_1 goes this way: $\rightarrow$. So the pair (E_0, d_0) does not determine d_1.

Definition. *A* double complex *is a bigraded object* $K^{\bullet\bullet} = \bigoplus_{p,q\in\mathbb{N}} K^{p,q}$*, together with differentials* $d^{p,q} : K^{p,q} \to K^{p,q+1}$ *and* $\delta^{p,q} : K^{p,q} \to K^{p+1,q}$ *(of bidegrees* $(0,1)$ *and* $(1,0)$ *respectively) such that* $d \circ d = 0, \delta \circ \delta = 0$ *and* $d \circ \delta + \delta \circ d = 0$.

Theorem 6.1. *A double complex $(K = \bigoplus K^{p,q}, d, \delta)$ canonically and functorially determines a spectral sequence $(E_r, d_r)_{r\in\mathbb{N}}$, such that $E_0^{p,q} = K^{p,q}$ for all $p, q \in \mathbb{N}$; $d_0 = d$, d_1 is induced by δ, and $d_2, d_3, \ldots$ are determined in a certain way to be described later.*

Definition. *Let $(K^\bullet; D)$ be a cocomplex of abelian groups. Then a* filtration *of $(K^\bullet; D)$ is an $\mathbb{N}$-graded filtration $K^n = F^0K^n \supseteq F^1K^n \supseteq \ldots$ of K^n for all $n \in \mathbb{N}$ such that $D(F^pK^n) \subseteq F^pK^{n+1}$ for all n, p. We also assume that for all n there exists p_0 (depending on N and $(K^\bullet; D)$) such that $F^pK^n = 0$ for all $p \geq p_0$.*

Definition. *A* filtered complex *is a complex $(K^\bullet; D)$ with a filtration.*

Notation. Given a double complex$K^{\bullet\bullet}$, let $n = p + q$, so $q = n - p$, and we let $K^{n;p} := K^{p,q}$. Then $K^n = \bigoplus_p K^{n;p}$ and $F^pK^n = \bigoplus_{p\geq p_0} K^{n;p}$. Also (after they are defined) $E_r^{n;p} = E_r^{pn-p}$ and d_r maps $E_r^{n;p}$ to $E_r^{n+1;p+r}$. Finally, d_0 is induced by D, in the sense that the diagram

$$\begin{array}{ccc} F^pK^n & \longrightarrow & \frac{F^pK^n}{F^{p+1}K^n} =: E_0^{n;p} \\ \downarrow & & \downarrow d_0 \\ F^pK^{n+1} & \longrightarrow & \frac{F^pK^{n+1}}{F^{p+1}K^{n+1}} =: E_0^{n+1;p} \end{array}$$

So if $(K^\bullet; D)$ comes from a double complex $(K^{\bullet\bullet}, d, \delta)$, then $E_0^{n;p} = K^{n;p}$ and $d_0 = d$

Theorem 6.2. *Let $(K; D)$ be a filtered complex, and assume that $F^pK^n = 0$ for all $p > n$. Also let $F^pK^n = K^n$ for all $n \in \mathbb{N}, p < 0$. For all $r, p, n \in \mathbb{N}$, let*

$$\begin{aligned} X_{-1}^{n;p} &= F^pK^n; \\ X_r^{n;p} &= F^pK^n \cap D^{-1}(F^{p+r}K^{n+1}); \\ Y_r^{n;p} &= D(X_{r-1}^{n-1;p-(r-1)}) + X_{r-1}^{n;p+1}; \\ E_r^{n;p} &= X_r^{n;p}/Y_r^{n;p} \end{aligned}$$

with $X, Y = 0$ if any of the values r, p, n are out of range. Then

(a) $Y_r^{n;p} \subseteq X_r^{n;p}$ (and $E_r^{n;p}$ is actually defined) for all r, n, p,

(b) D induces well-defined maps $d_r = d_r^{n;p} : E_r^{n;p} \to E_r^{n+1;p+r}$ for all r, n, p,

(c) letting $E_r^{p,q} = E_r^{n;p}$ and $d_r^{p,q} = d_r^{n;p}$ for all r, n, p, q with $n = p + q$, $\{E_r; d_r\}_{r\in\mathbb{N}}$ is a spectral sequence, and

(d) $F^{n+1}(H^n(K^\bullet)) = 0$ for all n and $\frac{F^pH^n(K^\bullet)}{F^{p+1}H^n(K^\bullet)} = E_\infty^{n;p}$ for all $n \in \mathbb{N}, p \leq n$.

Proof. See handout. □

Definition. *The* transpose *of a double complex $(K^{\bullet\bullet}, d, \delta)$ is the double complex $(\widetilde{K}^{\bullet\bullet}, \delta, d)$ where $\widetilde{K}^{p,q} = K^{q,p}$ for all p, q. This, too, is a double complex, and the transpose of the transpose is the original double complex.*

Key fact for spectral sequences. If the filtered complexes $(K^\bullet; D)$ and $(\widetilde{K}^\bullet; \widetilde{D})$ are obtained from a double complex and its transpose, respectively, then $K^\bullet = \widetilde{K}^\bullet$ and $D = \widetilde{D}$, therefore $H^n(K^\bullet) = H^n(\widetilde{K}^\bullet)$ for all n, only the filtrations differ.

Now we can re-prove:

Proposition 6.3 (Proposition 5.7). *Let X be a ringed space and let $\mathcal{F}, \mathcal{G} \in \mathbf{Mod}(X)$. Let $\cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$ be a finite-rank locally free resolution of $\mathcal{F}$. Then*

$$\mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \cong h^i(\mathcal{H}om(\mathcal{E}_\bullet, \mathcal{G})) \quad \forall i.$$

Proof. Let $0 \to \mathcal{G} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{G}$, and let $K^{p,q} = \mathcal{H}om(\mathcal{E}_p, \mathcal{I}^q)$ for all $p, q \in \mathbb{N}$ with $d : K^{p,q} \to K^{p,q+1}$ induced by $\mathcal{I}^q \to \mathcal{I}^{q+1}$, and $\delta : K^{p,q} \to K^{p+1,q}$ induced by $\mathcal{E}_{p+1} \to \mathcal{E}_p$ multiplied by $(-1)^q$ for all p, q. Then d and δ anticommute. Therefore $(K^{\bullet\bullet}, d, \delta)$ is a double complex. We have $E_0^{p,q} = \mathcal{H}om(\mathcal{E}_p, \mathcal{I}^q)$, therefore

$$\begin{aligned} E_1^{p,q} &= h^q(\mathcal{H}om(\mathcal{E}_p, \mathcal{I}^\bullet)) = \mathcal{E}xt^q(\mathcal{E}_p, \mathcal{G}) \cong \mathcal{E}xt^q(\mathcal{O}_X, \mathcal{E}_p^\vee \otimes \mathcal{G}) \\ &= \begin{cases} \mathcal{G} \otimes \mathcal{E}_p^\vee & \text{if } q = 0, \\ 0 & \text{if } q > 0. \end{cases} \end{aligned}$$

Here d^1 is induced by $\mathcal{E}_{p+1} \to \mathcal{E}_p$, so

$$E_2^{p,q} = h^p(E_1^{p,q}) = \begin{cases} h^p(\mathcal{H}om(\mathcal{E}, \mathcal{G})) & \text{if } q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now $d_r = 0$ for all $r > 1$ because the arrows have negative slope. Therefore $E_\infty^{p,q} = E_2^{p,q}$. Also, all but one of the subquotients of the filtration of $H^n(K^\bullet)$ are zero (for each n), so $H^n(K^\bullet) \cong E_\infty^{n,0} \cong h^n(\mathcal{H}om(\mathcal{E}_\bullet, \mathcal{G}))$.

Now do the same with the transpose of $(E^{\bullet\bullet}, d, \delta)$. Then $\widetilde{E}_0^{p,q} = E_0^{q,p} = \mathcal{H}om(\mathcal{E}_q, \mathcal{I}^p)$. Since $\mathcal{H}om(-, \mathcal{I}^p)$ is exact, we get

$$\begin{aligned} \widetilde{E}_1^{p,q} &= h^q(\mathcal{H}om(\mathcal{E}_q, \mathcal{I}^p)) = \mathcal{H}om(h^q(\mathcal{E}_\bullet), \mathcal{I}^p) \\ &= \begin{cases} \mathcal{H}om(\mathcal{F}, \mathcal{I}^p) & \text{if } q = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

and

$$\widetilde{E}_2^{p,q} = h^p(\widetilde{E}_1^{p,q}) = \begin{cases} h^p(\mathcal{H}om(\mathcal{F}, \mathcal{I}^\bullet)) & \text{if } q = 0 \\ 0 & \text{otherwise.} \end{cases} = \begin{cases} \mathcal{E}xt^p(\mathcal{F}, \mathcal{G}) & \text{if } q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Again, $\widetilde{E}_\infty^{p,q} = E_2^{p,q}$ for all p, q, and $H^n(\widetilde{K}^\bullet) \cong \widetilde{E}_2^{n,0} \cong \mathcal{E}xt^n(\mathcal{F}, \mathcal{G})$. Therefore

$$\mathcal{E}xt^n(\mathcal{F}, \mathcal{G}) \cong H^n(\widetilde{K}^\bullet) = H^n(K^\bullet) \cong h^n(\mathcal{H}om(\mathcal{E}_\bullet, \mathcal{G})) \quad \forall n.$$

□

Note. The proof does not work with Ext in place of $\mathcal{E}xt$. However it does work with $\mathrm{Ext}_A^i(M, N)$ (because this is the same as $\mathcal{E}xt$ over a one point space).

7 Higher Direct Images

Definition. *Let $f : X \to Y$ be a continuous map of topological spaces. Then the* higher direct images $R^i f_* : \mathbf{Ab}(X) \to \mathbf{Ab}(Y)$ *are the right derived functors of $f_* : \mathbf{Ab}(X) \to \mathbf{Ab}(Y)$.*

Note that f_* is left exact.

Key example. A morphism $f : X \to Y$ of schemes gives a family of schemes $(X_y)_{y \in Y}$, where $X_y = X \times_Y \{y\}$ for all $y \in Y$. Subexamples:

(i) **Moduli spaces.** A (fine) moduli space is a (nice) morphism $f : \mathcal{U} \to \mathcal{M}$ of schemes whose points in $\mathcal{M}$ are in canonical bijection with some iven family of curves, abelian varieties, etc., often with additional data. (or that represents a functor of families of such objects over S.) The bijection is $y \in \mathcal{M} \mapsto \mathcal{U}_y$.

(ii) A variety V over $\mathbb{Q}$ can be extended to a morphism $X \to \operatorname{Spec} \mathbb{Z}$; e.g. if V is $x^n + y^n = z^n \subseteq \mathbb{P}^2_{\mathbb{Q}}$, then X would be $x^n + y^n = z^n$ in $\mathbb{P}^2_{\mathbb{Z}}$. Then rational points in $V(\mathbb{Q})$ are in canonical bijection with sections of $\pi : X \to \operatorname{Spec} \mathbb{Z}$

In both cases, we are interested in how $H^i(X_y, \mathcal{F}_y)$ varies for a sheaf $\mathcal{F}$ on X, as $y \in Y$ varies.

Proposition 7.1. *For each $i \in \mathbb{N}$ and $\mathcal{F} \in \mathbf{Ab}(X)$, $R^i f_* \mathcal{F}$ is the sheafification of the presheaf $V \mapsto H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$ on Y.*

Proof. Since the $R^i f_*$ are derived functors, they form a universal δ-functor. On the other hand, for all i let $\mathcal{H}^i(\mathcal{F}) = (V \mapsto H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)}))^+$. Then $\mathcal{H}^0(\mathcal{F}) = (V \mapsto \Gamma(f^{-1}(V), \mathcal{F}))^+ = (f_*\mathcal{F})^+ = f_*\mathcal{F}$. Moreover, let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence in $\mathbf{Ab}(X)$. Then for all open $V \subseteq Y$ we get a long exact sequence

$$\cdots \to H^i(f^{-1}(V), \mathcal{F}'|_{f^{-1}(V)}) \to H^i(f^{-1}(V), \mathcal{F}_{f^{-1}(V)}) \to \ldots$$

Then the sequence of stalks of the presheaves is exact, so the sequence in $\mathcal{H}^i$ is exact. Therefore $(\mathcal{H}^i(-))$ is a δ-functor. It is effaceable because $\mathbf{Ab}(X)$ has enough injectives: If $\mathcal{I}$ is injective in $\mathbf{Ab}(X)$, then $\mathcal{I}|_{f^{-1}(V)}$ is injective for all open $V \subseteq Y$, so $H^i(f^{-1}(V), \mathcal{I}|_{f^{-1}(V)}) = 0$ for all $i > 0$, therefore $\mathcal{H}^i(\mathcal{I}) = 0$ for all $i > 0$. So $(\mathcal{H}^i(-))$ is effaceable, hence universal, so get our isomorphism. $\square$

Corollary 7.2. *Let $V \subseteq Y$ be an open subset, and let $f' : f^{-1}(V) \to V$ be the restriction of f. Then $R^i f_* \mathcal{F}|_V \cong R^i f'_*(\mathcal{F}|_{f^{-1}(V)})$.*

Corollary 7.3. *If the sheaf $\mathcal{F}$ is flasque, then $R^i f_* \mathcal{F} = 0$.*

Proof. Since $\mathcal{F}$ is flasque, so is $\mathcal{F}_U$ for all open $U \subseteq X$, so $\mathcal{H}^i(\mathcal{F}) = 0$. $\square$

Proposition 7.4 (Generalization of Proposition 1.20)**.** *Let $f : X \to Y$ be a morphism of ringed spaces. Then the higher direct images $R^i f_*$ can be computed on $\mathbf{Mod}(X)$ as the right derived functors of $f_* : \mathbf{Mod}(X) \to \mathbf{Mod}(Y)$ (using injectives or flasques in $\mathbf{Mod}(X)$).*

Proof. Same as for Proposition 1.20. □

Note. If k is a field and $f : X \to Y = \operatorname{Spec} k$ is a morphism of schemes, then $f_*\mathcal{F} = \Gamma(X, \mathcal{F})^\sim$, and $R^i f_*\mathcal{F} \cong H^i(X, \mathcal{F})^\sim$ for all i.

More generally:

Proposition 7.5. *Let $f : X \to Y$ be a morphism of schemes, where X is noetherian and $Y = \operatorname{Spec} A$ is affine (but not assumed to be noetherian). Then, for any quasi-coherent sheaf $\mathcal{F}$ on X, $R^i f_*\mathcal{F} \cong H^i(X, \mathcal{F})^\sim$.*

Proof. Think of both sides as functors from $\mathbf{QCoh}(X)$ to $\mathbf{Mod}(Y)$. We will use induction on i. When $i = 0$, both sides agree, since $f_*\mathcal{F}$ is quasi-coherent on Y. For the inductive step assume $i > 0$ and that $R^{i-1} f_*\mathcal{F} \cong H^{i-1}(X, \mathcal{F})^\sim$. Embed $\mathcal{F}$ into a quasi-coherent, flasque sheaf $\mathcal{G}$ on X. Then $R^j f_*\mathcal{G} = 0$ for all $j > 0$. Let $\mathcal{R} = \mathcal{G}/\mathcal{F}$, so $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{R} \to 0$ is a short exact sequence in $\mathbf{QCoh}(X)$. Then when $i = 1$ we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & f_*\mathcal{F} & \longrightarrow & f_*\mathcal{G} & \longrightarrow & f_*\mathcal{R} & \longrightarrow & R^1 f_*\mathcal{F} & \longrightarrow & R^1 f_*\mathcal{G} = 0 \\
 & & \downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\alpha} & & \\
0 & \longrightarrow & H^0(X, \mathcal{F})^\sim & \longrightarrow & H^0(X, \mathcal{G})^\sim & \longrightarrow & H^0(X, \mathcal{R})^\sim & \longrightarrow & H^1(X, \mathcal{F})^\sim & \to & H^0(X, \mathcal{G})^\sim = 0
\end{array}$$

with exact rows. Therefore there exists an isomorphism $\alpha : R^1 f_*\mathcal{F} \overset{\simeq}{\hookrightarrow} H^1(X, \mathcal{F})^\sim$ by uniqueness of the cokernel. The proof for $i > 1$ is similar but easier. □

This globalizes to:

Corollary 7.6. *Let $f : X \to Y$ be a morphism of schemes, with X noetherian. Then $R^i f_*\mathcal{F}$ is quasi-coherent on Y for all quasi-coherent sheaves $\mathcal{F}$ on X.*

Theorem 7.7. *Let $f : X \to Y$ be a projective morphism of noetherian schemes, let $\mathcal{O}_X(1)$ be a very ample line bundle on X over Y, and let $\mathcal{F}$ be a line bundle on X. Then*

(a) $R^i f_*\mathcal{F}$ is a coherent sheaf on Y for all $i \geq 0$.

(b) The natural map $f^* f_*(\mathcal{F}(n)) \to \mathcal{F}(n)$ is surjective for all $n \gg 0$.

(c) $R^i f_*(\mathcal{F}(n)) = 0$ for all $i > 0$, $n \gg 0$.

Proof. Since Y is quasi-compact and the question is local on Y, we may assume that Y is affine, say $Y = \operatorname{Spec} A$. Then A is noetherian. Parts (a) and (c) follow from Theorem 3.11 (a) and (b) respectively.

For (b) Let $M_n = H^0(X, \mathcal{F}(n))$. Then $f_*(\mathcal{F}(n)) = \widetilde{M}_n$ (on Y) for all $n \in \mathbb{Z}$. For all $n \gg 0$, $\mathcal{F}(n)$ is generated by global sections; for such n, the image of $M_n \to \mathcal{F}(n)_x$ generates $\mathcal{F}(n)_x$ for all $x \in X$. On open affines $U = \operatorname{Spec} B$ in X, $(f^* f_*(\mathcal{F}(n)))|_U = (f^* \widetilde{M}_n)|_U = (M_n \otimes_A B)\tilde{}$. Since the images of $M_n \to \mathcal{F}(n)_x$ generate $\mathcal{F}(n)_x$ for all $x \in U$, so does the image of $M_n \otimes_A B \to \mathcal{F}(n)_x = (M_n \otimes_A B)_{\mathfrak{p}}$ where $\mathfrak{p} \in \operatorname{Spec} B$ equals x $\square$

Additional comments on $R^i f_*$

1. To compare $R^i f_* \mathcal{F} \otimes k(y)$ with $H^i(X_y, \mathcal{F}_y)$ (with $y \in Y$), see [Har77, Theorem 12.8] and [Har77, Corollary 12.9].
2. For duality, there is a theory, but it is complicated.

8 Flatness

For the rest of this section, M and N are modules over a ring A.

Recall that the functor $M \otimes_A -$ from $\mathbf{Mod}(A)$ to $\mathbf{Mod}(A)$ is right exact and additive. Therefore it has left derived functors $L_i(M \otimes_A -)$ for all $i \geq 0$, denoted by $\mathrm{Tor}_i^A(M, -)$.

Recall:

Proposition 8.1. $\mathrm{Tor}_i^A(M,N) \cong \mathrm{Tor}_i^A(N,M)$ *for all* $i \geq 0$.

Proposition 8.2. *TFAE:*

(i) $M \otimes_A -$ *is an exact functor (i.e.* M *is* flat*);*

(ii) $\mathfrak{a} \otimes_A M \to M$ *is injective for all finitely generated ideals* $\mathfrak{a}$ *of* A*;*

(iii) $\mathrm{Tor}_1(M,N) = 0$ *for all* A*-modules* N*;*

(iv) $\mathrm{Tor}_i(M,N) = 0$ *for all* A*-modules* N*,* $i > 0$*.*

Definition. *Let* $f : X \to Y$ *be a morphism of schemes, and let* $\mathcal{F}$ *be an* $\mathcal{O}_X$*-module. Then*

1. $\mathcal{F}$ *is* flat over Y at $x \in X$ *if* $\mathcal{F}_x$ *is a flat* $\mathcal{O}_{Y,f(x)}$*-module (via* $f^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$*).*
2. $\mathcal{F}$ *is* flat over Y *if it is flat over* Y *at* x *for all* $x \in X$*.*
3. X *is* flat over Y *if* $\mathcal{O}_X$ *is flat over* Y*.*
4. f *is* flat *if* X *is flat over* Y*.*
5. X *is* flat over Y at x*, or* f is flat at x*, if* $\mathcal{O}_x$ *is flat over* Y *at* x*.*

The following follows from the corresponding commutative algebra facts.

Proposition 8.3.

(a) An open embedding is flat.

(b) Flatness is preserved by base change.

(c) A composition of flat morphisms is flat.

(d) Let $f : X \to Y$ *be a morphism, where* $X = \operatorname{Spec} B$ *and* $Y = \operatorname{Spec} A$ *are affine. Let* M *be a* B*-module. Then the sheaf* $\widetilde{M}$ *on* X *is flat over* Y *iff* M *is flat as an* A*-module.*

(e) Let $\mathcal{F}$ *be a coherent sheaf on a noetherian scheme* X*. Then* $\mathcal{F}$ *is flat over* X *iff* $\mathcal{F}$ *is locally free.*

(f) A product of flat morphisms is flat.

Some examples.

(1) $\operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}$ is not flat. More generally closed embeddings are usually not flat.

(2) Blowing up is usually not flat (e.g. the closure of the graph of $\frac{x}{y} : \mathbb{A}^2 \to \mathbb{P}^1$ is not flat). This follows from Proposition 8.5

(3) A free A-module is flat.

Theorem 8.4. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{u} & Y \end{array}$$

be a cartesian diagram of noetherian schemes, where f is separated of finite type. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then there are natural maps $u^*(R^i f_* \mathcal{F}) \to R^i f'_*(v^* \mathcal{F})$ of sheaves on Y' for all $i \geq 0$. Moreover, if u is flat, then these maps are isomorphisms.

Proof. The question is local on Y and Y', so by naturality we may assume that Y and Y' are affine, equal to $\operatorname{Spec} A$ and $\operatorname{Spec} A'$, respectively. Then A and A' are noetherian. Then also $R^i f_* \mathcal{F} = H^i(X, \mathcal{F})^{\sim}$, so $u^* R^i f_* \mathcal{F} = (H^i(X, \mathcal{F}) \otimes_A A')^{\sim}$. Also $R^i f'_*(v^* \mathcal{F}) = H^i(X', v^* \mathcal{F})^{\sim}$. So finding a map as above is equivalent to finding a natural map $H^i(X, \mathcal{F}) \otimes_A A' \to H^i(X', v^* \mathcal{F})$ of A'-modules. Since X, X' are noetherian, we can use Čech cohomology: Let $\mathcal{U}$ be an open affine cover of X, and let $\mathcal{U}'$ be the pull-back to X' (which is again an open affine cover of X' since the diagram is cartesian). So we want to construct $\check{H}^i(\mathcal{U}, \mathcal{F}) \otimes_A A' \to \check{H}^i(\mathcal{U}', v^* \mathcal{F})$; i.e.

$$h^i(C^\bullet(\mathcal{U}, \mathcal{F})) \otimes_A A' \to h^i(C^\bullet(\mathcal{U}', v^* \mathcal{F})) = h^i(C^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A A').$$

Define a map $C^\bullet(\mathcal{U}, \mathcal{F}) \to C^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A A'$ by $x \mapsto x \otimes 1$. This is a map of complexes of A-modules. It gives maps $h^i(C^\bullet(\mathcal{U}, \mathcal{F})) \to h^i(C^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A A')$ of A-modules for all i which in turn gives maps $h^i(C^\bullet(\mathcal{U}, \mathcal{F}))' \otimes_A A' \to h^i(C^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A A')$ of A'-modules. This is what we wanted. Moreover, if A' is flat over A, then this map is an isomorphism, which proves the second part. □

8.1 Dimension of Base and Fiber

Recall. If X is a scheme and $x \in X$, then $\dim_x X = \dim \mathcal{O}_{X,x} = \operatorname{codim}_X \overline{\{x\}}$.

Proposition 8.5. *Let k be a field, let X and Y be schemes of finite type over k, and let $f : X \to Y$ be a flat morphism. Let $x \in X$ and $y = f(x) \in Y$ be points. Then*

$$\dim_x X_y = \dim_x X - \dim_y Y.$$

Proof.

Step 1. Reduce to the case in which y is a closed point in Y, is the only closed point in Y, and $\dim_y Y = \dim Y$. Indeed, let $Y' = \operatorname{Spec} \mathcal{O}_{Y,y}$, and do a base change to Y'. Then $X' := X \times_Y Y'$ is flat over Y' and there exists $x' \in X'$ mapping to $x \in X$. For the latter, note that $x \in X_y$, which is unchanged by the base change. Also $\dim_y Y$ is unchanged, $\dim_x X_y$ is unchanged, and $\dim_x X = \dim_{x'} X'$. We need to relax our assumptions: X and Y are essentially of finite type over k, i.e. covered by open affines which are localizations of k-algebras of finite type.

Step 2. We may assume that Y is reduced. Indeed, base change via $Y_{\text{red}} \to Y$. This does not change the topological spaces of X and Y, and X_y is unchanged.

Step 3. The main step. We induct on $\dim Y = \dim_y Y$. If $\dim Y = 0$, then $Y = \operatorname{Spec} E$, where E is a field, finite over k. Also $X_y = X$. Therefore $\dim_X X_y = \dim_x X$ and $\dim_y Y = 0$. For the inductive step assume $\dim Y > 0$. Pick a non-zero divisor $t \in \mathfrak{m}_y$ (exists by [Eis95, 3.1b and 3.2]). Let $Y' = \operatorname{Spec} \mathcal{O}_{Y,y}/(t)$ and do base change by the closed embedding $Y' \hookrightarrow Y$. We have $\dim Y' = \dim Y - 1$ by Krull's Hauptidealsatz. By flatness $f^* t \in \mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ is also a non-zero divisor. Therefore $\dim_x X' = \dim_x X - 1$. Also $\dim_x X_y$ does not change, so we conclude by induction.

□

Note. We only used flatness to get $\dim_x X' = \dim_x X - 1$. Without flatness, we could have $\dim_x X' = \dim_x X$, and we would still get

$$\dim_x X_y \geq \dim_x X - \dim_y Y.$$

8.2 Flatness and Hilbert Polynomials

Recall Proposition 3.20. Let $\mathcal{F}$ be a coherent sheaf on a projective scheme X over a field k, and let $\mathcal{O}_X(1)$ be a very ample line bundle on X over k. Then there is a unique polynomial $P \in \mathbb{Q}[z]$ such that $P(n) = \chi(\mathcal{F}(n))$ for all $n \in \mathbb{Z}$. By Theorem 3.11, $P(n) = \dim_k H^0(X, \mathcal{F}(n))$ for all $n \gg 0$.

Theorem 8.6. *Let T be an integral noetherian scheme, let $f : X \to T$ be a projective morphism, and let $\mathcal{O}_X(1)$ be a very ample line bundle on X over T. For each $t \in T$, let $P_t \in \mathbb{Q}[z]$ be the Hilbert polynomial of $\mathcal{O}_{X_t}$ on the fiber X_t of f at t, relative to $\mathcal{O}(1)|_{X_t}$, and $k(t)$ (the residue field of $\mathcal{O}_{T,t}$). Then X is flat over T iff P_t is independent of t.*

Proof. Omitted (ran out of time). □

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