THE SPECTRAL THEOREM

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These are some notes for myself to remember some stuff around the proof of the spectral theorem. Most of this can be found in [Arv02] or [Fol15, Chapter 1], both books I really like! Throughout, $\mathcal{H}$ denotes a complex Hilbert space. If T is an operator on $\mathcal{H}$, we denote by $\sigma(T)$ its spectrum, i.e. the set of $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not invertible. It is considered as a measurable space equipped with its Borel σ-algebra.

The spectral theorem comes in different forms:

Theorem (Spectral Theorem)**.** *Let T be a bounded normal operator in $\mathcal{H}$. Then*

(i) ***Unitary equivalence to a multiplication operator.*** *T is unitarily equivalent to a multiplication operator in some L^2 space. This means: There is a measure space (X, μ), a function $f \in L^\infty(X, \mu)$, and a unitary isomorphism $U : \mathcal{H} \to L^2(X, \mu)$ such that $UT = M_f U$ where $M_f : L^2(X, \mu) \to L^2(X, \mu)$ is given by multiplication by f.*

(ii) ***Borel Functional Calculus.*** *There is a unique homomorphism $\Phi : L^\infty(\sigma(T))^1 \to \mathcal{B}(\mathcal{H})$ of $*$-algebras such that $\Phi(h) = T$ where $h : \sigma(T) \to \mathbb{C}, z \mapsto z$, and satisfying the following condition: If $(f_n)_n \subseteq L^\infty(\sigma(T))$ is a norm-bounded sequence converging pointwise to $f \in L^\infty(\sigma(T))$, then $\Phi(f_n) \to \Phi(f)$ in the strong operator topology.*

(iii) ***Spectral Decomposition.*** *There is a unique spectral projection-valued measure (see below) P on $\sigma(T)$ such that $T = \int_{\sigma(T)} \lambda \, \mathrm{d}P(\lambda)$.*

In (ii), $\Phi(f)$ will also be denoted by $f(T)$.

A projection-valued measure (or *resolution of the identity*) P on a measurable space $(X, \mathcal{B})$ is a map $P : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ such that

(1) $P(E)$ is an orthogonal projection for all $E \in \mathcal{B}$;

(2) $P(\emptyset) = 0, P(X) = \mathrm{id}_{\mathcal{H}}$;

(3) $P(E \cap F) = P(E)P(F)$ for all $E, F \in \mathcal{B}$;

(4) If $E_1, E_2, \dots \in \mathcal{B}$ is a sequence of pairwise disjoint sets, then $s{-}\sum_n P(E_n) = P\left(\bigcup_n E_n\right)$ where the limit in the infinite sum on the left is taken with respect to the strong operator topology.

Let P be such a measure. Then for any $x, y \in \mathcal{H}$, the association $P_{x,y}(E) = \langle P(E)x, y \rangle$ with $E \in \mathcal{B}$ defines a complex measure on $(X, \mathcal{B})$. We denote by $L^\infty(X, \mathcal{B}, P)$ to be set of essentially bounded measurable functions on X modulo those that are 0 P-almost everywhere. Given such a measure we can integrate bounded functions to get operators. More precisely, there is a unique isometric unital

[1] Here $\sigma(T)$ carries no measure, only the σ-algebra of Borel sets, so $L^\infty(\sigma(T))$ is the space of everywhere bounded Borel measurable functions. It is a C^*-algebra.

$*$-homomorphism $\Phi : L^\infty(X, \mathcal{B}, P) \to \mathcal{B}(\mathcal{H})$ such that $\langle \Phi(f)x, y\rangle = \int_X f \,\mathrm{d}P_{x,y}$. We then denote $\Phi(f)$ by $\int_X f \,\mathrm{d}P$. This is relatively easy to show: Show everything for simple functions and then extend by continuity.

In the finite-dimensional case, (i) corresponds to the fact that a normal matrix is unitarily similar to a diagonal matrix, while (iii) corresponds to writing a normal matrix as a weighted sum of the projections onto its eigenspaces.

Before going to the proof we sketch some relations between the different formulations:

"$(i) \Rightarrow (ii)$" We may assume that $\mathcal{H} = L^2(X)$ and $T = M_f$ for some function f. In this case the Borel functional calculus can be written out explicitly via $\Phi(g) := M_{g\circ f}$. This is well-defined as since $\operatorname{ess\,ran} f = \sigma(T)$. Uniqueness on polynomial functions in $z, \overline{z}$ on $\sigma(T)$ is clear by the requirement that the map be a $*$-homomorphism, and then the usual approximation argument gives uniqueness on $L^\infty(\sigma(T))$.

"$(ii) \Rightarrow (iii)$" The spectral measure is given by $P(E) = \mathbb{1}_E$ for Borel sets $E \subseteq \sigma(T)$ where $\mathbb{1}_E$ is the indicator function of E.

"$(iii) \Rightarrow (ii)$" Define $\Phi(f)$ by $\Phi(f) = \int_{\sigma(T)} f(\lambda)\,\mathrm{d}P(\lambda)$. In fact, this defines Φ on the quotient $L^\infty(\sigma(T), P)$ of $L^\infty(\sigma(T))$, and on $L^\infty(\sigma(T), P)$ with the essential sup norm, Φ is isometric.

"$(ii), (iii) \Rightarrow (i)$" I don't know if there is a direct way to do this direction without going over the proof of the spectral theorem.

We now sketch the proof of this, making use of the basic theory of commutative C^*-algebras.

Let $\mathcal{A}$ be the unital C^*-algebra generated by T. Since T is normal, $\mathcal{A}$ is commutative. Denote by $\widehat{\mathcal{A}}$ the Gelfand spectrum of $\mathcal{A}$, i.e. the space of (unital) algebra homomorphisms $\mathcal{A} \to \mathbb{C}$ with the w^*-topology. The space of continuous functions on $\widehat{\mathcal{A}}$ is denoted $C(\widehat{\mathcal{A}})$. There is a canonical map, the *Gelfand map*, $i : \mathcal{A} \to C(\widehat{\mathcal{A}})$ given by $i(x)(\varphi) = \varphi(x)$ for $x \in \mathcal{A}, \varphi \in \widehat{\mathcal{A}}$. We shall deduce the spectral theorem from the main result on commutative C^*-algebras, the little Gelfand-Naimark Theorem: The natural map $i : \mathcal{A} \to C(\widehat{\mathcal{A}})$ is an isometric isomorphism of $*$-algebras. In fact, in view of Proposition 1 below this already gives us part of (ii), namely the inverse map of i is the continuous functional calculus. Thus, the statement of (ii) is that this extends to all Borel functions. This can be done somewhat directly by proving a sort of dominated convergence theorem for strong convergence of operators, but we will go a different route. Essentially the idea is that if we know how continuous functions act, we get measures using the Riesz representation theorem, and once we have the measure, we can integrate more general functions, like those in L^∞. We will do this in the context of C^*-algebra representations.

For future reference we note the following proposition which follows from the general theory of commutative Banach algebras that will allow us to relate T and $\mathcal{A}$.

Proposition 1. *The map* $\widehat{\mathcal{A}} \to \sigma(T), \varphi \mapsto \varphi(T)$ *is a homeomorphism.*

We now first discuss some aspects of C^*-algebra representations that we will need for the proof.

Let $\mathcal{A}$ be a unital C^*-algebra. A representation of $\mathcal{A}$ is a $*$-homomorphism $\mathcal{A} \to \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. A homomorphism of C^*-algebras is automatically norm-decreasing, so we don't need to assume such homomorphisms to be bounded. In the case we are interested in, $\mathcal{A}$ is already given as a subalgebra of $\mathcal{B}(\mathcal{H})$ for some $\mathcal{H}$, so we get a representation through the inclusion $\mathcal{A} \hookrightarrow \mathcal{B}(\mathcal{H})$, but it will still be important to study other representations. How do we get representations of a general $\mathcal{A}$?

Definition. *Let $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a representation of $\mathcal{A}$. A vector $\xi \in \mathcal{H}$ is called* cyclic *for π if $\pi(\mathcal{A})\xi$ is dense in $\mathcal{H}$. π is called* cyclic *if $\mathcal{H}$ contains a cyclic vector.*

If $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a representation, then $\pi(\mathcal{A})$ is $*$-invariant, so if $X \subseteq \mathcal{H}$ is $\pi(\mathcal{A})$-invariant, then so is $X^{\perp}$. Using this a straightforward Zorn's Lemma argument shows that any representation decomposes as a direct sum of cyclic representations.

Let $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ a cyclic representation of $\mathcal{A}$ with cyclic vector ξ. Define $\varphi_\xi : \mathcal{A} \to \mathbb{C}$ by $\varphi_\xi(a) = \langle \pi(a)\xi, \xi\rangle$[2]. This is a positive functional, meaning that $\varphi_\xi(a) \geq 0$ for all $a \geq 0$ in $\mathcal{A}$. Positive functionals are automatically bounded, though in this case one can of course verify directly that φ_ξ is bounded. Thus, to any pair (π, ξ) of a cyclic representation π with cyclic vector ξ we can associate a positive functional φ_ξ. It turns out that this determines (π, ξ) uniquely up to unitary isomorphism.

Proposition 2. *Let $(\mathcal{H}, \pi), (\mathcal{K}, \rho)$ be two cyclic representations of $\mathcal{A}$ with cyclic vectors ξ, η respectively. Assume that $\varphi_\xi = \varphi_\eta$. Then there is a unitary isomorphism $U : (\mathcal{H}, \pi) \to (\mathcal{K}, \rho)$ of representations such that $U\xi = \eta$.*

Proof. If $a \in \mathcal{A}$, then

$$\|\pi(a)\xi\|^2 = \langle \pi(a)\xi, \pi(a)\xi\rangle = \varphi_\xi(a^*a) = \varphi_\eta(a^*a) = \|\rho(a)\eta\|^2 .$$

From this it follows that we can define a map $\pi(\mathcal{A})\xi \to \pi(\mathcal{A})\eta$ by $\pi(a)\xi \mapsto \pi(a)\eta$. It also shows this is isometric, hence extends to give the desired unitary isomorphism. □

We won't need this, but mention that conversely for every positive functional φ of $\mathcal{A}$ there is a cyclic representation (π, ξ) of $\mathcal{A}$ with $\varphi_\xi = \varphi$. This is called the *Gelfand-Naimark-Segal (GNS) construction*:

Proposition 3. *Let φ be a positive linear functional on $\mathcal{A}$. There is a cyclic representation $(\mathcal{H}, \pi)$ of $\mathcal{A}$ with cyclic vector ξ such that $\varphi = \varphi_\xi$.*

Proof. For $a, b \in \mathcal{A}$ define $\langle a, b\rangle_\varphi := \varphi(b^*a)$. This is a pre-inner product on $\mathcal{A}$, so if we let $\mathcal{N}_\varphi = \{a \in \mathcal{A} : \langle a, a\rangle_\varphi = 0\}$, then $\mathcal{N}_\varphi$ is a left ideal of $\mathcal{A}$, and $\langle -, -\rangle_\varphi$ descends to an inner product on $\mathcal{A}/\mathcal{N}_\varphi$. Let $\mathcal{H}$ be the completion of $\mathcal{A}/\mathcal{N}_\varphi$ with respect to this inner product. Since $\mathcal{N}_\varphi$ is a left ideal, $\mathcal{A}$ acts on $\mathcal{A}/\mathcal{N}_\varphi$ by left translations. Furthermore, this action is continuous, hence extends to an action π on $\mathcal{H}$. Let ξ be the image of $1 \in \mathcal{A}$ in $\mathcal{A}/\mathcal{N}_\varphi$. Then ξ is a cyclic vector and $\varphi_\xi(a) = \langle \pi(a)1, 1\rangle_\varphi = \varphi(a)$ for $a \in \mathcal{A}$. □

Now assume that $\mathcal{A}$ is commutative and let $(\mathcal{H}, \pi)$ be a cyclic representation with cyclic vector ξ. The Gelfand map $i : \mathcal{A} \to C(\widehat{\mathcal{A}})$ is an isomorphism, so we may view φ_ξ via i as a positive linear functional on $C(\widehat{\mathcal{A}})$. By the Riesz representation theorem this determines a regular Borel measure μ on $\widehat{\mathcal{A}}$ such that

$$\int_{\widehat{\mathcal{A}}} i(a) \,\mathrm{d}\mu = \varphi_\xi(a)$$

for $a \in \mathcal{A}$. Now consider $\mathcal{K} = L^2(\widehat{\mathcal{A}}, \mu)$. $C(\widehat{\mathcal{A}})$ acts on this via left multiplication and hence so does $\mathcal{A}$. The constant function $1 \in \mathcal{K}$ is a cyclic vector and we have $\varphi_1(a) = \langle i(a) \cdot 1, 1\rangle_{L^2} = \int_{\widehat{\mathcal{A}}} i(a) \,\mathrm{d}\mu = \varphi_\xi(a)$.

[2]For us the inner product is linear in the first and anti-linear in the second argument.

Therefore, Proposition 2 gives a unitary isomorphism $U : \mathcal{H} \to L^2(\widehat{\mathcal{A}}, \mu)$ such that $U\xi = 1$ and $U\pi(a)v = i(a) \cdot Uv$ for $v \in \mathcal{H}, a \in \mathcal{A}$.

We can now prove the spectral theorem. Let T be a normal operator in $\mathcal{H}$ and $\mathcal{A}$ the unital C^*-algebra generated by T. We have already noted that $\mathcal{A}$ is commutative. Let $\pi : \mathcal{A} \hookrightarrow \mathcal{B}(\mathcal{H})$ be the inclusion. Assume first that π is cyclic. By the above there is a regular Borel measure μ on $X := \widehat{\mathcal{A}}$ and an isometric isomorphism $U : \mathcal{H} \to L^2(X, \mu)$ of representations where $\mathcal{A}$ acts on $L^2(X, \mu)$ via i by leftmultiplication.. Then by the definitions and since U is an intertwining operator, we have

$$UTv = i(T) \cdot Uv$$

for $v \in \mathcal{H}_j$, so $UT = M_{i(T)}U$. Then $\|i(T)\|_{L^\infty(X,\mu)} \leq \|i(T)\|_\infty = \|T\| = \|M_{i(T)}\|$.[3] This proves part (i) of the spectral theorem in the case when $\mathcal{H}$ is cyclic for T (i.e. for the C^*-algebra generated by T). To get the general case, decompose $\mathcal{H} = \bigoplus_{j \in J} \mathcal{H}_j$ into a Hilbert space direct sum of cyclic subrepresentations of $\mathcal{A}$. Then for each $\mathcal{H}_j$ we can find a measure space (X_j, μ_j), a function $f_j \in L^\infty(X_j, \mu_j)$ and a unitary isomorphism $U_j : \mathcal{H}_j \to L^2(X_j, \mu_j)$ such that $U_j T|_{\mathcal{H}_j \to \mathcal{H}_j} = M_{f_j} U_j$. Now take (X, μ) to be the disjoint union of all these measure spaces. Since this may be an uncountable union, we briefly note the construction to avoid misunderstanding. The underlying set X is simply the disjoint union of all the X_j. A subset $A \subseteq X$ is measurable if $A \cap X_j$ is measurable in X_j for all j. For a measurable $A \subseteq X$ we define $\mu(A) := \sum_{j \in J} \mu_j(A \cap X_j)$. Then we have $L^2(X, \mu) = \bigoplus_{j \in J} L^2(X_j, \mu_j)$ for $1 \leq p \leq \infty$ via $g \mapsto (g|_{X_j})_{j \in J}$. Let $f : X \to \mathbb{C}$ be defined by $f(x) = f_j(x)$ if $x \in X_j$. Since $\|f_j\|_{L^\infty(X_j,\mu_j)} = \|T|_{\mathcal{H}_j \to \mathcal{H}_j}\| \leq \|T\|$, we have $f \in L^\infty(X, \mu)$. Finally, let $U = \bigoplus_{j \in J} U_j : \mathcal{H} = \bigoplus_j \mathcal{H}_j \to L^2(X, \mu)$. Then we have $UT = M_f U$ since this holds on every $\mathcal{H}_j$, and part (i) of the spectral theorem is proven.

We have already indicated after the statement of the theorem how parts (ii) and (iii) can be deduced from the first.

It seems we never needed Proposition 1, I thought we would... So everything would go through if we replace $\mathcal{A}$ with an arbitrary unital commutative C^*-subalgebra of $\mathcal{B}(\mathcal{H})$. If we would like to describe the Borel functional calculus directly in terms of the above construction, then we would need Proposition 1 (like we already mentioned at the beginning how the inverse of the Gelfand map is the continuous functional calculus).

Remarks.

- The proof shows that if $\mathcal{H}$ is separable, then the measure space (X, μ) in (i) can be chosen to be σ-finite. Using additional arguments one can show that (X, μ) can even chosen to be finite.
- The proof shows the following operator algebra version of part (i): Any commutative C^*-subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is equivalent to a subalgebra of $L^\infty(X, \mu) \subseteq \mathcal{B}(L^2(X, \mu))$ for some measure space (X, μ). If $\mathcal{A}$ is maximal abelian, then it is unitarily equivalent to $L^\infty(X, \mu)$ (because this is maximal abelian as it a von Neumann algebra admitting a cyclic vector).
- The image of the continuous functional calculus in $\mathcal{B}(\mathcal{H})$ is $\mathcal{A}$, the unital C^*-algebra generated by T. The image of the Borel functional calculus is contained in the von Neumann algebra generated by T, i.e. the closure of $\mathcal{A}$ in the weak operator topology or equivalently the double commutant $\mathcal{A}''$. If $\mathcal{H}$ is separable, this is an equality.

[3]Actually the first inequality is also an equality, since clearly $\|M_{i(T)}\| \leq \|i(T)\|_{L^\infty(X,\mu)}$ (this is also an equality for general $f \in L^\infty(X, \mu)$).

- For any non-empty open subset $U \subseteq \sigma(T)$, $P(U) \neq 0$.
- For $z \in \sigma(T)$, $P(\{z\})$ is the orthogonal projection onto $\ker(T - z)$, so in particular $P(\{z\}) \neq 0$ iff z is an eigenvalue of T.
- If $S \in \mathcal{B}(\mathcal{H})$, then S commutes with T iff S commutes with all $P(E)$, $E \in \mathcal{B}(\sigma(T))$, iff S commutes with all $f(T)$, $f \in L^\infty(\sigma(T))$.

References

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