QUALIFYING EXAM TRANSCRIPT

LEONARD TOMCZAK

Went pretty well. Started around 2:15 PM and ended at 3:50 PM. I was very nervous before, but then once in the exam it was actually quite enjoyable, after all I could talk about math which I like.

1. Algebraic Number Theory

Vojta: State the Chebotarev theorem.

Me: I explained the setup and stated the theorem.

Vojta: Can you give an application?

Me: I wasn't too sure at first what he meant (like a concrete example, or something else). I ended up talking about the density of the completely split primes and how it can be used to show that they determine the Galois extension.

Vojta: Prove this (that assuming $d(S_{L/K}) = \frac{1}{[L:K]}$ for Galois L/K, then $S_{L/K}$ determines L for Galois L/K)

Me: did so.

Sug Woo (or Vojta? not sure, may also have been asked earlier): Define the Frobenius element.

Me: did so.

Vojta: Consider $L = \mathbb{Q}(\sqrt[3]{5})$. In what ways can primes split in L?

Me: Wrote down the fundamental identity, and from that deduced the ways a prime could split in a general cubic extension. Some cases took a little longer than I would like to admit. For example, to find a prime mod which $f = t^3 - 5$ factors as quadratic $\times$ linear, I unecessarily checked manually that 5 is not a cube mod 7, so 7 doesn't work. After this I realized that of course for primes $p \equiv 2 \bmod 3$ the polynomial f splits in this desired form. After that the only case left to consider was whether there is a prime splitting as $\mathfrak{P}_1^2\mathfrak{P}_2$ with $\mathfrak{P}_1 \neq \mathfrak{P}_2$. Since 5 is clearly fully ramified, the only other candidate was 3. I mistakenly also included 2, since for some reason I thought $t^3 - 5$ was inseparable mod 2. To clear up doubts they asked me compute the discrimant. Since I forgot the formula for the discriminant of a cubic polynomial, Sug Woo suggested I compute the discriminant of $(1, \alpha, \alpha^2)$ with $\alpha = \sqrt[3]{5}$ via the definition as $\det(\mathrm{Tr}_{L/\mathbb{Q}}(\alpha^{i+j}))_{i,j}$. It turned out to be $-3^3 \cdot 5^2$. Therefore 3 ramifies in L (and 2 does not!) since the power of 3 dividing this is odd, hence 3 occurs in the discriminant of L even if $\mathbb{Z}[\alpha]$ is not the maximal order. Then I said that $f(t-1)$ is Eisenstein at 3, hence 3 is totally ramified, and thus there is no prime with the splitting behavior $\mathfrak{P}_1^2\mathfrak{P}_2$.

Vojta: Prove that a prime splits completely in an extension if and only if it splits completely in its Galois closure.

Me: did so (using completions).

Vojta: State global class field theory.

Me: Stated the ideal theoretic version of Artin reciprocity and Existence/Uniqueness theorem.

Vojta: How can you see from this which primes ramify and which primes split completely?

Me: The primes that ramify are those that divide the conductor (minimal admissible cycle), and the completely split primes are those that lie in the group associated to the extension (i.e. the kernel of the Artin map).

Sug Woo: State the idelic version.

Me: did so.

Sug Woo: How do you construct the map $\varphi_K : \mathbb{A}_K^\times / K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$?

Me: I mentioned two ways: Either if you already know the ideal theoretic Artin reciprocity law, you can get φ_K on finite extensions by identifying the ray class group with a certain quotient of the adelic class group. Otherwise, you can define it as the product of the local norm residue symbols. I mentioned that the factors are 1 at almost all primes (I didn't say that this only defined φ_K on $\mathbb{A}_K^\times$ a priori, and not yet on the quotient, which basically is the content of Artin reciprocity, but they didn't seem to mind).

After this section was finished, I cleaned the boards and we discussed how to properly use the eraser. I mentioned how in Germany there are containers with water and plastic sweeps in the lecture halls, that we clean the boards with water, and then dry them with the other towel-like side of the sweep.

2. Automorphic Forms

Sug Woo: What is an automorphic from for GL_1?

Me: Quasicharacters $\mathbb{A}_K^\times / K^\times \to \mathbb{C}^\times$.

Vojta: What is a quasicharacter?

Sug Woo: Give examples.

Me: $|\cdot|^s$, adelic lifts of Dirichlet characters or more generally characters of ray class groups.

Sug Woo: What is the L-function of such a form?

Me: Defined it as a product of the local L-factors, and defined the local L-factors for finite and infinite places (for the infinite places I only wrote down the L-factor in the unramified case).

Sug Woo: What about convergence of the infinite product?

Me: It converges if the exponent of the quasicharacter is > 1. This can be reduced to the case of $\mathbb{Q}$ where it is basically the convergence of $\prod_p (1 - p^{-s})^{-1}$ for $\operatorname{Re} s > 1$.

Sug Woo: What more can you say? (Analytic continuation, Functional Equation, ...)

Me: I talked about the general adelic approach to L-functions: Global/local zeta integrals, and how they relate to the L-factors and the epsilon factors.

Sug Woo: What can you say about the dimensions of M_k, S_k for level one forms? Define M_k, S_k first.

Me: Defined everything. I wrote down the dimensions of M_k for $k = 2, 4, 6, 8, 10, 12$ and S_{12} (defining the Eisenstein series and Δ), and then said that multiplication by Δ gives an isomorphism $M_k \cong S_{k+12}$. Then we know everything to compute the dimensions (though I didn't write down the exact formula).

Sug Woo: Why are $G_k = 0$ for k odd? (Here G_k is the weight k level 1 holomorphic Eisenstein series)

Me: All the terms in the defining sum cancel. I could already guess this wasn't what Sug Woo was after, and he asked for a more general argument that applies to showing $M_k = 0$ for k odd. So I said that it follows by applying the modularity condition to the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and mentioned that this doesn't work in higher level, and in fact there will be modular forms of odd weight of higher level.

Sug Woo: How can you turn a modular form into an adelic automorphic form?

Me: Stated strong approximation for GL_2 over $\mathbb{Q}$ and how it used that $\mathbb{Q}$ has class number one. Then defined the adelic lift using strong approximation.

Sug Woo: How can you get an automorphic representation from this?

Me: I first introduced the space of cuspidal automorphic forms to define a cuspidal automorphic representation and stated that $\mathcal{A}_0$ is completely reducible. Then I was about to say that we get an irreducible representation from a modular form, Sug Woo said we needed some condition. I corrected myself and said that a modular form does not necessarily generate an irreducible representation, but an eigenform does.

Sug Woo: Talk about the tensor product theorem.

Me: Stated it.

Sug Woo: What goes into its proof?

Me: I just mentioned that the main part is to show that irreducible admissible representations of a product $G \times H$ are outer tensor products of such representations of G, H resp. (the proof wasn't technically on the syllabus, so we didn't go into detail)

Sug Woo: What can you say about the local components of an automorphic representation coming from a modular form?

Me: Since we were only talking about level 1 forms, if we have an eigenform all the nonarchimedean components are spherical, and the infinite component is the weight k discrete series. I mentioned that for higher level, we only get local spherical representations at primes not dividing the conductor, for the primes dividing the conductor, we may get others.

Sug Woo: Classify all the spherical representations of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Me: did so.

3. Functional Analysis

Voiculescu: State the Krein-Milman theorem.

Me: did so.

Voiculescu: Let K be a compact space, consider in the dual space of $C(K)$, the set of probability measures. Show it is compact, find its extreme points.

Me: Clarified that he meant the compactness with respect to the weak-$*$ topology. I then said that by Banach-Alaoglu the unit ball is compact, and the set P of probability measures is a closed subset (for the w^*-topology).

Voiculescu: Why is it closed?

Me: I explained that the conditions for a measure to be a probability measure are given by w^*-continuous functionals. I then continued saying that the extreme points of P are the Dirac measures δ_k for $k \in K$.

Voiculescu: What is the idea in showing that the δ_k are extreme points?

Me: (I actually wasn't too sure, I tried to remember how to do it for the unit ball) I said that we consider the intersection $X = \bigcap_{F \ni \delta_k} F$ of all the faces F of P containing δ_k. Then show using Hahn-Banach separation that X contains only one element, which must be δ_k. Since X is still a face, δ_k must be an extreme point. Looking back at it now I am not sure if that really works, however the idea of taking an intersection of faces does lead to a possible solution. He said something about a different more direct way he had in mind, but seemed fine with it.

Voiculescu: What is the Gelfand Mazur theorem?

Me: Stated it.

Voiculescu: Prove it.

Me. Proved it.

Voiculescu: How do you prove the spectrum is non-empty?

Me: Usual proof using Liouville.

Voiculescu: How do you prove Banach space-valued Liouville?

Me: Either build up complex analysis for Banach space-valued functions, or deduce it from the classical version via functionals.

Voiculescu: Talk about the spectral theory of normal operators.

Me: I said there are different versions of the spectral theorem: I first stated that every normal operator unitarily equivalent to a multiplication operator on an L^2-space. Then I said that there is a spectral measure P on $\sigma(T)$ such that $T = \int_{\sigma(T)} \lambda \mathrm{d}P(\lambda)$.

Voiculescu: What is a spectral measure?

Me: I defined it (actually I didn't give the complete definition, he stopped to go on to the next question)

Voiculescu: What kind of functional calculus do you get from such a measure?

Me: Borel functional calculus. You can "integrate" bounded Borel measureable functions with respect to the measure to get bounded operators.

Voiculescu: What can you say when you have three commuting self-adjoint operators.

Me: At first I was confused about the number three and had to clarify (lol). Then I said they are simultaneously unitarily equivalent to a multiplication operator.

Voiculescu: What about if there are arbitrarily many?

Me: Doesn't change things: You just look at the (commutative!) C^*-algebra they generate and do spectral theory there.

Department of Mathematics, Evans Hall, University of California, Berkeley, CA 94720, USA

Email address: `leonard.tomczak@berkeley.edu`