SPHERICAL REPRESENTATIONS OF p-ADIC GROUPS

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Contents

Notes still in progress.

1. Preliminaries

Let G be a locally compact totally disconnected group. The Hecke algebra $C_c^\infty(G)$ is denoted $\mathcal{H} := \mathcal{H}(G)$, and if $K \subseteq G$ is a compact open subgroup, we denote by $\mathcal{H}_K := \mathcal{H}(G, K) := C_c^\infty(K \backslash G / K)$ the spherical Hecke algebra, the subalgebra consisting of the compactly supported locally constant K-biinvariant functions.

Definition. *A smooth representation (V, π) of G is* spherical *(or* unramified*) with respect to K if V is irreducible and $V^K \neq 0$. A non-zero vector in V^K is called* spherical.

Proposition 1 ([CKD73, p. 33])**.** *Let $(V_1, \pi_1), (V_2, \pi_2)$ be smooth representations of G and assume that V_1^K generates V_1 over G and V_2^K cogenerates V_2 over G. The latter condition means that every G-submodule of V_2 intersects V_2^K non-trivially. Then the natural map*

$$\mathrm{Hom}_{\mathcal{H}}(V_1, V_2) \to \mathrm{Hom}_{\mathcal{H}_K}(V_1^K, V_2^K)$$

is an isomorphism.

Note the condition of the proposition is satisfied if for example both V_1, V_2 are irreducible.

Corollary 2. *Let $(V_1, \pi_1), (V_2, \pi_2)$ be spherical irreducible admissible representations of G. Then $V_1 \cong V_2$ if and only if $V_1^K \cong V_2^K$ as $\mathcal{H}_K$-modules.*

Proposition 3. *If M is a simple $\mathcal{H}_K$ module, then there is an irreducible smooth representation of (V, π) of G such that $V^K \cong M$.*

Let $\mathbf{G}$ be a reductive group over a nonarchimedean local field F. We will assume that $\mathbf{G}$ is *split*, meaning that $\mathbf{G}$ admits a maximal torus that splits over F. Let $\mathbf{T}$ denote such a split maximal torus, $\mathbf{B}$ a Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$ and $\mathbf{N}$ the unipotent radical of $\mathbf{B}$. We will usually denote algebraic groups by boldface letters $\mathbf{G}, \mathbf{B}, \dots$ and their topological groups of F-valued points by normal letters $G = \mathbf{G}(F), B = \mathbf{B}(F), \dots$. Assume that K is a compact open subgroup of G such that $G = BK$, $B \cap K = (T \cap K)(N \cap K)$ and that $K_T := K \cap T$ is a maximal open subgroup of T.

The group of characters (resp. cocharacters) of T is denoted by $X^*(T)$ (resp. $X_*(T)$). There is a surjective map $\mathrm{ord} : T \to X_*(T)$ satisfying $\langle \mathrm{ord}(t), \lambda \rangle = \mathrm{ord}(\lambda(t))$ for $t \in T, \lambda \in X^*(T)$. The kernel $\ker \mathrm{ord}$ is the largest compact subgroup of T, hence $\ker \mathrm{ord} = K_T$. The set of roots of (G, T) is denoted by Φ and Φ^+ denotes the subset of positive roots corresponding to the Borel subgroup B. The Weyl group is $W = N(T)/T$.

The Haar measures on G, T will be normalized such that K, K_T have measure 1.

Lemma 4. *$\mathcal{H}_K$ is commutative, hence any finite-dimensional simple module of $\mathcal{H}_K$ is one-dimensional and given by a character $\mathcal{H}_K \to \mathbb{C}$.*

Proof. This follows from Theorem 7. □

Theorem 5. *The spherical representations of G are in bijection with $\mathrm{Hom}(\mathcal{H}_K, \mathbb{C})$, the set of characters of $\mathcal{H}_K$.*

2. The Satake Isomorphism

Let $(\widehat{G}, \widehat{T})$ be the complex dual group to $\mathbf{G}$. Let $\widehat{W} = N(\widehat{T})/\widehat{T}$ be its Weyl group. We have canonical isomorphisms (recall that we assumed that $\mathbf{T}$ is split)

$$\mathcal{H}(T, K_T) \cong \mathbb{C}[X_*(\mathbf{T})] \cong \mathbb{C}[X^*(\widehat{T})] \cong \mathcal{O}(\widehat{S}).$$

These isomorphisms respect the action of W and $\widehat{W}$, so they induce isomorphisms on the respective invariant rings.

Let $f \in \mathcal{H}_K$. The *constant term* of f along B is a function $f^B : T \to \mathbb{C}$ defined by

$$f^B(t) := \delta^{1/2}(t) \int_N f(tn)\,\mathrm{d}n.$$

Here $\delta : B \to \mathbb{R}_{>0}$ is the modular function of B. Note that $f^B \in \mathcal{H}(T, K_T)$. Indeed, left invariance under K_T is clear from the formula and right invariance follows from the equivalent description $f^B(t) = \delta^{-1/2}(t) \int_N f(nt)\,\mathrm{d}n$. We will also denote f^B by $\mathcal{S}f$. This defines an algebra homomorphism

$$\begin{aligned} \mathcal{H}(G, K) &\longrightarrow \mathcal{H}(T, K_T), \\ f &\longmapsto \mathcal{S}f. \end{aligned}$$

Lemma 6 ([Sat63, Lemma 4.3])**.** *Let $f \in \mathcal{H}(G, K)$ and $t \in T$. Define*

$$D(t) := \left|\det\left(1 - \mathrm{Ad}(t) : \mathfrak{g}/\mathfrak{t} \to \mathfrak{g}/\mathfrak{t}\right)\right|^{1/2}.$$

If $D(t) \neq 0$, then

$$\mathcal{S}f(t) = D(t) \int_{G/T} f(gtg^{-1})\,\mathrm{d}g \qquad (*)$$

Here the measure $\mathrm{d}g$ is a measure on G/T that is invariant for the multiplication action of G and compatible with the measures on G and T. The integral on the right is called an *orbital integral.*

If C_t^0 denotes the centralizer of t in G, then $D(t) \neq 0$ if and only if $C_t^0 = T$ and these elements are dense in T [GH, Proposition 8.7.3]. E.g. if $\mathbf{G} = \mathrm{GL}_n$, then the elements $t \in T$ with $D(t) \neq 0$ are precisely those with distinct eigenvalues. The expression on the right of $(*)$ is easily seen to be invariant under elements in $N(T)$, hence the image of the homomorphism $\mathcal{S} : \mathcal{H}(G,K) \to \mathcal{H}(T,K_T)$ is contained in $\mathcal{H}(T,K_T)^W$. In fact:

Theorem 7 (Satake isomorphism [Gro98, Proposition 6.3], [Car79, Theorem 4.1])**.** *$\mathcal{S}$ induces an isomorphism*

$$\mathcal{S} : \mathcal{H}(G,K) \xrightarrow{\simeq} \mathcal{H}(T,K_T)^W \simeq \mathbb{C}[X_*(\mathbf{T})]^W \simeq \mathcal{O}(\widehat{T})^{\widehat{W}}.$$

The use of the dual group $\widehat{G}, \widehat{T}$ is useful for the generalization from the split to the unramified case.

This allows us to describe the homomorphisms $\mathcal{H}(G,K) \to \mathbb{C}$ and hence the spherical representations of G by Theorem 5. The map $\mathrm{Hom}(\mathcal{H}(T,K_T),\mathbb{C}) \to \mathrm{Hom}(T/K_T,\mathbb{C}^\times)$ given by $\varphi \mapsto (t \mapsto \varphi(\mathbb{1}_{tK_T}))$ is a bijection. The inverse is given by

$$\chi \longmapsto \big(\varphi_\chi : f \mapsto \int_T f(t)\chi(t)\,\mathrm{d}t\big).$$

We call a character $\chi : T \to \mathbb{C}^\times$ (not assumed to be unitary) *unramified* if it is trivial on K_T. Hence unramified characters correspond to characters of the algebra $\mathcal{H}(T,K_T)$. What do the characters of $\mathcal{H}(T,K_T)^W$ look like? Let $\chi : T \to \mathbb{C}$ be an unramified character, so we get a character φ_χ as above which we can restrict to $\mathcal{H}(T,K_T)^W$ Conversely, given a character $\widetilde{\varphi}$ of $\mathcal{H}(T,K_T)^W$, by commutative algebra[1] we can extend $\widetilde{\varphi}$ to a character φ of $\mathcal{H}(T,K_T)$. Hence we get an unramified character χ of T such that $\widetilde{\varphi} = \varphi_\chi|_{\mathcal{H}(T,K_T)^W}$. Furthermore, two unramified characters χ, χ' restrict to the same character of $\mathcal{H}(T,K_T)^W$ if and only if $\chi' = w\chi$ for some $w \in W$.[2] Here $(w\chi)(t) := \chi(x^{-1}tx)$ where $x \in N(T)$ is a representative of w.

By pulling back along $\mathcal{S}$ we can thus describe all the characters of $\mathcal{H}(G,K)$. Given an unramified character χ of T, define $\xi_\chi : \mathcal{H}(K,G) \to \mathbb{C}$ by

$$\xi_\chi(f) = \varphi_\chi(\mathcal{S}f) = \int_T \mathcal{S}f(t)\chi(t)\,\mathrm{d}t.$$

The discussion above gives the following:

Theorem 8 ([Car79, Corollary 4.2])**.** *All characters of $\mathcal{H}(G,K)$ are of the form ξ_χ for some unramified character χ of T. If χ, χ' are unramified characters of T, then $\xi_\chi = \xi_{\chi'}$ if and only if $\chi' = w\chi$ for some $w \in W$.*

Definition. *A character χ of T is called* regular *if $w\chi \neq \chi$ for all $w \in W$.*

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[1] I remember having to prove this on an exam, see 2d here.

[2] If A is a commutative ring and G a finite group acting on A, then G acts transitively on every fiber of $\operatorname{Spec} A \to \operatorname{Spec} A^G$.

3. Spherical Representations via Principal Series

Recall that one way to get representations of G is via parabolic induction, i.e. we start with a parabolic subgroup P with Levi decomposition $P = MN$ and a representation (U, σ) of M. Then we view σ as a representation of P by inflation and then induce to get a representation of G. We do this in our setting with $P = B$ to get the unramified principal series. Let $\chi : T \to \mathbb{C}^\times$ be a character which we view as a one-dimensional representation of T.

Definition. *The* principal series representation *corresponding to* χ *is*

$$I(\chi) := \operatorname{Ind}_B^G \chi$$

where we view χ *as a character of* B *via* $B \to B/N = T$*.*

Here Ind is the normalized induction, meaning that $\operatorname{Ind}_B^G \chi$ is the space of functions $f : G \to \mathbb{C}$ that are right invariant under some compact open subgroup of G and satisfy

$$f(bg) = \delta^{1/2}(b)\chi(b)f(g) \tag{$*$}$$

for $b \in B, g \in G$. G acts on this space by right translation. By general results on induced representations we have:

- $I(\chi)$ is admissible,
- $\widehat{I(\chi)} \cong I(\chi^{-1})$, where $\widehat{I(\chi)}$ denotes the contragredient of $I(\chi)$,
- $I(\chi)$ is unitarizable if χ is unitary.

By the Iwasawa decomposition $G = BK$, a function $f \in I(\chi)$ is uniquely determined by its restriction to K. Conversely, if $g : K \to \mathbb{C}$ is smooth and satisfies $g(bk) = \chi(b)g(k)$ for $b \in B \cap K, k \in K$, then it is the restriction of a unique element in $I(\chi)$.

Furthermore, note that $B = TN$ and both δ and χ are trivial on N, hence $(*)$ can be rewritten as $f(tng) = \delta^{1/2}(t)\chi(t)f(g)$ for all $t \in T, n \in N, g \in G$.

Proposition 9. $I(\chi)$ *contains spherical vectors if and only if* χ *is unramified, in which case* $\dim I(\chi)^K = 1$*.*

Note that we don't know yet whether $I(\chi)$ is irreducible (and indeed that need not be the case), so according to our definition this does say that $I(\chi)$ is spherical.

Proof. Clearly $0 \neq f \in I(\chi)$ is spherical if and only if f is constant on K. This satisfies $(*)$ precisely when $\chi|_{K_T} = 1$, i.e. when χ is unramified. If that is the case, then $\Phi_{K,\chi}$ defined by $\Phi_{K,\phi}(tnk) := \chi(t)\delta^{1/2}(t)$ is the unique function (up to scaling) satisfying that condition. □

Proposition 10 ([GH, Lemma 7.6.6])**.** $I(\chi)$ *admits a a unique spherical subquotient which we will denote by* $J(\chi)$*.*

Proof. It is a fact that $I(\chi)$ has finite length. Then the claim follows easily from $\dim I(\chi)^K = 1$ (and the fact that $V \mapsto V^K$ is an exact functor). □

What does the character ξ of $J(\chi)$ look like? We can determine this from $I(\chi)$. Indeed, we must have $\pi(f)\Phi_{K,\chi} = \xi(f)\Phi_{K,\chi}$ for $f \in \mathcal{H}(G,K)$. To determine ξ is suffices to evaluate both sides at 1. The calculation in (†) in 4 shows that $\xi(f) = \int_T \mathcal{S}f(t)\chi(t)\,\mathrm{d}t = \xi_\chi(f)$. Hence $J(\chi)$ is the spherical representation associated to the character χ by Theorem 8 and Theorem 5.

It is useful to know how the Jacquet module of $I(\chi)$ looks like. Recall that the Jacquet module of a smooth representation (V,π) of G with respect to N is given by $V_N := V/V(N)$ where $V(N)$ is the subspace generated by elements of the form $v - \pi(n)v$, $n \in N, v \in V$. It is a T-module.

Let χ be an unramified character of T.

Theorem 11 ([Car79, Theorem 3.5])**.** *The semisimplification of $I(\chi)_N$ is isomorphic to $\bigoplus_{w\in W} w\chi \otimes \delta^{1/2}$. In particular, if χ is regular, then $I(\chi)_N \cong \bigoplus_{w\in W} w\chi\otimes\delta^{1/2}$.*

Corollary 12. *Let $w \in W$. Then there is a non-zero intertwining operator $I(\chi) \to I(w\chi)$. If χ is regular, then this operator is unique up to scalars. If $I(\chi)$ is irreducible, then $I(\chi) \cong I(w\chi)$.*

Proof. By Frobenius reciprocity we have

$$\mathrm{Hom}_G(I(\chi), I(w\chi)) = \mathrm{Hom}_B(I(\chi), w\chi\otimes\delta^{1/2}) = \mathrm{Hom}_T(I(\chi)_N, w\chi\otimes\delta^{1/2}),$$

and the result follows from this. □

Taking $w = 1$ gives $\mathrm{End}_G(I(\chi)) = \mathbb{C}$ if χ is regular. This implies:

Corollary 13. *Assume χ is unitary and regular. Then $I(\chi)$ is irreducible.*

Assume χ is regular and $w \in W$. Then by Corollary 12 there is a non-zero functional $\Lambda_w : I(\chi) \to \mathbb{C}$ satisfying $\Lambda_w(\pi(tn)f) = \delta^{1/2}(t)(w\chi)(t)\Lambda_w(f)$ for $t \in T, n \in N$ and it is unique up to scalar. Then the associated intertwining operator $T_w^\chi : I(\chi) \to I(w\chi)$ is given by $T_w^\chi(f)(g) = \Lambda_w(\pi(g)f)$. We can normalize T_w^χ such that

$$T_w^\chi(f)(1) = \int_{wNw^{-1}\cap N\backslash N} f(w^{-1}n)\,\mathrm{d}n$$

for functions $f \in I(\chi)$ supported in $Bw^{-1}B$. Here the measure is normalized such that the orbit of 1 under $N\cap K$ has measure 1 [Cas80, p. 397]. Note that T_w^χ must take $I(\chi)^K$ to $I(w\chi)^K$, hence there is a scalar $c_w(\chi)$ such that $T_w^\chi(\Phi_{K,\chi}) = c_w(\chi)\Phi_{K,w\chi}$. The composition $T_{w^{-1}}^{w\chi}T_w^\chi$ is an intertwining operator $I(\chi) \to I(\chi)$, and thus a scalar multiple of the identity. This scalar must be $c_{w^{-1}}(w\chi)c_w(\chi)$.

Given a root $\alpha \in \Phi(G,T) \subseteq X^*(T)$, denote by $\alpha^\vee \in X_*(T)$ its coroot. Let $a_\alpha \in T$ be any element such that $\mathrm{ord}(a_\alpha) = \alpha^\vee$ under the map $\mathrm{ord} : T \to X_*(T)$. Define

$$c_\alpha(\chi) := \frac{1 - q^{-1}\chi(a_\alpha)}{1-\chi(a_\alpha)}.$$

Theorem 14 ([Cas80, Theorem 3.1])**.** *$c_w(\chi) = \prod_\alpha c_\alpha(\chi)$ where the product runs over the positive roots $\alpha \in \Phi^+$ such that $w\alpha \in \Phi^-$.*

Theorem 15 ([Car79, Theorem 3.10], [Cas80, Proposition 3.5])**.** *The representation $I(\chi)$ is irreducible if and only if $c_w(\chi) \neq 0, c_{w^{-1}}(w\chi) \neq 0$ for the $w \in W$ that takes Φ^+ to Φ^-.*

Let us spell this out in the case $\mathbf{G} = \mathrm{GL}_2$ where we take $\mathbf{B}$ to be the standard Borel subgroup, the group of upper triangular matrices, and $\mathbf{T}$ the diagonal matrices. Under the usual identification $X^*(T) \cong \mathbb{Z}^2$, corresponding to our choice of $\mathbf{B}$ we have $\Phi^+ = \{\alpha\}$ with $\alpha = e_1 - e_2$. Then $\alpha^\vee = e_1^\vee - e_2^\vee$. We may choose $a_\alpha = \begin{pmatrix} \varpi & 0 \\ 0 & \varpi^{-1} \end{pmatrix}$. Let χ be a regular unramified character of $T = F^\times \oplus F^\times$. It is of the form $\chi_1 \otimes \chi_2$ with characters χ_1, χ_2 of $F^\times$. Then

$$c_\alpha(\chi) = \frac{1 - q^{-1}(\chi_1\chi_2^{-1})(\varpi)}{1 - (\chi_1\chi_2^{-1})(\varpi)}.$$

Let w denote the non-trivial element in the Weyl group. Then $c_w(\chi) = c_\alpha(\chi)$ is non-zero iff $\chi_1\chi_2^{-1} \neq |\cdot|^{-1}$. We have $w\chi = \chi_2 \otimes \chi_1$, and then similarly $c_{w^{-1}}(w\chi) \neq 0$ iff $\chi_1\chi_2^{-1} \neq |\cdot|$. Hence, we get that $I(\chi)$ is irreducible if and only if $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}$. This result is also true without either of the assumptions that χ be unramified or regular, see [Bum97, Theorem 4.5.1].

4. Spherical Representations via Zonal Spherical Functions

Definition. *A* zonal spherical function *is a smooth function $\omega : G \to \mathbb{C}$ satisfying*

(i) $\omega(kgk') = \omega(g)$ *for* $g \in G$, $k, k' \in K$;

(ii) $\omega(1) = 1$;

(iii) *For every $f \in \mathcal{H}_K$ there is a constant $\lambda_f \in \mathbb{C}$ such that*

$$f * \omega = \omega * f = \lambda_f f.$$

Given $\omega \in C^\infty(G)$, define the linear map $\xi : \mathcal{H}_K \to \mathbb{C}$ by

$$\xi_\omega(f) = \int_G \omega(g) f(g) \,\mathrm{d}g$$

Then condition (iii) above is equivalent to each of the following [Car79, p. 149][3]

(iii') ξ_ω is an algebra homomorphism;

(iii'') $\omega(g_1)\omega(g_2) = \int_K \omega(g_1 k g_2) \,\mathrm{d}k$ for all $g_1, g_2 \in G$.

If $\xi : \mathcal{H}_K \to \mathbb{C}$ is a linear map, then $\omega : G \to \mathbb{C}$ defined by

$$\omega(g) = \frac{1}{\mathrm{vol}(KgK)} \xi(\mathbb{1}_{KgK})$$

satisfies (i) and (ii) above and we have $\xi = \xi_\omega$. If furthermore ξ is an algebra homomorphism, then ω is a zonal spherical function by the equivalence of (iii') and (iii). Hence, if we denote by Ω the space of zonal spherical functions, we get an isomorphism

$$\begin{aligned} \mathrm{Hom}(\mathcal{H}(G,K), \mathbb{C}) &\longrightarrow \Omega \\ \xi &\longmapsto \left(\omega : g \mapsto \frac{1}{\mathrm{vol}(KgK)} \xi(\mathbb{1}_{KgK})\right) \\ \xi_\omega &\longleftarrow\!\shortmid\ \omega \end{aligned}$$

[3] Warning: I call the zonal spherical functions ω as in [Sat63], but [Car79] denotes them by Γ and instead writes ω for what I call ξ (I think it would have been clearer if I adopted the notation in [Car79]).

So by Theorem 5 the spherical representations of G are in bijections with zonal spherical functions. We also know by Theorem 8 they are in bijection with unramified characters of T modulo W. We make this explicit. Let χ be an unramified character of T. Then consider the spherical vector $\Phi_{K,\chi}$ in the principal series representation $I(\chi)$, defined by

$$\Phi_{K,\chi}(tnk) = \chi(t)\delta^{1/2}(t)$$

where $t \in T, n \in N, k \in K$. Then as in [Sat63, 5.11] and [Car79, p. 150] let

$$\omega_\chi(g) = \int_K \Phi_{K,\chi}(kg)\,\mathrm{d}k.$$

Using the notation from 2 we have for $f \in \mathcal{H}_K$:

$$\begin{aligned}
\xi_\chi(f) &= \int_T \mathcal{S}f(t)\chi(t)\,\mathrm{d}t \\
&= \int_T \chi(t)\delta^{1/2}(t) \int_N f(tn)\,\mathrm{d}t\,\mathrm{d}n \\
&= \int_T \int_N \Phi_{K,\chi}(tn) f(tn)\,\mathrm{d}t\,\mathrm{d}n \\
&= \int_K \int_T \int_N \Phi_{K,\chi}(tnk) f(tnk)\,\mathrm{d}t\,\mathrm{d}n\,\mathrm{d}k \\
&= \int_G \Phi_{K,\chi}(g) f(g)\,\mathrm{d}g \qquad (\dagger) \\
&= \int_K \int_G \Phi_{K,\chi}(g) f(k^{-1}g)\,\mathrm{d}g\,\mathrm{d}k \\
&= \int_G \omega_\chi(g) f(g)\,\mathrm{d}g = \xi_{\omega_\chi}(f).
\end{aligned}$$

Now Theorem 8 translates to:

Theorem 16 ([Car79, Theorem 4.2])**.** *The zonal spherical functions on G are exactly the functions ω_χ for unramified characters χ of T. Two such functions $\omega_\chi, \omega_{\chi'}$ are equal if and only if $\chi' = w\chi$ for some $w \in W$.*

Next we look at how this translates to the spherical representations. Let ω be a zonal spherical function on G. Consider $C^\infty(G)$ as a representation of G via right translation. Let (V_ω, π_ω) denote the subrepresentation of $C^\infty(G)$ generated by ω. In other words, elements of V_ω are $\mathbb{C}$-linear combinations of functions of the form $g \mapsto \omega(gg')$ with $g' \in G$. Then property (iii'') implies $\omega(g_1)f(g_2) = \int_K f(g_1kg_2)\,\mathrm{d}k$ for $g_1, g_2 \in G$ and $f \in V_\omega$.

Theorem 17. *The representation (V_ω, π_ω) is spherical.*

Proof. V_ω is smooth since $C^\infty(G)$ is smooth. By definition we have $\omega \in V_\omega^K$. Using the functional equation $\omega(g_1)f(g_2) = \int_K f(g_1kg_2)\,\mathrm{d}k$ it is easy to see that any non-zero subrepresentation of V_ω contains ω, hence is equal to V_ω. Therefore V_ω is irreducible. $\square$

The spherical character of π_ω is of course given by ξ_ω, i.e. $\pi_\omega(f)\omega = \xi_\omega(f)\omega$ for $f \in \mathcal{H}_K$. Indeed, the left side is a multiple of ω, so it suffices to evaluate at 1 (recall $\omega(1) = 1$) and we get

$$(\pi_\omega(f)\omega)(1) = \int_G f(g)\omega(g)\,\mathrm{d}g = \xi_\omega(f).$$

Since zonal spherical functions are in bijection with the characters of $\mathcal{H}_K$ under $\omega \leftrightarrow \xi_\omega$, we deduce:

Theorem 18. *If (V, π) is any spherical representation of G, there is a unique zonal spherical function ω such that $(V, \pi) \cong (V_\omega, \pi_\omega)$.*

5. Relation with the Principal Series

We now compare the two versions of the spherical representations constructed in 3 and 4. Let χ be an unramified character of T. Then we have two representations attached to it, the principal series representation $I(\chi)$ (which may be reducible) and the irreducible representation V_ω where $\omega = \omega_\chi$. Suppose $I(\chi)$ is irreducible. Given $f \in I(\chi)$ define $Q(f) : G \to \mathbb{C}$ by

$$Q(f)(g) = \int_K f(kg)\,\mathrm{d}k.$$

By definition we have $Q(\Phi_{K,\chi}) = \omega$. Since we assumed $I(\chi)$ to be irreducible, it is generated by $\Phi_{K,\chi}$ as a G-representation, hence Q maps $I(\chi)$ into V_ω. It preserves the G-action and hence gives an isomorphism $I(\chi) \cong V_\omega$.

For general unramified χ we know that $J(\chi) \cong V_\omega$ since they both have to the same character:

$$\xi_{J(\chi)}(f) = \int_T \mathcal{S}f(t)\chi(t) = \xi_\omega(f),$$

see (†) and the discussion after Proposition 10.

References

[Bum97] D. Bump. *Automorphic Forms and Representations*. Vol. 55. Cambridge Studies in Advanced Mathematics. Cambridge, GBR: Cambridge University Press, 1997.

[Car79] P. Cartier. "Representations of p-adic groups -A survey". In: *Proc.Symp.Pure Math* 33 (1979).

[Cas80] W. Casselman. "The Unramified Principal Series of P-adic Groups. 1. The Spherical Function". In: *Compositio mathematica* 40.3 (1980).

[CKD73] W. Casselman, W. Kuijk, and P. Deligne. "An Assortment of Results on Representations of GL2(k)". In: *Modular Functions of One Variable II*. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer Berlin Heidelberg, 1973.

[GH] J. Getz and H. Hahn. *An Introduction to Automorphic Representations.*

[Gro98] B. H. Gross. "On the Satake isomorphism". In: *Galois Representations in Arithmetic Algebraic Geometry* (1998).

[Sat63] I. Satake. "Theory of spherical functions on reductive algebraic groups over p-adic fields". In: *Publications Mathématiques de l'IHÉS* 18 (1963).

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