AUTOMORPHIC NOTES

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Contents

Some notes for Qualifying exam preparation. Mainly GL_2 stuff (automorphic forms/representations, local representations), and also some Algebraic Number theory, and GL_1 theory. Unfortunately, the notation is not consistent throughout (sometimes not even whithin a a single section), since I wrote different parts at different times, and used many different sources, etc.

Part 1. GL_1 Theory (Tate's Thesis and More)

1. Local Theory

1.1. Haar Measures

Let F be a local field (which we allow to be archimedean in this section).

The absolute value $|\cdot|$ on F is defined by the property $\mathrm{d}(ax) = |a|\,\mathrm{d}x$ where $\mathrm{d}x$ is a fixed Haar measure on F. We have:

- $|\cdot|$ is the usual absolute value if $F = \mathbb{R}$,
- $|\cdot|$ is the square of the usual absolute value if $F = \mathbb{C}$,
- $|x| = q^{-v(x)}$ if F is nonarchimedean, where q is the cardinality of the residue field and $v(x)$ is the valuation of x.

Fix a non-trival additive character $\psi : F \to \mathbb{C}^\times$.

Definition. *For "nice" functions $f : F \to \mathbb{C}$, define their* Fourier transform $\widehat{f} : F \to \mathbb{C}$ *by*

$$\widehat{f}(\xi) = \int_F f(x)\psi(x\xi)\,\mathrm{d}x.$$

For example, f could be in the Bruhat Schwartz space $S(F)$, which is the usual Schwartz space if $F = \mathbb{R}$ or $F = \mathbb{C} \cong \mathbb{R}^2$, and the space $C_c^\infty(F)$ of compactly supported locally constant functions for nonarchimedean F.

The association $a \mapsto \psi_a$ gives a topological isomorphism $F \to \widehat{F}$, where $\psi_a(x) = \psi(ax)$. Therefore the above definition of the Fourier transform coincides with the one from general abstract harmonic analysis. We let $\mathrm{d}x$ denote the self-dual measure on F with respect to this identification, in other words for nice functions f, we have

$$f(x) = \int_F \widehat{f}(\xi)\psi(-x\xi)\,\mathrm{d}\xi.$$

Here "nice" could mean that $f, \widehat{f} \in L^1(F)$. We note that an easy computation shows that the self-dual measure for ψ_a is $|a|^{1/2}\,\mathrm{d}x$.

We introduce an explicit choice of ψ (which we will call the standard character) and give the corresponding self-dual measures. Let K be the closure of $\mathbb{Q}$ in F. We first define a map $\lambda : K \to \mathbb{R}/\mathbb{Z}$. If $K = \mathbb{R}$, then $\lambda(x) = -x \bmod 1$. If $K = \mathbb{Q}_p$, then we let λ be the composition $\mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \cong \mathbb{Z}[1/p]/\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{R}/\mathbb{Z}$.[1] Then we define $\psi(x) = e^{2\pi i\lambda(\mathrm{Tr}_{F/K} x)}$ for $x \in F$.

[1] $\lambda(x)$ for $x \in \mathbb{Q}_p$ can be characterized as follows: It is the unique rational number z with the property that z has only a p power in the denominator, and $z - x \in \mathbb{Z}_p$.

Proposition 1.1. *For this particular choice of character the self-dual measure* $\mathrm{d}x$ *is given as follows:*

- *usual Lebesgue measure if* $F = \mathbb{R}$,
- *twice the Lebesgue measure if* $F = \mathbb{C}$,
- *the Haar measure that gives* $\mathcal{O}_F$ *the measure* $N\mathfrak{d}^{-\frac{1}{2}}$ *if* F *nonarchimedean. Here* $\mathfrak{d}$ *is the absolute different of* F.

If ψ has conductor $\mathfrak{p}^0$, then $\mathcal{O}_F$ has measure 1.

We normalize the multiplicative Haar measure on $F^\times$ as follows:

- $\mathrm{d}^\times x = \frac{\mathrm{d}x}{|x|}$ if F is archimedean,
- $\mathrm{d}^\times x = \frac{q}{q-1}\frac{\mathrm{d}x}{|x|}$ if F is nonarchimedean and q the cardinality of the residue field.

We note that in the nonarchimedean case the volume of $\mathcal{O}_F^\times$ with respect to this choice of multiplicative Haar measure is $N\mathfrak{d}^{-\frac{1}{2}}$. In particular if $F/\mathbb{Q}_p$ is unramified, the volume of $\mathcal{O}_F^\times$ is 1.

We will also need multiplicative characters. Let U be the subgroup of F consisting of the elements x with $|x| = 1$. χ is called *unramified* if χ is trivial on U, otherwise it is *ramified.* There is a surjective map $F^\times \to U$ given by $x \mapsto \widetilde{x}$ where $\widetilde{x} = \frac{x}{|x|}$ if $F = \mathbb{R}$, $\widetilde{x} = \frac{x}{\sqrt{|x|}}$ if $F = \mathbb{C}$, and $\widetilde{x} = x/\varpi^{v(x)}$ if F is nonarchimedean for some fixed choice of uniformizer ϖ. This map splits:

- $F \cong U \times \mathbb{R}_{>0}$ if F is archimedean,
- $F \cong U \times \mathbb{Z}$ if F is nonarchimedean.

An unramified character χ is of the form $\chi(x) = |x|^s$ for some $s \in \mathbb{C}$. s is uniquely determined if F is archimedean, if F is nonarchimedean, s is only uniquely determined mod $2\pi i/\log q$. Let χ be any quasi-character of $F^\times$, i.e. a continuous homomorphism $F^\times \to \mathbb{C}^\times$. From the above we see, that χ can be written as $\chi(x) = \chi_0(\widetilde{x})\,|x|^s$ where χ_0 is a character of U and $s \in \mathbb{C}$. Note that in this way we get a complex coordinate s on the set of quasi-characters of $F^\times$, hence we can speak of holomorphic or meromorphic functions on the set of quasi-characters.

We have $|\chi(x)| = |x|^{\operatorname{Re} s}$, and we call $\sigma = \operatorname{Re} s$ the *exponent* of χ.

1.2. **Zeta Functions**

We consider the class $\mathfrak{Z}$ of functions $f : F \to \mathbb{C}$ on F

(1) f and $\widehat{f}$ are in $L^1(F, \mathrm{d}x)$,
(2) $f\,|\cdot|^\sigma$ and $\widehat{f}\,|\cdot|^\sigma$ are in $L^1(F^\times, \mathrm{d}^\times x)$ for $\sigma > 0$.

A function $f : F \to \mathbb{C}$ is *Bruhat-Schwartz* if f is a Schwartz function in the ordinary sense for F archimedean, and f is compactly supported locally constant if F is nonarchimedean. We denote the space of Bruhat Schwarz functions on F by $S(F)$. Note that $S(F) \subseteq \mathfrak{Z}$.

We define the local Zeta functions as Mellin transforms of functions in $\mathfrak{Z}$:

Definition. *For $f \in \mathfrak{Z}$ and χ a quasi-character of $F^\times$ we define*
$$Z(f,\chi) = \int_{F^\times} f(x)\chi(x)\mathrm{d}^\times x.$$

We might also write
$$Z(f,\chi,s) = \int_{F^\times} f(x)\chi(x)\left|x\right|^s \mathrm{d}^\times x.$$

Note that $Z(f,\chi,s) = Z(f,\chi\left|\cdot\right|^s)$, so this doesn't give us anything new, it is just notational convenience.

It is easy to see that the integral defining $Z(f,\chi)$ converges absolutely if the exponent of χ is > 0 and defines a holomorphic function of χ there.

Given a quasi-character χ of $F^\times$ we let $\check{\chi} = \left|\cdot\right|\chi^{-1}$. Note that $\widecheck{\chi\left|\cdot\right|^s} = \chi^{-1}\left|\cdot\right|^{1-s}$. If σ is the exponent of χ, then the exponent of $\check{\chi}$ is $1-\sigma$.

Lemma 1.2 ([Tat67a, Lemma 2.4.2])**.** *Let χ be a quasi-character with exponent $0 < \sigma < 1$. Then for $f, g \in \mathfrak{Z}$, we have*
$$Z(f,\chi)Z(\widehat{g},\check{\chi}) = Z(\widehat{f},\check{\chi})Z(g,\chi).$$

Proof. Write all the integrals out and do a change of variables, pretty straightforward computation. □

Theorem 1.3 ([Tat67a, Theorem 2.4.1], [Bum97, Proposition 3.1.5])**.** *Fix $f \in \mathfrak{Z}$. Then $Z(f,\chi)$ has an meromorphic continuation to all quasi-characters χ and satifies a functional equation*
$$Z(\widehat{f},\check{\chi}) = \gamma(\chi,\psi)Z(f,\chi) \tag{$*$}$$
where γ does not depend on f, but on χ and the choice of additive character ψ defining the self-dual measure.

As usual we write $\gamma(\chi,s,\psi) = \gamma(\chi\left|\cdot\right|^s,\psi)$. If ψ is fixed we also write $\gamma(\chi) = \gamma(\chi,\psi)$. Note that $\gamma(\chi,\psi) = \chi(-1)\rho(\chi,\psi)^{-1}$ in the notation of [Tat67a], the factor $\chi(-1)$ coming from the different definition of the Fourier transform (in Tate it is $\int f(x)\psi(-x\xi)\mathrm{d}x$.)

If ψ gets replaced by ψ_a, then $Z(\widehat{f},\check{\chi})$ becomes $\chi(a)Z(\widehat{f},\check{\chi})$ and $Z(f,\chi)$ becomes $\left|a\right|^{1/2}Z(f,\chi)$, hence $\gamma(\chi,\psi_a) = \chi(a)\left|a\right|^{-1/2}\gamma(\chi,\psi)$.

Proof of Theorem 1.3. Let σ denote the exponent of χ. Choose a function f such that $Z(f,\chi)$ is not identically 0. Then let $\gamma(\chi,\psi)$ be the quotient
$$\frac{Z(\widehat{f},\check{\chi})}{Z(f,\chi)}$$
By the lemma above we then have $Z(\widehat{g},\check{\chi}) = \gamma(\chi,\psi)Z(g,\chi)$ for all $g \in \mathfrak{Z}$. It is defined in $0 < \sigma < 1$. There are different ways to continue and show that γ admits a meromorphic continuation to all quasi-characters.

- Proof in [Tat67a]. The point is to explicitly exhibit for each χ a function f such that $Z(f,\chi)$ is not identically 0 and compute $\gamma(\chi)$ in this case. The explicit computation will show that $\gamma(\chi)$ is meromorphic in all χ, since $Z(\widehat{f},\check{\chi})$ is defined for all χ is defined for all χ of exponent < 1, the functional equation then extens $Z(f,\chi)$ to all χ.
- Proof in [Bum97]. TODO

□

Proposition 1.4 ([Bum97, Exercise 3.1.9])**.** *Assume F is nonarchimedean. If the exponent of χ is < 1, and N is sufficiently large, then*

$$\gamma(\chi,\psi) = \int_{\mathfrak{p}^{-N}} \chi^{-1}(x)\psi(x)\mathrm{d}x.$$

Proof. For convenience we consider $\gamma(\chi,s,\psi)$ with χ unitary and $\operatorname{Re} s < 1$. We need to compute

$$\gamma(\chi,s,\psi) = \frac{Z(\widehat{\Phi},\chi^{-1},1-s)}{Z(\Phi,\chi,s)}.$$

Note that both integrals on the right converge, the numerator by $\operatorname{Re} s < 1$ and the denominator by the next calculation. By definition,

$$Z(\Phi,\chi,s) = \int_{F^\times} \Phi(x)\chi(x)|x|^s \,\mathrm{d}^\times x = \int_{1+\mathfrak{p}^N} \chi(x)\,\mathrm{d}^\times x$$

Since χ is continuous, $\chi|_{1+\mathfrak{p}^N}$ is trivial for N large enough, hence for such N we have $Z(\Phi,\chi,s) = \int_{1+\mathfrak{p}^N} \mathrm{d}^\times x = (1-q^{-1})^{-1}\int_{1+\mathfrak{p}^N}\mathrm{d}x = (1-q^{-1})^{-1}\operatorname{vol}_{\mathrm{d}x}(\mathfrak{p}^N)$. For the other integral, let m denote the conductor of ψ. Then we have

$$\widehat{\Phi}(y) = \int_F \Phi(x)\psi(xy)\,\mathrm{d}x = \int_{1+\mathfrak{p}^N} \psi(xy)\,\mathrm{d}x = \psi(y)\int_{\mathfrak{p}^N}\psi(xy)\,\mathrm{d}x = \psi(y)\operatorname{vol}_{\mathrm{d}x}(\mathfrak{p}^N)\mathbb{1}_{\mathfrak{p}^{m-N}}(y).$$

Then

$$\begin{aligned} Z(\widehat{\Phi},\chi^{-1},1-s) &= \int_{F^\times} \widehat{\Phi}(x)\chi^{-1}(x)\,|x|^{1-s}\,\mathrm{d}^\times x \\ &= \operatorname{vol}_{\mathrm{d}x}(\mathfrak{p}^N)\int_{\mathfrak{p}^{m-N}-\{0\}} |x|^{1-s}\,\chi^{-1}(x)\psi(x)\,\mathrm{d}^\times x \\ &= \operatorname{vol}_{\mathrm{d}x}(\mathfrak{p}^N)(1-q^{-1})^{-1}\int_{\mathfrak{p}^{m-N}-\{0\}} |x|^{-s}\,\chi^{-1}(x)\psi(x)\,\mathrm{d}x. \end{aligned}$$

We can also include 0 as it is of measure 0, so

$$\gamma(s,\chi,\psi) = \frac{Z(\widehat{\Phi},\chi^{-1},1-s)}{Z(\Phi,\chi,s)} = \int_{\mathfrak{p}^{m-N}} |x|^s\,\chi^{-1}(x)\psi(x)\,\mathrm{d}x.$$

Note for this calculation it was not important that $\mathrm{d}x$ is self-dual. I believe Bump only mentions this because it was assumed in the definition of γ? □

Proposition 1.5. *Some properties of the local gamma factor:*

(1) $\gamma(\chi, s, \psi)\gamma(\chi^{-1}, 1-s, \psi) = \chi(-1)$.

(2) $\gamma(\chi, s, \psi_a) = \chi(a)\,|a|^{s-1/2}\,\gamma(\chi, s, \psi)$.

(3) $|\gamma(\chi, \psi)| = 1$ *if* χ *has exponent* $\frac{1}{2}$.

Proof.

(1) $\gamma(\check{\chi}, \psi)\gamma(\chi, \psi)Z(f, \chi) = \gamma(\check{\chi}, \psi)Z(\widehat{f}, \check{\chi}) = Z(\widehat{\widehat{f}}, \check{\check{\chi}})$. Now note that $\widehat{\widehat{f}}(x) = f(-x)$ and $\check{\check{\chi}} = \chi$.

(2) If ψ gets replaced by ψ_a, then $Z(\widehat{f}, \check{\chi})$ becomes $\chi(a)Z(\widehat{f}, \check{\chi})$ and $Z(f, \chi)$ becomes $|a|^{1/2}\,Z(f, \chi)$, hence $\gamma(\chi, \psi_a) = \chi(a)\,|a|^{-1/2}\,\gamma(\chi, \psi)$.

□

The real and complex Gamma functions are as follows:

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right), \quad \Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = 2(2\pi)^{-s}\Gamma(s)$$

Definition. *We define* L*-functions as follows.*

- *If* $F = \mathbb{R}$ *and* $\chi(x) = \operatorname{sgn}(x)^{\varepsilon}\,|x|^s$, *then*
$$L(\chi) = \Gamma_{\mathbb{R}}(s+\varepsilon) = \pi^{-(s+\varepsilon)/2}\Gamma\left(\frac{s+\varepsilon}{2}\right).$$
$L(\chi)$ *has poles at the even (resp. odd) nonpositive integers for* $\varepsilon = 0$ *(resp.* $\varepsilon = 1$*).*
- *If* $F = \mathbb{C}$ *and* $\chi(x) = \left(\frac{x}{\sqrt{|x|}}\right)^n |x|^s$, *then*
$$L(\chi) = \Gamma_{\mathbb{C}}(s + \frac{|n|}{2}) = 2(2\pi)^{-s-|n|/2}\Gamma\left(s + \frac{|n|}{2}\right).$$
$L(\chi)$ *has poles at* $s = l - \frac{|n|}{2}$ *for nonpositive integers* l.
- *If* F *is nonarchimedean and* χ *is unramified, then*
$$L(\chi) = (1 - \chi(\varpi))^{-1}$$
where ϖ *is a uniformizer of* F. *If* χ *is ramified, then we set* $L(\chi) = 1$.

We also set $L(\chi, s) = L(\chi\,|\cdot|^s)$. In every case $L(\chi, s)$ is a meromorphic function without zeros.

Theorem 1.6 ([Bum97, Proposition 3.1.8])**.** *For any* $f \in S(F)$, *the quotient*
$$\frac{Z(f, \chi)}{L(\chi)}$$
defines an analytic function in χ. *Moreover for fixed* χ, $L(\chi, s)$ *has a pole at* $s = s_0$ *if and only if* $Z(f, \chi, s)$ *has a pole there for some* $f \in S(F)$. *For any* χ *there is* $f \in S(F)$ *such that* $Z(f, \mathrm{ch}, s) = L(\chi, s)$ *for all* s. *If* F *is nonarchimedean, then* $Z(f, \chi, s)$ *is a rational function in* q^{-s} *(for fixed* χ*).*

Thus, in some way $L(\chi, s)$ is a greatest common denominator of the $Z(f, \chi, s)$ for $f \in S(F)$. [Bum97] only says that there is $f \in S(F)$ such that $\frac{Z(f,\chi,s)}{L(\chi,s)}$ is of exponential type (i.e. of the form ab^s with constants $a \in \mathbb{C}^\times, b \in \mathbb{R}$), but it seems we can actually make this 1.

Proof. First suppose that F is nonarchimedean. Then

$$\begin{aligned} Z(f, \chi, s) &= \int_{F^\times} f(x)\chi(x)\,|x|^s\,\mathrm{d}^\times x \\ &= \sum_{n\in\mathbb{Z}} \int_{u\in U} f(\varpi^n u)\chi(\varpi^n u)\,|\varpi^n u|^s\,\mathrm{d}^\times u \\ &= \sum_{n\in\mathbb{Z}} \chi(\varpi^n)q^{-ns} \int_U f(\varpi^n u)\chi(u)\mathrm{d}^\times u. \end{aligned}$$

Since f has compact support, there is $n_0 \in \mathbb{Z}$ such that $f(\omega^n u) = 0$ for all $u \in U, n < n_0$. Also f is constant in a neighboorhood of 0, so there is $n_1 > n_0$ such that $f(\omega^n u) = f(0)$ for all $u \in U, n > n_1$. So

$$Z(f, \chi, s) = \sum_{n=n_0}^{n_1} \chi(\varpi^n)q^{-ns} \int_U f(\varpi^n u)\chi(u)\mathrm{d}^\times u + \sum_{n>n_1} \chi(\varpi^n)q^{-ns} f(0) \int_U \chi(u)\mathrm{d}^\times u.$$

The first summand is certainly an entire function in χ and moreover easily seen to be rational in q^{-s}. In the second summand note that $\int_U \chi(u)\mathrm{d}^\times x$ is 0 if χ is ramified and $\mathrm{vol}_{\mathrm{d}^\times x}(U)$ if χ is unramified. This shows that $\frac{Z(f,\chi)}{L(\chi)} = L(f, \chi)$ is entire if χ is ramified. Suppose χ is unramified, then the second term is

$$\begin{aligned} \sum_{n>n_1} \chi(\varpi^n)q^{-ns} f(0) \int_U \chi(u)\mathrm{d}^\times u &= \sum_{n>n_1} \chi(\varpi^n)q^{-ns} f(0)\,\mathrm{vol}_{\mathrm{d}^\times x}(U) \\ &= f(0)\,\mathrm{vol}_{\mathrm{d}^\times x}(U)(\chi(\varpi)q^{-s})^{n_1+1} \frac{1}{1-\chi(\varpi)q^{-s}} \\ &= f(0)\,\mathrm{vol}_{\mathrm{d}^\times x}(U)(\chi(\varpi)q^{-s})^{n_1+1} L(\chi, s). \end{aligned}$$

We see that this second term is also a rational function in q^{-s} and moreover, $\frac{Z(f,\chi)}{L(\chi)}$ is entire.

Examining the calculation shows that taking

$$f = \begin{cases} \mathrm{vol}_{\mathrm{d}^\times x}(U)^{-1} \mathbb{1}_{\mathcal{O}_K} & \text{if } \chi \text{ is unramified} \\ \mathrm{vol}_{\mathrm{d}^\times x}(U)^{-1} \chi^{-1} \mathbb{1}_U & \text{if } \chi \text{ is ramified} \end{cases}$$

gives $Z(f, \chi, s) = L(\chi, s)$.

Now suppose F is archimedean. For that use [Bum97, Proposition 3.1.7] and the explicit description of the poles of $L(\chi, s)$ above, though note that in the assumption of that Proposition, we can only assume that $\sum_{\nu\in\Sigma} a(\nu)x^\nu$ is an asymptotic expansion of f near 0, not that it converges to f. In the complex case one also needs to work slightly more.

We now give explicit functions f such that $Z(f, \chi) = L(\chi)$. First suppose F is real. Write $\chi(x) = \mathrm{sgn}(x)^\varepsilon\,|x|^s$ with $\varepsilon \in \{0, 1\}$. Take

$$f(x) = x^\varepsilon e^{-\pi x^2}.$$

Then

$$Z(f,\chi) = \pi^{-(s+\varepsilon)/2}\Gamma\left(\frac{s+\varepsilon}{2}\right) = L(\chi).$$

If F is complex, write $\chi(x) = \left(\frac{x}{\sqrt{|x|}}\right)^n |x|^s$. Take

$$f(z) = \begin{cases} \frac{1}{\pi}\overline{z}^{|n|}e^{-2\pi|z|} & \text{if } n \geq 0, \\ \frac{1}{\pi}z^{|n|}e^{-2\pi|z|} & \text{if } n \leq 0. \end{cases}$$

(Recall $|z|$ is the square of the usual absolute value on $\mathbb{C}$.) Then

$$Z(f,\chi) = 2(2\pi)^{-(s+\frac{|n|}{2}}\Gamma\left(s\frac{|n|}{2}\right) = L(\chi).$$

□

We also introduce the ε-factors.

Definition. *For any multiplicative quasi-character we define*

$$\varepsilon(\chi,\psi) = \frac{\gamma(\chi,\psi)L(\chi)}{L(\check{\chi})}.$$

And as usual we set $\varepsilon(\chi, s, \psi) = \varepsilon(\chi\,|\cdot|^s, \psi)$. So the epsilon factors measure how far $L(\chi)$ is from satisfying the functional equation $(*)$. Note from the corresponding fact for γ, we have $\varepsilon(\chi,\psi_a) = \chi(a)\,|a|^{-1/2}\,\varepsilon(\chi,\psi)$.[2] Also note that if $f \in S(F)$ is such that $Z(f,\chi) = L(\chi)$, then

$$\varepsilon(\chi,\psi) = \frac{Z(\widehat{f},\check{\chi})}{L(\check{\chi})}.$$

Here is the most important information summarized, for the standard character ψ:

F	χ	L-factor	$f \in S(F)$ such that $Z(f,\chi) = L(\chi)$	$\gamma(\chi,\psi)$	$\varepsilon(\chi,\psi)$
$\mathbb{R}$	$(\operatorname{sgn} x)^\varepsilon\,|x|^s$	$\Gamma_\mathbb{R}(s)$	$x^\varepsilon e^{-\pi x^2}$	$i^\varepsilon \frac{L(\check{\chi})}{L(\chi)}$	i^ε
$\mathbb{C}$	$\left(\frac{z}{\sqrt{z\overline{z}}}\right)^n |z|^s$	$\Gamma_\mathbb{C}(s+\frac{|n|}{2})$	$\begin{cases} \frac{1}{\pi}\overline{z}^{|n|}e^{-2\pi z\overline{z}} & \text{if } n \geq 0, \\ \frac{1}{\pi}z^{|n|}e^{-2\pi z\overline{z}} & \text{if } n \leq 0. \end{cases}$	$i^{|n|}\frac{L(\check{\chi})}{L(\chi)}$	$i^{|n|}$
nonarch.	$|\cdot|^s$	$\frac{1}{1-q^{-s}}$	$N\mathfrak{d}^{\frac{1}{2}}\mathbb{1}_{\mathcal{O}_F}$	$N\mathfrak{d}^{\frac{1}{2}-s}\frac{L(\check{\chi})}{L(\chi)}$	$N\mathfrak{d}^{\frac{1}{2}-s}$
nonarch.	$\chi_0|\cdot|^s$ χ_0 ramified, with $\chi_0(\varpi) = 1$	1	$N\mathfrak{d}^{\frac{1}{2}}\chi^{-1}\mathbb{1}_{\mathcal{O}_F^\times}$	$N(\mathfrak{d}\mathfrak{f})^{\frac{1}{2}-s}\rho_0(\chi_0)^{-1}$	$\gamma(\chi,\psi)$

TABLE 1. Local data

[2] In some sources (e.g. [Tat79] or [RV99]), the ε-factors are denoted to be dependent of the additive measure dx, but for us we only consider the Haar measure that is self-dual with respect to ψ, so it is determined by ψ.

Here,

$$\rho_0(\chi_0) = N\mathfrak{f}^{-\frac{1}{2}} \sum_{\varepsilon \in \mathcal{O}_F^\times/(1+\mathfrak{f})} \chi_0(\varepsilon)\psi\left(\frac{\varepsilon}{\varpi^{v(\mathfrak{d}\mathfrak{f})}}\right).$$

where we write $\chi = \chi_0 |\cdot|^s$ with $\chi_0(\varpi) = 1$, and $\mathfrak{f}$ is the conductor of χ_0. Also in the row with $F = \mathbb{C}$, when we write $|n|$, we mean the usual absolute value, so $|n| = \pm n$.

Proposition 1.7 ([Bum97, Proposition 3.1.9])**.**

(1) $\varepsilon(\chi, s, \psi)\varepsilon(\chi^{-1}, 1-s, \psi) = \chi(-1)$.

(2) $\varepsilon(\chi, s, \psi_a) = \chi(a)|a|^{s-\frac{1}{2}} \varepsilon(\chi, s, \psi)$.

(3) *For fixed χ, $\varepsilon(\chi, s, \psi)$ is a function of exponential type, i.e. of the form ab^s with $a \in \mathbb{C}^\times, b \in \mathbb{R}$.*

(4) *If F is nonarchimedean, χ unramified and the conductor of ψ is $\mathcal{O}_F$, then $\varepsilon(\chi, s, \psi) = 1$.*

(5) *If F is nonarchimedean and χ is ramified, then $\varepsilon(\chi, s, \psi) = \gamma(\chi, s, \psi)$.*

Proof.

(1) Follows from Proposition 1.5

(2) Follows from Proposition 1.5

(3)

(4) From the table.

(5) In this case both L-factors in the definition of the ε-factor are 1.

□

1.3. **Viewpoint of Invariant Distributions**

TODO (reference: [Kud04])

2. Background on Adeles

We fix some notation. F is a number field, and $\mathbb{A}$ (resp. $\mathbb{A}^\times$) the ring of adeles (resp. group of ideles) of F. S_∞ denotes the set of infinite places of F. For a finite set S of places of F, containing S_∞, we let

$$\mathbb{A}_S = \prod_{v \in S} F_v \times \prod_{v \notin S} \mathcal{O}_v,$$
$$\mathbb{A}_S^\times = \prod_{v \in S} F_v^\times \times \prod_{v \notin S} \mathcal{O}_v^\times,$$
$$\mathbb{A}^S = \mathbb{A} \cap \prod_{v \notin S} F_v = \{(x_v) \in \prod_{v \notin S} F_v \mid x_v \in \mathcal{O}_v \text{ for almost all } v\},$$
$$\mathbb{A}^{\times,S} = \mathbb{A}^\times \cap \prod_{v \notin S} F_v^\times = \{(x_v) \in \prod_{v \notin S} F_v^\times \mid x_v \in \mathcal{O}_v^\times \text{ for almost all } v\},$$

$$\widehat{\mathcal{O}}^S = \prod_{v \notin S} \mathcal{O}_v,$$
$$\mathcal{O}^S = F \cap \widehat{\mathcal{O}}^S = \{x \in F \mid x \in \mathcal{O}_v \text{ for all } v \notin S\}.$$

We also set $\mathbb{A}_{\mathrm{f}} = \mathbb{A}^{S_\infty}$ and

$$F_\infty = \prod_{v \in S_\infty} F_v = F \otimes_{\mathbb{Q}} \mathbb{R},$$

so that $\mathbb{A} = F_\infty \times \mathbb{A}_{\mathrm{f}}$.

Recall that F sists discretely inside $\mathbb{A}$. If we drop just one of the places, the following happens:

Theorem 2.1 (Strong Approximation Theorem, [Cas67, § 15 Theorem])**.** *Let v_0 be any place of F. Let $V = \prod_{v \neq v_0}(F_v, \mathcal{O}_v)$ be the restricted direct product over all places except v_0. Then the image of the diagonal map $F \to V$ is dense.*

Explicitly, this means: Let places $v_1, \dots, v_n$ and $x_1 \in F_{v_1}, \dots, x_v \in F_{v_n}$ be given. For any $\varepsilon > 0$ there exists $x \in F$ such that $|x - x_i|_{v_i} < \varepsilon$ for $i = 1, \dots, n$, and $x \in \mathcal{O}_v$ for all $v \neq v_0, v_1, \dots, v_n$.

For comparison the *weak approximation theorem* says that the image of F in $\prod_{v \in S} F_v$ is dense for any finite set S of places. In the following will ever only need the weak version (I think).

Lemma 2.2 (Adelic Minkowski Lattice Theorem)**.** *There is a constant $c > 0$ such that for any $a \in \mathbb{A}^\times$ with $|a| > c$, there exists $x \in F^\times$ such that $|x|_v \leq |a_v|_v$ for all v.*

Proof. The same proof as the usual Minkowski lattice point theorem works, using $\mathbb{A}/F$ has finite volume and the measure-theoretic pigeonhole principle. □

Proof of Theorem 2.1. Let places $v_1, \dots, v_n$ and $x_1 \in F_{v_1}, \dots, x_v \in F_{v_n}$ be given. Let $\varepsilon > 0$. There are $\delta_v > 0$ such that $\delta_v = 1$ for almost all v and the set $X = \{x \in \mathbb{A} \mid |x_v|_v \leq \delta_v \, \forall v\}$ surjects onto $\mathbb{A}/F$. It follows from Lemma 2.2 that there exists $\lambda \in F^\times$ such that $|\lambda|_v < \delta_{v_i}^{-1}\varepsilon$ for all $i = 1, \dots, n$ and $|\lambda|_v < \delta_v^{-1}$ for all $v \neq v_0, \dots, v_n$. Let y be any adele with $y_{v_i} = x_i$ for $i = 1, \dots, n$, and integral components elsewhere. Then since X surjects onto $\mathbb{A}/F$ there exists $z \in F$ such that $\lambda^{-1}y = a + z$ where $a \in X$. Then $y = \lambda a + \lambda z$. Let $x = \lambda z$. This works because $|(\lambda a)_v|_v < 1$ for $v \neq v_0, \dots, v_n$ and $|(\lambda a)_v| < \varepsilon$ for $v = v_1, \dots, v_n$. □

The absolute value $|x|$ of $x \in \mathbb{A}$ is $|x| = \prod_v |x_v|_v$, it is trivial on $F^\times$ by the product formula. We denote the subgroup of $\mathbb{A}^\times$ consisting of the elements x with $|x| = 1$ by $\mathbb{A}^{\times,1}$. The idele class group of F is $C = C_F = \mathbb{A}^\times/F^\times$ and we denote $C^1 = \mathbb{A}^{\times,1}/F^\times$.

Let I be the set of nonzero fractional ideals of F, and P the subgroup of fractional principal ideals, so that $I/P = \mathrm{Cl}_F$ is the ideal class group of F. There is a surjective map $\mathrm{Id} : \mathbb{A}^\times \to I$ given by $\mathrm{Id}(x_v) = \prod_{v \nmid \infty} \mathfrak{p}_v^{v(x_v)}$ where $\mathfrak{p}_v$ is the prime ideal of F corresponding to v. The kernel of this map is $\mathbb{A}^\times_{S_\infty}$, so we get an isomorphism $\mathbb{A}^\times/(F^\times \mathbb{A}^\times_{S_\infty}) \cong \mathrm{Cl}_F$.

Theorem 2.3. *C^1 is compact.*

This implies that $\mathbb{A}_S^{\times,1}/(\mathbb{A}_S^{\times,1} \cap F^\times)$ is compact as this embeds as an open and closed subgroup of $\mathbb{A}^{\times,1}/F^\times$. Also note that $\mathbb{A}_S^{\times,1} \cap F^\times = (\mathcal{O}^S)^\times$.

We show how to deduce the unit theorem and the finiteness of the class number from this. The class number is easy: $\mathbb{A}^{\times,1}$ maps continuously onto Cl (where Cl has the discrete topology) via Id, and we get a surjective continuous map $C^1 \to \mathrm{Cl}$, hence Cl is compact and discrete, so finite. For the unit theorem, we consider the usual logarithmic map. Let $S \supseteq S_\infty$ be a finite set of places. Consider the map

$$\begin{aligned} l : \mathbb{A}_S^\times &\longrightarrow \mathbb{R}^S \\ (x_v)_v &\longmapsto (\log |x_v|_v)_{v \in S}. \end{aligned}$$

Then $\mathbb{A}_S^{\times,1} := \mathbb{A}_S^\times \cap \mathbb{A}^{\times,1}$ maps surjectively onto the trace 0 hyperplane H consisting of elements $(z_v)_{v\in S} \in \mathbb{R}^S$ satisfying the equation

$$\sum_{v\in S} z_v = 0.$$

The image of $l((\mathcal{O}^S)^\times)$ is a discrete subgroup of H (this is not difficult to see). It is easy to see that $\ker l \cap (\mathcal{O}^S)^\times$ is the group of roots of unity, so it suffices to show that $l((\mathcal{O}^S)^\times)$ is a lattice of full rank $(\#S-1)$ in H. By the theorem $(\mathcal{O}^S)^\times$ is cocompact in $\mathbb{A}_S^{\times,1}$, hence $l((\mathcal{O}^S)^\times)$ is cocompact in H, which implies that $l((\mathcal{O}^S)^\times)$ is a lattice of full rank in H. We get $(\mathcal{O}^S)^\times \cong \mathbb{Z}^{\#S-1} \times \mu(F)$.

More succinctly, the equivalence of Theorem 2.3 and the combination of finiteness of class group and unit theorem, is expressed in the exact sequence

$$0 \to \mathbb{A}_\infty^{\times,1}/\mathcal{O}_F^\times \to \mathbb{A}^{\times,1}/F^\times \to \mathrm{Cl}_F \to 0.$$

Direct proof of Theorem 2.3 (from [Gar18]). Let $X \subseteq \mathbb{A}$ be compact with measure larger than the one of $\mathbb{A}/F$ (i.e. 1 in our case). For $\alpha \in \mathbb{A}^{\times,1}$, αX has the same measure as X, hence there are $x_1 \neq x_2 \in X$ such that $\alpha x_1, \alpha x_2$ have the same image in $\mathbb{A}/F$, i.e. there is $a \in F^\times$ such that $a = \alpha(x_1 - x_2) \in \alpha(X - X) \cap F^\times$. Similarly there is $b \in \alpha^{-1}(X-X) \cap F^\times$. We have

$$ab = (a\alpha^{-1})(b\alpha) \in (X-X)^2 \cap F^\times.$$

Let $Z = (X-X)^2 \cap F^\times$. $(X-X)^2$ is compact in $\mathbb{A}$, and $F^\times$ is discrete in $\mathbb{A}$, hence Z is finite. For $z \in Z$, let $Y_z \subseteq \mathbb{A}^\times$ be the subset

$$Y_z = \{\beta \in \mathbb{A}^\times \mid \beta \in X - X, \beta^{-1} \in z^{-1}(X-X)\} = i^{-1}\Big((X-X) \times z^{-1}(X-X)\Big)$$

Here i is the map $i : \mathbb{A}^\times \to \mathbb{A} \times \mathbb{A}$, given by $i(x) = (x, x^{-1})$. i is a homoeomorphism onto its image, and the image is closed in $\mathbb{A} \times \mathbb{A}$. In particular Y_z is a compact set in $\mathbb{A}^\times$. Hence, the finite union $\bigcup_{z\in Z} Y_z$ is compact. We show that it surjects onto $\mathbb{A}^{\times,1}/F^\times$, finishing the proof. Let $\beta \in \mathbb{A}^{\times,1}$. Let $\alpha = \beta^{-1}$. As in the beginning there are $a, b \in F^\times$ such that $a\alpha^{-1} \in X - X$ and $b\alpha \in X - X$. Let $z = ab \in Z$. Then $a\beta \in X - X$ and $(a\beta)^{-1} = bz^{-1}\alpha \in z^{-1}(X-X)$, hence $a\beta \in Y_z$, as desired. □

2.1. **Adelic Realization of Ray Class Groups**

Let $\mathfrak{m}$ be a cycle of F, i.e. a formal finite product of places of F, where each place v occurs with finite nonnegative multipicity $m(v)$. Real primes occur with multiplicity at most 1, while complex primes have multiplcity 0. If $a = (a_v)_v \in \mathbb{A}^\times$, we write $a \equiv 1 \bmod \mathfrak{m}$ if $a_v \in 1 + \mathfrak{p}_v^{m(v)}\mathcal{O}_v$ for nonarchimedean

$v \mid \mathfrak{m}$ and $a_v > 0$ for $v \mid \mathfrak{m}$ real. The subgroup of ideles $\equiv 1 \bmod \mathfrak{m}$ is denoted $\mathbb{A}_{\mathfrak{m}}^{\times}$. Let $F_{\mathfrak{m}} = F \cap \mathbb{A}_{\mathfrak{m}}^{\times}$. We denote by $S(\mathfrak{m})$ the set of places v of F such that $m(v) > 0$.

Some more notation. For $v \mid \mathfrak{m}$ we let $W_{\mathfrak{m}}(v)$ be the preimage of $\mathbb{A}_{\mathfrak{m}}^{\times}$ in $F_v^{\times}$, so that $\mathbb{A}_{\mathfrak{m}}^{\times} = \prod_{v \mid \mathfrak{m}} W_{\mathfrak{m}}(v) \times \mathbb{A}^{\times, S(\mathfrak{m})}$. Wor $v \nmid \mathfrak{m}$ we let $W_{\mathfrak{m}}(v) = \mathcal{O}_v^{\times}$ (which we define to be $F_v^{\times}$ for archimedean v). We also define the subgroup $W_{\mathfrak{m}} \subseteq \mathbb{A}_{\mathfrak{m}}^{\times}$ by $W_{\mathfrak{m}} = \prod_{v \mid \mathfrak{m}} W_{\mathfrak{m}}(v) \times \prod_{v \nmid \mathfrak{m}} \mathcal{O}_v^{\times} = \prod_{\mathrm{all} v} W_{\mathfrak{m}}(v)$.

By the approximation theorem the map $\mathbb{A}_{\mathfrak{m}}^{\times} \to \mathbb{A}^{\times}/F^{\times}$ is surjective, hence $C = \mathbb{A}^{\times}/F^{\times} \cong \mathbb{A}_{\mathfrak{m}}^{\times}/F_{\mathfrak{m}}$.

For a finite set S of primes let I^S be the subgroup of fractional ideals that have no prime divisor in S. If $\mathfrak{m}$ is a cycle we define the subgroup $P_{\mathfrak{m}}$ of $I^{S(\mathfrak{m})}$ as the image of $K_{\mathfrak{m}}$ under Id, i.e. it is the set of principal ideals that have a generator in $K_{\mathfrak{m}}$.

The quotient $\mathrm{Cl}_{\mathfrak{m}} := I^{S(\mathfrak{m})}/P_{\mathfrak{m}}$ is called the *ray class group* of F modulo $\mathfrak{m}$.

The map $\mathrm{Id} : \mathbb{A}^{\times} \to I$ restricts to a surjective map $\mathbb{A}_{\mathfrak{m}}^{\times} \to I^{S(\mathfrak{m})}$ with kernel $W_{\mathfrak{m}}$. The preimage of $P_{\mathfrak{m}}$ under this map is $F_{\mathfrak{m}} W_{\mathfrak{m}}$, hence we get an isomorphism

$$\mathbb{A}^{\times}/F^{\times} W_{\mathfrak{m}} \cong \mathbb{A}_{\mathfrak{m}}^{\times}/F_{\mathfrak{m}} W_{\mathfrak{m}} \cong I^{S(\mathfrak{m})}/P_{\mathfrak{m}} = \mathrm{Cl}_{\mathfrak{m}} .$$

2.2. **Admissible Maps**

Let S be a finite set of places of F containing S_∞. We define the map $\mathrm{Id}^S : \mathbb{A}^{\times} \to I^S$ by $\mathrm{Id}^S(a) = \prod_{v \notin S} \mathfrak{p}_v^{v(a_v)}$, so basically $\mathrm{Id}(a)$ without the primes in S. Let G be a commutative topological group. A homomorphism $\phi : I^S \to G$ is called *admissible* if for every neighborhood N of 1 in G there is $\varepsilon > 0$ such that $\varphi(\mathrm{Id}^S a) \in N$ whenever $a \in F^{\times}$ is such that $|a-1|_v \leq \varepsilon$ for all $v \in S$.

Proposition 2.4 ([Tat67b, Proposition 4.1])**.** *Assume in addition to the above that G is complete. Let $\phi : I^S \to G$ be an admissible map. There is a unique homomorphism $\psi : \mathbb{A}^{\times} \to G$ such that*

(i) *ψ is continuous,*

(ii) *ψ is trivial on $F^{\times}$,*

(iii) *$\psi(a) = \phi(\mathrm{Id}^S a)$ for all $a \in \mathbb{A}^{\times, S}$.*

Conversely, suppose $\psi : \mathbb{A}^{\times} \to G$ is a continuous homomorphism that is trivial on $F^{\times}$. If G has no small subgroups, then there is a finite set $S \supseteq S_\infty$ of places and an admissible map $\phi : I^S \to G$ such that ψ is the map associated to ϕ as above.

Proof. Conditions (ii) and (iii) define ψ uniquely on $F^{\times}\mathbb{A}^{\times, S}$. By the approximation theorem $F^{\times}\mathbb{A}^{\times, S}$ is dense in $\mathbb{A}^{\times}$. Using the admissibility of ϕ and the completeness of G we can extend ψ to a continuous homomorphism on all of $\mathbb{A}^{\times}$. Uniqueness is clear.

For the converse, let $\psi : \mathbb{A}^{\times} \to G$ be a continuous homomorphism that is trivial on $F^{\times}$, and assume G has no small subgroups. Let N be a neighborhood of 1 in G such that N contains no nontrivial subgroups (this is the no small subgroups hypothesis). Then for $S \supseteq S_\infty$ large enough we have $\psi(U^S) \subseteq N$, where $U^S = \prod_{v \notin S} \mathcal{O}_v^{\times}$, since ψ is continuous. As U^S, and hence $\psi(U^S)$, is a subgroup, we have $\psi(U^S) = \{1\}$. Then ψ descends to a map $\mathbb{A}^{\times, S}/U^S \to G$. We know that $\mathbb{A}^{\times, S}/U^S$ is isomorphic to I^S via Id^S, hence we get a map $\phi : I^S \to G$ satisfying $\phi(\mathrm{Id}^S a) = \psi(a)$ for all $a \in \mathbb{A}^{\times, S}$. To show that ϕ is admissible, let N' be a neighborhood of 1 in G. Suppose $\varepsilon > 0$ and $a \in F^{\times}$

is such that $|a-1|_v < \varepsilon$ for $v \in S$. Write $a = a_S a^S$ with $a_S \in \prod_{v\in S} F_v^\times$ and $a^S \in \mathbb{A}^{\times,S}$. Then $\varphi(\mathrm{Id}^S a) = \varphi(\mathrm{Id}^S a^S) = \psi(a^S) = \psi(a_S)^{-1}\psi(a) = \psi(a_S)^{-1}$. Since ψ is continuous, $\psi(a_S)^{-1} \in N'$ for ε small enough (independently of a). $\square$

Note as an important special case take G discrete. Then $\phi : I^S \to G$ is admissible if and only if ϕ factors through $I^S/P_\mathfrak{m}$ for some cycle $\mathfrak{m}$ with $S(\mathfrak{m}) = S$. In this case the map ψ is given by the composition

$$\mathbb{A}^\times \to \mathbb{A}^\times/F^\times \cong \mathbb{A}_\mathfrak{m}^\times/F_\mathfrak{m} \to \mathbb{A}_\mathfrak{m}^\times/F_\mathfrak{m} W_\mathfrak{m} \cong I^S/P_\mathfrak{m} \to G.$$

3. Global Theory

Let F be a number field. A place of F is usually denoted v and the corresponding completion F_v.

A quasi-character on $\mathbb{A}^\times$ will always be assumed to be trivial on $F^\times$.

3.1. **Haar Measures**

Fix a nontrivial character ψ of $\mathbb{A}$ that is trivial on F. Via the inclusion $F_v \hookrightarrow \mathbb{A}$, we get in this way a nontrivial[3] character ψ_v on F_v for each place v of F. Then for $x = (x_v)_v \in \mathbb{A}$ we have $\psi_v(x_v) = 1$ for almost all v, and $\psi = \bigotimes_v \psi_v$.

It is a fact that $\mathbb{A} \to \widehat{\mathbb{A}}, a \mapsto \psi_a$ where $\psi_a(x) = \psi(ax)$ is an isomorphism. Under this isomorphism we have $\widehat{A/F} = F^\perp \cong F$, i.e. ψ_a is trivial on F if and only if $a \in F$.[4]

For each place v we get a self-dual Haar measure $\mathrm{d}x_v$ on F_v with respect to ψ_v. We can then define a self-dual Haar measure $\mathrm{d}x$ on $\mathbb{A}$ by the formula $\mathrm{d}x = \bigotimes_v \mathrm{d}x_v$. Note that as in the local case the self-dual measure for ψ_a is $|a|^{1/2}\,\mathrm{d}x$. Since $|a| = 1$ for $a \in F^\times$, we see that the self-dual measure is indepedendent of the choice of ψ as long as ψ is trivial on F.

If for every place v of F we choose the standard character ψ_v as in Section 1.1, then it is easy to check that $\psi = \bigotimes_v \psi_v$ defines a non-trivial character on $\mathbb{A}$ that is trivial on F. We again call this the standard character.

Suppose F is given the counting measure. Then we get an induced Haar measure on the compact quotient A/F such that

$$\int_{\mathbb{A}/F} \sum_{\xi\in F} f(\xi + x)\mathrm{d}(x+F) = \int_{\mathbb{A}} f(x)\mathrm{d}x$$

for $f \in C_c(\mathbb{A})$. Note that with this definition if $X \subseteq \mathbb{A}$ is such that the map $X \to \mathbb{A}/F$ is injective then the measure of X in $\mathbb{A}$ is the same as that in $\mathbb{A}/F$.

[3] That this character is nontrivial seems somewhat nontrivial (...) to me. One way to see this is as follows: Let φ_v be the standard character on F_v and $\varphi = \bigotimes_v \varphi_v$. For this character we know that every character of $\mathbb{A}$ is of the form $\varphi_a(x) = \varphi(ax)$ with $a \in \mathbb{A}$ (because φ_v is nontrivial for all v), and φ_a is trivial on F iff $a \in F$. Hence $\psi = \varphi_a$ for some $a \in F$ and $a \neq 0$ as ψ is nontrivial, then ψ_v is nontrivial for all v.

[4] Note this part is different from the analogous situation $\mathbb{R}, \mathbb{Z}$. If ψ is a non-trivial character of $\mathbb{R}$ trivial on $\mathbb{Z}$, then under the identification $\mathbb{R} \cong \widehat{\mathbb{R}}$ induced by ψ, $\mathbb{Z}^\perp$ may be strictly larger than $\mathbb{Z}$. The reason is that F is an infinite field, so it cannot be a finite index proper subgroup of $F^\perp$ (which is also a field), while $\mathbb{Z}$ can be.

Proposition 3.1. *A/F has measure* 1.

Proof. There are several ways of doing this. Here we give an explicit computation using a fundamental domain, but later in Proposition 3.2 we will see it also follows from the Poisson summation formula. Basically the point is roughly that the dual measure of a group is compatible with quotients and the dual measure of a discrete group gives the compact dual group the measure 1.

We noted above that the Haar measure on $\mathbb{A}$ is independent of the choice of ψ (under the restriction that ψ be trivial on F), so we might as well choose the standard character.

To compute the volume of A/F we find a fundamental domain of F in $\mathbb{A}$. This is accomplished as follows. Suppose $\omega_1, \dots, \omega_n$ is a basis for the ring of integers $\mathcal{O}_F$ over $\mathbb{Z}$. Let $D_\infty \subseteq F_\infty = F \otimes \mathbb{R}$ be the (half-open) parallelotope spanned by the ω_i. Then it is easy to see that $D := D_\infty \times \widehat{\mathcal{O}}_F$, where $\widehat{\mathcal{O}}_F = \prod_{v\nmid\infty} \mathcal{O}_{F_v}$ is a fundamental domain for F in $\mathbb{A}$. Thus the measure of A/F is

$$\int_D \mathrm{d}x = \int_{D_\infty} \mathrm{d}x_\infty \int_{\widehat{\mathcal{O}}_F} \mathrm{d}x^\infty.$$

We have

$$\int_{\widehat{\mathcal{O}}_F} \mathrm{d}x^\infty = \prod_{v\nmid\infty} \int_{\mathcal{O}_{F_v}} \mathrm{d}x_v = \prod_{v\nmid\infty} N\mathfrak{d}_v^{-1/2} = N\mathfrak{d}^{-1/2} = |d|^{-1/2},$$

where d is the absolute discriminant of F. The computation of the volume fo D_∞ is very classical from Minkowski theory. We briefly recall the argument. We identify $F_\infty = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. Let $\omega_i^{(j)}$, $j = 1, \dots, n$ be the conjugates of ω_i such that $\omega_i^{(j+r_2)} = \overline{\omega_i^{(j)}}$ for $r_1 < j \le r_1 + r_2$. In $\mathbb{R}^n$ the volume of D_∞ would be $|\det A|$ where A is the matrix with columns $(\omega_i^{(j)})_{i=1,\dots,n}$ for $1 \le j \le r_1$ and columns $(\operatorname{Re}\omega_i^{(j)})_{i=1,\dots,n}, (\operatorname{Im}\omega_i^{(j)})_{i=1,\dots,n}$ for $j = r_1+1, \dots, r_1+r_2$. Note however that in our case we take on the complex factors twice the usual Lebesgue measure, hence the volume of D_∞ is $2^{r_2}|\det A|$. By elementary column operations we see that $|\det A| = 2^{-r_2}|\det(\omega_i^{(j)})_{i,j}| = 2^{-r_2}\sqrt{|d|}$, hence the result. $\square$

3.2. **Poisson Summation Formula**

Definition. *For "nice" functions $f : \mathbb{A} \to \mathbb{C}$ define the Fourier transform by*

$$\widehat{f}(\xi) = \int_{\mathbb{A}} f(x)\psi(\xi x)\mathrm{d}x.$$

Proposition 3.2 (Poisson Summation Formula)**.** *Let $f \in C(\mathbb{A}) \cap L^1(\mathbb{A})$ such that $\sum_{\xi\in F} f(x+\xi)$ is locally uniformly convergent and $\sum_{\xi\in F} |\widehat{f}(\xi)| < \infty$. Then*

$$\sum_{\eta\in F} f(\eta) = \sum_{\xi\in F} \widehat{f}(\xi)$$

Proof. The proof is the usual one, the point is to consider the F-periodic function $\varphi(x) = \sum_{\xi\in F} f(x+\xi)$ and compute its Fourier series. If we had not computed the volume of A/F earlier to be 1, this computation would only give

$$\mathrm{vol}(A/F)\sum_{\xi\in F} f(\xi) = \sum_{\eta\in F}\widehat{f}(\eta)$$

and then iteration of this formula using $\widehat{\widehat{f}}(\xi) = f(-\xi)$ (for suitable functions) would give $\mathrm{vol}(A/F)^2 = 1$, hence $\mathrm{vol}(A/F) = 1$. So we get the promised second proof of $\mathrm{vol}(A/F) = 1$. □

Corollary 3.3. *Let $f \in C(\mathbb{A})\cap L^1(\mathbb{A})$ such that $\sum_{\xi\in F} f(a(x+\xi))$ converges for all $a \in \mathbb{A}^\times, x \in \mathbb{A}$, locally uniformly in x, and $\sum_{\xi\in F} |\widehat{f}(a\xi)| < \infty$ for all $a \in \mathbb{A}^\times$. Then*

$$\sum_{\eta\in F} f(a\eta) = \frac{1}{|a|}\sum_{\xi\in F}\widehat{f}(\xi/a)$$

for $a \in \mathbb{A}^\times$.

3.3. **Zeta Functions**

We consider a class $\mathfrak{Z}$ of functions $f : \mathbb{A} \to \mathbb{C}$ satisfying

(1) f and $\widehat{f}$ are continuous,

(2) $\sum_{\xi\in F} f(\alpha(x+\xi))$ and $\sum_{\xi\in F}\widehat{f}(\alpha(x+\xi))$ are convergent for each $\alpha \in \mathbb{A}^\times$ and $x \in \mathbb{A}$. The convergence is assumed to be locally uniform in x and a.

(3) $f(\alpha)\,|\alpha|^\sigma$ and $\widehat{f}(\alpha)\,|\alpha|^\sigma$ are in $L^1(\mathbb{A}^\times)$ for all $\sigma > 1$

A Bruhat Schwartz function is a finite linear combination of functions of the form $\bigotimes_v f_v$ with $f_v \in S(F_v)$ and $f_v = \mathbb{1}_{\mathcal{O}_v}$ for almost all v. In other words, the Bruhat Schwartz space $S(\mathbb{A})$ is the restricted tensor product of the local Bruhat Schwartz spaces $S(F_v)$.

Proposition 3.4. *The Schwartz functions lie in $\mathfrak{Z}$.*

Proof. See [RV99, Lemma 7.6] for the convergence of the sums. As for the integrals, if $f = \bigotimes_v f_v$, then

$$\int_{\mathbb{A}^\times} |f(x)|\,|x|^\sigma\,\mathrm{d}^\times x = \prod_v \int_{F_v^\times} |f_v(x_v)|\,|x_v|_v^\sigma\,\mathrm{d}^\times x_v.$$

For almost all v we have $f_v = \mathbb{1}_{\mathcal{O}_v}$ in which case the integral evaluates to $\frac{1}{1-q_v^{-\sigma}}$. The convergence therefore boils down to the convergence of the infinite product

$$\prod_{v\text{ finite}} \frac{1}{1-q_v^{-\sigma}}.$$

This can be reduced to the case of $\mathbb{Q} = F$, where it is classical. □

Let χ be a quasi-character on $\mathbb{A}^\times$. There is a unique $s \in \mathbb{C}$ such that $\chi = \chi_0\,|\cdot|^s$ where χ_0 is unitary. $\sigma := \mathrm{Re}\,s$ is called the *exponent* of χ.

Definition. *We define the zeta distribution (or function) $Z(f,\chi)$ for $f \in \mathfrak{Z}$ by*
$$Z(f,\chi) = \int_{\mathbb{A}^\times} f(\alpha)\chi(\alpha)\,\mathrm{d}^\times\alpha$$

The integral defining $Z(f,\chi)$ converges in the domain of quasi-characters of exponent > 1.

As in the local theory we define $\check{\chi} = |\cdot|\,\chi^{-1}$. We have $\widecheck{\chi\,|\cdot|^s} = \chi^{-1}\,|\cdot|^{1-s}$. The exponent of $\check{\chi}$ is $1-\sigma$.

Theorem 3.5 ([Tat67a, Theorem 4.4.1])**.** *$Z(f,\chi)$ can be extended to a meromorphic function on the space of quasi-characters of $\mathbb{A}^\times$ and satisfies the functional equation*
$$Z(\widehat{f},\check{\chi}) = Z(f,\chi)$$
$Z(f,\chi)$ is holomorphic except for simple poles at $\chi = 1$ and $\chi = |\cdot|$ with residues $-\kappa f(0)$ and $\kappa\widehat{f}(0)$ where κ is the volume of $\mathbb{A}^{\times,1}/F^\times$.

Proof. We have
$$Z(f,\chi) = \int_{\mathbb{A}^\times} f(a)\chi(a)\mathrm{d}^\times a = \int_{\substack{\mathbb{A}^\times\\|a|<1}} f(a)\chi(a)\mathrm{d}^\times a + \int_{\substack{\mathbb{A}^\times\\|a|>1}} f(a)\chi(a)\mathrm{d}^\times a.$$
The second integral is no problem. It converges for all χ and defines an entire function. We have to examine the first integral. We write
$$\begin{aligned}\int_{\substack{\mathbb{A}^\times\\|a|<1}} f(a)\chi(a)\mathrm{d}^\times a &= \int_{\substack{\mathbb{A}^\times/F^\times\\|a|<1}} \sum_{\xi\in F^\times} f(a\xi)\chi(a\xi)\mathrm{d}^\times a\\ &= \int_{\substack{\mathbb{A}^\times/F^\times\\|a|<1}} \sum_{\xi\in F} f(a\xi)\chi(a)\mathrm{d}^\times a - f(0)\int_{\substack{\mathbb{A}^\times/F^\times\\|a|<1}} \chi(a)\mathrm{d}^\times a\end{aligned}$$
We consider these two pieces separately. For the first we can apply Corollary 3.3 to get
$$\sum_{\xi\in F} f(a\xi) = \frac{1}{|a|}\sum_{\xi\in F}\widehat{f}(\xi/a),$$
so
$$\begin{aligned}\int_{\substack{\mathbb{A}^\times/F^\times\\|a|<1}} \sum_{\xi\in F} f(a\xi)\chi(a)\mathrm{d}^\times a &= \int_{\substack{\mathbb{A}^\times/F^\times\\|a|<1}} \frac{1}{|a|}\sum_{\xi\in F}\widehat{f}(\xi/a)\chi(a)\mathrm{d}^\times a\\ &= \int_{\substack{\mathbb{A}^\times/F^\times\\|a|>1}} \sum_{\xi\in F}\widehat{f}(a\xi)\chi(a)^{-1}\,|a|\,\mathrm{d}^\times a\\ &= \int_{\substack{\mathbb{A}^\times/F^\times\\|a|>1}} \sum_{\xi\in F^\times}\widehat{f}(a\xi)\check{\chi}(a)\mathrm{d}^\times a + \widehat{f}(0)\int_{\substack{\mathbb{A}^\times/F^\times\\|a|>1}} \check{\chi}(a)\mathrm{d}^\times a\\ &= \int_{\substack{\mathbb{A}^\times\\|a|>1}} \widehat{f}(a)\check{\chi}(a)\mathrm{d}^\times a + \widehat{f}(0)\int_{\substack{\mathbb{A}^\times/F^\times\\|a|>1}} \check{\chi}(a)\mathrm{d}^\times a\end{aligned}$$
We now compute the second integral. We pick a splitting, which we just denote by $t \mapsto t$, of $|\cdot| : \mathbb{A}^\times \to \mathbb{R}_{>0}$. Note the splitting then gives $\mathbb{R}_{>0}$ the measure $\frac{\mathrm{d}t}{t}$, regardless if we choose a real or complex place

(in which case $t \mapsto \sqrt{t} \in \mathbb{C}^\times$). Then we have

$$\int_{\substack{\mathbb{A}^\times/F^\times \\ |a|<1}} \chi(a)\mathrm{d}^\times a = \int_0^1 \int_{\substack{\mathbb{A}^\times/F^\times \\ |a|=1}} \chi(ta)\mathrm{d}^\times a \frac{\mathrm{d}t}{t} = \int_0^1 \chi(t)\frac{\mathrm{d}t}{t} \int_{\mathbb{A}_1^\times/F^\times} \chi(a)\mathrm{d}^\times a$$

If χ is nontrivial on $\mathbb{A}^{\times,1}$, the second integral vanishes. Otherwise $\chi = |\cdot|^s$, and we have

$$\int_{\substack{\mathbb{A}^\times/F^\times \\ |a|<1}} \chi(a)\mathrm{d}^\times a = \frac{\kappa}{s}.$$

The same calculation shows

$$\int_{\substack{\mathbb{A}^\times/F^\times \\ |a|>1}} \check{\chi}(a)\mathrm{d}^\times a = \begin{cases} 0 & \text{if } \chi \text{ is nontrivial on } \mathbb{A}^{\times,1}, \\ -\frac{\kappa}{1-s} & \text{if } \chi = |\cdot|^s. \end{cases}$$

Hence putting things together:

$$Z(f,\chi) = \int_{\substack{\mathbb{A}^\times \\ |a|>1}} \widehat{f}(a)\check{\chi}(a)\mathrm{d}^\times a + \int_{\substack{\mathbb{A}^\times \\ |a|>1}} f(a)\chi(a)\mathrm{d}^\times a + \left\{ -f(0)\frac{\kappa}{s} + \widehat{f}(0)\frac{\kappa}{s-1} \right\}$$

where the $\{\}$ term is only there in the case $\chi = |\cdot|^s$. The integrals both converge for all χ, hence we get the analytic continuation of Z. The right side is evidently invariant under $(f,\chi) \to (\widehat{f}, \check{\chi})$, hence we get the functional equation. The residue at $\chi = 1$ is $-\kappa f(0)$ and at $\chi = |\cdot|$ is $\kappa \widehat{f}(0)$. □

Proposition 3.6. *The volume of* $\mathbb{A}_1^\times/F^\times$ *is*

$$\kappa = \frac{2^{r_1}(2\pi)^{r_2} hR}{\sqrt{|d|}w}$$

where r_1, r_2 *are the number of real and complex places of* F, h *is the class number,* R *is the regulator, and* w *is the number of roots of unity in* F*. In particular,* $\kappa = 1$ *if* $F = \mathbb{Q}$.

Proof. This is the proof in [RV99] which I found a little more intuitive to follow than the one in [Tat67a]. Recall the discussion after Theorem 2.3. We have a short exact sequence

$$0 \to \mathbb{A}_\infty^{\times,1}/\mathcal{O}_F^\times \to \mathbb{A}^{\times,1}/F^\times \to \mathrm{Cl}_F \to 0.$$

Hence, $\mathrm{vol}_{\mathrm{d}^\times x}(\mathbb{A}^{\times,1}/F^\times) = h\,\mathrm{vol}_{\mathrm{d}^\times x}(\mathbb{A}_\infty^{\times,1}/\mathcal{O}_F^\times)$. Now consider the logarithmic map

$$\begin{aligned} l : \mathbb{A}_\infty^{\times,1} &\longrightarrow \mathbb{R}^{S_\infty}, \\ (x_v)_v &\longmapsto (\log |x_v|_v)_{v\in S_\infty}. \end{aligned}$$

Then l surjects onto the trace 0 hyperplane H, and has kernel $B = \{x \in \mathbb{A}^\times, |x_v|_v = 1\,\forall v\}$. The subset $F^\times \cap \mathbb{A}_\infty^{\times,1}$ maps onto a complete lattice Λ in H, so we get an exact sequence

$$0 \to B\mathcal{O}_F^\times/\mathcal{O}_F^\times \to \mathbb{A}_\infty^{\times,1}/\mathcal{O}_F^\times \to H/\Lambda \to 0.$$

Now note that $B\mathcal{O}_F^\times/\mathcal{O}_F^\times \cap B/(B\cap\mathcal{O}_F^\times) = B/\mu_F$. Hence we get

$$\mathrm{vol}_{\mathrm{d}^\times x}(\mathbb{A}_\infty^{\times,1}/\mathcal{O}_F^\times) = \mathrm{vol}(B/\mu_F)\,\mathrm{vol}(H/\Lambda).$$

We have $B = \prod_v F_v^{\times,1}$, where $F_v^{\times,1}$ denotes the subset of elements of F_v of absolute value 1. Then

$$\operatorname{vol} F_v^{\times,1} = \begin{cases} 2 & v \text{ real}, \\ 2\pi & v \text{ complex}, \\ N\mathfrak{d}_v^{-\frac{1}{2}} & v \text{ finite}. \end{cases}$$

Therefore

$$\operatorname{vol}_{\mathrm{d}^\times x}(\mathbb{A}_\infty^{\times,1}/\mathcal{O}_F^\times) = \operatorname{vol}(B/\mu_F)\operatorname{vol}(H/\Lambda) = \frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|d|}w}R.$$

The result follows.

Note: need to check that the different Haar measures on the sub- and quotient groups are compatible for this to hold □

3.4. L-Functions

Let $\chi : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$ be a quasi-character. Write χ_v for the induced local quasi-character of $F_v^\times$ for places v of F. Then $\chi = \bigotimes_v \chi_v$.

Definition. *Define*

$$L(\chi) = \prod_v L_v(\chi_v)$$

where $L_v(\chi_v)$ is the local L-factor defined before Theorem 1.6.

As usual we set $L(\chi, s) = L(\chi\,|\cdot|^s)$.

Also recall that we defined certain epsilon factors. We define

$$\varepsilon(\chi) = \prod_v \varepsilon_v(\chi_v, \psi_v).$$

Again we set $\varepsilon(\chi, s) = \varepsilon(\chi\,|\cdot|^s)$. Note that for almost all nonarchimedean places v, ψ_v has conductor $\mathcal{O}_v$ and χ_v is unramified, so that for those v we have $\varepsilon_v(\chi_v, \psi_v) = 1$, hence the product is finite.

Furthermore we note that the product is independent of the choice of ψ (hence we omitted it from the notation). Indeed, for $a \in F^\times$ we have

$$\prod_v \varepsilon_v(\chi_v, (\psi_a)_v) = \prod_v (\chi(a)\,|a|^{-1/2}\,\varepsilon_v(\chi_v, \psi_v)) = \prod_v \varepsilon_v(\chi_v, \psi_v),$$

by the product formula and using $\chi(a) = 1$ for $a \in F^\times$.

Theorem 3.7. *$L(\chi, s)$ admits a meromorphic continuation to the whole complex plane. Poles only occur if $\chi = |\cdot|^\lambda$, in which case the poles are simple and at $s = \lambda, 1 - \lambda$. L satisfies the functional equation*

$$L(\chi, s) = \varepsilon(\chi, s)L(\chi^{-1}, 1 - s).$$

Proof. Let $S \supseteq S_\infty$ be a finite set of places such that for all $v \notin S$, χ_v is unramified and the conductor of ψ_v is $\mathcal{O}_v$ (so $\mathfrak{d}_v = \mathcal{O}_v$). Define

$$L_S(\chi, s) = \prod_{v \notin S} L_v(\chi_v, s)$$

Pick a Bruhat Schwatz function $f \in S(\mathbb{A})$ of the form $f = \bigotimes_v f_v$ with $f_v \in S(F_v)$ such that $f_v = \mathbb{1}_{\mathcal{O}_v}$ for all $v \notin S$. For those v we then have $\widehat{f_v} = f_v$ since ψ_v has conductor $\mathcal{O}_v$. At the remaining $v \in S$ we just require that f_v is a function such that its zeta integral and that of its Fourier transform is non-zero. Then we have

$$Z_v(f_v, \chi_v, s) = (1 - \chi_v(\varpi_v) q_v^{-s})^{-1} = L_v(\chi_v, s),$$
$$Z_v(\widehat{f_v}, \chi_v^{-1}, 1-s) = (1 - \chi_v(\varpi_v)^{-1} q_v^{1-s})^{-1} = L_v(\chi_v^{-1}, 1-s).$$

Therefore

$$L_S(\chi, s) = \prod_{v \notin S} L_v(\chi_v, s) = Z(f, \chi, s) \prod_{v \in S} Z_v(f_v, \chi_v, s)^{-1}.$$

This shows that $L_S(\chi, s)$ admits a meromorphic continuation to all of $\mathbb{C}$, since $Z(f, \chi, s)$ does. Given a point s_0 which is not $\lambda, 1-\lambda$ if $\chi = |\cdot|^\lambda$, then $L_S(\chi, s)$ has no pole at s_0 since $Z(f, \chi, s)$ does not and we can choose f_v so that $Z_v(f_v, \chi_v, s)$ has no zero there.

By the global functional equation Theorem 3.5 we have

$$\begin{aligned} L_S(\chi, s) =& Z(\widehat{f}, \chi^{-1}, 1-s) \prod_{v \in S} Z_v(f_v, \chi_v, s)^{-1} \\ &= L_S(\chi^{-1}, 1-s) \prod_{v \in S} Z_v(f_v, \chi_v, s)^{-1} Z_v(\widehat{f_v}, \chi_v^{-1}, 1-s) \\ &= L_S(\chi^{-1}, 1-s) \prod_{v \in S} \gamma_v(\chi_v, s, \psi_v) \end{aligned}$$

Now multiply this by the remaining L-factors:

$$\begin{aligned} L(\chi, s) &= L_S(\chi, s) \prod_{v \in S} L_v(\chi, s) \\ &= L_S(\chi^{-1}, 1-s) \prod_{v \in S} \gamma_v(\chi_v, s, \psi_v) L_v(\chi, s) \\ &= L_S(\chi^{-1}, 1-s) \prod_{v \in S} \varepsilon_v(\chi_v, s, \psi_v) L_v(\chi_v^{-1}, 1-s, \psi_v) \\ &= L(\chi^{-1}, 1-s) \prod_{v \in S} \varepsilon_v(\chi_v, s, \psi_v) \end{aligned}$$

It remains to notice that $\prod_{v \in S} \varepsilon_v(\chi_v, s, \psi_v) = \varepsilon(\chi_v, s)$ since $\varepsilon_v(\chi_v, s) = 1$ for $v \notin S$ by Proposition 1.7. Finally we have

$$L(\chi, s) = Z(f, \chi, s) \prod_{v \in S} \frac{L_v(\chi_v, s)}{Z_v(f_v, \chi_v, s)}.$$

Since we can choose the f_v such that the local zeta integral is the L function, we get the statement about the poles of L from the corresponding result on the poles of Z. □

3.5. **Examples**

Let F be a number field and ζ_F its Dedekind zeta function defined by

$$\zeta_F(s) = \sum_{\mathfrak{a} \subseteq \mathcal{O}_F} \frac{1}{N\mathfrak{a}^s} = \prod_{\mathfrak{p}} \frac{1}{1 - N\mathfrak{p}^{-s}} = L_{S_\infty}(|\cdot|^s).$$

We complete it to have nicer functional equations:

$$L(|\cdot|^s) = \zeta_F(s) \prod_{v|\infty} L_v(|\cdot|^s) = \zeta_F(s)\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2},$$

where of course r_1, r_2 are the number of real resp. pairs of complex places, and

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2), \quad \Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = 2(2\pi)^{-s}\Gamma(s).$$

Note from the list in Section 1.2 that $\varepsilon(|\cdot|^s) = N\mathfrak{d}^{\frac{1}{2}-s}$. Hence Theorem 3.7 gives the functional equation

$$L(|\cdot|^s) = N\mathfrak{d}^{\frac{1}{2}-s}L(|\cdot|^{1-s}).$$

If we let $\Lambda_F(s) = N\mathfrak{d}^{s/2}L(|\cdot|^s)$, then we have the more symmetrical form

$$\Lambda_F(s) = \Lambda_F(1-s).$$

Theorem 3.8 (Analytic Class Number Formula)**.** *ζ_F has a meromorphic continuation to $\mathbb{C}$ with only a simple pole at $s=1$ with residue*

$$\kappa = \frac{2^{r_1}(2\pi)^{r_2}hR}{\sqrt{|d|}w}.$$

ζ_F has a zero at $s=0$ of order r_1+r_2-1 and leading coefficient

$$-\frac{hR}{w}.$$

Proof. For each place v of F let $f_v \in S(F_v)$ be the function as in the proof of Theorem 1.6 so that $Z_v(f_v, |\cdot|^s) = L_v(|\cdot|^s)$. Then let $f = \bigotimes_v f_v$, so that $L(|\cdot|^s) = Z(f, |\cdot|^s)$. By Theorem 3.7, $Z(f, |\cdot|^s)$ has simple poles at $s=0$ and $s=1$ with residues $-\kappa f(0)$ and $\kappa\widehat{f}(0)$. We have $\widehat{f}(0) = \pi^{-r_2}$. Since

$$\zeta_F(s) = Z(f, |\cdot|^s) \prod_{v|\infty} Z_v(f_v, |\cdot|^s)^{-1} = Z(f, |\cdot|^s) \prod_{v|\infty} L_v(|\cdot|^s)^{-1} = Z(f, |\cdot|^s)\Gamma_{\mathbb{R}}(s)^{-r_1}\Gamma_{\mathbb{C}}(s)^{-r_2},$$

and $\Gamma_{\mathbb{R}}(1) = 1$, $\Gamma_{\mathbb{C}}(1) = \pi^{-1}$, the residue of $\zeta_F(s)$ at $s=1$ is indeed κ. $\Gamma_{\mathbb{R}}, \Gamma_{\mathbb{C}}$ both have simple poles at $s=0$, hence $\zeta_F(s)$ has a zero of order r_1+r_2-1 at $s=0$. To compute the leading coefficient note that $f(0) = \pi^{-r_2}N\mathfrak{d}^{1/2}$, and the residues of $\Gamma_{\mathbb{R}}(s)$ and $\Gamma_{\mathbb{C}}(s)$ at $s=0$ are both 2. Hence the leading coefficient is

$$-\kappa f(0)2^{-r_1-r_2} = -\frac{hR}{w}.$$

$\square$

Next consider the more general situation where we have a cycle $\mathfrak{m}$ of F and a character $\chi_r : I^{S(\mathfrak{m})}/P_{\mathfrak{m}} \to \mathbb{C}^\times$ of the ray class group. We define the L-function of χ_r by

$$L_{\mathfrak{m}}(\chi_r, s) = \sum_{\mathfrak{a}\in I^{S(\mathfrak{m})}} \frac{\chi_r(\mathfrak{a})}{N\mathfrak{a}^s} = \prod_{\mathfrak{p}\nmid\mathfrak{m}} \frac{1}{1-\chi_r(\mathfrak{p})N\mathfrak{p}^{-s}}.$$

Let $\chi : \mathbb{A}^\times/F^\times$ be the idelic lift of χ_r as described in Section 2.2. Let $S = S(\mathfrak{m}) \cup S_\infty$. Then

$$L_{\mathfrak{m}}(\chi_r, s) = \prod_{v\notin S} L_v(\chi_v, s).$$

The completed L-function is

$$L(\chi, s) = L_{\mathfrak{m}}(\chi_r, s) \prod_{v \in S} L_v(\chi_v, s)$$

as in the last section. Note that for finite note that if a finite prime v divides the conductor of χ_r, i.e. if χ_v is ramified, then $L_v(\chi_v, s) = 1$. At the infinite places $L_v(\chi_v, s)$ is again some kind of Gamma function, see Section 1.2.

By Theorem 3.7 we have the functional equation

$$L(\chi, s) = \varepsilon(\chi, s) L(\chi^{-1}, 1 - s).$$

From the list in Section 1.2 we see that

$$\varepsilon(\chi, s) = \prod_{v \in S} \varepsilon_v(\chi_v, s) = ..?..$$

If χ_r is trivial we more or less get the Dedekind zeta function above, if χ_r is non-trivial, then $L_{\mathfrak{i}}(\chi_r, s)$ extends to an entire function. Indeed, by Theorem 3.7 $L(\chi, s)$ is entire,

$$L_{\mathfrak{m}}(\chi_r, s) = L(\chi, s) \prod_{v \in S} L_v(\chi_v, s)^{-1},$$

and the local L functions $L_v(\chi_v, s)$ have no zeros.

4. Class Field Theory

4.1. **Local Class Field Theory**

Let K be a local field. If K is nonarchimedean and L/K is a finite unramified extension, the Frobenius automorphism $\mathrm{Frob}_{L/K}$ of L/K is the unique element in $\mathrm{Gal}(L/K)$ such that

$$\mathrm{Frob}_{L/K}(x) \equiv x^{q_K} \bmod \mathcal{O}_l$$

for all $x \in \mathcal{O}_L$. If L/K is a finite extension, let $\mathcal{N}_L = N_{L/K}(L^\times)$ be its *norm group*.

Theorem 4.1 (Local Reciprocity Law, [Mil20, Theorem I 1.1])**.** *There is a unique homomorphism*

$$\phi_K : K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

such that

(1) If K is nonarchimedean, then for every uniformizer ϖ of K and every finite unramified extension L/K, $\phi_K(\varpi)$ is the Frobenius automorphism $\mathrm{Frob}_{L/K}$.

(2) For every finite abelian extension L/K, $\phi_K(a)|_L$ is trivial for $a \in \mathcal{N}_L$ and ϕ_K induces an isomorphism

$$\phi_{L/K} : K^\times/\mathcal{N}_L \to \mathrm{Gal}(L/K).$$

The image $\phi_{L/K}(a) \in \mathrm{Gal}(L/K)$ of $a \in K^\times$ under the reciprocity homomorphism is also denoted

$$(a, L/K) := \phi_{L/K}(a),$$

and called the *norm residue symbol*.

Abbreviate $\mathrm{Gal}(K^{\mathrm{alg}}/K) = G_K$.

Theorem 4.2 ([Ser67, 2.4])**.** *Let K'/K be a finite separable extension. The follwing diagrams commute:*

$$\begin{tikzcd} K^\times \arrow[r, "\phi_K"] \arrow[d, hook, "incl."] & G_K^{\mathrm{ab}} \arrow[d, "V"] \\ K'^\times \arrow[r, "\phi_{K'}"] & G_{K'}^{\mathrm{ab}} \end{tikzcd} \qquad \begin{tikzcd} K'^\times \arrow[r, "\phi_{K'}"] \arrow[d, "N_{K'/K}"] & G_{K'}^{\mathrm{ab}} \arrow[d] \\ K^\times \arrow[r, "\phi_K"] & G_K^{\mathrm{ab}} \end{tikzcd}$$

Here the map V is the Verschiebung (or transfer).

Theorem 4.3 (Existence Theorem, [Neu99, Theorem V 1.4])**.** *The assignment*

$$L \mapsto \mathcal{N}_L$$

gives a one-to-one correspondence between the finite abelian extensions of K and the open subgroups $\mathcal{N}$ of finite index in $K^\times$. It satisfies

$$L_1 \subseteq L_2 \Longleftrightarrow \mathcal{N}_{L_1} \supseteq \mathcal{N}_{L_2}, \qquad \mathcal{N}_{L_1 L_2} = \mathcal{N}_{L_1} \cap \mathcal{N}_{L_2}, \qquad \mathcal{N}_{L_1 \cap L_2} = \mathcal{N}_{L_1}\mathcal{N}_{L_2}.$$

The field corresponding to an open finite index subgroup $\mathcal{N}$, is called its *class field*.

Proof. Most of the assertions follow easily from the Reciprocity Law and Galois theory. The only nontrivial part is the fact that every open finite index subgroup $\mathcal{N}$ of $K^\times$ is the norm group of some finite abelian extension. □

The following theorem shows that the norm groups can "see" only abelian extensions:

Theorem 4.4 (Norm Limitation Theorem, [Mil20, Theorem III 3.5], [Ser67, Proposition 4])**.** *Let L/K be a finite extension of K and $E = K^{\mathrm{ab}} \cap L$ the largest abelian extension of K in L. Then*

$$N_{L/K}(L^\times) = N_{E/K}(E^\times)$$

Assume K nonarchimedean.

Recall that we have a filtration of the unit group U_K by the subgroups $U_K^{(n)} = 1 + \mathfrak{p}_K^n$ for $n \geq 1$ and $U_K^{(0)} = U_K = \mathcal{O}_K^\times$.

Theorem 4.5. *Let L/K be a finite abelian extension. Then for any $n \geq 0$, $\phi_{L/K} : K^\times \to \mathrm{Gal}(L/K)$ maps $U_K^{(n)}$ to $G^n(L/K)$, the n-th higher ramification group in in the upper numbering.*

Corollary 4.6. *Let L/K be a finite abelian extension. Then $e_{L/K} = [U_K : N_{L/K}(U_L)]$.*

Proof. This follows from the reciprocity law and the short exact sequence

$$0 \to U_K/N_{L/K}(U_L) \to K^\times/N_{L/K}(L^\times) \xrightarrow{v_K} \mathbb{Z}/f_{L/K}\mathbb{Z} \to 0.$$

□

Definition. *Let L/K be a finite abelian extension. Let n be the smallest integer such that $\phi_{L/K}$ is trivial on $U_K^{(n)}$. The* conductor *of L/K, denoted $\mathfrak{f}_{L/K}$, is the ideal $\mathfrak{p}_K^n$.*

Can also define this in the archimedean case.

Corollary 4.7. *L/K is unramified if and only if $\mathfrak{f}_{L/K} = \mathcal{O}_K$.*

Proof. The unramified part is immediate from the corollary. □

Proposition 4.8. *The class field corresponding to $\mathcal{N} = \langle \varpi^f \rangle \times U_K$ is the unique unramified extension of K of degree f.*

Proof. Let L be the class field. Since $U_K \subseteq \mathcal{N}$, the L/K is unramified. The degree is $[L : K] = \#(K^\times/\mathcal{N}) = f$. □

Recall that the unramified extension of K of degree f is $K(\zeta_{q^f-1})$.

Proposition 4.9. *Let L/K be ramified. The following are equivalent:*

(1) L/K is tamely ramified,

(2) $v_L(\mathfrak{d}_{L/K}) = e_{L/K} - 1$.

If L/K is Galois, they are in addition equivalent to

(3) $G_1(L/K) = 1$,

And if L/K is abelian, they are in addition equivalent to

(3) $\mathfrak{f}_{L/K} = \mathfrak{p}_K$.

Proof. □

4.1.1. *Example $K = \mathbb{Q}_p$.*

Theorem 4.10. *Let $L = \mathbb{Q}_p(\zeta)$ where $\zeta = \zeta_m$ is a primitive m-th root of unity.*

- *If $(m,p) = 1$, then*
$$(a, L/\mathbb{Q}_p)(\zeta) = \zeta^{p^{v_p(a)}}.$$
- *If $m = p^n$, then*
$$(a, L/\mathbb{Q}_p)(\zeta) = \zeta^{u^{-1}},$$
where u is the "angular component" of a, i.e. $a = up^{v_p(a)}$.

Proof. The first case is clear since then $L/\mathbb{Q}_p$ is unramified. The second case requires more work, see e.g. [Neu99, Theorem V 2.4]. □

Proposition 4.11 ([Neu99, Proposition V 1.8]). *The norm group of $\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p$ is $\langle p\rangle \times U_{\mathbb{Q}_p}^{(n)}$*

Corollary 4.12 (Local Kronecker-Weber). *Every finite abelian extension L of $\mathbb{Q}_p$ is contained in a cyclotomic extension of $\mathbb{Q}_p$.*

Proof. Since the sets $U_{\mathbb{Q}_p}^{(n)}$ form a basis of neighborhoods at the identity in U_F, $\mathcal{N}_L$ must contain a set of the form $\langle p^f\rangle \times U_{\mathbb{Q}_p}^{(n)}$. We have

$$\langle p^f\rangle \times U_{\mathbb{Q}_p}^{(n)} = (\langle p^f\rangle \times U_{\mathbb{Q}_p}) \times (\langle p\rangle \times U_{\mathbb{Q}_p}^{(n)})$$

The class field of $\langle p^f\rangle \times U_{\mathbb{Q}_p}$ is $\mathbb{Q}(\zeta_{p^f-1})$ by Proposition 4.8, and the class field of $\langle p\rangle \times U_{\mathbb{Q}_p}^{(n)}$ is $\mathbb{Q}_p(\zeta_{p^n})$ by Proposition 4.11, hence L is contained in $\mathbb{Q}_p(\zeta_{p^n(p^f-1)})$. □

4.2. Global Class Field Theory

Let K be a number field. We first give the ideal theoretic version of global class field theory.

Let L/K be a finite abelian extension. Let S be the set containg the infinite places, and the finite places of K that ramify in L. Then we have a well-defined map $F_{L/K} : I^S \to \mathrm{Gal}(L/K)$ such that $F_{L/K}(\mathfrak{p}) = \mathrm{Frob}_{L/K}(\mathfrak{p}) = (\mathfrak{p}, L/K)$, the Frobenius at $\mathfrak{p}$.

If $K \subseteq E \subseteq L$, then the diagrams

$$\begin{tikzcd} I_E^S \arrow[r, "F_{L/E}"] \arrow[d, "N_{E/K}"] & \mathrm{Gal}(L/E) \arrow[d, hook] \\ I_K^S \arrow[r, "F_{L/K}"] & \mathrm{Gal}(L/K) \end{tikzcd} \qquad \begin{tikzcd} I_K^S \arrow[r, "F_{L/K}"] \arrow[d, dash, "="] & \mathrm{Gal}(E/K) \arrow[d, two heads] \\ I_K^S \arrow[r, "F_{E/K}"] & \mathrm{Gal}(L/K) \end{tikzcd}$$

commute. This is immediate from $(N_{E/K}\mathfrak{P}, L/K) = (\mathfrak{p}^{f_{\mathfrak{P}/\mathfrak{p}}}, L/K) = (\mathfrak{p}, L/K)^{f_{\mathbb{P}/\mathfrak{p}}} = (\mathfrak{P}, L/E)$ for primes $\mathfrak{P}$ in E coprime to S that with $\mathfrak{p}$ the prime of K lying below. Note even though S is a set of primes in K, we use the notation I_L^S in the obvious way to denote the set of primes of L coprime to the primes of L lying above the primes in S.

In particular taking $E = L$ gives that $N_{L/K}(I_L^S) \subseteq \ker F_{L/K}$.

Theorem 4.13 (Reciprocity Law, [Mil20, Theorem V 3.5]). *The map $F_{L/K} : I^S \to \mathrm{Gal}(L/K)$ is admissible, i.e. it admits a cycle $\mathfrak{m}$ with $S(\mathfrak{m}) = S$ and $P_\mathfrak{m} \subseteq \ker F_{L/K}$. It defines an isomorphism*

$$I_K^S/(P_\mathfrak{m} N_{L/K} I_L^S) \xrightarrow{\simeq} \mathrm{Gal}(L/K).$$

The group $T(L/K, \mathfrak{m}) = P_\mathfrak{m} N_{L/K}(I_L^\mathfrak{m})$ is also called the *Takagi* group of L/K.

Definition. *The* conductor *of L/K, denoted $\mathfrak{p}_{L/K}$, is the smallest possible cycle for $F_{L/K}$.*

If $\mathfrak{m}$ is a cycle for K, a subgroup of $H \subseteq I^\mathfrak{m} := I^{S(\mathfrak{m})}$ is called a congruence subgroup mod $\mathfrak{m}$ if $P_\mathfrak{m} \subseteq H$.

Theorem 4.14 (Existence Theorem, [Mil20, Theorem V 3.6]). *For every congruence subgroup H modulo $\mathfrak{m}$, there is a unique abelian extension L/K, unramified at the primes not dividing $\mathfrak{m}$, such that $H = P_\mathfrak{m} N_{L/K}(I_L^\mathfrak{m})$. In particular, $F_{L/K}$ gives an isomorphism*

$$I_K^\mathfrak{m}/H \xrightarrow{\simeq} \mathrm{Gal}(L/K)$$

Note we can in particular take $H = P_\mathfrak{m}$. This gives the *ray class field* modulo $\mathfrak{m}$. We denote it by $K^\mathfrak{m}$.

By Galois theory we then get an inclusion reversing bijection between abelian extensions E/K contained in $K^\mathfrak{m}$ and subgroups $H \subseteq I_K^\mathfrak{m}/P_\mathfrak{m} = \mathrm{Cl}_\mathfrak{m}$ via

$$E \longmapsto P_\mathfrak{m} N_{E/K}(I_E^\mathfrak{m}).$$

It is kind of awkard to have to fix a cycle and then work in $\mathrm{Cl}_\mathfrak{m}$. This is where the idelic formulation comes into play and simplify things!

Recall the notation from Section 2.1. We have an isomorphism

$$\mathbb{A}_K^\times/K_\mathfrak{m} W_\mathfrak{m} \cong I_K^\mathfrak{m}/P_\mathfrak{m},$$

under which the class of an idele $a = (a_v)_v \in \mathbb{A}_{K,\mathfrak{m}}^\times$, i.e. an idele satisfying $a \equiv 1 \bmod \mathfrak{m}$, corresponds to the ideal $\mathrm{Id}\, a = \prod_{v\nmid\infty} \mathfrak{p}_v^{v(a_v)}$.

Proposition 4.15 ([Mil20, Proposition V 5.2]). *There is a unique homomorphism $\phi_K : \mathbb{A}_K^\times \to G_K^{\mathrm{ab}} = \mathrm{Gal}(K^{\mathrm{ab}}/K)$ such that for every finite abelian extension L/K and any prime w of L lying over a prime v of K, the diagram*

$$\begin{tikzcd} K_v^\times \arrow[r, "\phi_v"] \arrow[d] & \mathrm{Gal}(L_w/K_v) \arrow[d] \\ \mathbb{A}_K^\times \arrow[r, "a\mapsto\phi_K(a)|_L"] & \mathrm{Gal}(L/K) \end{tikzcd}$$

commutes.

Here ϕ_v denotes the local norm residue symbol defined as in Section 4.1. Since the extension is abelian, the map $\mathrm{Gal}(L_w/K_v) \hookrightarrow \mathrm{Gal}(L/K)$ does not depend on w. We let $\phi_{L/K}$ be the composition of ϕ_K with the restriction $\mathrm{Gal}(K^{\mathrm{ab}}/K) \to \mathrm{Gal}(L/K)$. The map $\phi_{L/K}$ is also denoted $(-, L/K)$.

Proof. This is relatively easy. Define $\phi_{L/K} : \mathbb{A}_K \to \mathrm{Gal}(L/K)$ as the $\phi_{L/K}(a) = \prod_v \phi_v(a_v)$ where $\phi_v(a_v) \in \mathrm{Gal}(L_w/K_v) \hookrightarrow \mathrm{Gal}(L/K)$. Then patch all these maps for different L together. $\square$

For abelian extensions $L'/L/K$ we have a commuting diagram:

$$\begin{tikzcd} \mathbb{A}_K^\times \arrow[r, "\phi_{L'/K}"] \arrow[dr, "\phi_{L/K}"] & \mathrm{Gal}(L'/K) \arrow[d, "\mathrm{res}"] \\ & \mathrm{Gal}(L/K) \end{tikzcd}$$

Let L/K be a finite abelian extension. Let $\mathfrak{f}_{L/K} = \prod_v \mathfrak{f}_{L_w/K_v}$ be the product of the local conductors, possibly including real places, so that $\mathfrak{f}_{L/K}$ is really a cycle. By Corollary 4.7 $\mathfrak{f}$ is precisely divisible by the ramified places. Then the following diagram commutes

$$\begin{tikzcd}
\mathbb{A}_{K,\mathfrak{f}}^{\times} \arrow[r, "\phi_{L/K}"] \arrow[d, "\mathrm{Id}"] & \mathrm{Gal}(L/K) \\
I_K^{\mathfrak{m}} \arrow[ur, "F_{L/K}"'] &
\end{tikzcd}$$

Indeed, $\phi_{L/K}(a) = \prod_v \phi_v(a_v)$. If $a \in \mathbb{A}_{K,\mathfrak{f}}^{\times}$, then $\phi_v(a_v) = 1$, so

$$\begin{aligned}
\phi_{L/K}(a) &= \prod_{v \nmid \mathfrak{f}} \phi_v(a_v) = \prod_{v \nmid \mathfrak{f}} \mathrm{Frob}_{L_w/K_v}^{v(a_v)} \\
&= \prod_{v \nmid \mathfrak{f}} \mathrm{Frob}_{L/K}^{v(a_v)} = F_{L/K}(\mathrm{Id}\, a).
\end{aligned}$$

So by Proposition 2.4, the existence of a cycle for $F_{L/K}$ is essentially equivalent to ϕ_K being trivial on $K^{\times}$:

Theorem 4.16 (Reciprocity Law, [Mil20, Theorem V 5.3])**.** $\phi_K : \mathbb{A}_K^{\times} \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ *is trivial on* $K^{\times}$ *and for every finite abelian extension* L/K*,* ϕ_K *induces an isomorphism*

$$\phi_{L/K} : \mathbb{A}_K^{\times}/(K^{\times} N_{L/K}(\mathbb{A}_L^{\times})) \xrightarrow{\simeq} \mathrm{Gal}(L/K).$$

We can also relate the second part of the statement to the ideal theoretic version: Let L/K be a finite Galois extension. An admissible cycle $\mathfrak{m}$ is one such that $W_{\mathfrak{m}}(v) \subseteq N_{L_w/K_v}(L_w^{\times})$. Equivalently, $W_{\mathfrak{m}} \subseteq N_{L/K}(\mathbb{A}_L^{\times})$. In the abelian case this is the case iff $\mathfrak{f} \mid \mathfrak{m}$. Note we could also define the local conductor for nonabelian Galois extensions and this would still be true. However, the local conductor would only depend on the maximal abelian subextension by Theorem 4.4

Proposition 4.17 ([Lan94, Theorem VII 7])**.** *Let* L/K *be a finite Galois extension. Let* $\mathfrak{n} \mid \mathfrak{m}$ *be admissible cycles for* L/K*. Then the inclusion* $I^{\mathfrak{n}} \hookrightarrow I^{\mathfrak{m}}$ *induces an isomorphism*

$$I^{\mathfrak{m}}/(P_{\mathfrak{m}} N_{L/K}(I_L^{\mathfrak{m}})) \xrightarrow{\simeq} I^{\mathfrak{n}}/(P_{\mathfrak{n}} N_{L/K}(I_L^{\mathfrak{n}})).$$

If $\mathfrak{n}, \mathfrak{m}$ *are divisible by the same primes, then* $P_{\mathfrak{m}} N_{L/K}(I_L^{\mathfrak{m}}) = P_{\mathfrak{n}} N_{L/K}(I_L^{\mathfrak{n}})$*.*

Proposition 4.18. *Let* L/K *be a finite Galois extension and* $\mathfrak{m}$ *an admissible cycle for* L/K*. Then there is an isomorphism*

$$\mathbb{A}_K^{\times}/(K^{\times} N_{L/K}(\mathbb{A}_L^{\times})) \xrightarrow{\simeq} I_K^{\mathfrak{m}}/(P_{\mathfrak{m}} N_{L/K}(I_L^{\mathfrak{m}})).$$

This isomorphism takes any $a \in \mathbb{A}_{K,\mathfrak{m}}^{\times}$ *to the class of* $\mathrm{Id}\, a$*.*

Proof. We proceed in two steps. First we show that $\psi : \mathbb{A}_{K,\mathfrak{m}}^{\times} \to I_K^{\mathfrak{m}}$, $a \mapsto \mathrm{Id}(a)$ induces an isomorphism

$$\mathbb{A}_{K,\mathfrak{m}}^{\times}/(K_{\mathfrak{m}} W_{\mathfrak{m}} N_{L/K}(\mathbb{A}_L^{\times, S(\mathfrak{m})})) \xrightarrow{\simeq} I_K^{\mathfrak{m}}/(P_{\mathfrak{m}} N_{L/K}(I_L^{\mathfrak{m}})),$$

where $\mathbb{A}_L^{\times, S(\mathfrak{m})}$ is the group of ideles in $\mathbb{A}_L$ whose components at places lying over places in $S(\mathfrak{m})$ are 1. That this is an isomorphism is equivalent to

$$K_{\mathfrak{m}} W_{\mathfrak{m}} N_{L/K}(\mathbb{A}_L^{\times, S(\mathfrak{m})}) = \psi^{-1}(P_{\mathfrak{m}} N_{L/K}(I_L^{\mathfrak{m}})).$$

The inclusion "$\subseteq$" is obvious. For the reverse, let $a \in \mathbb{A}_{K,\mathfrak{m}}^\times$ such that $\operatorname{Id} a = \psi(a) = (\alpha)N_{L/K}\mathfrak{a}$ with $\alpha \in K_\mathfrak{m}, \mathfrak{a} \in I_L^\mathfrak{m}$. First note that there is $A \in \mathbb{A}_L^{\times,S(\mathfrak{m})}$ such that $N_{L/K}(\operatorname{Id} A) = N_{L/K}\mathfrak{a}$. Then $\operatorname{Id} a = \operatorname{Id}(\alpha N_{L/K} A)$, hence $a(\alpha N_{L/K}A)^{-1} \in \ker \psi = W_\mathfrak{m}$. This establishes the isomorphism. Next we need to verify that inclusion $i : \mathbb{A}_{K,\mathfrak{m}}^\times \to \mathbb{A}_K^\times$ induces an isomorphism

$$\mathbb{A}_{K,\mathfrak{m}}^\times/(K_\mathfrak{m} W_\mathfrak{m} N_{L/K}(\mathbb{A}_L^{\times,S(\mathfrak{m})})) \xrightarrow{\simeq} \mathbb{A}_K^\times/(K^\times N_{L/K}(\mathbb{A}_L^\times)).$$

It is surjective since $\mathbb{A}_{K,\mathfrak{m}}^\times \to \mathbb{A}_K^\times/K^\times$ is already surjective (approximation theorem). So we need to establish

$$K_\mathfrak{m} W_\mathfrak{m} N_{L/K}(\mathbb{A}_L^{\times,S(\mathfrak{m})}) = i^{-1}(K^\times N_{L/K}(\mathbb{A}_L^\times)) = \mathbb{A}_{K,\mathfrak{m}}^\times \cap (K^\times N_{L/K}(\mathbb{A}_L^\times)).$$

The inclusion "$\subseteq$" holds because $W_\mathfrak{m} \subseteq N_{L/K}(\mathbb{A}_L^\times)$ as $\mathfrak{m}$ is admissible. For the other inclusion Let $a \in \mathbb{A}_{K,\mathfrak{m}}^\times \cap (K^\times N_{L/K}(\mathbb{A}_L^\times))$. For every $v \in S(\mathfrak{m})$ fix one place w_0 of L lying above v. There is $\gamma_{w_0} \in L_{w_0}$ such that $N_{L_{w_0}/K_v}(\gamma_{w_0}) = a_v$ since $a_v \in W_\mathfrak{m}(v)$. Choose $\gamma \in L^\times$ such that γ is very close to γ_{w_0} at w_0 and very close to 1 at the other $w \mid v$, for all $v \in S(\mathfrak{m})$. Then $N_{L/K}\gamma$ will be very close to x at all $v \in S(\mathfrak{m})$. Write $a = \alpha N_{L/K} A$ with $\alpha \in K^\times$, $A \in \mathbb{A}_L^\times$. Let $\delta \in L^\times$ such that δ is very close to A. Then

$$a = \Big(\frac{\alpha N_{L/K}\delta}{N_{L/K}\gamma}\Big)\Big(\frac{N_{L/K}A)}{N_{L/K}\delta}\Big)_S\Big(\frac{N_{L/K}A}{N_{L/K}\delta}\Big)^S.$$

By the subscript x_S we mean only that part of the idele with support in S and by x^S with support in the complement. Note that since $N_{L/K}\delta$ is really close to $N_{L/K}A$ at $v \in S$, $\alpha N_{L/K}\delta$ will be very close to a at $v \in S$, hence the first term will in the above expression will be very close to 1 for $v \in S$, in other words it is in $K_\mathfrak{m}$. Similarly the second term is in $W_\mathfrak{m}$ and the last term in $N_{L/K}\mathbb{A}_L^{\times,S}$. $\square$

We can also rephrase things in terms of the idelic class group. The map ϕ_K descends to a continuous homomorphism

$$\phi_K : C_K = \mathbb{A}_K^\times/K^\times \to \operatorname{Gal}(K^{\mathrm{ab}}/K).$$

Let E/K be any finite extension. We have a commuting diagram:

$$\begin{tikzcd} C_E \arrow[r, "\phi_E"] \arrow[d, "N_{E/K}"] & G_E^{\mathrm{ab}} \arrow[d, "\mathrm{res}"] \\ C_K \arrow[r, "\phi_K"] & G_K^{\mathrm{ab}} \end{tikzcd}$$

A less obvious compatibility property is:

Proposition 4.19 ([Neu99, Proposition IV 5.9])**.** *Let L/K be a finite Galois extension and K' an intermediate field. The diagram*

$$\begin{tikzcd} \mathbb{A}_K^\times \arrow[r, "\phi_{L/K}"] \arrow[d] & \operatorname{Gal}(L/K)^{\mathrm{ab}} \arrow[d, "\mathrm{Ver}"] \\ \mathbb{A}_{K'}^\times \arrow[r, "\phi_{L/K'}"] & \operatorname{Gal}(L/K')^{\mathrm{ab}} \end{tikzcd}$$

commutes, where the map on the left is the inclusion, and the map on the right is the Verlagerung (or transfer) map.

For a finite extension L/K we let

$$\mathcal{N}_L = N_{L/K} C_L,$$

so that $C_K/\mathcal{N}_L = \mathbb{A}_K^\times/(K^\times N_{L/K}(\mathbb{A}_L^\times))$.

Theorem 4.20 (Existence Theorem, [Mil20, Theorem V 5.5])**.** *The association*

$$L \mapsto \mathcal{N}_L$$

is a bijection between finite abelian extensions L of K, and open finite index subgroups of C_K. Moreover,

$$L_1 \subseteq L_2 \Longleftrightarrow \mathcal{N}_{L_1} \supseteq \mathcal{N}_{L_2}, \qquad \mathcal{N}_{L_1 L_2} = \mathcal{N}_{L_1} \cap \mathcal{N}_{L_2}, \qquad \mathcal{N}_{L_1 \cap L_2} = \mathcal{N}_{L_1} \mathcal{N}_{L_2}.$$

The field L corresponding to a given open finite index subgroup $H \subseteq C_K$ is called the *class field* belonging to H. For a cycle $\mathfrak{m}$ let $C_K(\mathfrak{m}) = (K^\times W_\mathfrak{m})/K^\times$, so that $C/C(\mathfrak{m}) = I^\mathfrak{m}/P_\mathfrak{m} = \mathrm{Cl}_\mathfrak{m}$. The *ray class field* for $\mathfrak{m}$ is the class field L corresponding to the subgroup $C(\mathfrak{m}) \subseteq C_K$. The conductor $\mathfrak{f}_{L/K}$ of L/K is a divisor of $\mathfrak{m}$, possibly proper, see the example after Proposition 4.28.

Corollary 4.21 ([Lan94, p. 211 Corollary])**.** *If L is any finite extension of K, then $\mathcal{N}_L = \mathcal{N}_E$ where $E = K^{\mathrm{ab}} \cap L$ is the maximal abelian subextension of L/K.*

Proof. Let F be the class field to $\mathcal{N}_L$. We wish to show that $F \subseteq L$, or equivalently $FL = L$. If $a \in \mathbb{A}_L^\times$ then by the above commutative diagram we have $\phi_{FL/L}(a)|_F = \phi_{F/K}(N_{L/K}(a)) = 1$ since $N_{L/K}(a) \in \mathcal{N}_L = \mathcal{N}_F$. Hence $\phi_{FL/L}(a) = 1$, so $\phi_{FL/L}$, and therefore $FL = L$. Then $F \subseteq E$, and therefore $\mathcal{N}_E \subseteq \mathcal{N}_F = \mathcal{N}_L$, but $E \subseteq L$ also gives the other inclusion $\mathcal{N}_L \subseteq \mathcal{N}_E$. □

Proposition 4.22. *The Artin map $\phi_K : C_K \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ is surjective.*

Proof. Since it is surjective on finite extensions, its image is dense. Choose a splitting $C_K = \mathbb{R}_{>0} \times C_K^1$. $\mathbb{R}_{>0}$ is infinitely divisible, hence so is its image under ϕ_K, but it is easy to see that in a profinite group the only infinitely divisible element is the identity element, hence the restriction of $\phi_K|_{C_K^1}$ is surjective. Since C_K^1 is compact, the range of ϕ_K is closed, implying the assertion. □

Let $D_K = \bigcap_{L/K} \mathcal{N}_L$ be the intersection of all $\mathcal{N}_L$, where L ranges over all the finite extensions of K. It is called the group of *universal norms.* Then by the proposition

$$\phi_K : C_K/D_K \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

is an isomorphism.

Theorem 4.23 ([Lan94, Theorem XI 6], [AT59, Theorem 7])**.** *$D_K \subseteq C_K$ is infinitely divisible.*

Theorem 4.24 ([AT59, pp. 69, 70])**.** *D_K is the connected component of the identity in C_K.*

4.2.1. *The Hilbert Class Field.* Let K be a number field. The Hilbert class field of K is the ray class field for the trivial cycle, in other words it is the unique abelian extension L/K such that

$$N_{L/K}(\mathbb{A}_L^\times) = K^\times \mathbb{A}_{K,1}^\times = K^\times \mathbb{A}_{K,S_\infty}^\times = K^\times (K_\infty \times \widehat{\mathcal{O}_K}).$$

Note that $C_K/C_K(1)$ is just the class group, hence $\mathrm{Cl}_K \cong \mathrm{Gal}(L/K)$ via the Artin map and $[L:K] = h_K$ is the class number.

Recall by Corollary 4.7 that a finite abelian extension E/K is unramified at a place v (possibly infinite), if and only if $i_v(U_v) \subseteq N_{L/K}\mathbb{A}_L^\times$, where i_v is the inclusion at the place v, and U_v the local units ($= K^\times$ if K is archimedean). This easily implies the following characterization of the Hilbert class field:

Theorem 4.25. *The Hilbert class field of K is the largest abelian unramified extension of K.*

Here "unramified" includes the infinite places.

Theorem 4.26 (Principal Ideal Theorem)**.** *Let L be the Hilbert class field of K. The natural map $\mathrm{Cl}_K \to \mathrm{Cl}_L$ is trivial, in other words, every ideal in K becomes principal in L.*

Proof. Let L_1 be the Hilbert class field of L. Then L_1 is Galois over L. Indeed, if $\sigma : L_1 \to K^{\mathrm{ab}}$ is an embedding over K, then $\sigma(L_1)$ will be class field to $\sigma(1)$ of $\sigma(L)$. But $\sigma(L) = L$, $\sigma(1) = 1$, whence $\sigma(L_1) = L_1$. Let $G_1 = \mathrm{Gal}(L_1/K)$. Since L_1/K is unramified, L must be the maximal abelian subextension of K in L_1, so $G = \mathrm{Gal}(L/K) = G_1^{\mathrm{ab}}$. Under the Artin isomorphism the map $\mathrm{Cl}_K \to \mathrm{Cl}_L$ corresponds to a certain map $G \to G_1'$, where G_1' is the commutator subgroup. It follows from Proposition 4.19 that this is the Verlagerung. Then the problem reduces to the following problem in group theory (in our case $G_1 = H, G = H/H', H' = \mathrm{Gal}(L_1/L), H'' = 1$):

Theorem. *Let H be a finitely generated group such that H' is of finite index in H. Then the map*

$$\mathrm{Ver} : H/H' \to H'/H''$$

is the trivial map.

See e.g. [AT59, p. 140] or [Neu99, Theorem VI 7.6]. $\square$

4.2.2. *Example $K = \mathbb{Q}$.*

Proposition 4.27 (Quadratic Reciprocity)**.** *Let p, q be distinct odd primes. Then*

$$\left(\frac{p}{q}\right) = (-1)^{\frac{(p-1)(q-1)}{2}} \left(\frac{q}{p}\right)$$

Here $\left(\frac{-}{-}\right)$ is the Legendre symbol.

Proof. Let $L = \mathbb{Q}(\sqrt{q^*})$ where $q^* = (-1)^{(q-1)/2}q$. We compute $\phi_{L/\mathbb{Q}}(p)$. Of course we know that $\phi_Q(p) = 1$ since $p \in \mathbb{Q}^\times$. On the other hand we can write $\phi_{L/\mathbb{Q}}(p) = \phi_{L/\mathbb{Q},\infty}(p) \prod_\ell \phi_{L/\mathbb{Q},\ell}(p)$. Since $p > 0$, we have $\phi_{L/\mathbb{Q},\infty}(p)$. Likewise for all $\ell \neq p, q$ we have $\phi_{L/\mathbb{Q},\ell}(p) = 1$. For $\ell = p$, we have

$$\phi_{L/\mathbb{Q},p}(p) = \left(\frac{q^*}{p}\right),$$

since L is unramified at p, so that the Artin symbol is just the power of the Frobenius corresponding to the valuation of p at p which is 1. Here the identity means that the left side is the identity (resp. nonidentity) element in the group if the right side is. It remains to compute $\phi_{L/\mathbb{Q},q}(p)$. We know that the local Artin map is an isomorphism

$$\phi_{L/\mathbb{Q},q} : \mathbb{Q}_q^\times / N_{L_q/\mathbb{Q}_q}(L_q^\times) \cong \mathrm{Gal}(L_q/\mathbb{Q}_p).$$

Here L_q is the completion of L at the unique prime lying above q. $\mathcal{N}_{L_q} = N_{L_q/\mathbb{Q}_q}(L_q^\times)$ is an index 2 subgroup of $\mathbb{Q}_q^\times$ which does not contain $\mathbb{Z}_q^\times$ since the extension is ramified. Hence $\mathcal{N}_{L_q} \cap \mathbb{Z}_q^\times$ is an open index 2 subgroup. The unique such subgroup is the preimage of the index two subgroup, the group of squares, in $(\mathbb{Z}/q\mathbb{Z})^\times$. Hence the image of p in $\mathrm{Gal}(L_q/\mathbb{Q}_p)$ trivial iff $\left(\frac{p}{q}\right)$. We get

$$1 = \phi_{L/\mathbb{Q}}(p) = \left(\frac{q^*}{p}\right)\left(\frac{p}{q}\right) = (-1)^{\frac{(p-1)(q-1)}{4}}\left(\frac{q}{p}\right)\left(\frac{p}{q}\right).$$

□

Proposition 4.28. *Let $n \geq$ be an integer. The ray class field corresponding to the cycle $\mathfrak{m} = \infty(n)$ of $\mathbb{Q}$ is given by $\mathbb{Q}(\zeta_n)$. The ray class field for (n) is $\mathbb{Q}(\zeta_n + \zeta_n^{-1})$.*

Proof. Let $L = \mathbb{Q}(\zeta_n)$. It suffices to prove that $F_{L/\mathbb{Q}} : I^\mathfrak{m}/P_\mathfrak{m} \to \mathrm{Gal}(L/\mathbb{Q})$ is an isomorphism. It is easily seen that $(\mathbb{Z}/n\mathbb{Z})^\times \cong I^\mathfrak{m}/P_\mathfrak{m}$ where a prime $p \nmid n$ corresponds to the ideal $(p) \in I^\mathfrak{m}$. Finally the Frobenius above p is the Galois automorphism taking ζ_n to ζ_n^p, hence under the usual identification $\mathrm{Gal}(L/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$, we see that the map $(\mathbb{Z}/n\mathbb{Z})^\times \cong I^\mathfrak{m}/P_\mathfrak{m}\,\mathrm{Gal}(L/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$ is the identity.

The second part follows easily from the first. □

This shows that the ray class field does not determine the cycle. Indeed, if $n = 2$, then $\mathbb{Q}^{(\infty)(2)} = \mathbb{Q}^{(2)} = \mathbb{Q}^{(1)} = \mathbb{Q}$ (because $(\mathbb{Z}/2\mathbb{Z})^\times$ is trivial).

Corollary 4.29 (Kronecker-Weber)**.** *Every finite abelian extension of $\mathbb{Q}$ is contained in a cyclotomic field.*

Proposition 4.30. *Let $L/\mathbb{Q}$ be a finite abelian extension. Let n be the minimal integer such that $L \subseteq \mathbb{Q}(\zeta_n)$. Then the conductor $\mathfrak{f}_{L/\mathbb{Q}}$ of $L/\mathbb{Q}$ is (n) if L is purely real, and $\infty(n)$, otherwise.*

Proposition 4.31. *The conductor $\mathfrak{f} = \mathfrak{d}_{L/\mathbb{Q}}$ is the smallest cycle $\mathfrak{m}$ such that $W_\mathfrak{m} \subseteq N_{L/K}(\mathbb{A}_L^\times)$ or equivalently, that $\mathcal{N}_L \supseteq C(\mathfrak{m})$. This is the case iff $L \subseteq \mathbb{Q}^\mathfrak{m}$, hence the claim follows from Proposition 4.28.*

Thanks to unique factorization we have a (direct product) decomposition

$$\mathbb{A}_\mathbb{Q}^\times \cong \mathbb{Q}^\times \times \mathbb{R}_{>0} \times \prod_p \mathbb{Z}_p^\times.$$

We can use this to describe the Artin map

$$\phi_\mathbb{Q} : \mathbb{A}_\mathbb{Q}^\times \to \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}).$$

Note first by Kronecker-Weber, $\mathbb{Q}^{\mathrm{ab}}$ is the composite of all the cyclotomic extensions, hence it suffices to say what $\phi_Q(a)$ does to each ζ.

Proposition 4.32. *The Artin map $\phi_{\mathbb{Q}} : \mathbb{A}_{\mathbb{Q}}^{\times} \to \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ has the following description: Let $a \in \mathbb{A}_{\mathbb{Q}}^{\times}$ and write $a = \gamma\rho u$ with $\gamma \in \mathbb{Q}^{\times}, \rho \in \mathbb{R}_{>0}$ and $u \in \widehat{\mathbb{Z}}^{\times}$. Then for any root of unity ζ, we have*

$$\phi_{\mathbb{Q}}(a)\zeta = \zeta^{u^{-1}}.$$

Note that power $\zeta^{u^{-1}}$ is (for example) defined as follows: Assume $\zeta^n = 1$. Let x be the projection of u^{-1} in $(\mathbb{Z}/n\mathbb{Z})^{\times} \cong (\widehat{\mathbb{Z}}/n\widehat{\mathbb{Z}})^{\times}$. Then $\zeta^{u^{-1}} := \zeta^x$.

Proof. It follows from Theorem 4.10. Alternatively here is a direct way:Fix a root of unity $\zeta = \zeta_n$ and let $L = \mathbb{Q}(\zeta)$. Let $\widetilde{\phi} : \mathbb{A}_Q^{\times} \to \mathrm{Gal}(L/\mathbb{Q})$ be map as indicated, i.e. $\widetilde{\phi}(a)\zeta = \zeta^{u^{-1}}$ where $a = \gamma\rho u$. We have to show $\widetilde{\phi} = \phi_{L/\mathbb{Q}}$. Since ϕ is trivial on $\mathbb{Q}^{\times}$ and continuous, by the uniqueness part in Proposition 2.4, it suffices to prove $\widetilde{\phi}(a) = F_{L/\mathbb{Q}}(\mathrm{Id}^S a)$ for all $a \in \mathbb{A}_{\mathbb{Q}}^{\times,S}$, where $S = \{\infty\} \cup \{p \mid n\}$. So let p be a finite prime not dividing n. Let $a = i_p(p)$ be the idele with p in the p-component, and 1 every else. Then the decomposition $a = \gamma\rho u$ is

$$a = p \cdot 1 \cdot \underbrace{(p^{-1}, \dots p^{-1}, 1, p^{-1}, \dots)}_{=u}.$$

Note the projection of u^{-1} in $\mathbb{Z}/n\mathbb{Z}$ is p, hence

$$\zeta^{u^{-1}} = \zeta^{(p,\dots,p,1,p,\dots)} = \zeta^p = F_{L/\mathbb{Q}}((p)) = F_{L/\mathbb{Q}}(\mathrm{Id}^S a).$$

$\square$

Example. [a] There is no S_3 extension of $\mathbb{Q}$ that is unramified outside $\{7, \infty\}$. Indeed, suppose there is such an extension L. It has a quadratic subfield K. Since K is unramified outside $\{7, \infty\}$, we must have $K = \mathbb{Q}(\sqrt{-7})$. It is easily seen that K has class number 1. Hence,

$$\mathbb{A}_K^{\times} = (K^{\times} \times \mathbb{C}^{\times} \times \prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^{\times})/\pm 1.$$

L is a degree 3 abelian extension of K, therefore corresponds to an open index 3 subgroup of

$$\mathbb{A}_K^{\times}/(K^{\times}\mathbb{C}^{\times}) = \Big(\prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^{\times}\Big)/\pm 1.$$

Note that $\mathbb{C}^{\times}$ doesn't have any finite index subgroups, hence it corresponds to an index 3 subgroup of $\prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^{\times}$. Moreover, since L/K is unramified at all primes not lying above 7, the subgroup must contain $\prod_{\mathfrak{p}\nmid 7} \mathcal{O}_{\mathfrak{p}}^{\times}$. Let $\mathfrak{q}$ be the prime of K lying above 7. Then we conclude that L must correspond to an index 3 subgroup of $\mathcal{O}_{\mathfrak{p}}^{\times}$. We have isomorphisms

$$\mathcal{O}_{\mathfrak{p}}^{\times} \cong \mathbb{F}_{7^2}^{\times} \times U_{K_{\mathfrak{p}}}^{(1)} \cong \mathbb{F}_{7^2}^{\times} \times \mathcal{O}_{\mathfrak{p}}.$$

Since $3 \mid 7^2 - 1$, we see that there is precisely one index 3 subgroup of $\mathcal{O}_{\mathfrak{p}}^{\times}$. Consequently, K has exactly one degree 3 extension that is unramified outside $\mathfrak{p}$. Clearly $\mathbb{Q}(\zeta_7)$ is one such extension. But $\mathbb{Q}(\zeta_7)/\mathbb{Q}$ has Galois group $C_6 \not\cong S_3$, and the result follows.

[a]This question is from the exam here.

4.2.3. *Applications of L-series to Class Field Theory.* Let K be a number field. Let $\mathcal{P}_K$ denote the set of finite places of K, and $\mathcal{P}_K^1$ the subset of those primes having absolute inertia degree 1. In the following, density refers to Dirichlet density. For a set of primes S, we write $\delta(S)$ for its density (if it exists). It is defined by

$$\delta(S) = \lim_{s \to 1^+} \frac{\sum_{\mathfrak{p} \in S} N\mathfrak{p}^{-s}}{\sum_{\mathfrak{p} \in \mathcal{P}_K} N\mathfrak{p}^{-s}}.$$

Proposition 4.33. $\lim_{s \to 1^+} \frac{\sum_{\mathfrak{p} \in \mathcal{P}_K} N\mathfrak{p}^{-s}}{-\log(s-1)} = 1.$

Hence we may also compute the Dirichlet density as

$$\delta(S) = \lim_{s \to 1^+} \frac{\sum_{\mathfrak{p} \in S} N\mathfrak{p}^{-s}}{-\log(s-1)}.$$

In the following we write $f \sim g$ if $f(s) - g(s)$ stays bounded as $s \to 1+$. Similarly write $f \succeq g$ if $f(s) - g(s)$ stays bounded from below as $s \to 1+$.

Proof. Recall from Section 3.5 the Dedekind zeta function ζ_K has a simple pole at $s = 1$. Hence

$$\begin{aligned} -\log(s-1) &\sim \log \zeta_K(s) \\ &= \sum_{\mathfrak{p} \in \mathcal{P}_K} -\log(1 - N\mathfrak{p}^{-s}) \\ &\sim \sum_{\mathfrak{p} \in \mathcal{P}_K} N\mathfrak{p}^{-s}, \end{aligned}$$

and the result follows. $\square$

Proposition 4.34. $\delta(\mathcal{P}_K^1) = 1$.

Proof.

$$\begin{aligned} -\log(s-1) &\sim \sum_{\mathfrak{p} \in \mathcal{P}_K} N\mathfrak{p}^{-s} \\ &= \sum_{\mathfrak{p} \in \mathcal{P}_K^1} N\mathfrak{p}^{-s} + \sum_{\mathfrak{p} \notin \mathcal{P}_K^1} N\mathfrak{p}^{-s} \end{aligned}$$

Above every prime $p \in \mathbb{Z}$ there are at most $[K : \mathbb{Q}]$ primes $\mathfrak{p}$ in K, hence we can bound the second some as

$$\sum_{\mathfrak{p} \notin \mathcal{P}_K^1} N\mathfrak{p}^{-s} \leq [L : K] \sum_{p} p^{-2s} < [L : K]\zeta(2s)$$

This stays bounded as $s \to 1^+$, and the result follows. $\square$

For a finite extension L/K denote by $S_{L/K}$ the set of primes of K that split completely in L.

Proposition 4.35. *Let L/K be a finite Galois extension. Then $\delta(S_{L/K}) = \frac{1}{L:K}$.*

Proof. Over every prime in $S_{L/K}$ there are exactly $n := [L : K]$ primes in L. For prime $\mathfrak{P} \in \mathcal{P}_L$, $\mathfrak{p} = \mathfrak{P} \cap K \in \mathcal{P}_K$ lies in $S_{L/K}$ if and only if $f_{\mathfrak{P}/\mathfrak{p}} = 1$. In this case $N\mathfrak{p} = N\mathfrak{P}$. Hence,

$$\begin{aligned} -\log(s-1) &\sim \sum_{\mathfrak{P}\in\mathcal{P}_L} N\mathfrak{P}^{-s} \\ &= n \sum_{\mathfrak{p}\in S_{L/K}} N\mathfrak{p}^{-s} + \sum_{\mathfrak{P}: f_{\mathfrak{P}/\mathfrak{P}\cap K}>1} N\mathfrak{P}^{-s} \end{aligned}$$

The second sum is bounded as in the previous proposition, hence the result. □

Recall that if E/K is the Galois closure of L/K, then $S_{L/K} = S_{E/K}$.

Corollary 4.36. *Let L/K be a finite extension, and E its Galois closure. Then $\delta(S_{L/K}) = \frac{1}{[E:K]}$.*

Corollary 4.37. *Let L/K be a finite extension. Almost every prime of K splits completely in L if and only if $L = K$.*

Corollary 4.38. *Let L/K be an abelian extension and S a finite set of primes of K containing the ramified ones. Then the Artin map $F_{L/K} \to \mathrm{Gal}(L/K)$ is surjective.*

Proof. Let H be the image of $F_{L/K}$. Let $E = L^H$ its fixed field. If $\mathfrak{p}$ is a prime of K, not contained in S, then $F_{E/K}(\mathfrak{p}) = F_{L/K}(\mathfrak{p})|_E = \mathrm{id}_E$, so $\mathfrak{p}$ splits completely in E. Hence $E = K$ be the corollary, and therefore $H = \mathrm{Gal}(L/K)$. □

For two sets S, T of primes write $S \preceq T$ if $S \setminus T$ has density 0. Write $S \approx T$ if $S \preceq T, T \preceq$, i.e. if they differ by a set of density 0.

Theorem 4.39. *Let $L_1, L_2/K$ be finite extensions with L_1/K Galois. Then $S_{L_1/K} \preceq S_{L_2/K}$ if and only if $L_2 \subseteq L_1$.*

Proof. If $L_2 \subseteq L_1$, then obviously $S_{L_1/K} \subseteq S_{L_2/K}$, so assume $S_{L_1/K} \preceq S_{L_2/K}$. We may assume L_2/K is also Galois. Let $L = L_1L_2$. Let $S = S_{L_1/K} \cap S_{L_2/K}$. Then $S = S_{L/K}$, hence $\delta(S) = [L : K]^{-1}$. On the other hand since $S_{L_1/K} \preceq S_{L_2/K}$, we have $\delta(S) = \delta(S_{L_1/K}) = [L_1 : K]^{-1}$, hence $L_1 = L$. □

Corollary 4.40. *Let $L_1, L_2/K$ be finite Galois extensions. Then $L_1 = L_2$ if and only if $S_{L_1/K} \approx S_{L_2/K}$.*

Theorem 4.41 (Universal Norm Index Inequality, Second Fundamental Inequality)**.** *Let L/K be a finite extension, $\mathfrak{m}$ a cycle of K divisible by all the primes ramifying in L. Then*

$$[I_K^{\mathfrak{m}} : P_{\mathfrak{m}} N_{L/K}(I_L^{\mathfrak{m}})] \leq [L:K].$$

Furthermore, if χ is a non-trivial character of $I_K^{\mathfrak{m}}/(P_{\mathfrak{m}} N_{L/K}(I_L^{\mathfrak{m}}))$, the L-series $L_{\mathfrak{m}}(\chi, s)$ is non-vanishing at $s = 1$.

Proof. For every character χ of $I_K^{\mathfrak{m}}/(P_{\mathfrak{m}} N_{L/K}(I_L^{\mathfrak{m}}))$ let $m(\chi)$ be the order of the zero of $L_{\mathfrak{m}}(\chi, s)$ at $s = 1$. We have

$$\begin{aligned}
\log L_{\mathfrak{m}}(\chi, s) &= \sum_{\mathfrak{p} \in I_K^{\mathfrak{m}}} -\log(1 - \chi(\mathfrak{p}) N\mathfrak{p}^{-s}) \\
&\sim \sum_{\mathfrak{p} \in I_K^{\mathfrak{m}}} \chi(\mathfrak{p}) N\mathfrak{p}^{-s} \\
&= \sum_{\mathfrak{A} \in I_K^{\mathfrak{m}}/(P_{\mathfrak{m}} N_{L/K}(I_L^{\mathfrak{m}}))} \chi(\mathfrak{A}) \sum_{\mathfrak{p} \in \mathfrak{A}} N\mathfrak{p}^{-s}
\end{aligned}$$

On the other hand, we know $\log L_{\mathfrak{m}}(\chi, s) \sim m(\chi) \log(1 - s)$. Therefore summing over all χ and using the orthogonality relations we get

$$\begin{aligned}
\sum_{\chi} m(\chi) \log(1 - s) &\sim \sum_{\chi \in (I_K^{\mathfrak{m}}/P_{\mathfrak{m}} N_{L/K}(I_L^{\mathfrak{m}}))^\frown} \sum_{\mathfrak{A} \in I_K^{\mathfrak{m}}/(P_{\mathfrak{m}} N_{L/K}(I_L^{\mathfrak{m}}))} \chi(\mathfrak{A}) \sum_{\mathfrak{p} \in \mathfrak{A}} N\mathfrak{p}^{-s} \\
&= [I_K^{\mathfrak{m}} : P_{\mathfrak{m}} N_{L/K}(I_L^{\mathfrak{m}})] \sum_{\mathfrak{p} \in P_{\mathfrak{m}} N_{L/K}(I_L^{\mathfrak{m}})} N\mathfrak{p}^{-s}
\end{aligned}$$

Now note that all the primes $P_{\mathfrak{m}} N_{L/K}$ contains all the primes that split completeley in L, hence by similar arguments as in the proof of Proposition 4.35

$$\begin{aligned}
\sum_{\chi} m(\chi) \log(1 - s) &\succeq [I_K^{\mathfrak{m}} : P_{\mathfrak{m}} N_{L/K}(I_L^{\mathfrak{m}})] \sum_{\mathfrak{p} \in S_{L/K}} N\mathfrak{p}^{-s} \\
&\succeq \frac{[I_K^{\mathfrak{m}} : P_{\mathfrak{m}} N_{L/K}(I_L^{\mathfrak{m}})]}{[L:K]} \sum_{\mathfrak{P} \in \mathcal{P}_L^1} N\mathfrak{P}^{-s} \\
&\sim -\frac{[I_K^{\mathfrak{m}} : P_{\mathfrak{m}} N_{L/K}(I_L^{\mathfrak{m}})]}{[L:K]} \log(s - 1)
\end{aligned}$$

Next note that if χ_0 is the trivial character, then $m(\chi_0) = -1$. So

$$\sum_{\chi} m(\chi) \log(s - 1) = -\log(1 - s) + \sum_{\chi \neq \chi_0} m(\chi) \log(s - 1).$$

Since we also know that $m(\chi) \geq 0$ for $\chi \neq \chi_0$, this forces $m(\chi) = 0$ for all $\chi \neq \chi_0$ and $\frac{[I_K^{\mathfrak{m}} : P_{\mathfrak{m}} N_{L/K}(I_L^{\mathfrak{m}})]}{[L:K]} \leq 1$. The claim follows. □

This and Corollary 4.38 show that if we can show that the Artin map admits an admissible cycle, then the rest of the reciprocity law Theorem 4.13 follows.

4.2.4. *Applications of Class Field Theory to L-series and Primes.* Let K be a number field.

Theorem 4.42. *Let $\mathfrak{m}$ be a cycle of K and χ a non-trivial character of $I_K^{\mathfrak{m}}/P_{\mathfrak{m}}$. Then $L(\chi, 1) \neq 0$.*

Proof. Apply Theorem 4.41 to the ray class field for $\mathfrak{m}$. □

Corollary 4.43 (Dirichlet's Theorem on Primes in Arithmetic Progressions)**.** *Let $\mathfrak{m}$ be a cycle, $\mathfrak{A}$ a class in $I_K^{\mathfrak{m}}/P_{\mathfrak{m}}$. Then the set of primes of K that lie in $\mathfrak{A}$ has density $\frac{1}{[I_K^{\mathfrak{m}}:P_{\mathfrak{m}}]}$.*

Theorem 4.44 (Chebotarev density theorem)**.** *Let L/K be a finite Galois extension. Let $C \subseteq \mathrm{Gal}(L/K)$ be closed under conjugation. Let S be the set of primes in K that have Frobenius in C. Then $\delta(S) = \frac{\#S}{[L:K]}$.*

Proof. □

Part 2. Archimedean Local Theory

Let $G = \mathrm{GL}_2(\mathbb{R})$, $K = O(2)$. $\mathfrak{g}$ denotes the Lie algebra of G. We define the following elements of $\mathfrak{g}$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In Bump's notation $H = \widehat{H}, X = \widehat{R}, Y = \widehat{L}, Z = \widehat{Z}$. Note that the basis elements in [GH11] are a little different. Let

$$\Delta = -\frac{1}{4}(H^2 + 2XY + 2YX).$$

Note that the products are being taken in $\mathcal{U}(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$. Then $Z(\mathcal{U}(\mathfrak{g})) = \mathbb{C}[Z, \Delta]$.

They form a basis for $\mathfrak{g}$. We also let

$$k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in K$$

for $\theta \in \mathbb{R}$. Let $\delta_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in K$. It is slightly annoying that K is not connected like $\mathrm{SO}(2)$, so for its action we have to consider not only the k_θ, but also δ_1.

5. $(\mathfrak{g}, K)$-modules

Let V be an irreducible admissible module for $(\mathfrak{g}, K)$. A version of Schur's lemma implies that $Z(\mathcal{U}(\mathfrak{g}))$ acts via scalars on V, hence there are $\mu, \lambda \in \mathbb{C}$ such that

$$\Delta v = \lambda v, \qquad Zv = \mu v,$$

for all $v \in V$.

5.1. K-types

For each $m \in \mathbb{Z}$ we have a one-dimensional representation of $K^+ = \mathrm{SO}(2)$, given by $k_\theta \mapsto e^{im\theta}$.

Given an admissible $(\mathfrak{g}, K)$-module V, let V_m the corresponding isotypic subspace, i.e.

$$V_m = \{v \in V \mid \pi(k_\theta) = e^{im\theta} v \, \forall \theta \in \mathbb{R}\}.$$

The set of *K-types* of V is the set Σ_V of integers m such that $V_m \neq 0$.

Proposition 5.1 ([GH11, Proposition 7.5.7])**.** *Let V be an irreducible admissible $(\mathfrak{g}, K)$-module. Let $\Sigma = \Sigma_V$ be its set of K-types. Then Σ is one of the following:*

$$\begin{aligned} &\{k \in \mathbb{Z} \mid k \equiv 0 \bmod 2\}, \\ &\{k \in \mathbb{Z} \mid k \equiv 1 \bmod 2\}, \\ &\{k \in \mathbb{Z} \mid k \equiv m \bmod 2, \, |k| \leq m\} \qquad \text{for some } m \geq 0, \\ &\{k \in \mathbb{Z} \mid k \equiv m \bmod 2, \, |k| > m\} \qquad \text{for some } m \geq 0. \end{aligned}$$

6. Principal Series Representation

Let χ_1, χ_2 be quasi-characters of $\mathbb{R}^\times$. We denote by $\mathcal{B}(\chi_1, \chi_2)$ the space of functions $f : G \to \mathbb{C}$ satisfying

(1) $f\left(\begin{pmatrix} y_1 & x \\ 0 & y_2 \end{pmatrix} g\right) = \chi_1(y_1)\chi_2(y_2)\left|\frac{y_1}{y_2}\right|^{1/2} f(g)$ for $y_1, y_2 \in \mathbb{R}^\times, x \in \mathbb{R}, g \in G$.

(2) f is K-finite on the right.

(In other words, it is the space of K-finite vectors in an induction of the quasi-character $\chi_1 \boxtimes \chi_2$ of B)

The elements in $\mathcal{B}(\chi_1, \chi_2)$ are determined by their restriction to $K^+ = \mathrm{SO}(2)$ (by the Iwasawa decomposition), and automatically smooth. $V = \mathcal{B}(\chi_1, \chi_2)$ is a $(\mathfrak{g}, K)$-module, where K acts via right translation, and $\mathfrak{g}$ via differentiation.

Note that if we write $\chi_i = \xi_i |\cdot|^{s_i}$ with ξ_i unitary, then in the notation of [GH11], we have $\mathcal{V}_\infty((s_1 - \frac{1}{2}, s_2 + \frac{1}{2}), (\xi_1, \xi_2)) = \mathcal{B}(\chi_1, \chi_2)$.

Every element $g \in G$ may be uniquely written as $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} k_\theta$. For $n \in \mathbb{Z}$ define the function

$$f_n(g) = \chi_1(y_1)\chi_2(y_2)\left|\frac{y_1}{y_2}\right|^{1/2} e^{in\theta}.$$

Write

$$\chi_i = (\mathrm{sgn})^{m_i} |\cdot|^{s_i},$$

for $i = 1, 2$, where $m_i \in \{0, 1\}$ and $s_i \in \mathbb{C}$. Let $s = s_1 - s_2$ and $m = |m_1 - m_2|$. Note $(-1)^m = \chi_1(-1)\chi_2(-1)$.

If $n \equiv m \bmod 2$, we have

$$V_n = \mathbb{C} f_n$$

The functions f_n for $n \equiv m \bmod 2$ form a basis for V.

The action of various elements in K and $\mathfrak{g}$ on V is given as follows:

$$\begin{aligned} \pi(k_\theta) v_l &= e^{il\theta} v_l, \\ \pi(\delta_1) v_l &= \chi_1(-1) v_{-l}, \\ Z v_l &= (s_1 + s_2) v_l, \\ \Delta v_l &= (s_1^2 + s_2^2 - s_1 + s_2) v_l. \end{aligned}$$

TODO

Theorem 6.1 ([JL70, Lemma 5.7]).

(1) $V = \mathcal{B}(\chi_1, \chi_2)$ is irreducible as a $\mathfrak{g}$-module, except when $s - m$ is an odd integer.

(2) If $s - m$ is an odd integer and $s \geq 0$, then the proper nontrivial $\mathfrak{g}$-invariant subspaces are by

$$\mathcal{B}_1(\chi_1, \chi_2) = \mathrm{span}\{f_{s+1}, f_{s+3}, f_{s+5}, \dots\},$$

$$\mathcal{B}_1(\chi_1, \chi_2) = \operatorname{span}\{\dots, f_{s-5}, f_{s-3}, f_{s-1}\},$$

and $\mathcal{B}_s(\chi_1, \chi_2) = \mathcal{B}_1(\chi_1, \chi_2) + \mathcal{B}_2(\chi_1, \chi_2)$. *The latter only when* $\mathcal{B}_s \neq V$.

(3) *If* $s - m$ *is an odd integer and* $s <$, *then the proper nontrivial* $\mathfrak{g}$*-invariant subspaces are by*

$$\mathcal{B}_1(\chi_1, \chi_2) = \operatorname{span}\{f_{s+1}, f_{s+3}, f_{s+5}, \dots\},$$
$$\mathcal{B}_1(\chi_1, \chi_2) = \operatorname{span}\{\dots, f_{-s-5}, f_{-s-3}, f_{-s-1}\},$$

and $\mathcal{B}_f(\chi_1, \chi_2) = \mathcal{B}_1(\chi_1, \chi_2) \cap \mathcal{B}_2(\chi_1, \chi_2)$.

7. Classification of Irreducible $(\mathfrak{g}, K)$-modules

Theorem 7.1 ([Bum97, Theorem 2.5.5])**.** *The following is a complete list of all the irreducible admissible* $(\mathfrak{g}, K)$*-modules:*

(1) *The finite-dimensional representations are the twists of the symmetric powers of the standard representation.*

(2) *If* χ_1, χ_2 *are quasi-characters of* $\mathbb{R}^\times$ *such that* $\chi_1\chi_2^{-1} \neq (\operatorname{sgn})^\varepsilon \left|\cdot\right|^{k-1}$ *for some* $\varepsilon \in \{0, 1\}$ *and* $k \in \mathbb{Z}$ *with* $k \equiv \varepsilon \bmod 2$, *then we have the principal series representation* $\pi(\chi_1, \chi_2)$.

(3) *For* $\mu \in \mathbb{R}$ *and* $k \in \mathbb{Z}_{\geq 1}$, *we have the representations* $\mathcal{D}_\mu(k)$.

8. Whittaker Models

Part 3. Nonarchimedean Local Theory

9. Generalities on Representations of Totally Disconnected Locally Compact Groups

We abbreviate “totally disconnected locally compact” by *tdlc*. Let G be a *tdlc* group.

9.1. Algebraic Representations

Definition. *An* algebraic representation (V, π)[a] *of G is a complex vector space V, together with a group homomorphism $\pi : G \to \mathrm{Aut}_{\mathbb{C}}(V)$. If $K \subseteq G$ is a subgroup, let V^K be the set of K-fixed vectors. A vector $v \in V$ is called* smooth *if it is fixed by an open compact subgroup. The set of smooth vectors in V is denoted $V^\infty = \bigcup_K V^K$. (V, π) is called*

- smooth *if every vector is smooth, i.e. $V = V^\infty$,*
- admissible *if it is smooth and V^K is finite-dimensional for every open compact subgroup K,*
- irreducible *if V has no proper G-invariant non-zero subspace.*

[a] It seems in the literature the order (π, V) is prevalent, but I somehow got used to the opposite order. I also prefer it since we need to define V first to be able to talk about the action π of G.

Of course we also have the usual definitions of homomorphisms of representations, subrepresentations, quotients...

Note that V^∞ is always a smooth subreprepresentation of V.

A function on a totally disconnected space is called *smooth* if it is locally constant. Thus, a vector $v \in V$ is smooth if and only if the map $G \to V, g \mapsto \pi(g)v$ is smooth.

When talking about irreducible representations, we will always exclude the 0 representation.

Theorem 9.1. *Let K be a* compact *tdlc group. Let (V, π) be a smooth representation of K. Then π is semisimple, i.e. V is the direct sum of irreducible subrepresentations. Any irreducible smooth representation of K is finite-dimensional.*

Proposition 9.2 (Schur's Lemma)**.** *Let G be a tdlc group and (V, π) a smooth irreducible representation. Assume that one of the following holds:*

(1) G/K is countable for some compact open subgroup K, or

(2) π is admissible.

Then $\mathrm{End}_G(V) = \mathbb{C}$, i.e. if $T : V \to V$ is an intertwining operator, there is $\lambda \in \mathbb{C}$ such that $T = \lambda\, \mathrm{id}_V$.

Corollary 9.3. *Assumptions as in the previous proposition. There is a quasi-character $\omega : Z(G) \to \mathbb{C}^\times$, called the* central quasi-character of π, *such that $\pi(z)v = \omega(z)v$ for all $z \in Z(G), v \in V$.*

Corollary 9.4. *Assumptions as in the previous proposition. If G is abelian, then* $\dim V = 1$.

Remark. Unlike in the case of finite groups (or more generally unitarizable representations), the converse of Schur's lemma does not hold, i.e. $\mathrm{End}_G(V) = \mathbb{C}$ does not imply that V is irreducible. For example if F is a local nonarchimedean field, χ a quasi-character of $F^\times$, then the principal series representation $(V,\pi) = \mathcal{B}(\chi, \chi\,|\cdot|)$ is reducible (Theorem 15.7), but $\dim_{\mathbb{C}} \mathrm{End}_G V = 1$ (Theorem 15.11).

Definition. *Let G be a tdlc group, K a compact open subgroup. We denote by $\widehat{K}$ the set of equivalence classes of irreducible smooth representations of K. Let (V,π) be a smooth representation of G. If $\rho \in \widehat{K}$, denote by V^ρ the sum of all subspaces of V which are isomorphic to ρ as K-representations. We call it the* ρ-isotypic component *of (V,π).*

Theorem 9.5 ([Bum97, Proposition 4.2.2])**.** *In the setup as in the definition we have*

$$V = \bigoplus_{\rho \in \widehat{K}} V^\rho.$$

V is admissible if and only if each V^ρ is finite-dimensional.

Definition. *Let G be tdlc and (V,π) a smooth representation. The contragredient representation $(\widehat{V}, \widehat{\pi})$ is the representation of G where $\widehat{V}$ is the space of smooth vectors in the algebraic dual of (V,π), i.e.*

$$\begin{aligned}\widehat{V} &= (\mathrm{Hom}(V,\mathbb{C}))^\infty \\ &= \{f : V \to \mathbb{C} \text{ linear} : \exists \text{compact open subgroup } K \subseteq G \text{ with } f(kv) = f(v)\, \forall k \in K, v \in V\},\end{aligned}$$

and $\widehat{\pi}$ acts on this space by $(\widehat{\pi}(g)f)(v) = f(\pi(g^{-1})v)$.

We might also write $\widetilde{V}$ for $\widehat{V}$.

If $f \in \widehat{V}, v \in V$ we also denote $f(v)$ by $\langle v, f\rangle$. Then $\langle \pi(g^{-1})v, f\rangle = \langle v, \pi(g) f\rangle$.

Definition. *Let (V,π) be a smooth representation of G. A* matrix coefficient *of π is a function $G \to \mathbb{C}$ of the form $g \mapsto \langle \pi(g)v, \widehat{v}\rangle$ with $v \in V, \widehat{v} \in \widehat{V}$.*

Proposition 9.6. *Let (V,π) be an admissible representation of G and K a compact open subgroup. Then the pairing between $V, \widehat{V}$ induces a non-degenerate pairing between V^K and $\widehat{V}^K$, so we can naturally identify $(V^K)^* = \widehat{V}^K$. $\widehat{V}$ is admissible and the natural map $V \to \widehat{\widehat{V}}$ is an isomorphism.*

Let G be a *tdlc* group and $H \subseteq G$ a closed subgroup. Denote by δ_G, δ_H the modular quasi-characters of G, H respectively (see Appendix A). Let (U,σ) be a smooth representation of H. This induces two representations of G in the following way: Let V the vector space of functions $f : G \to U$ satisfying

(i) $f(hg) = \delta_G(h)^{-1/2}\delta_H(h)^{1/2} f(g)$ for $h \in H, g \in G$.

(ii) There is a compact open subgroup $K \subseteq G$ such that $f(gk) = f(g)$ for $g \in G, k \in K$.

Let V_c be the subspace of V of functions f additionally satisfying

(iii) f has compact support modulo H, i.e. the image of the support of f is compact in G/H.

Letting G act on V (resp. V_c) gives us a representation, denoted $\operatorname{Ind}_H^G \sigma$ (resp. $c\text{-}\operatorname{Ind}_H^G \sigma$) and called the *induced representation* (resp. *induced representation with compact support*). Both $\operatorname{Ind}_H^G \sigma$ and $c\text{-}\operatorname{Ind}_H^G \sigma$ are smooth representations of G. Note that if G/H is compact, then they coincide.

Remark. In [BH06] the notation is slightly different, there this would be denoted

$$\boldsymbol{\iota}_H^G \sigma = \operatorname{Ind}_H^G(\delta_G^{-1/2}|_H \otimes \delta_H^{1/2} \otimes \sigma).$$

The inclusion of the modular quasi-characters has the advantage that $c\text{-}\operatorname{Ind}_H^G \sigma$ will be unitarizable if σ, see Theorem 9.15, and it behaves nicely under taking the contragredient, see Theorem 9.9.

Proposition 9.7. *Assume $H\backslash G$ is compact. If (U,σ) is an admissible representation of H, then $\operatorname{Ind}_H^G \sigma = c\text{-}\operatorname{Ind}_H^G \sigma$ is an admissible representation of G.*

Proof. Let K be a compact open subgroup of G. Then any element $f \in (\operatorname{Ind}_H^G \sigma)^K$ satisfies

$$f(hgk) = (\delta_G^{-1/2}\delta_H^{1/2}\sigma)(h)f(g)$$

for $h \in H, g \in G, k \in K$. In particular it is determined by its values on a set of coset representatives for $H\backslash G/K$. Since $H\backslash G$ is compact, $H\backslash G/K$ is finite and the result follows. □

Theorem 9.8 (Frobenius reciprocity, [BH06, 2.4, 2.5], [BZ76, 2.29])**.** *Let $(V,\pi),(U,\sigma)$ be smooth representations of G, H respectively. Then there is are canonical isomorphisms*

$$\operatorname{Hom}_G(\pi, \operatorname{Ind}_H^G \sigma) \cong \operatorname{Hom}_H(\pi|_H, \sigma \otimes \delta_G^{-1/2}\delta_H^{1/2}),$$

$$\operatorname{Hom}_G(c\text{-}\operatorname{Ind}_H^G \sigma, \widetilde{\pi}) \cong \operatorname{Hom}_H(\delta_G^{-1/2}\delta_H^{1/2} \otimes \sigma, \widetilde{(\pi|_H)}).$$

If H is also open in G, there also is a canonical isomorphism:

$$\operatorname{Hom}_G(c\text{-}\operatorname{Ind}_H^G \sigma, \pi) \cong \operatorname{Hom}_H(\sigma \otimes \delta_G^{-1/2}\delta_H^{1/2}, \pi|_H).$$

Theorem 9.9 ([BH06, 3.5], [BZ76, 2.25 (c)])**.** *Let (U,σ) be a smooth representation of H. Then there is an isomorphism*

$$\widehat{c\text{-}\operatorname{Ind}_H^G \sigma} \cong \operatorname{Ind}_H^G \widehat{\sigma}.$$

Proof. Let $f \in c\text{-}\operatorname{Ind}_H^G \sigma, \phi \in \operatorname{Ind}_H^G \widehat{\sigma}$. Then $\langle f(hg), \phi(hg)\rangle = (\delta_G^{-1}\delta_H)(h)\langle f(g), \phi(g)\rangle$. Therefore the function $g \mapsto \langle f(g), \phi(g)\rangle$ is in $C_c(H\backslash G, \delta_G|_H^{-1}\delta_H)$ and so we can define

$$(f,\phi) = \int_{H\backslash G} \langle f(g), \phi(g)\rangle \mathrm{d}\mu_{H\backslash G}(g)$$

by Theorem A.6. This satisfies $(\pi(g)f, \pi'(g)\phi) = (f,\phi)$ for $g \in G$, where π, π' are the actions of G on $c\text{-}\operatorname{Ind}_H^G \sigma, \operatorname{Ind}_H^G \widehat{\sigma}$ respectively. That way we get a map $\operatorname{Ind}_H^G \widehat{\sigma} \to \widehat{c\text{-}\operatorname{Ind}_H^G \sigma}$ given by $\phi \mapsto (-,\phi)$.

To show it is an isomorphism one explicitly describes $(c\text{-}\mathrm{Ind}_H^G \sigma)^K$ for open compact subgroups $K \subseteq G$. $\square$

Let G be a locally profinite unimodular group and fix a Haar measure $\mathrm{d}g = \mu$.

Definition. *The Hecke algebra of G is $\mathcal{H} = \mathcal{H}(G) = C_c^\infty(G)$ the space of compactly supported locally constant functions on G. The algebra operation is given by convolution: If $f_1, f_2 \in \mathcal{H}$, define the convolution $f_1 * f_2$ by*

$$(f_1 * f_2)(g) = \int_G f_1(h) f_2(h^{-1}g)\,\mathrm{d}h.$$

If (V, π) is a smooth representation of G, then V becomes a module over $\mathcal{H}$ via

$$\pi(f)v := \int_G f(g)\pi(g)v\,\mathrm{d}g$$

where $f \in \mathcal{H}(G), v \in V$. To make sense of the integral one can note that that it is really a finite sum, since f is locally constant of compact support and v is fixed by an open compact subgroup of G. If K is a compact open subgroup, let $\mathcal{H}_K = \mathcal{H}(G, K)$ be the subalgebra of $\mathcal{H}$ consisting of those functions that are biinvariant under K. Given $\rho \in \widehat{K}$, define a function $e_\rho \in \mathcal{H}$ by $e_\rho(k) = \frac{\dim \rho}{\mu(K)} \operatorname{Tr} \rho(k^{-1})$ when $k \in K$ and $e_\rho(k) = 0$ otherwise. If ρ is the trivial representation we also denote e_ρ by e_K. It is $\frac{1}{\mu(K)}$ times the characteristic function of K. We then have $\mathcal{H}_K = e_K * \mathcal{H} * e_K$ and $\mathcal{H}_K$ is a unital algebra with unit e_K.

Theorem 9.10 ([BH06, Proposition 4.4])**.** *Let (V, π) be a smooth representation of G. Then $\pi(e_\rho)$ is the projection $V = \bigoplus_{\rho' \in \widehat{K}} V^{\rho'} \to V^\rho$.*

For the next definition, note that if $T : V \to V$ is an endomorphism of a vector space V with finite-dimensional image W, then we may define the trace of T by $\operatorname{Tr} T := \operatorname{Tr} T|_{W' \to W'}$ where $W' \subseteq V$ is any finite-dimensional subspace containing W. A *distribution* on a locally compact totally disconnected space X is a linear functional $C_c^\infty(X) \to \mathbb{C}$.[5]

Definition. *Let (V, π) be an admissible representation of G. The* character *of π is the distribution $\operatorname{Tr} \pi : C_c^\infty(G) \to \mathbb{C}$, defined by*

$$\operatorname{Tr} \pi(f) = \operatorname{Tr}(\pi(f) : V \to V).$$

Note that if $f \in C_c^\infty(G)$, then $\pi(f)$ has finite rank, so the trace is well-defined by the remark before the definition.

Theorem 9.11 ([JL70, Lemma 7.1])**.** *Let $(V_1, \pi_1), \ldots, (V_n, \pi_n)$ be pairwise non-isomorphic irreducible admissible representation of G. Then their characters $\operatorname{Tr} \pi_1, \ldots, \operatorname{Tr} \pi_n$ are linearly independent.*

Note that if $0 \to \pi' \to \pi \to \pi'' \to 0$ is a short exact sequence of admissible representations, then $\operatorname{Tr} \pi = \operatorname{Tr} \pi' + \operatorname{Tr} \pi''$. Together with the theorem this easily implies

[5] Note that unlike in the analytic case no continuity restriction is placed on such functionals.

Corollary 9.12 ([Cas+08, Corollary 2.3.3])**.** *Let $(V,\pi),(V',\pi')$ be admissible representations of G of finite length. Then the irreducible composition factors and their multiplicities of π,π' coincide (i.e. π,π' have isomorphic semisimplifications) if and only if $\operatorname{Tr}\pi=\operatorname{Tr}\pi'$.*

Definition. *A representation (V,π) is* unitary *if it is equipped with a G-invariant inner product $(-,-)$.*

Note in the next section we might call these representations *preunitary* to distinguish them from those acting on a Hilbert space.

Proposition 9.13. *Let (V,π) be a unitary admissible representation. Then the map $v\mapsto(-,v),V\mapsto\widehat{V}$ is an anti-linear isomorphism.*

Proof. First it is easy to see that $(-,v)$ is indeed smooth. Let $f\in\widehat{V}$. We want to show that $f=(-,v)$ for some vector $v\in V$. If $f=0$, take $v=0$. Let K be a compact open subgroup such that $f\in\widehat{V}^K\cong(V^K)^*$. Then there is a vector $v\in V^K$ such that $f(w)=(w,v)$ for all $w\in V^K$, since V^K is finite-dimensional. Then for arbitrary $w\in V$ we have $f(w)=f(\pi(e_K)w)=(\pi(e_K)w,v)=(w,\pi(e_K)v)=(w,v)$. □

This could also be proven by passing to the completion, using the Riesz representation theorem, and then show that the representing vector is smooth.

Proposition 9.14. *Let (V,π) be a unitarizable admissible representation. Then its matrix coefficients are bounded.*

Proof. Let $\langle -,-\rangle$ be an invariant inner product. Then via the inner product we can identify V with $\widehat{V}$, see Proposition 9.13, so a matrix coefficient ϕ is given by $\phi(g)=\langle\pi(g)v,w\rangle$ with $v,w\in V$. Then by Cauchy Schwarz:

$$|\langle\pi(g)v,w\rangle|\leq\|\pi(g)v\|\,\|w\|=\|v\|\,\|w\|\,.$$

□

Theorem 9.15. *Let G be tdlc and H a closed subgroup. If (U,σ) is a unitary representation of H, then $c\text{-}\mathrm{Ind}_H^G\sigma$ is unitarizable.*

Proof. Let $(-,-)$ be an H-invariant inner product on U. Let $f_1,f_2\in c\text{-}\mathrm{Ind}_H^G\sigma$. Then the function $g\mapsto(f_1(g),f_2(g))$ is in $C_c(H\backslash G,\delta_G|_H^{-1}\delta_H)$ and we can define

$$(f_1,f_2)=\int_{H\backslash G}(f_1(g),f_2(g))\mathrm{d}\mu_{H\backslash G}(g).$$

This is a G-invariant inner product. It is positive by the last remark after Theorem A.6. □

9.2. **Unitary Hilbert Space Representations**

The following definition is completely general for an arbitrary topological group G.

Definition. *A* unitary Hilbert space representation *of a topological group G is a Hilbert space V with a homomorphism $\pi : G \to \mathrm{Aut}(V)$ such that $\pi(g)$ is unitary and for every $v \in V$, the map $g \mapsto \pi(g)v$ is continuous (i.e. π is continuous for the strong operator topology on $\mathcal{B}(H)$). V is* irreducible *if there are no closed proper nontrivial invariant subspaces.*

Definition. *A unitary Hilbert space representation (V, π) of a locally compact group G is admissible if for some compact subgroup K every irreducible representation of K occurs with finite multiplicity in V.*

If the condition holds for some K, it holds for all compact $K' \supseteq K$, see [Dei12, Lemma 7.5.22].

We will just say Hilbert space representation or unitary representation.

A lot of the results from the previous section carry over to this setting. Notably, let $\mathcal{H} = (C_c^\infty(G), *)$ be the Hecke algebra of G. It carries an involution $f \mapsto f^*$ where $f^*(g) = \overline{f(g^{-1})}$, in this way it becomes a $*$-algebra. By a unitary representation (or $*$-representation) of $\mathcal{H}$ we mean a Hilbert space V together with a $*$-homomorphism $\pi : \mathcal{H} \to \mathcal{B}(V)$ such that V is non-degenerate, meaning $\pi(\mathcal{H})V$ is dense in V.

Given a unitary representation (V, π) of G, we get a unitary (i.e. $*$-) representation of $\mathcal{H}$ via

$$\pi(f)v = \int_G f(x)\pi(x)f\mathrm{d}x.$$

Conversely, given a $*$-representation (V, π) of $\mathcal{H}$, we get a unitary representation (requires a little bit of explanation) of G via $\pi(g)v = \lim_K \pi(\delta_g * e_K)v$ where the limit runs through a neighbhorhood basis filter of compact open subgroups. That way as before we get a bijection between unitary representations of G and of the Hecke algebra.

We now describe the relationship between smooth and Hilbert space representations for *tdlc* groups. Let G be a *tdlc* group and suppose (V, π) is a Hilbert space representation of G. Let V^∞ be the subspace of smooth vectors, i.e. the set of vectors fixed by an open subgroup of G.

Proposition 9.16. *If (V, π) is a Hilbert space representation, the subspace V^∞ is dense and G-invariant.*

The density part is actually the statement that V is non-degenerate as a $\mathcal{H}$-module as asserted above.

Hence from a Hilbert space representation we get a smooth algebraic representation of G. Conversely, if we have a smooth algebraic representation (V, π) of G that is *preunitarizable*, i.e. there exists a G-invariant inner product on V, then we can consider the completion $\overline{V}$ with respect to this inner product, and obtain in this way a unitary Hilbert space representation of G.

Question: Are these operators inverse to each other? One way is the proposition, but conversely, if V is an algebraic smooth representation with G-invariant inner product, is $(\overline{V})^\infty = V$?

I thought of the following argument in case V is admissible: Fix an open compact subgroup K. Then $V = \bigoplus_{\rho \in \widehat{K}} V^\rho$. If V is admissible, each V^ρ is finite-dimensional, hence

$$\overline{V} = \widehat{\bigoplus_{\rho \in \widehat{K}}} V^\rho.$$

Now if $v = (v_\rho)_\rho \in \overline{V}$ is smooth, then there is a compact open subgroup K_0, wich we may take to be normal of finite index in K, such that v is fixed by K_0. If $v_\rho \neq 0$, then $\pi(K)v_\rho = \rho(K)v_\rho = V^\rho$ by irreducibility of V^ρ. Hence K_0 acts trivially on V^ρ. But then ρ must come from one of the finitely many representations of K/K_0 (a finite group), hence v is only non-zero in finitely many components and we get

$$v \in \bigoplus_{\rho \in \widehat{K}} V^\rho = V \subseteq \overline{V}.$$

So $(\overline{V})^\infty = V$.

In [Car79, 2.8] however this is deduced in the case of reductive groups using a nontrivial fact (the multiplicity of K-representations in an admissible representation is uniformly bounded). Not sure if I missed something.

10. General Results

Notation: F nonarchimedean local field. $G = \mathrm{GL}_2(F)$, B the upper triangular matrices, T the diagonal matrices, N the upper triangular unipotent matrices, M the matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, $K = \mathrm{GL}_2(\mathcal{O}_F)$.

We define the matrices

$$\begin{aligned} w &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ w_0 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ w_1 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

Theorem 10.1. *If (V, π) is an irreducible smooth representation of G, then V is admissible.*

Proof. Either V is one-dimensional, a subrepresentation of a principal series representation, or a supercuspidal representation, see Theorem 23.1. In the first two cases the result is clear. In the last case, the result follows from Proposition 22.1. □

Theorem 10.2. *Let (V, π) be an irreducible smooth representation of G. Then its contragredient $(\widetilde{V}, \widetilde{\pi})$ is isomorphic to the following two representations on V:*

(1) (V, π_1) where $\pi_1(g) = \pi(g^T)^{-1}$.

(2) (V, π_2) *where* $\pi_2 = \omega^{-1} \otimes \pi$ *where* ω *is the central quasi-character of* G.

Proof. (Proof from [Bum97]). The second statement can be easily deduced from the first using the identity

$$g^{-T} = \begin{pmatrix} \det g & 0 \\ 0 & \det g \end{pmatrix}^{-1} w_0^{-1} g w_0$$

where $w_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. As for the first, it suffices to show that $\widetilde{\pi}$ and π_1 have the same character. If $\phi \in C_c^\infty(G)$, we let $\check{\phi}, \phi^T$ be defined by the formulas $\check{\phi}(g) = \phi(g^{-1}), \phi^T(g) = \phi(g^T)$. For $\phi \in C_c^\infty(G)$ we have $\operatorname{Tr} \pi_1(\phi) = \operatorname{Tr} \pi(\check{\phi}^T)$. A straightforward computation shows that $\widetilde{\pi}(\check{\phi}) : \widetilde{V} \to \widetilde{V}$ is the adjoint of $\pi(\phi)$, hence $\operatorname{Tr} \widetilde{\pi}(\check{\phi}) = \operatorname{Tr} \pi(\phi)$. To show that $\operatorname{Tr} \pi_1 = \operatorname{Tr} \widetilde{\pi}$, it suffices to show that $\operatorname{Tr} \pi_1(\phi^T) = \operatorname{Tr} \pi_1(\phi)$ for all $\phi \in C_c^\infty(G)$, i.e. that $\operatorname{Tr} \pi_1$ is transpose-invariant. If $\operatorname{Tr} \pi_1$ was a function on G, this would be easy since it is conjugation-invariant and any matrix is similar to its transpose. But a priori $\operatorname{Tr} \pi_1$ is only defined as a distribution, so we need to be more careful. It can be proved that any distribution on G that is conjugation-invariant is transpose-invariant. This uses the involution method, see [BZ76] and [Bum97].

Alternatively one can show that $\operatorname{Tr} \pi$ is actually (or rather represented by) a continuous function, see [JL70, Theorem 7.7] for a proof. □

Lemma ([Bum97, Exercise 4.4.2])**.** *If* $\gamma \in \mathrm{SL}_2(F) - B$, *then* γ *and* N *together generate* $\mathrm{SL}_2(F)$.

Proof. Let H be the subgroup generated by N and g. Write $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \neq 0$. Then

$$\begin{pmatrix} 1 & -a/c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -d/c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1/c \\ c & 0 \end{pmatrix} =: w \in H.$$

We have

$$w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} w = \begin{pmatrix} -1 & 0 \\ xc^2 & -1 \end{pmatrix},$$

hence $-N^T \subseteq H$. Since $w^2 = -I$, we have $N^T \subseteq H$. This means we can apply the following row operations to matrices: Add a multiple of one of the rows to the other (same with columns). It is easily seen that we can reduce any matrix in $\mathrm{SL}_2(F)$ with such operations to a matrix of the form $\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$, so it suffices to show these are in H. We apply the row operations:

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \xrightarrow{R_2 + (x-1)R_1} \begin{pmatrix} x & 0 \\ x-1 & x^{-1} \end{pmatrix} \to \begin{pmatrix} 1 & -x^{-1} \\ x-1 & x^{-1} \end{pmatrix} \to \begin{pmatrix} 1 & -x^{-1} \\ 0 & 1 \end{pmatrix} \in H.$$

□

Proposition 10.3. *Suppose* (V, π) *is a finite-dimensional irreducible smooth representation of* G. *Then* V *is one-dimensional and* $\pi = \chi \circ \det$ *for some quasi-character* χ *of* $F^\times$.

Proof. Since V is finite-dimensional and smooth, the kernel of π is an open normal subgroup. We show that this implies $\mathrm{SL}_2(F) \subseteq \ker\pi$. Indeed, we have $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in \ker\pi$ for $|x|$ small enough. Since $\ker\pi$ is normal we have

$$\begin{pmatrix} 1 & tx \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in \ker\pi$$

for all $t \in F^\times$. But then $N \subseteq \ker\pi$. Again since $\ker\pi$ is open, it must also contain a matrix that is not upper triangular, hence $\mathrm{SL}_2(F) \subseteq \ker\pi$ by the lemma above.[6] So π factors through $\det : \mathrm{GL}_2(F) \to F^\times$. Then V is one-dimensional since $F^\times$ is abelian and we have $\pi = \chi \circ \det$ for some quasi-character χ of $F^\times$. □

Proposition 10.4. *Suppose (V,π) is an irreducible smooth representation of G. If V contains a nonzero vector fixed by N, then V is one-dimensional.*

Proof. Suppose $0 \neq v \in V$ is fixed by N. Its stabilizer then contains both N and an open subgroup of G. By the lemma it contains $\mathrm{SL}_2(F)$. Let W be the one-dimensional subspace spanned by v. Then W is invariant under $\mathrm{SL}_2(F)$ and Z. Note that $\mathrm{GL}_2(F)/(Z\,\mathrm{SL}_2(F)) \cong F^\times/(F^\times)^2$ is finite (F is a local nonarchimedean field of characteristic 0[7]). Hence the the subrepresentation spanned by W is finite-dimensional. As V is irreducible, we must have $V = W$ and the result follows follows from the previous proposition. □

11. Jacquet Modules

Definition. *Let (V,π) be a smooth representation of N. The Jacquet module of V, denoted V_N is $V_N = V/V(N)$. $V(N)$ is invariant under B, hence V_N is a B-module.*

If ψ is a character of N, the ψ-twisted Jacquet module of V is $V_\psi = V/V(\psi)$.

Proposition 11.1. *Let (V,π) be a smooth representation of N and $v \in V$. Then $v \in V(\psi)$ if and only if*

$$\int_{N_0} \overline{\psi(n)}\pi(n)v\,\mathrm{d}n = 0$$

some compact open subgroup $N_0 \subseteq N$.

Note that if the condition holds for some compact open $N_0 \subseteq N$, then it holds for all N_1 containing N_0.

Proposition 11.2. *The functor $V \mapsto V_\psi$ is exact.*

Proof. Right exactness is obvious and left exactness follows from the previous proposition. □

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[6]In this case the proof can be made a liittle simpler than in the lemma, since we could directly show $N^T \subseteq \ker\pi$

[7]It also works in characteristic $\neq 2$. In characteristic 2 the proposition still holds, but this part in the proof requires modification.

Lemma 11.3 ([BH06, Restriction-Induction Lemma 9.3])**.** *Let (U,σ) be a smooth representation of T which we view as a representation of B via inflation. Let $(V,\pi) = \operatorname{Ind}_B^G \sigma$. Then we have a short exact sequence of representations of T:*

$$0 \to \sigma^w \otimes \delta_B|_T^{1/2} \to \pi_N \to \sigma \otimes \delta_B|_T^{1/2} \to 0, \text{ [a]}$$

where $\sigma^w(t) = \sigma(wtw^{-1})$ with $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The right map is given by $f \mapsto f(1)$.

[a]Note that in [BH06] they use another convention for the modular quasi-character, our δ_B is their δ_B^{-1}.

Proof. (From [BH06]).[8]

There is a canonical map $\alpha : V \to U$ given by $\alpha(f) = f(1)$. It is surjective[9] and a map of B-representations. Let $W = \ker \alpha$. Since $U_N = U$, taking the Jacquet module gives the exact sequence

$$0 \to W_N \to V_N \xrightarrow{\alpha} U \to 0$$

Since $G = B \cup BwN$, $f \in V$ is in W if and only iff $\operatorname{supp} f \subseteq BwN$. Suppose this is the case. There is a compact subgroup $N_0' \subseteq N' = \left\{ \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \right\}$ such that f is invariant under N_0'. So if zero on B, then f is also 0 on BN_0'. The identity

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -x^{-1} & 0 \\ 0 & x \end{pmatrix} w \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix}$$

for $x \neq 0$ then shows that $\operatorname{supp} f \subseteq BwN_0$ for some compact subgroup $N_0 \subseteq N$.

So for $f \in W$ we can define $f_N : T \to U$ by

$$f_N(x) = \int_N f(xwn)\,\mathrm{d}n = \delta_B(x)^{1/2}\sigma(x) f_N(1).$$

The map $\Psi : W \to U, f \mapsto f_N(1)$ is surjective.[10] We claim that the kernel of this map is $W(N)$. Indeed, first note that $f_N(1) = 0$ iff $f_N = 0$. If $f_N(1) = 0$, then since $\operatorname{supp} f \subseteq BwN_0$ for some compact open N_0, we may restrict the integral to N_0 and see that $f \in W(N)$ by Proposition 11.1. Conversely, if $f \in W(N)$, then $\int_{N_1} f(gn)\,\mathrm{d}n = 0$ for some compact open $N_1 \subseteq N$ and all $g \in G$. We may assume $N_1 \subseteq N_0$. Then

$$\begin{aligned} f_N(x) &= \int_N f(xwn)\,\mathrm{d}n = \int_{N_0} f(xwn)\,\mathrm{d}n \\ &= \sum_{y \in N_0/N_1} \int_{N_1} f(xwyn)\,\mathrm{d}n = \sum_{y \in N_0/N_1} \int_{N_1} f(xy^w wn)\,\mathrm{d}n \\ &= 0. \end{aligned}$$

Or directly by the comment after the proof of Proposition 11.1.

[8]Note here and in the following my notation is slightly different than in [BH06]. The meaning of V, W is different.

[9]I think this deserves a brief justification which I have not seen in [BH06]. To show surjectivity, let $u \in U$ and $\varphi \in C_c^\infty(G)$. We define $f : G \to U$ by $f(g) = \int_B f(b^{-1}g)\delta^{1/2}\sigma(b)u\,\mathrm{d}b$. It is easy to see that $f \in \operatorname{Ind}_B^G(\sigma)$. If we choose $f = \frac{1}{\mu_B(B\cap H)}\mathbb{1}_H$ where $H \subseteq G$ is a compact open subgroup such that $u \in U^{B\cap H}$, then we have $f(1) = u$.

[10]I suppose one could argue that $\Phi : W \to C_c^\infty(N,U)$ given by $\Phi(f)(n) = f(wn)$ is an isomorphism. Then Ψ is the composition of Φ with the integration map $C_c^\infty(N,U) \to U$ and the latter is clearly surjective.

Therefore Ψ maps W_N isomorphically onto U. It remains to show that the induced action is the one claimed. For $t \in T$ and $v \in W$ we have:

$$\Psi(tf) = \int_N f(wnt)\,\mathrm{d}n = \delta_B(t)\int_N f(wtn)\,\mathrm{d}n = \delta_B(t)\int_N f(t^w wn)\,\mathrm{d}n = \delta_B(t)^{1/2}\sigma(t^w)\Psi(f).$$

Hence $W_N \cong \sigma^w \otimes \delta_B^{1/2}$ as claimed.

□

Theorem 11.4. *Let (V, π) be an admissible representation of G. Then V_N is admissible as a T-representation.*

Proof. The proof is taken from [BZ76], but it is basically the same as in [Bum97]. Let Δ be the set of matrices of the form $\delta = \begin{pmatrix} \varpi^n & 0 \\ 0 & \varpi^m \end{pmatrix}$ with $m \geq n$ where ϖ is a uniformizer. Set $t(\delta) = m - n$.

For $n \geq 1$ let $K_n = I + \mathfrak{p}^n M_{2\times 2}(\mathcal{O}_F)$ and $K_n^- = K_n \cap N^T, K_n^0 = K_n \cap T, K_n^+ = K_n \cap N$. Note that the K_n form a basis of open compact subgroups around the identity.

Lemma. *We have $K_n = K_n^+ K_n^0 K_n^-$.*

Denote the $p : V \to V_N$ the quotient map.

Lemma. *We have $T(V^{K_n}) \subseteq (V_N)^{K_n^0}$. For $\eta \in V_N^{K_n^0}$, there is a $t \in \mathbb{Z}$ such that $\pi_N(\delta^{-1})\eta \in p(V^{K_n})$ for all $\delta \in \Delta$ satisfying $t(\delta) > t$.*

Proof. The inclusion is obvious, p is a map of T-representations. For the second part let $\eta \in V_N^{K_n^0}$ and $\xi \in V$ with $p(\xi) = \eta$. Then also $\pi_N(\delta^{-1})\eta \in V_N^{K_n^0}$ for any $\delta \in \Delta$. We have $\pi_N(\delta^{-1})\eta = p(\pi(\varepsilon_{K_n^+ K_n^0})\pi(\delta^{-1})\xi)$ since $K_n^+ \subseteq N$ acts trivially on V_N. Let $k = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \in K_n^-$ and $\delta = \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix} \in \Delta$. Then

$$\delta k \delta^{-1} = \begin{pmatrix} 1 & 0 \\ \varpi^{b-a}x & 1 \end{pmatrix}.$$

So for $t(\delta) = b - a$ large enough, $\delta k \delta^{-1}$ will be in the stabilizer of ξ for all $k \in K_n^-$, so that $\pi(\delta^{-1})\xi = \pi(\varepsilon_{K_n^{-1}})\pi(\delta^{-1})\varepsilon$. Then for such δ we have

$$\pi_N(\delta^{-1})\eta = p(\pi(\varepsilon_{K_n^+ K_n^0})\pi(\delta^{-1})\xi) = p(\pi(\varepsilon_{K_n^+ K_n^0})\pi(\varepsilon_{K_n^-})\pi(\delta^{-1})\xi) = p(y)$$

with $y = \pi(\varepsilon_{K_n^+ K_n^0 K_n^-})\pi(\delta^{-1})\xi \in V^{K_n}$. □

We now prove that $T(V^{K_n}) = (V_N)^{K_n^0}$. It then follows that V_N is admissible. Since we know that $T(V^{K_n}) \subseteq (V_N)^{K_n^0}$, it suffices to show that $\dim(V_N)^{K_n^0} \leq \dim V^{K_n}$. If $\eta_1, \ldots, \eta_l$ are linearly independent vectors in $(V_N)^{K_n^0}$, then by the lemma there is a t such that $\pi_N(\delta^{-1})\eta_i \in p(V^{K_n})$ for all $\delta \in \Delta$ with $t(\delta) > t$. $\pi_N(\delta^{-1})\eta_1, \ldots, \pi_N(\delta^{-1})\eta_l$ are still linearly independent, so $l \leq \dim p(V^{K_n}) \leq \dim V^{K_n}$, hence the claim. □

Proposition 11.5. *If (V, π) is an irreducible smooth representation of G, then V_N is at most two-dimensional.*

Proof. If $V_N = 0$, there is nothing to show. Otherwise by Theorem 22.3, V embeds into a principal series representation and the result follows from Proposition 15.3. □

Proposition 11.6. *If (V, π) is one-dimensional, of the form $\chi \circ \det$ for some quasi-character χ of $F^\times$, then $V_N \cong \chi \circ \det = \chi \boxtimes \chi$.*

Note the equality $\chi \circ \det = \chi \boxtimes \chi$ of course only makes sense as T-representations.

Proof. Immediate. □

12. Representations of M, N

Let ψ be a nontrivial character of F. Then M acts on $C^\infty(F^\times)$ and $C_c^\infty(F^\times)$ by

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\phi(x) = \phi(ax), \tag{$*$}$$

$$\pi\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right)\phi(x) = \psi(bx)\phi(x) \tag{$**$}$$

Denote by V the subspace of $C^\infty(F^\times)$ consisting of functions ϕ satisfying $\phi(x) = 0$ for $|x| > c$ where c is a constant depending on ϕ. V is an M-subrepresentation of $C^\infty(F^\times)$ containing $C_c^\infty(F^\times)$.

Proposition 12.1 ([BH06, §8.2 Gloss.])**.** *There are isomorphisms of M-representations $V \cong \mathrm{Ind}_N^M \psi$ and $C_c^\infty(F^\times) \cong c\text{-}\mathrm{Ind}_N^M \psi$.*

Proof. This basically follows from the fact that $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ is a bijection $F^\times \to M/N$. If $f \in \mathrm{Ind}_N^M \psi$, we define the function $\varphi_f : F^\times \to \mathbb{C}$ by

$$\varphi_f(a) = f\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right).$$

Then $f \mapsto \varphi_f$ is an isomorphism of M-representations $\mathrm{Ind}_N^M \psi \cong V$. The same argument works for the compact induction.

□

12.1. Irreducibility of $C_c^\infty(F^\times)$ as an M-representation

Proposition 12.2. *The representation of M on $C_c^\infty(F^\times)$ is irreducible.*

12.1.1. *Proof in [Bum97].* Let U be a nontrivial invariant subspace of $C_c^\infty(F^\times)$. Fix $a \in F^\times$. Let $\phi \in U$ be nonzero. By $(*)$ we may assume $\phi(a) \neq 0$. For any $f \in C_c^\infty(F)$, we define

$$\phi_1 = \int_F f(x)\pi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\phi \,\mathrm{d}x.$$

We have $\phi_1 \in U$ and

$$\phi_1(y) = \int_F f(x)\psi(xy)\phi(y)\,\mathrm{d}x = \widehat{f}(y)\phi(y).$$

Now if V is a small neighboorhood of a such that ϕ is constant on V, then we chooose f so that $\widehat{f}$ is the characteristic function of V. This implies that U contains all characteristic functions of arbitrarily small neighborhoods of a. Letting a vary, these functions span $C_c^\infty(F^\times)$, so $U = C_c^\infty(F^\times)$ as desired.

12.2. **Twisted Jacquet modules determine elements**

Proposition 12.3. *Let (V,π) be a smooth representation of N. We have $\bigcap_\psi V(\psi) = 0$ where the intersection is taken over all characters ψ of F.*

Proposition 12.4. *If (V,π) is a smooth representation of M, we have $V_\psi \cong V_{\psi'}$ as vector spaces for all nontrivial characters ψ, ψ' of F.*

Proof. Indeed, let ψ,ψ' be nontrivial characters of F. Then there is $a \in F^\times$ such that $\psi'(x) = \psi(ax)$ for all x. Then the map $V(\psi') \to V(\psi)$ given by $v \mapsto \pi\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right)v$ is an isomorphism. □

So if ψ is a nontrivial character of F and (V,π) a smooth representation of M, we get $V_{\psi'} = 0$ for all nontrivial characters ψ' of F. By Proposition 12.3 we then must have $V(N) = 0$, in other words N acts trivially on V. Now if V is actually a representation of G, then also $\mathrm{SL}_2(F)$ acts trivially on V since it is generated by N and its conjugates. So we have shown:

Corollary 12.5 ([Bum97, Theorem 4.4.3])**.** *If (V,π) is a smooth representation of G such that $V_\psi = 0$ for some nontrivial character ψ of F, then the action of G factors through $\mathrm{GL}_2(F)/\mathrm{SL}_2(F) \cong F^\times$. In particular, if π is admissible and irreducible, then $\dim V = 1$ and we have $\pi = \chi \circ \det$ for some quasi-character χ of $F^\times$.*

12.2.1. *Proof in [BH06].* Let $v \neq 0$ be in V. We show that there is a character ψ such that $v \notin V(\psi)$. Let $N_0 \subseteq N$ be a compact open subgroup such that $v \in V^{N_0}$. Let $N_0 \subseteq N_1 \subseteq N_2 \subseteq \dots$ be a filtration of compact subgroups such that $N = \bigcup_j N_j$ (e.g. explicitly $N_j = \begin{pmatrix}1 & \mathfrak{p}^{k-j}\\ 0 & 1\end{pmatrix}$ for some fixed k). For each $j \geq 1$ we may view V^{N_0} as a representation of the finite group N_j/N_0. Since $v \in V^{N_0}$ is nonzero, there is a character ψ_j of N_j/N_0 such that the projection of v onto the ψ_j-isotypic component of V^{N_0} is nonzero, i.e. $v \notin V^{N_0}(\psi_j)$. This means hat $\int_{N_j} \psi_j(n)^{-1}\pi(n)v\,\mathrm{d}n \neq 0$. In fact we may choose the ψ_j inductively in a compatible way so that ψ_{j+1} extends ψ_j. Then let $\psi = \bigcup_j \psi_j$ be the corresponding character of N. Since for all $j \geq 1$ we have $\int_{N_j} \psi(n)^{-1}\pi(n)v\,\mathrm{d}n \neq 0$, we have $v \notin V(\psi)$.

12.2.2. *Proof in [Bum97].* This is quite a different view on the matter. Fix a nontrivial character ψ of F. Then the map $F \to \widehat{F}$, $a \mapsto \psi_a : x \mapsto \psi(ax)$ is an isomorphism. We equip F with the self-dual Haar measure so that the Fourier transform gives an isomorphism of rings $(C_c^\infty, *) \cong (C_c^\infty, \cdot)$. We define an action ρ of $C_c^\infty(F)$ on V by $\rho(x) = \pi\left(\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}\right)$. Then the Fourier transform gives V the structure of a smooth (or cosmooth in the terminology of [Bum97]) module of $(C_c^\infty(F), \cdot)$. This smooth module corresponds to a sheaf $\mathcal{F}$ on F such that $V = \mathcal{F}_c$.

Lemma 12.6 ([Bum97, Proposition 4.4.5])**.** *Let $a \in F$. The stalk of $\mathcal{F}$ at $a \in F$ satisfies*

$$\mathcal{F}_a \cong V_{\psi_a}.$$

Note for $a = 0$ we have $V_{\psi_a} = V_N$. Then Proposition 12.3 follows immediately from the sheaf properties.

Proof of Lemma 12.6. We have $\mathcal{F}_a = V/V(a)$ where $V(a)$ is the subspace of all $v \in V$ such that $\mathbb{1}_U \cdot v = 0$ for some open neighborhood U of a. It remains to show have $V(a) = V(\psi_a)$. We have $v \in V(\psi_a)$ if and only if for some compact open subgroup $F_0 = \mathfrak{p}^k \subseteq F$ we have

$$\rho(\psi_a \mathbb{1}_{F_0})v = \int_{F_0} \psi_a(-x)\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) v \,\mathrm{d}x = 0.$$

Now we have $\widehat{\mathbb{1}_{a+\mathfrak{p}^{n-k}}} = \mathrm{vol}(\mathfrak{p}^{n-k})\psi_a \mathbb{1}_{\mathfrak{p}^k}$, where n is the exponent of the conductor of ψ, and the result follows. $\square$

13. Whittaker Models

Let ψ be a nontrivial character of F. We denote the corresponding character on N again by ψ. Let $\mathcal{W}(\psi)$ the space of smooth functions $W : G \to \mathbb{C}$ such that $W(ng) = \psi(n)W(g)$ for all $n \in N$ and $g \in G$. Then G acts on $\mathcal{W}(\psi)$ via right translation. Note that $\mathcal{W}(\psi) = \mathrm{Ind}_N^G \psi$.

Definition. *A Whittaker model of a representation (V, π) is an injective G-homomorphism $V \to \mathcal{W}(\psi)$. A Whittaker functional on (V, π) is a nonzero linear map $\Lambda : V \to \mathbb{C}$ such that $\Lambda(nv) = \psi(n)\Lambda(v)$ for all $n \in N, v \in V$.*

By Frobenius reciprocity, we have

$$\mathrm{Hom}_N(\pi|_N, \psi) \cong \mathrm{Hom}_G(\pi, \mathcal{W}(\psi)).$$

So if π is irreducible, giving a Whittaker model is equivalent to giving a Whittaker functional.

Theorem 13.1. *If (V, π) is an irreducible smooth representation of G, then $\dim V_\psi \leq 1$. If V is infinite-dimensional, then $\dim V_\psi = 1$.*

Suppose (V, π) is irreducible and smooth. The space of the Whittaker model is denoted $\mathcal{W}(\pi, \psi)$. Explicitly, if Λ is a Whittaker functional on V, then $W(\pi, \psi)$ is the space of functions W_v for $v \in V$ where

$$W_v(g) = \Lambda(\pi(g)v).$$

Proof. The proof in Bump uses some the geometric theory and the involution method to show that certain Bessel distributions are invariant under a suitable involution. If we assume the existence of a Kirillov model, we can give a short proof (see next section).

It follows from Theorem 10.2 that if V admits a Whittaker functional, then so does $\widetilde{V}$. Let Λ be a Whittaker functional for V. Then there is also a Whittaker functional $\widetilde{\Lambda}$ for $\widetilde{V}$. We show that Λ and $\widetilde{\Lambda}$ determine each other up to scalar multiple.

13.0.1. *Proof adapted from [Ngo].* Note that Λ is functional on V, but it is not smooth. However, we can convolve it with smooth functions, to get a smooth functional. Define:

$$\begin{aligned}\Phi_\Lambda : \mathcal{H}(G) &\longrightarrow \widetilde{V}\\ \varphi &\longmapsto \Lambda * \varphi := \int_G \varphi(g)(\Lambda\circ\pi(g))\mathrm{d}g = \left(v \mapsto \int_G \varphi(g)\Lambda(\pi(g)v)\mathrm{d}g = \Lambda(\pi(\varphi)v)\right).\end{aligned}$$

Similarly define

$$\begin{aligned}\Phi_{\widetilde{\Lambda}} : \mathcal{H}(G) &\longrightarrow \widetilde{\widetilde{V}}\\ \varphi &\longmapsto \widetilde{\Lambda} * \varphi = \left(\widetilde{v} \mapsto \widetilde{\Lambda}(\pi(\varphi)\widetilde{v})\right).\end{aligned}$$

It is easily seen that these satisfy

$$\begin{aligned}\Phi_\Lambda(\lambda(n)\varphi) &= \psi(n)\Phi_\Lambda(\varphi),\\ \Phi_{\widetilde{\Lambda}}(\lambda(n)\varphi) &= \psi(n)\Phi_{\widetilde{\Lambda}(\varphi)},\\ \Phi_\Lambda(\rho(g)\varphi) &= \pi(g)\Phi_\Lambda(\varphi),\\ \Phi_{\widetilde{\Lambda}}(\rho(g)\varphi) &= \pi(g)\Phi_{\widetilde{\Lambda}}(\varphi),\end{aligned}$$

for $n \in N, g \in G$. Let $\Phi : \mathcal{H}(G)\otimes\mathcal{H}(G) \to \widetilde{V}\otimes\widetilde{\widetilde{V}}$ be the tensor product of the maps $\Phi_\Lambda, \Phi_{\widetilde{\Lambda}}$. Let $B : \mathcal{H}(G)\otimes\mathcal{H}(G) \to \mathbb{C}$ be the composition of Φ with the natural pairing $\widetilde{V}\otimes\widetilde{\widetilde{V}} \to \mathbb{C}$. Then

$$B(\varphi_1\otimes\varphi_2) = \langle \Phi_\Lambda(\varphi_1), \Phi_{\widetilde{\Lambda}}(\varphi_2)\rangle.$$

It satisfies

$$\begin{aligned}B(\lambda(n)\varphi_1\otimes\lambda(n')\varphi_2) &= \psi(n)\psi(n')B(\varphi_1\otimes\varphi_2),\\ B(\rho(g)\varphi_1\otimes\rho(g)\varphi_2) &= B(\varphi_1\otimes\varphi_2),\end{aligned}$$

for $n \in N, g \in G$. There is an isomorphism $\mathcal{H}(G)\otimes\mathcal{H}(G) \to \mathcal{H}(G\times G)$, induced by the map $\varphi_1\otimes\varphi_2 \mapsto \varphi_1\otimes\varphi_2$, where the tensor product on the right means $(\varphi_1\otimes\varphi_2)(g,h) = \varphi_1(g)\varphi_2(h)$. Hence we get a map $\mathcal{H}(G\times G) \to \mathbb{C}$, still denoted B, such that

$$\begin{aligned}B(\lambda_1(n)\lambda_2(n')\psi) &= \psi(n)\psi(n')B(\psi),\\ B(\rho_1(g)\rho_2(g)\psi) &= B(\psi),\end{aligned}$$

where $n, n' \in N, g \in G$. By λ_1, λ_2 we mean left translation in the first or second component, similarly for ρ_1, ρ_2.

Consider the action of G on the space $X = G\times G$ via $g\cdot(g_1,g_2) = (g_1g^{-1}, g_2g^{-1})$. Then we get an induced map

$$P : C_c^\infty(X) \longrightarrow C_c^\infty(G\backslash X)$$

$$\phi \longmapsto \left(P\phi : Gx \mapsto \int_G \phi(g\cdot x)\mathrm{d}g\right).$$

Lemma 13.2. *P is surjective. If $B : C_c^\infty(X) \to \mathbb{C}$ satisfies $B(\rho(g)\phi) = B(\phi)$ for all $g \in G$, then B factors through P.*

Proof. The surjectivity of P is basically Proposition A.5. $\square$

Note that the map $G \to G\backslash X$, $g \mapsto (1, g)$ is a bijection, and hence we get an isomorphism $C_c^\infty(G\backslash X) \to C_c^\infty(G)$, $\psi \mapsto (g \mapsto \psi(1, g))$. Then the map P becomes

$$\begin{aligned} P : C_c^\infty(X) &\longrightarrow C_c^\infty(G) \\ \phi &\longmapsto \left(h \mapsto \int_G \phi(g^{-1}, hg^{-1})\mathrm{d}g = \int_G \phi(g, hg)\mathrm{d}g\right) \end{aligned}$$

Apply this to our B. We get a distribution $\Delta : \mathcal{H}(G) \to \mathbb{C}$ satisfying $\Delta(P\phi) = B\phi$ for $\phi \in \mathcal{H}(G\times G)$. It is easily seen that $\lambda(n)P(\phi) = P(\lambda_2(n)\phi)$, and $\rho(n)P(\phi) = P(\lambda_1(n)\phi)$. Hence,

$$\Delta(\lambda(n)\rho(n')\varphi) = \psi(n)\psi(n')\Delta(\varphi),$$

for $n, n' \in N$, i.e. $\lambda(n)^{-1}\Delta = \psi(n)\Delta$ and $\rho(n)\Delta = \psi(n)\Delta$ for $n \in N$. By Theorem 26.1, this gives that Δ is invariant under ι, where $\iota : G \to G$, $g \mapsto wg^Tw$. Let $\varphi_1, \varphi_2 \in \mathcal{H}(G)$. Then

$$(P\varphi_1 \otimes \varphi_2)(h) = \int_G \varphi_1(g)\varphi_2(hg)\mathrm{d}g$$

and

$$(\iota(P\varphi_1 \otimes \varphi_2))(h) = \int_G \varphi_1(g)\varphi_2(wh^Twg)\mathrm{d}g.$$

It is easily seen that $\iota(P\varphi_1 \otimes \varphi_2) = P(\widetilde{\iota\varphi_1} \otimes \widetilde{\iota\varphi_2})$. Then $B(\varphi_1, \varphi_2) = B(\widetilde{\iota\varphi_2}, \widetilde{\iota\varphi_1})$. Now if $\varphi_1 \in \ker \Phi_\Lambda$, then $B(\widetilde{\iota\varphi_2}, \widetilde{\iota\varphi_1}) = B(\varphi_1, \varphi_2) = 0$ for all $\varphi_2 \in \mathcal{H}(G)$. This implies that $\Phi_{\widetilde{\Lambda}}(\widetilde{\iota\varphi_1}) = 0$.

This shows that $\ker \Phi_\Lambda$ determines $\ker \Phi_{\widetilde{\Lambda}}$. Of course, we could swap the roles and get the reverse. Hence, if Λ' is another Whittaker functional for V, then $\ker \Phi_\Lambda = \ker \Phi_{\Lambda'} =: U$. Then $\Phi_\Lambda, \Phi_{\Lambda'}$ both induces G-equivariant isomorphisms $\mathcal{H}(G)/U \to V$. Since V is irreducible, they are differ by a scalar. It is then easy to see that Λ, Λ' differ by the same scalar.

13.0.2. *Proof adapted from [Bum97].* Here is a slightly different version in the spirit of Bump's proof. Define the map Φ_Λ as before. Define $\Delta : \mathcal{H}(G) \to \mathbb{C}$ by

$$\Delta(\varphi) = \widetilde{\Lambda}(\Phi_\Lambda(\varphi)).$$

Then

$$\begin{aligned} (\lambda(n)\Delta)(\varphi) &= \Delta(\lambda(n^{-1})\varphi) = \widetilde{\Lambda}(\Phi_\Lambda(\lambda(n^{-1})\varphi)) = \psi(n)^{-1}\widetilde{\Lambda}(\Phi_\Lambda(\varphi)) = \psi(n)^{-1}\Delta(\varphi), \\ (\rho(n)\Delta)(\varphi) &= \Delta(\rho(n^{-1})\varphi) = \widetilde{\Lambda}(\Phi_\Lambda(\rho(n^{-1})\varphi)) = \widetilde{\Lambda}(\pi(n)\Phi_\Lambda(\varphi)) = \psi(n)^{-1}\widetilde{\Lambda}(\Phi_\Lambda(\varphi)) = \psi(n)^{-1}\Delta(\varphi), \end{aligned}$$

for $n \in N$. This is not quite the condition in Theorem 26.1, but its proof shows that still $\iota(\Delta) = \Delta$.

Lemma 13.3. *If* $\varphi \in \mathcal{H}(G)$ *is such that* $\Phi_\Lambda(\varphi) = 0$, *then* $\Phi_{\widetilde{\Lambda}}(\widetilde{\iota\varphi}) = 0$.

Here $\check{f}(g) = f(g^{-1})$.

Proof. First note that $\Phi_\Lambda(\rho(g)\varphi) = \pi(g)\Phi_\Lambda(\varphi) = 0$ for all $g \in G$. Next we have

$$0 = \widetilde{\Lambda}(\Phi_\Lambda \rho(g)\varphi) = \Delta(\rho(g)\varphi) = \Delta(\iota(\rho(g)\varphi)) = \Delta(\lambda(\iota(g))\iota(\varphi)).$$

Hence also $\Phi_\Lambda(\lambda(g)\iota(\varphi)) = 0$ for all $g \in G$. Let $\phi \in \mathcal{H}(G)$. Then

$$(\phi * \iota(\varphi))(h) = \int_G \phi(g)(\lambda(g)\iota(\varphi))(h)\mathrm{d}g.$$

Then we get $\widetilde{\Lambda}(\Phi_\Lambda(\phi * \iota(\varphi))) = 0$. We have $\Phi_\Lambda(\phi * \iota(\varphi))(v) = \Lambda(\pi(\phi)\pi(\iota\varphi)v) = (\pi(\widetilde{\iota\varphi})\Phi_\Lambda(\phi))(v)$. Now it is easy to see that the functions $\Phi_\Lambda(\phi)$, $\phi \in \mathcal{H}(G)$, span $\widetilde{V}$ (since they generate an invariant non-zero subspace, and $\widetilde{V}$ is irreducible). This show that $\widetilde{\Lambda}(\pi(\widetilde{\iota\varphi})\widehat{v}) = 0$ for every $\widehat{v} \in \widetilde{V}$, hence $\Phi_{\widetilde{\Lambda}}(\widetilde{\iota\varphi}) = 0$. ☐

We can switch the roles of $\Lambda, \widetilde{\Lambda}$ (using $\widetilde{\widetilde{V}} = V$) and obtain that the kernels of $\Phi_\Lambda, \Phi_{\widetilde{\Lambda}}$ determine each other. As in the previous proof we see that this suffices to establish the claim.

☐

14. Kirillov Models

Let (V, π) be an infinite-dimensional irreducible smooth representation. In the previous section we used the isomorphism $\mathrm{Hom}_N(\pi|_N, \psi) \cong \mathrm{Hom}_G(\pi, W(\psi))$ to realize π in $W(\psi) \subseteq C^\infty(G)$. We also have

$$\mathrm{Hom}_N(\pi|_N, \psi) \cong \mathrm{Hom}_M(\pi, \mathrm{Ind}_N^M \psi).$$

Explicitly, if Λ is a Whittaker functional on V, then $f : v \mapsto (m \mapsto \Lambda(\pi(m)v))$ is the unique (up to scalar) nonzero M-homomorphism $\pi \to \mathrm{Ind}_N^M \psi$.

Proposition 14.1. f *is injective, so* V *embeds into* $\mathrm{Ind}_N^M \psi$.

Proof. This is the argument in [BH06, p. 227], but it is basically the same as in [Bum97]. The composition of $V \to \mathrm{Ind}_N^M \psi \to \psi$, where the second map is given by $g \mapsto g(1)$ is just Λ, and thus induces an isomorphism $V_\psi \cong \psi$. So by exactness of the twisted Jacquet functor, $(\ker f)_\psi = 0$, hence $(\ker f)(N) = 0$ by Proposition 12.3 and Proposition 12.4, in other words, N acts trivially on $\ker f$. This forces $\ker f = 0$ by Proposition 10.4. ☐

Note that by Proposition 12.1 we have $\mathrm{Ind}_N^M \psi \cong W$ as M-representations where W is the subspace of $C^\infty(F^\times)$ of bounded support. Hence we get an embedding

$$V \hookrightarrow \mathrm{Ind}_N^M \psi \cong W \hookrightarrow C^\infty(F^\times).$$

as M-representations.

Definition. *This (or also the image in* $\operatorname{Ind}_N^M \psi$*) is the Kirillov model of* (V, π).

So there is a space of functions of bounded support in $C^\infty(F^\times)$ with a G-action (extending the natural M-action) that is isomorphic to V. This space and the action is unique, essentially by uniqueness of the Whittaker model, of which we can now give a (second) proof (though this is not particular useful in our treatment, since our proof of existence of the Kirillov model relied on $\dim V_\psi = 1$. But there are other ways to prove it, e.g. [JL70].):

Proof of Theorem 13.1. (Assuming the existence of a Kirillov model). Let $i : V \hookrightarrow C^\infty(F^\times)$ be a Kirillov model. Define $\Lambda : V \to \mathbb{C}$ by $\Lambda(v) = i(v)(1)$. Then Λ is a Whittaker functional. It suffices to show that $V(\psi) = \ker \Lambda$. Clearly $V(\psi) \subseteq \ker \Lambda$. Conversely, if $v \in \ker \Lambda$, then $i(v)(t) = 0$ for $t \in F^\times$ close to 1. Let $w = \int_{\mathfrak{p}^{-n}} \psi(-x)\pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v \mathrm{d}x$. We have

$$i(w)(t) = \int_{\mathfrak{p}^{-n}} \psi(-x) i(\pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v)(t) \mathrm{d}x = \int_{\mathfrak{p}^{-n}} \psi(x(t-1)) i(v)(t) \mathrm{d}x.$$

If $|t-1|$ is bounded away from zero we can choose n large enough such that $\int_{\mathfrak{p}^{-n}} \psi(x(t-1)) \mathrm{d}x = 0$ for all such t. Since also $i(v)(t) = 0$ in a neighborhood of 1, we see that we can choose n such that $i(w)(t) = 0$ for all t, hence $i(w) = 0$, so $w = 0$, i.e. $v \in V(\psi)$. □

In fact we can describe the space explicitly:

Theorem 14.2. *Suppose V is equal to its Kirillov model, so that $V \subseteq C^\infty(F^\times)$. Then $V(N) = C_c^\infty(F^\times)$. Moreover:*

(1) If V is isomorphic to a principal series representation $\pi(\chi_1, \chi_2)$, define $\phi_j : F^\times \to \mathbb{C}$ by $\phi_j = |\cdot|^{1/2} \chi_j \mathbb{1}_{\mathcal{O}_F \setminus \{0\}}$. Then

$$V = \mathbb{C}\phi_1 + \mathbb{C}\phi_2 + C_c^\infty(F^\times),$$

if $\chi_1 \neq \chi_2$ and

$$V = \mathbb{C}\phi_1 + Cv\phi_2 + C_c^\infty(F^\times),$$

if $\chi_1 = \chi_2$ where $v : F^\times \to \mathbb{C}$ is the valuation.

(2) If V is isomorphic to a special representation $\sigma(\chi_1, \chi_2)$ with $\chi_1 \chi_2^{-1} = |\cdot|^{-1}$, then

$$V = \mathbb{C}\phi_2 + C_c^\infty(F^\times),$$

where ϕ_2 is as in the previous case.

(3) If V is a supercuspidal representation, then $V = C_c^\infty(F^\times)$.

Proof. If $n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in N$ and $f \in V$, then $(\pi(n)f - f)(y) = (\psi(yx) - 1) f(y)$. For $|y|$ small, this is 0, so $V(N) \subseteq C_c^\infty(F^\times)$. Since $V(N)$ is nonzero (e.g. as V is infinite dimensional and V_N finite dimensional) and $C_c^\infty(F^\times)$ is irreducible as an M-representation, Proposition 12.2, we get $C_c^\infty(F^\times) = V(N)$. We now use the explicit description of V_N for the different types of representations, see Theorem 15.10, Proposition 16.2 and Theorem 22.3. In the supercuspidal case there is nothing to do. Assume we are in the case of the principal series representation. Let $f \in V$ is such that $\pi_N(t)\overline{f} = \eta(t)\overline{f}$ where $\overline{f}$ is the image of f in V_N, $t \in T$ and $\eta = \delta^{1/2}\chi_1$. Fix $t_0 \in \varpi\mathcal{O}^\times$. Let $t = \begin{pmatrix} t_0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\pi(t)f - \eta(t)f \in V(N)$, so by the first part it is in $C_c^\infty(F^\times)$ and there is $\varepsilon = \varepsilon(t_0) > 0$ such that

$$0 = (\pi(t)f - \eta(t)f)(x) = f(t_0x) - |t_0|^{1/2}\chi_1(t_0)f(x)$$

for $|x| < \varepsilon$. By local constancy, this also holds in a neighborhood of t_0. Since $\varpi\mathcal{O}^\times$ is compact, there is a $\varepsilon > 0$ such that this holds for all $t_0 \in \varpi\mathcal{O}^\times$. It then holds for all $t_0 \in F^\times$ with $|t_0| < 1$ since any such element can be written as a finite product of elements in $\varpi\mathcal{O}^\times$. In other words, $f(t_0x) = |t_0|^{1/2}\chi_1(t_0)f(x)$ for all $t_0, x \in F^\times$ with $|t_0| < 1, |x| < \varepsilon$. Therefore f differes from ϕ_1 by a function in $C_c^\infty(F^\times)$ and we get $\phi_1 \in V$. If $\chi_1 \neq \chi_2$, we get similarly $\phi_2 \in V$. Since $\dim V_N = 2$, ϕ_1, ϕ_2 must span V modulo $V(N) = C_c^\infty(F^\times)$. If $\chi_1 = \chi_2$ adjust this slightly, and for the special representation it works the same. □

15. Principal Series Representations

Definition. *Let $\chi_1, \chi_2 : F^\times \to \mathbb{C}^\times$ be quasi-characters. Let $\chi = \chi_1 \boxtimes \chi_2$ and $(V, \pi) = \mathrm{Ind}_B^G(\chi)$ be the principal series representation. We will also denote $V = \pi(\chi_1, \chi_2)$ or $\mathcal{B}(\chi_1, \chi_2)$.*

Note that $\mathrm{Ind}_B^G(\chi) = c\text{-}\mathrm{Ind}_B^G(\chi)$ is admissible by Proposition 9.7 since G/B is compact.

Proposition 15.1. *The central quasi-character of V is $\chi_1 \otimes \chi_2$. If μ is any quasi-character of $F^\times$, then $\mu \otimes \mathcal{B}(\chi_1, \chi_2) \cong \mathcal{B}(\mu\chi_1, \mu\chi_2)$.*

Proof. Immediate. □

Proposition 15.2. $\mathcal{B}\widehat{(\chi_1, \chi_2)} \cong \mathcal{B}(\chi_1^{-1}, \chi_2^{-1})$.

Proof. By general properties of induction, Theorem 9.9, and using $\mathrm{Ind}_B^G(\chi) = c\text{-}\mathrm{Ind}_B^G(\chi)$. □

Proposition 15.3. *There is a short exact sequence of representations of T:*

$$0 \to \chi^w \otimes \delta_B|_T^{1/2} \to V_N \to \chi \otimes \delta_B|_T^{1/2} \to 0.$$

Proof. Immediate from Lemma 11.3. □

Note in particular that if $\chi_1 \neq \chi_2$, then $\chi^w \neq \chi$ and the above sequence splits.

We might not use it, but we note:

Proposition 15.4. *There is a surjective map $P : C_c^\infty(G) \to \mathcal{B}(\chi_1, \chi_2)$ of G-representations, defined by*

$$(P\phi)(g) = \int_B \phi(b^{-1}g)(\delta^{1/2}\chi)(b)\mathrm{d}b.$$

Proof. In the notation of Appendix A we have $\mathcal{B}(\chi_1,\chi_2) = C_c^\infty(B\backslash G, \delta^{1/2}\chi)$ and $P = P^{\delta^{1/2}\chi}$. The result follows from Proposition A.5, or rather its smooth analogue, see e.g. [Bum97, Proposition 4.5.3] or [BH06, 3.4]. $\square$

15.1. **Irreducibility of Principal Series Representations**

Let ψ be a nontrivial character of F. Let $(V,\pi) = \mathrm{Ind}_B^G(\chi)$.

Proposition 15.5. *The principal series representation (V,π) has a unique Whittaker functional, i.e.* $\dim V_\psi = 1$.

Proposition 15.6. *The principal series representation has an invariant one-dimensional (resp. codimension one) subspace if and only if $\chi_1\chi_2^{-1} = |\cdot|^{-1}$ (resp. $\chi_1\chi_2^{-1} = |\cdot|$). In this case the invariant one-dimensional (resp. codimension one) subspace is unique.*

Proof. TODO (not difficult). The case $\chi_1\chi_2^{-1} = |\cdot|$ follows by dualizing using Proposition 15.2 $\square$

In the case $\chi_1\chi_2^{-1} = |\cdot|^{-1}$, the invariant one-dimensional subspace is spanned by $f(g) = \widetilde{\chi}(\det g)$ where $\widetilde{\chi} = \chi_1\,|\cdot|^{1/2} = \chi_2\,|\cdot|^{-1/2}$. G acts on it via $\widetilde{\chi}$ (viewed as usual as a character of G via det). By dualizing in the case $\chi_1\chi_2^{-1} = |\cdot|^{-1}$, the one-dimensional quotient is $\chi_1\,|\cdot|^{-1/2} = \chi_2\,|\cdot|^{1/2}$.

Theorem 15.7. *The principal series representation (V,π) is irreducible if and only if $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}$. In the case $\chi_1\chi_2^{-1} = |\cdot|^{\pm 1}$, (V,π) has length 2.*

Definition. *In the case $\chi_1\chi_2^{-1} = |\cdot|^{\pm 1}$, the unique infinite-dimensional irreducible quotient or submodule of $\mathcal{B}(\chi_1,\chi_2)$ is denoted $\sigma(\chi_1,\chi_2)$. It is called a* special representation.

If $\chi_1\chi_2^{-1} = |\cdot|^{-1}$ it fits into the exact sequence

$$0 \to \chi_1\,|\cdot|^{1/2} = \chi_2\,|\cdot|^{-1/2} \to \mathcal{B}(\chi_1,\chi_2) \to \sigma(\chi_1,\chi_2) \to 0$$

Similarly in the case $\chi_1\chi_2^{-1} = |\cdot|$ we have

$$0 \to \sigma(\chi_1,\chi_2) \to \mathcal{B}(\chi_1,\chi_2) \to \chi_1\,|\cdot|^{-1/2} = \chi_2\,|\cdot|^{1/2} \to 0$$

By uniqueness of the quotient/submodule we also get that $\sigma\widehat{(\chi_1,\chi_2)} \cong \sigma(\chi_1^{-1},\chi_2^{-1})$.

15.1.1. *Proof in [BH06].* We have a short exact sequence of B-representations

$$0 \to W \to V \to \chi \otimes \delta_B^{1/2} \to 0$$

where the map on the right is given by $f \mapsto f(1)$. Part of the proof of Lemma 11.3 shows that the map $\Phi : W \to C_c^\infty(N)$, given by $\Phi(f) : n \mapsto f(wn)$ with $f \in W$ is an isomorphism. Note that it is an isomorphism of N-modules when N acts in the usual way via (right) translation on $C_c^\infty(N)$. It is clear that $\dim C_c^\infty(N)_\psi = 1$ (twisted Haar measure on N). Note also that $\chi \otimes \delta_B^{1/2}$ is trivial as an N-module. Hence applying the ψ-twisted Jacquet functor to the above sequence gives

$$0 \to W_\psi \to V_\psi \to 0$$

which shows Proposition 15.5.

Let $U = W(N)$. From the proof of Lemma 11.3 we know that the map $W \to \chi^w \otimes \delta_B^{1/2}$, $f \mapsto \int_N f(wn)\,\mathrm{d}n$ is surjective with kernel U. So we have a short exact sequence

$$0 \to U \to W \to \chi^w \otimes \delta_B^{1/2} \to 0$$

which gives $U_\psi \cong W_\psi$.

The following lemma isn't verbatim in [BH06], but we use some ideas from there (I didn't fully understand all of their proofs first and rewrote it myself in a different way, and this came out of it). It is also related to some calculations around the Kirillov model, see [Bum97, Proposition 4.7.2].

Lemma 15.8. *$U \cong C_c^\infty(F^\times)$ as M-representations.*

Proof. Define $\widetilde{\Phi} : U \to C_c^\infty(F^\times)$ via

$$\widetilde{\Phi}(f)(a) = \int_N \psi^{-1}(n)\Phi\left(\pi\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} f\right)(n)\,\mathrm{d}n = \int_N \psi^{-1}(n) f\left(wn\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\right)\mathrm{d}n.$$

Idea: Corresponding to the canonical projection $U \to U_\psi$ in $\mathrm{Hom}_N(U, U_\psi)$ we have an element in $\mathrm{Hom}_M(U, \mathrm{Ind}_N^M U_\psi)$. That element has in fact image in $c\text{-}\mathrm{Ind}_N^M U_\psi$. Then we compose this with $U_\psi \xrightarrow{\simeq} W_\psi \xrightarrow{\simeq} \psi$ where the last map is given by $f \mapsto \int_N \psi^{-1}(n) f(n)\,\mathrm{d}n$, and finally identify $C_c^\infty(F^\times)$ with $c\text{-}\mathrm{Ind}_N^M \psi$ via $\varphi \mapsto (a \mapsto \varphi\left(\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\right))$.

First we check that $\widetilde{\Phi}$ is well-defined, i.e. that indeed $\widetilde{\Phi}(f) \in C_c^\infty(F^\times)$. Clearly $\widetilde{\Phi}(f)$ is smooth, so we only need to check that its support is compact, i.e. bounded away from 0 and bounded. If we write $n = \begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}$, then we have

$$f\left(wn\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\right) = f\left(\begin{pmatrix} 1 & 0 \\ 0 & a\end{pmatrix} w \begin{pmatrix} 1 & x/a \\ 0 & 1\end{pmatrix}\right) = \delta_B\left(\begin{pmatrix} 1 & 0 \\ 0 & a\end{pmatrix}\right)^{1/2} \chi_2(a) f\left(w\begin{pmatrix} 1 & x/a \\ 0 & 1\end{pmatrix}\right).$$

Since the support of f is contained in BwN_0 for some compact open $N_0 \subseteq N$, we have

$$f\left(wn\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\right) = 0$$

unless $|x|$ is small enough (i.e. so that $\begin{pmatrix} 1 & x/a \\ 0 & 1\end{pmatrix} \in N_0$). Pick a sufficiently small compact open subgroup N_1 of N so that $\psi(n) = 1$ for $n \in N_1$, say $N_1 = \begin{pmatrix} 1 & \mathfrak{p}^j \\ 0 & 1\end{pmatrix}$ for some $j \in \mathbb{Z}$. If $|a|$ is sufficiently small, we have $f\left(wn\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\right) = 0$ unless $n \in N_1$. Then (write $n = \begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}$)

$$\begin{aligned}\widetilde{\Phi}(f)(a) &= \delta_B\left(\begin{pmatrix} 1 & 0 \\ 0 & a\end{pmatrix}\right)^{1/2} \chi_2(a) \int_N \psi^{-1}(n) f\left(w\begin{pmatrix} 1 & x/a \\ 0 & 1\end{pmatrix}\right)\mathrm{d}n \\ &= \delta_B\left(\begin{pmatrix} 1 & 0 \\ 0 & a\end{pmatrix}\right)^{1/2} \chi_2(a) \int_{\mathfrak{p}^j} f\left(w\begin{pmatrix} 1 & x/a \\ 0 & 1\end{pmatrix}\right)\mathrm{d}x\end{aligned}$$

$$= |a|\, \delta_B \left(\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \right)^{1/2} \chi_2(a) \int_{a^{-1}\mathfrak{p}^j} f \left(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \mathrm{d}n$$
$$= 0.$$

If $|a|$ is sufficiently small, the last integral is 0 since $f \in W(N)$, see Proposition 11.1. So the support of $\widetilde{\Phi}(f)$ is bounded away from 0. Next it is easy to see that $\widetilde{\Phi} \left(\pi \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} f \right)(a) = \psi(ay)\widetilde{\Phi}(f)(a)$. If $|y|$ is small enough, but nonzero, we have $\pi \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} f = f$, so $\widetilde{\Phi}(f)(a) = \psi(ay)\widetilde{\Phi}(f)(a)$. Then if $|a|$ is large enough, $\psi(ay) \neq 1$, so $\widetilde{\Phi}(f)(a) = 0$. This shows that the support of $\widetilde{\Phi}(f)$ is bounded and therefore compact. This shows that $\widetilde{\Phi}$ is well-defined. It is also clear that it is a map of M-representations. Suppose $f \in U$ is in the kernel of $\widetilde{\Phi}$. Let $f_N = \Phi(f)$. Then by the above computation we have

$$0 = \widetilde{\Phi}(f)(a) = \delta_B \left(\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \right)^{1/2} \chi_2(a) \int_N \psi^{-1}(n) f \left(w \begin{pmatrix} 1 & x/a \\ 0 & 1 \end{pmatrix} \right) \mathrm{d}n$$
$$= \delta_B \left(\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \right)^{1/2} \chi_2(a) \int_N \psi^{-1} \left(\begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix} \right) f \left(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \mathrm{d}x$$

This shows that the function f_N satisfies $\widehat{f_N}(a) = 0$ (Fourier transform) for all $a \neq 0$. But this is also true at 0, since f is in the kernel of the map $W \to \chi^w \otimes \delta^{1/2}$. So $\widehat{f_N} \equiv 0$, and then $f \equiv 0$. Therefore $\widetilde{\Phi}$ is injective. It is also surjective since $C_c^\infty(F^\times)$ is irreducible as an M-representation by Proposition 12.2. $\square$

We could extend this to an isomorphism of B-representations if we defined an action of B on $C_c^\infty(F^\times)$ by letting the center act via the central character.

We note that it is easy to see that $V(N) \subseteq U$, so that $V(N) = U$.

Corollary 15.9. *As an M-module (or B-module) a composition series of V is given by*

$$0 \subseteq U \subseteq W \subseteq V$$

with quotients $V/W \cong \chi \otimes \delta_B^{1/2}$, $W/U \cong \chi^w \otimes \delta_B^{1/2}$ and $U \cong C_c^\infty(F^\times)$. So $V|_M$ has length 3.

TODO: finish up proof (should be quick, basically given what we have done, it is like the one given next)

15.1.2. *Proof in [Bum97].* First prove Proposition 15.5 using the sheaf and involution method (todo).

Suppose $V' \subsetneq V$ is a nontrivial proper subspace. Let $V'' = V/V'$ so that we have a short exact sequence

$$0 \to V' \to V \to V'' \to 0.$$

By Proposition 15.5 we have $\dim V_\psi = 1$. So by exactness of the twisted Jacquet functor we have either $V'_\psi = 0$ or $V''_\psi = 0$. Suppose $V'_\psi = 0$. By replacing V by $\widetilde{V}$ (which replaces $\chi_1\chi_2^{-1}$ by $\chi_2\chi_1^{-1}$) we may assume $V'_\psi = 0$. Then by Corollary 12.5 the action of π on V' factors through $F^\times$. Then V' has a one-dimensional invariant subspace, so that $\chi_1\chi_2^{-1} = |\cdot|^{-1}$ by Proposition 15.6.

It remains to show that in the case $\chi_1\chi_2^{-1} \neq |\cdot|^{-1}$, V has length 2. Bump leaves this as an exercise, I am not sure what he intended. I will make use of Proposition 15.3, see also my question on Math StackExchange. Denote by X_- the quotient by the invariant one-dimensional subspace of V if $\chi_1\chi_2^{-1} = |\cdot|^{-1}$ and X_+ the invariant codimension one subspace if $\chi_1\chi_2^{-1} = |\cdot|$. We have to show that $X_\pm$ is irreducible. We know that $X_+ = \widetilde{X}_-$. First note that $\dim(X_\pm)_\psi = 1$. Assume U is a proper invariant subspace of X_-. Then we have a short exact sequence

$$0 \to U \to X_- \to X_-/U \to 0.$$

Hence either $U_\psi = 0$ or $(X_-/U)_\psi = 0$. If $U_\psi = 0$, then by the same logic as before U has a one-dimensional invariant subspace. We can pull this back to to V under the quotient map $V \to X_-$ which has one-dimensional kernel. Hence V has a two-dimensional invariant subspace W. We show that this is impossible. Indeed, by Proposition 15.3 (which is proven later in Bump), we have $V_N \cong \chi\delta^{1/2} \oplus \chi^w\delta^{1/2}$ as $F^\times$-modules. Since W is finite-dimensional, N must act trivially on it, so that W embeds into V_N. But then W has two distinct invariant one-dimensional subspaces. This is a contradiction since we showed in Proposition 15.6 that one-dimensional invariant subspace in V (in the case $\chi_1\chi_2^{-1} = |\cdot|^{-1}$) is unique. So $U_\psi = 0$ is impossible. If $(X_-/U)_\psi = 0$, then dualize and we see that X_+ contains an invariant nonzero subspace W with $W_\psi = 0$. Then again W contains a one-dimensional invariant subspace, but $W \subseteq X_+ \subseteq V$ does not have any invariant one-dimensional subspace by Proposition 15.6 for $\chi_1\chi_2^{-1} = |\cdot|$.

15.2. **Jacquet Modules of Principal Series Representations**

Theorem 15.10. *The Jacquet module of $V = \mathcal{B}(\chi_1, \chi_2)$ is two-dimensional and T acts as*

$$\chi\delta^{1/2} \oplus \chi^w\delta^{1/2}$$

if $\chi_1 \neq \chi_2$ and as

$$(\chi\delta^{1/2})(t)\begin{pmatrix} 1 & v(t_1/t_2) \\ 0 & 1 \end{pmatrix}$$

if $\chi_1 = \chi_2$, where $v : F^\times \to \mathbb{C}$ is the valuation.

Proof. The comment after Proposition 15.3 shows assertion in the case $\chi_1 \neq \chi_2$. If $\chi_1 = \chi_2$, we have the exact sequence

$$0 \to \chi \otimes \delta_B^{1/2} \to V_N \to \chi \otimes \delta_B^{1/2} \to 0,$$

and we need to work a little more. Abbreviate $\xi = \chi \otimes \delta_B^{1/2}$. Let v_1 be a non-zero vector in V_N coming from the inclusion $\xi \to V_N$ and v_2 a vector that maps to a non-zero vector under $V_N \to \xi$. Then for $t \in T$ we have we have

$$\pi_N(t)v_1 = \xi(t)v_1,$$
$$\pi_N(t)v_2 = \xi(t)v_2 + \eta(t)v_1,$$

for some scalar $\eta(t) \in \mathbb{C}$. Note that $t \mapsto \eta(t) \in \mathbb{C}$ is continuous. Computing $\pi_N(t_1t_2)$ gives $\eta(t_1t_2) = \xi(t_2)\eta(t_1) + \xi(t_1)\eta(t_2)$ which shows that $\lambda(t) = \frac{\eta(t)}{\xi(t)}$ defines a continuous homomorphism $T \to \mathbb{C}$. Consider the composition $\phi : F^\times \hookrightarrow T \to \mathbb{C}$ where the inclusion is on the first component, i.e. $t_1 \mapsto \begin{pmatrix} t_1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\phi : F^\times \to \mathbb{C}$ is a continuous homomorphism, which is necessarily trivial on the compact subgroup $\mathcal{O}_F^\times$, hence $\phi = cv$ for some $c \in \mathbb{C}$. Then for $t = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \in T$ we have

$$\lambda(t) = \lambda\left(t_2^{-1}\begin{pmatrix} t_1/t_2 & 0 \\ 0 & 1 \end{pmatrix}\right) = \lambda(t_2^{-1}I_2) + cv(t_1/t_2).$$

Clearly λ is trivial on the center, so $\lambda(t) = cv(t_1/t_2)$. If $c \neq 0$ we are done, the representation will be in the desired matrix from when replacing v_1 by $c^{-1}v_1$. We need to argue why $c = 0$ is not possible. Indeed, if $c = 0$, then $V_N \cong \xi \oplus \xi$, hence, by Schur's lemma and Frobenius duality,

$$\begin{aligned} \mathbb{C} = \operatorname{Hom}_G(V, V) &\cong \operatorname{Hom}_B(V|_B, \chi \otimes \delta_B^{1/2}) \\ &\cong \operatorname{Hom}_T(V_N, \xi) \cong \mathbb{C}^2, \end{aligned}$$

a contradiction. □

15.3. **Homomorphisms between Principal Series Representations**

Suppose $\mu_1, \mu_2 : F^\times \to \mathbb{C}^\times$ is another pair of quasi-characters. Let $\mu = \mu_1 \boxtimes \mu_2$.

Theorem 15.11. *We have*

$$\dim \operatorname{Hom}_G(\operatorname{Ind}_B^G(\chi), \operatorname{Ind}_B^G(\mu)) = \begin{cases} 1 & \text{if } \mu = \chi \text{ or } \mu^w = \chi \\ 0 & \text{otherwise} \end{cases}$$

Corollary 15.12. $\mathcal{B}(\chi_1, \chi_2) \cong \mathcal{B}(\chi_2, \chi_1)$ *whenever they are irreducible.*

See Theorem 16.1 for what happens when they are reducible.

15.3.1. *Proof of Theorem 15.11 in [BH06].* We make use of Proposition 15.3. By Frobenius reciprocity and the proposition, we have

$$\begin{aligned} \operatorname{Hom}_G(\operatorname{Ind}_B^G(\chi), \operatorname{Ind}_B^G(\mu)) &\cong \operatorname{Hom}_B(\operatorname{Ind}_B^G(\chi)|_B, \mu \otimes \delta_B^{1/2}) \\ &\cong \operatorname{Hom}_T(\operatorname{Ind}_B^G(\chi)_N, \mu \otimes \delta_B^{1/2}) \end{aligned}$$

Now we noted that if $\chi \neq \chi^w$, then $\operatorname{Ind}_B^G(\chi)_N \cong (\chi^w \otimes \delta_B|_T^{1/2}) \oplus (\chi \otimes \delta_B|_T^{1/2})$. The assertion follows immediately in this case. Suppose that $\chi = \chi^w$. If $\mu \neq \chi$, then clearly $\operatorname{Hom}_T(\operatorname{Ind}_B^G(\chi)_N, \mu \otimes \delta_B^{1/2}) = 0$, and if $\mu = \chi$, then both $\operatorname{Ind}_B^G(\chi)$ and $\operatorname{Ind}_B^G(\mu)$ are irreducible, so $\dim \operatorname{Hom}_G(\operatorname{Ind}_B^G(\chi), \operatorname{Ind}_B^G(\mu)) = 1$ by Schur's lemma.

15.3.2. *Proof in [Bum97].* Bump first proves $\operatorname{Hom}_G(\operatorname{Ind}_B^G(\chi), \operatorname{Ind}_B^G(\mu)) = 0$ if $\mu \neq \chi$ and $\mu^w \neq \chi$ using distributions. In the remaining cases a nonzero homomorphism is constructed explicitly as an integral and analytic continuation. In the case when $\operatorname{Ind}_B^G(\chi)$ and $\operatorname{Ind}_B^G(\mu)$ are both irreducible the result $\dim \operatorname{Hom}_G(\operatorname{Ind}_B^G(\chi), \operatorname{Ind}_B^G(\mu)) = 1$ can then be deduced from Schur's lemma, but he doesn't prove that the dimension is 1 in the remaining cases.

We define the intertwining integral. First we introduce some notation. Write $\chi_i = \xi_i \left|\cdot\right|^{s_i}$ with ξ_i unitary. We will eventually want to vary the s_i. We will then write V_{s_1,s_2} for $V = \mathcal{B}(\chi_1, \chi_2)$. Let V_0 be the space of functions $f_0 : K = \mathrm{GL}_2(\mathcal{O}_F) \to \mathbb{C}$ satisfying

$$f\left(\begin{pmatrix} y_1 & x \\ 0 & y_2 \end{pmatrix} k\right) = \xi_1(y_1)\xi_2(y_2)f(k) \tag{†}$$

for $y_1, y_2 \in \mathcal{O}_F^\times, x \in \mathcal{O}_F, k \in K$. Then it is not difficult to see that the restriction $V_{s_1,s_2} \to V_0, f \mapsto f|_K$ is an isomorphism. So for any s_1, s_2 there is a unique $f_{s_1,s_2} \in V_{s_1,s_2}$ such that $f_{s_1,s_2}|_K = f_0$. We call this a *flat section*.

We forget about this for the moment and come back to it later.

For $f \in V$ define $Mf : G \to \mathbb{C}$ by

$$(Mf)(g) = \int_F f\left(w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \mathrm{d}x,$$

where $w_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We have

$$f\left(w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = f\left(\begin{pmatrix} x^{-1} & -1 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix} g\right) = |x|^{-1} (\chi_1^{-1}\chi_2)(x) f\left(\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix} g\right).$$

Since f is locally constant, we have $f\left(\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix} g\right) = f(g)$ for all x with $|x| > c$ for some constant c (depending on g, but if g varies in a small neigbhorhood we can choose c uniformly). Hence we have

$$\int_{|x|>c} \left| f\left(w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)\right| \mathrm{d}x = |f(g)| \int_{|x|>c} |x|^{-1} \left|\chi_1^{-1}\chi_2\right|(x) \mathrm{d}x = |f(g)| \int_{|x|>c} |x|^{-1-s_1+s_2} \mathrm{d}x. \quad (*)$$

By writing it out as a geometric series, it is easy to see that this integral converges if and only if $\mathrm{Re}(s_1 - s_2) > 0$, which we will then assume for the moment. Hence the integral defining Mf converges absolutely in this region.

Clearly M commutes with right translation, so since f is invariant on the right by some open compact, so is Mf. It is not too difficult to see that $(Mf)(bg) = (\delta^{1/2}\chi^w)(b)(Mf)(g)$. Indeed, for $b \in N$ this is immediate, and for $b = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}$ it follows from writing

$$w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} b = \begin{pmatrix} y_2 & 0 \\ 0 & y_1 \end{pmatrix} w_0 \begin{pmatrix} 1 & y_2 y_1^{-1} x \\ 0 & 1 \end{pmatrix}.$$

Hence $Mf \in V' = \mathcal{B}(\chi_2, \chi_1)$, and M defines a G-equivariant homomorphism $V \to V'$. We have to show that this is non-zero. To that end we compute it for a concrete f. We define f on $Bw_0(K \cap N)$ by

$$f(bw_0 n) = (\delta^{1/2}\chi)(b)$$

and 0 everywhere else. One can check that $f \in V$. We have

$$(Mf)(1) = \int_{\mathcal{O}_F} \mathrm{d}x \neq 0.$$

Hence M is non-zero.

We have thus constructed a non-zero intertwining operator, hence isomorphism by irreducibility, $M : V \to V'$ if $\mathrm{Re}(s_1 - s_2) > 0$. We now indicate how to analytically continue this to a homomorphism $V \to V'$ for all s_1, s_2 (except to $\chi_1 = \chi_2$ where we get a pole). To show the analytic continuation we make a more precise computation in $(*)$. Fix $f_0 \in V_0$. For $\mathrm{Re}(s_1 - s_2) > 0$ we consider f_{s_1,s_2}. Fix $g \in G$. For N large enough we have

$$\int_{|x|>q^N} f_{s_1,s_2}\left(w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \mathrm{d}x = f_{s_1,s_2}(g) \int_{|x|>q^N} |x|^{-1} (\chi_1^{-1}\chi_2)(x) \mathrm{d}x$$

$$= f_{s_1,s_2}(g) \int_{|x|>q^N} |x|^{-1-s_1+s_2} (\xi_1^{-1}\xi_2)(x)\mathrm{d}x$$
$$= f_{s_1,s_2}(g) \sum_{k=N+1}^{\infty} q^{k(-s_1+s_2)} \int_{|x|=q^k} (\xi_1^{-1}\xi_2)(x)\mathrm{d}^\times x$$

Now we have

$$\int_{|x|=q^k} (\xi_1^{-1}\xi_2)(x)\mathrm{d}^\times x = \begin{cases} 0 & \xi_1^{-1}\xi_2 \text{ ramified,} \\ \mathrm{vol}_{\mathrm{d}^\times x}(\mathcal{O}_F^\times)\alpha^k & \xi_1^{-1}\xi_2 \text{ unramified.} \end{cases}$$

Here in the unramified case α is the complex number such that $(\xi_1\xi_2^{-1})(x) = \alpha^{\mathrm{ord}\, x}$ for all $x \in F^\times$. In this case we get

$$\int_{|x|>q^N} f_{s_1,s_2}\left(w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \mathrm{d}x = \mathrm{vol}_{\mathrm{d}^\times x}(\mathcal{O}_F^\times) f_{s_1,s_2}(g) \sum_{k=N+1}^{\infty} \alpha^k q^{k(-s_1+s_2)}$$
$$= \mathrm{vol}_{\mathrm{d}^\times x}(\mathcal{O}_F^\times) f_{s_1,s_2}(g)(\alpha q)^{-N-1}(1-\alpha q^{-s_1+s_2})^{-1}.$$

Note that $\alpha q^{-s_1+s_2} = 1$ precisely when $\chi_1 = \chi_2$. This shows that $(s_1,s_2) \mapsto Mf_{s_1,s_2}(g)$ can be analytically continued to the domain of (s_1,s_2) where $\chi_1 \neq \chi_2$. By the identity principle, Mf_{s_1,s_2} is still contained in V'_{s_2,s_1} and M defines an intertwining operator $V_{s_1,s_2} \to V'_{s_2,s_1}$. Let f_0 be the restriction of the f we used before to show that M is non-zero, and let f_{s_1,s_2} be its flat section. Then for $\mathrm{Re}(s_1 - s_2) > 0$ computed $(Mf_{s_1,s_2})(1) = \mathrm{vol}(\mathcal{O}_F)$. This is independent of s_1, s_2, hence by the identity principle, $Mf_{s_1,s_2}(1) = \mathrm{vol}(\mathcal{O}_F)$ for all s_1, s_2. This shows that M is non-zero for all s_1, s_2.

We note:

Theorem 15.13 ([Bum97, Proposition 4.5.10])**.** *Fix a non-trivial character $\psi : F \to \mathbb{C}^\times$. Let $M : \mathcal{B}(\chi_1,\chi_2) \to \mathcal{B}(\chi_2,\chi_1)$ and $M' : \mathcal{B}(\chi_2,\chi_1) \to \mathcal{B}(\chi_1,\chi_2)$ be the intertwining operators constructed above. Assume that the Haar measure on F used to construct these operators is normalized so that it is self-dual with respect to the character ψ. Them $M' \circ M : \mathcal{B}(\chi_1,\chi_2) \to \mathcal{B}(\chi_1,\chi_2)$ is the operator given by multiplication by*

$$\gamma(1-s_1+s_2, \xi_1^{-1}\xi_2, \psi)\gamma(1+s_1-s_2, \xi_1\xi_2^{-1}, \psi).$$

Here the γ terms are the γ-factors from Theorem 1.3.

15.4. **Unitarizable Principal Series Representations**

Theorem 15.14 ([Bum97, Theorem 4.6.7])**.** *If $V = \mathcal{B}(\chi_1,\chi_2)$ is an irreducible principal series representation, then V is unitarizable if and only if either χ_1, χ_2 are unitary, or $\chi_1 = \chi_0 |\cdot|^s, \chi_2 = \chi_0^{-1} |\cdot|^{-s}$ with χ_0 unitary and $-\frac{1}{2} < s < \frac{1}{2}$.*

Proof. If χ_1, χ_2 are both unitary, then V is unitary by Theorem 9.15. Suppose V is unitary. Then the inner product gives an isomorphism $\widehat{V} \cong \overline{V}$ where $\overline{V}$ is the complex conjugate of V which is easily seen to be identifiable with $\mathcal{B}(\overline{\chi_1}, \overline{\chi_2})$. Hence $\mathcal{B}(\overline{\chi_1}, \overline{\chi_2}) \cong \mathcal{B}(\chi_1^{-1}, \chi_2^{-1})$. Therefore either $\chi_1^{-1} = \overline{\chi_1}, \chi_2^{-1} = \overline{\chi_2}$, in which case χ_1, χ_2 are unitary, or $\chi_2^{-1} = \overline{\chi_1}, \chi_1^{-1} = \overline{\chi_2}$. Assume the latter. Then we can write $\chi_1 = \chi_0 |\cdot|^s, \chi_2 = \chi_0 |\cdot|^{-s}$ for some unitary character χ_0 and a real number s. So it remains to show that in this case V is unitarizable if and only if $-\frac{1}{2} < s < \frac{1}{2}$. Note that we are assuming $s \neq \pm\frac{1}{2}$, since V is irreducible. We may also assume $s \neq 0$.

Since $\mathcal{B}(\chi_1, \chi_2) = \chi_0 \otimes \mathcal{B}(|\cdot|^s, |\cdot|^{-s})$, we may assume $\chi_0 = 1$. Denote $\chi_s = |\cdot|^s$. Since $\mathcal{B}(\chi_s, \chi_s^{-1})$ is irreducible, Schur's lemma implies that an invariant non-degenerate sesquilinear form is unique up to scalar multiple, if it exists. The strategy is then to exhibit somewhat explicitly such a form and inspect when it is hermitian and positive definite (up to scalar multiple).

Let $M_s : \mathcal{B}(\chi_s, \chi_s^{-1}) \to \mathcal{B}(\chi_s^{-1}, \chi_s)$ be the isomorphism constructed in Section 15.3.2. The proof of Theorem 9.15 and Proposition A.8 shows that the duality pairing between $\mathcal{B}(\mu_1, \mu_2)$ and $\mathcal{B}(\mu_1^{-1}, \mu_2^{-1})$ is, up to scalar, given by

$$(f_1, f_2) = \int_K f_1(k) f_2(k) \mathrm{d}k.$$

By composing this with $\mathcal{B}(\chi_s, \chi_s^{-1}) \xrightarrow{M_s} \mathcal{B}(\chi_s^{-1}, \chi_s)$ and complex conjugation we get an invariant non-degenerate sesquilinear form on $\mathcal{B}(\chi_s, \chi_s^{-1})$ given by

$$\langle f_1, f_2 \rangle = \int_K (M_s f_1)(k) \overline{f_2(k)} \mathrm{d}k.$$

Note that the formula $(f_1, f_2) = \overline{\langle f_1, f_2 \rangle}$ also defines an invariant non-degenerate sesquilinear form, hence it differs from $\langle -, - \rangle$ by a scalar. Thus to show that it is Hermitian it suffices to show this scalar is one for a single pair of functions.

Let f_0 (my notation here differs slightly from Bump's) be the indicator function of $K_0(\mathfrak{p})$ on K. Then f_0 satisfies (†) and hence we get a flat section $f_s := f_{s,-s} \in \mathcal{B}(\chi_s, \chi_s^{-1})$ defined by $f_s(g) = \delta^{s+\frac{1}{2}}(b)$ if $g = bk$ with $b \in B, k \in K_0(\mathfrak{p})$ and $f_s(g) = 0$ otherwise. Assume first that $s > 0$ so that we can use the integral definition of M. The other case then follows from analytic continuation. We will compute $\langle f_s, f_s \rangle$. First note that

$$\langle f_s, f_s \rangle = \int_K (M_s f_s)(k) \overline{f_s(k)} \mathrm{d}k = \int_{K_0(\mathfrak{p})} (M_s f_s(k) \overline{f_s(k)} \mathrm{d}k = \mathrm{vol}(K_0(\mathfrak{p}))(M_s f_s)(1).$$

Next we have

$$(M_s f_1)(1) = \int_F f_s \left(w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \mathrm{d}x$$

From the disjoint Iwahori decomposition $G = BK_0(\mathfrak{p}) \sqcup Bw_0K_0(\mathfrak{p})$ we see that $w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in BK_0(\mathfrak{p})$ if and only if $x \notin \mathcal{O}_F$, in which case we have

$$w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x^{-1} & -1 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}.$$

Hence

$$\begin{aligned}
(M_s f_1)(1) &= \int_{|x|>1} f_s \left(\begin{pmatrix} x^{-1} & -1 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix} \right) \mathrm{d}x = \int_{|x|>1} |x|^{-2s-1} \mathrm{d}x \\
&= \sum_{n=1}^{\infty} q^{-2ns} \mathrm{vol}_{\mathrm{d}^\times x}(\mathcal{O}_F^\times) \\
&= \mathrm{vol}_{\frac{\mathrm{d}x}{x}}(\mathcal{O}_F^\times) q^{-2s}(1 - q^{-2s})^{-1}.
\end{aligned}$$

If we normalize the Haar measures so that $\mathcal{O}_F$ and K have volume 1, then $\mathcal{O}_F^\times$ and $K_0(\mathfrak{p})$ have volume $1-q^{-1}, (1+q)^{-1}$ respectively.[11] Though these normalizations aren't really relevant in determining whether the form is Hermitian or positive definite of course. Hence we have

$$\langle f_s, f_s \rangle = \frac{1-q^{-1}}{1+q} \frac{q^{-2s}}{1-q^{-2s}}.$$

By analytic continuation this expression is also valid for $s < 0$.[12] Note in particular that $\langle f_s, f_s\rangle$ is real, hence $\langle f_s, f_s\rangle = \overline{\langle f_s, f_s\rangle}$ and $\langle -,-\rangle$ is Hermitian for all s. To figure out if it can be made positive definite we will need to compute $\langle f, f\rangle$ for another function f. We will take the standard spherical function $\phi_{K,s}$ from Section 20. By Proposition 20.7 we have

$$M\phi_{K,s} = \frac{1-q^{-1}\alpha_1\alpha_2^{-1}}{1-\alpha_1\alpha_2^{-1}} \phi_{K,-s}.$$

Therefore

$$\langle \phi_{K,s}, \phi_{K,s}\rangle = \frac{1-q^{-1}\alpha_1\alpha_2^{-1}}{1-\alpha_1\alpha_2^{-1}} \int_K \phi_{K,-s}(k)\overline{\phi_{K,s}(k)}\mathrm{d}k = \frac{1-q^{-1}\alpha_1\alpha_2^{-1}}{1-\alpha_1\alpha_2^{-1}} = \frac{1-q^{-1-2s}}{1-q^{-2s}}.$$

where in the last equality we used that $\alpha_1 = \chi_s(\varpi) = q^{-s}$ and similarly $\alpha_2 = q^s$. Assume $s > 0$ (the other case can be handled similarly or deduced from this one through symmetry). Then $\langle f_s, f_s\rangle > 0$, but $\langle \phi_{K,s}, \phi_{K,s}\rangle > 0$ if and only if $s < \frac{1}{2}$. Hence a necessary condition for $\langle -,-\rangle$ to be positive definite is $s < \frac{1}{2}$. It remains to show that this is also sufficient.

The trick is to start at $s = 0$ where we already know that $\mathcal{B}(\chi_s, \chi_s^{-1})$ is unitarizable. Then we will deform this and show that it remains unitary up to $s < \frac{1}{2}$. Since M_s has a pole at $s = 0$, we modify it as follows:

$$M_s^* = (1-q^{-2s})M_s$$

The proof in Section 15.3.2 shows that M_s^* extends also to $s = 0$. Similarly we extend the inner product $\langle -,-\rangle$ to $s = 0$ by

$$\langle f_1, f_2\rangle^* = (1-q^{-2s})\langle f_1, f_2\rangle.$$

By Proposition 20.7 and Schur's lemma, M_0^* is the scalar $1-q^{-1}$. Hence $\langle f_1, f_2\rangle^*$ at $s=0$ is the usual inner product on $\mathcal{B}(1,1)$ from Theorem 9.15 and Proposition A.8, therefore positive definite. We have to show that $\langle -,-\rangle^*$ is positive definite for $s < \frac{1}{2}$.

The idea is the following: Restrict to the isotypic subspaces $V(\rho)$ for $\rho \in \widehat{K}$ which are finite-dimensional. If the restriction of $\langle -,-\rangle^*$ to $V(\rho)$ was not positive-definite for some $s \in (0, \frac{1}{2})$, by continuity it would be degenerate at some s contradicting that $\langle -,-\rangle^*$ is non-degenerate for all $s \in (0, \frac{1}{2})$ at $s = 0$. $\square$

16. Steinberg and Special Representations

We know from the preceeding section that $\mathcal{B}(|\cdot|^{\frac{1}{2}}, |\cdot|^{-\frac{1}{2}})$ has a unique irreducible subrepresentation:

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[11] Note for the latter it is the reciprocal of the index of $K_0(\mathfrak{p})$ in K which is the same as the index of the upper triangular matrices over $\mathrm{GL}_2(\mathbb{F}_q)$ in $\mathrm{GL}_2(\mathbb{F}_p)$, which is easily seen to be $1+q$.

[12] Note for this argument to work we would also need to allow complex s as there is a pole at $s = 0$.

Definition. *This is called the Steinberg representation, denoted* St.

It fits into the short exact sequence:

$$0 \to \mathrm{St} \to \mathcal{B}(|\cdot|^{\frac{1}{2}}, |\cdot|^{-\frac{1}{2}}) \to 1_G \to 0.$$

Here 1_G denotes the trivial representation. We have $\mathrm{St} = \sigma(|\cdot|^{\frac{1}{2}}, |\cdot|^{-\frac{1}{2}})$.

Theorem 16.1. $\sigma(\chi_1, \chi_2) \cong \sigma(\chi_2, \chi_1)$ *whenever* $\chi_1\chi_2^{-1} = |\cdot|^{\pm 1}$*. These are the only non-trivial isomorphisms among special representations.*

Proof. We may assume $\chi_1\chi_2^{-1} = |\cdot|^{-1}$. Consider the unique (up to scalar) non-zero intertwining map $M : \mathcal{B}(\chi_1, \chi_2) \to \mathcal{B}(\chi_2, \chi_1)$ from Theorem 15.11. $\mathcal{B}(\chi_1, \chi_2)$ has a one-dimensional invariant subspace V_0. Its image under M is again an invariant subspace, of dimension at most 1. But $\mathcal{B}(\chi_2, \chi_1)$ does not have a one-dimensional invariant subspace, hence M maps V_0 to 0. We know that $\mathcal{B}(\chi_1, \chi_2)/V_0 \cong \sigma(\chi_1, \chi_2)$ is irreducible, hence its image in $\mathcal{B}(\chi_1, \chi_2)$ must be the unique irreducible subrepresentation $\sigma(\chi_2, \chi_1)$ and the first part follows.

For the second part assume that $\sigma(\chi_1, \chi_2) \cong \sigma(\mu_1, \mu_2)$. By the first part we may assume that $\chi_1\chi_2^{-1} = |\cdot|^{-1}$ and $\mu_1\mu_2^{-1} = |\cdot|$. Then consider the composition

$$\mathcal{B}(\chi_1, \chi_2) \twoheadrightarrow \sigma(\chi_1, \chi_2) \cong \sigma(\mu_1, \mu_2) \hookrightarrow \mathcal{B}(\mu_1, \mu_2).$$

We get a non-zero intertwining map $\mathcal{B}(\chi_1, \chi_2) \to \mathcal{B}(\mu_1, \mu_2)$, hence $\chi_1 = \mu_2, \chi_2 = \mu_1$ by Theorem 15.11. $\square$

Therefore

$$\widehat{\mathrm{St}} = \sigma(|\cdot|^{-1/2}, |\cdot|^{1/2}) \cong \sigma(|\cdot|^{1/2}, |\cdot|^{-1/2}) = \mathrm{St}.$$

If χ is an arbitrary quasi-character of $F^\times$, then

$$\chi \otimes \mathrm{St} = \sigma(\chi\,|\cdot|^{1/2}, \chi\,|\cdot|^{-1/2}),$$

and every special representation is of this form for some χ.

Proposition 16.2. *Assume* $\chi_1\chi_2 = |\cdot|^{-1}$*. Then the Jacquet module of* $\sigma(\chi_1, \chi_2)$ *is* $\delta^{1/2} \otimes \chi^w$ *where* $\chi = \chi_1 \boxtimes \chi_2$*.*

Need to fix the notation, sometimes χ is quasi-character of $F^\times$, sometimes of T,...

Proof. $\mathcal{B}(\chi_1, \chi_2)$ has a one-dimensional invariant subspace V_0 with quasi-character $\chi_1\,|\cdot|^{1/2} = \chi_2\,|\cdot|^{-1/2}$. Note that $(V_0)_N \cong \chi_1\,|\cdot|^{1/2} \boxtimes \chi_2\,|\cdot|^{-1/2} = \delta^{1/2} \otimes \chi$ by Proposition 11.6. Then taking Jacquet modules in the exact sequence $0 \to V_0 \to \mathcal{B}(\chi_1, \chi_2) \to \sigma(\chi_1, \chi_2) \to 0$ gives (using Theorem 15.10)

$$0 \to \delta^{1/2} \otimes \chi \to \delta^{1/2} \otimes (\chi \oplus \chi^w) \to \sigma(\chi_1, \chi_2)_N \to 0.$$

The result follows. $\square$

Hence if η is a quasi-character of $F^\times$, the Jacquet module of $\eta \otimes \mathrm{St}$ is $\delta \otimes (\eta \boxtimes \eta) = \delta \otimes (\eta \circ \det)$.

17. Matrix Coefficients

Let (V, π) be a smooth representation of G. Recall that a *matrix coefficient* of π is a function $G \to \mathbb{C}$ of the form

$$\phi_{v,\widetilde{v}}(g) = \langle \pi(g)v, \widetilde{v} \rangle$$

where $v \in V, \widetilde{v} \in \widetilde{V}$. Clearly $\phi_{\widetilde{v},v} \in C^\infty(G)$. We denote the vector space spanned by these functions by $\mathcal{C}(\pi)$.

Definition. (V, π) *is* quasi-cuspidal *if its matrix coefficients are compactly supported modulo Z. If its central quasi-character is unitary*[a]*, then V is called* square integrable *(resp.* tempered*) if its matrix coefficients are in $L^2(G/Z)$ (resp. $L^{2+\varepsilon}(G/Z)$ for all $\varepsilon > 0$).*

[a] This guaranties that $|\phi_{v,\widetilde{v}}|^2$ is well-defined modulo Z, so that $\int_{G/Z} |\phi_{v,\widetilde{v}}|^2 \, dg$ makes sense.

A quasi-cuspidal representation that is admissible is called *supercuspidal*. Note that what we call square integrable is sometimes called square integrable modulo the center in the literature.

Square integrable representations are also called *discrete series*. This terminology is justified since in the unitary dual, square integrable representations have positive Plancherel measure (TODO: reference?).

Definition. *Let (V, π) be a smooth representation. V is* essentially square integrable *(resp.* essentially tempered*) if there is a quasi-character χ of $F^\times$ such that $\chi \otimes \pi$ is square integrable (resp. tempered).*

In [GH24] a slightly different definition is given, they are equivalent by the following:

Proposition 17.1. *Let (V, π) be a smooth representation. V is essentially square integrable (resp. tempered) if and only if the restriction of any matrix coefficient to G^1 lies in $L^2(G^1)$ (resp. $L^{2+\varepsilon}(G)$ for all $\varepsilon > 0$).*

Here $G^1 = \{g \in G \mid \det g \in \mathcal{O}_F^\times\}$.

Proof. "Only if" is immediate from the fact that the restriction of a quasi-character to G^1 is unitary. "If" follows since $G^1 \cap Z$ is compact and $G^1/(G^1 \cap Z)$ is of index 2 in G/Z. $\square$

Proposition 17.2. *Suppose (V, π) is an irreducible admissible representation of G. If one non-zero matrix coefficient of G is compactly supported (resp. square integrable, $L^{2+\varepsilon}$ for all ε) modulo Z, then V is quasi-cuspidal (resp. square integrable, tempered).*

In the case of square integrable, tempered, we assume that the central quasi-character is unitary.

Proof. One can show, see [BH06, Proposition 10.1] that $V \boxtimes \widehat{V}$ is an irreducible representation for $G \times G$, hence the map $V \boxtimes \widehat{V} \to \mathcal{C}(\pi), v \otimes \widetilde{v} \mapsto \phi_{v,\widetilde{v}}$ is an isomorphism. Therefore if $\phi \in \mathcal{C}(\pi)$ is any fixed non-zero matrix coefficient, then any other $\phi' \in \mathcal{C}(\pi)$ is a finite sum of terms of the form $(\pi \boxtimes \widehat{\pi})(g, h)\phi$ with $(g, h) \in G \times G$, and the result is immediate. $\square$

18. Square Integrable Representations

Proposition 18.1. *Let (V,π) be a square integrable irreducible smooth representation of G. Then V is unitarizable.*

Proof. For fixed $0 \neq \widetilde{v} \in \widehat{V}$ define an inner product by

$$\langle v, w\rangle = \int_{G/Z} \phi_{v,\widetilde{v}}(g)\overline{\phi_{w,\widetilde{v}}(g)}\mathrm{d}g.$$

Since π is square integrable, this is defined. It is straightforward to check that this gives an invariant inner product on V. For positive definite if $\langle v, v\rangle = 0$, then $\phi_{v,\widetilde{v}} = 0$ and since V is irreducible, if v was non-zero, then its orbit under G would span V, hence $\phi_{v,\widetilde{v}} = 0$ would imply $\widetilde{v} = 0$, contradiction. □

Corollary 18.2. *If (V,π) is square integrable irreducible admissible representation, then it is tempered.*

Proof. By the previous result, V is unitarizable. By Proposition 9.14 the matrix coefficients are bounded. Bounded L^2 functions are in $L^{2+\varepsilon}$ for all $\varepsilon > 0$. □

Theorem 18.3 (Schur Orthogonality Relations)**.** *Let $(V_1,\pi_1),(V_2,\pi_2)$ be admissible irreducible square integrable representations of G with the same central character. Then for any $v_i \in V_i, \widetilde{v}_i \in \widetilde{V}_i$ we have*

$$\int_{G/Z} \phi_{v_1,\widetilde{v}_1}(g)\phi_{v_2,\widetilde{v}_2}(g^{-1})\mathrm{d}g = \begin{cases} d(\pi)^{-1}\langle v_2, \widetilde{v}_1\rangle\langle v_1, \widetilde{v}_2\rangle & \text{if } \pi := \pi_1 = \pi_2, \\ 0 & \text{if } \pi_1 \ncong \pi_2. \end{cases}$$

Here $d(\pi)$ is a positive constant only depending on π and the choice of Haar measure on G/Z.

Proof. Fix $\widetilde{v}_1 \in \widetilde{V}_1$ and $v_2 \in V_2$. Consider the operator $T = T_{\widetilde{v}_1,v_2} : V_1 \to V_2$, given by $v \mapsto \langle v, \widetilde{v}_1\rangle v_2$. We can symmetrize this to get an intertwining operator $S = S_{\widetilde{v}_1,v_2} : V_1 \to V_2$, defined by

$$Sv = \int_{G/Z} \pi_2(g^{-1})T\pi_1(g)v\mathrm{d}g = \int_{G/Z} \langle \pi_1(g)v, \widetilde{v}_1\rangle \pi_2(g^{-1})v_2\mathrm{d}g.$$

This integral is supposed to be understood in a weak sense: For $\widetilde{v}_2 \in \widetilde{V}_2$, the integral

$$\int_{G/Z} \langle \pi_1(g)v, \widetilde{v}_1\rangle\langle \pi_2(g^{-1})v_2, \widetilde{v}_2\rangle\mathrm{d}g$$

converges since the matrix coefficients are L^2, thus we may view Sv as a functional on $\widetilde{V}_2$ which is smooth, hence defines an element in $\widetilde{\widetilde{V}}_2$ which is canonically identified with V_2. Now if $\pi_1 \ncong \pi_2$, we must have $S = 0$, hence the claim follows in this case. Thus, assume for the remainder that $\pi = \pi_1 = \pi_2$. Then by Schur's lemma, S is a multiple of the identity, say $S = c_{\widetilde{v}_1,v_2}\,\mathrm{id}_V$. Next varying $\widetilde{v}_1, v_2$ gives a smooth bilinear pairing $\widetilde{V} \times V \to \mathbb{C}, (\widetilde{v}_1, v_2) \mapsto c_{\widetilde{v}_1,v_2}$, hence $c_{\widetilde{v}_1,v_2} = c_\pi\langle \widetilde{v}_1, v_2\rangle$ for some constant c_π. Then

$$\int_{G/Z} \phi_{v_1,\widetilde{v}_1}(g)\phi_{v_2,\widetilde{v}_2}(g^{-1})\mathrm{d}g = \langle Sv_1, \widetilde{v}_2\rangle = c_{\widetilde{v}_1,v_2}\langle v_1, \widetilde{v}_2\rangle = c_\pi\langle \widetilde{v}_1, v_2\rangle\langle v_1, \widetilde{v}_2\rangle.$$

So if we can show $c_\pi > 0$, the theorem will follow with $d(\pi) = c_\pi^{-1}$. By Proposition 18.1 there is an invariant inner prudct $(-,-)$ on V. Let $0 \neq \widetilde{v} \in \widetilde{V}$. Then by Proposition 9.13, there is $v \in V$ such that $\langle u, \widetilde{v}\rangle = (u, v)$ for all $u \in V$. Then

$$c_\pi \langle v, \widetilde{v}\rangle^2 = \int_{G/Z} \phi_{v,\widetilde{v}}(g)\phi_{v,\widetilde{v}}(g^{-1})\mathrm{d}g = \int_{G/Z} (\pi(g)v, v)(\pi(g^{-1})v, v)\mathrm{d}g = \int_{G/Z} |(\pi(g)v, v)|^2 \,\mathrm{d}g > 0.$$

Since $\langle v, \widetilde{v}\rangle = (v, v) > 0$, the result follows. □

The theorem also holds for essentially square integrable representations. Indeed, we can reduce to the case of unitary central quasi-character by twisting by $|\cdot|^s$ for an appropriate s.

Theorem 18.4. *[BH06, Theorem 17.5] Let (V, π) be an irreducible admissible representation of G. V is square integrable if and only if either V is supercuspidal with unitary central character, or if $V \cong \chi\mathrm{St}$ with χ a unitary character of $F^\times$, and* St *the Steinberg representation.*

Proof. We have to show that the representations in question are square integrable and that all others are not. By Theorem 23.1 the other admissible irreducible representations with unitary central character are either one dimensional or principal series representations. One dimensional representations are obviously not square integrable. So let χ_1, χ_2 be quasi-characters of $F^\times$ such that $\chi_1\chi_2$ is unitary. We have to show that $(V, \pi) = \mathcal{B}(\chi_1, \chi_2)$ is not square integrable. Let $\chi = \chi_1 \boxtimes \chi_2$ be the quasi-character on T. Let r be large enought so that χ is trivial on $T \cap K_r$ where $K_r = K(\mathfrak{p}^r) = I_2 + M_{2\times 2}(\mathfrak{p}^r)$. Define $f \in V$ by $f(bk) = (\delta^{1/2}\chi)(b)$ for $b \in B, k \in K_r$ and $f(x) = 0$ for $x \notin BK_r$. Define $\widetilde{f} \in \mathcal{B}(\chi_1^{-1}, \chi_2^{-1})$ in the same way (with χ^{-1} in place of χ). Consider the matrix coefficient

$$\phi = \phi_{f,\widetilde{f}}(g) = \langle \pi(g)f, \widetilde{f}\rangle = \int_K f(kg)\widetilde{f}(k)\mathrm{d}k.$$

Then follow [BH06] to show this is not square integrable mod Z.

Then one has to show that the representations in the statement are indeed square integrable. For the supercuspidals that is clear. For the Steinberg twists this requires some work. □

Theorem 18.5 ([GH11, Proposition 9.2.8])**.** *Let (V, π) be an irreducible admissible representation of G. V is tempered if and only if V is one if the representations in the previous theorem (i.e. if V is square integrable), or if V is a principal series representation $\mathcal{B}(\chi_1, \chi_2)$ with χ_1, χ_2 unitary.*

19. Unitary Representations

Let (V, π) be an irreducible admissible representation of G, ω its central character. We will determine when V is unitarizable.

Proposition 19.1. *If V is one-dimensional, of the form $\chi \circ \det$, then V is unitarizable if and only if χ is unitary*

Proof. Obvious. □

Theorem 19.2. *If $V = \mathcal{B}(\chi_1, \chi_2)$ is an irreducible principal series representation, then V is unitarizable if and only if either χ_1, χ_2 are unitary, or $\chi_1 = \chi_0 |\cdot|^s, \chi_2 = \chi_0^{-1} |\cdot|^{-s}$ with χ_0 unitary and $-\frac{1}{2} < s < \frac{1}{2}$.*

Proof. See Section 15.4. □

Theorem 19.3 ([GH11, Proposition 9.3.1])**.** *If V is either supercuspidal or special, then V is unitarizable if and only if ω is unitary.*

Proof. Necessity of the condition is clear. For sufficiency, note that if ω is unitary, then by Theorem 18.4, (V, π) is square integrable, hence unitarizable by Proposition 18.1. □

20. Spherical Representations

Let (V, π) be an irreducible admissible representation of G. Recall that $K = \mathrm{GL}_2(\mathcal{O}_F)$ is the standard maximal compact subgroup of G.

Definition. *π is called* spherical *if there is a non-zero K-fixed vector, i.e. if $V^K \neq 0$. Such a vector is called* spherical.

Proposition 20.1. *If π is spherical, so is $\widehat{\pi}$.*

Proof. This follows from Theorem 10.2. □

Recall that $\mathcal{H}_K$ is the space of locally constant compactly supported K-biinvariant functions $G \to \mathbb{C}$. It is an algebra under convolution. Matrix involution induces an involution on $\mathcal{H}_K$ and the Cartan decomposition $G = BK$ implies that it must be the identity, so $\mathcal{H}_K$ is commutative. As a $\mathbb{C}$-algebra, $\mathcal{H}_K$ is generated by T, R, R^{-1} where T, R are the characteristic functions of $K \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix} K$ and $K \begin{pmatrix} \varpi & 0 \\ 0 & \varpi \end{pmatrix} K$ respectively. More generally let $T(\mathfrak{p}^k)$ be the characteristic function of the set of matrices A in $M_{2\times 2}(\mathcal{O}_F)$ such that $\mathrm{ord}(\det A) = k$. Then $T = T(\mathfrak{p})$.

Theorem 20.2 (Hecke relations, [Bum97, Proposition 4.6.4])**.** *For $k \geq 1$ we have*
$$T(\mathfrak{p})T(\mathfrak{p}^k) = T(\mathfrak{p}^{k+1}) + qRT(\mathfrak{p}^{k-1}).$$

Theorem 20.3. *If π is spherical,* $\dim V^K = 1$*, so a spherical vector is unique up to scalars.*

Proof. V^K is a finite dimensional simple module for the commutative ring $\mathcal{H}_K$. □

Denote by v_K any spherical vector in V (unique up to scalars by the theorem). Then there is a homomorphism $\xi : \mathcal{H}_K \to \mathbb{C}$ such that $\pi(\phi)v_K = \xi(\phi)v_K$ for $\phi \in \mathcal{H}_K$. This is the *character* of $\mathcal{H}_K$ associated to π.

Theorem 20.4. *Two irreducible admissible spherical representations are isomorphic if and only if the corresponding characters of $\mathcal{H}_K$ coincide.*

Example. A finite-dimensional irreducible admissible representation of G is one-dimensional and of the form $\chi \circ \det$ for a quasi-character χ of $F^\times$. It is spherical if and only if χ is unramified, i.e. trivial on $\mathcal{O}_F^\times$.

Example. Let χ_1, χ_2 be unitary unramified quasi-characters of $F^\times$ (hence of the form $|\cdot|^s$). Assume that $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}$, so that $(V, \pi) = \mathcal{B}(\chi_1, \chi_2)$ is irreducible. Write χ for the character of B. Consider the function $\phi_K : G \to \mathbb{C}$ defined by

$$\phi_K(g) = (\delta^{1/2}\chi)(b)$$

where we write $g = bk$ with $b \in B, k \in K$. This is independent of the choice of b, k. Then ϕ_K is a spherical vector in V.

Let $\alpha_1 = \chi_1(\varpi), \alpha_2 = \chi_2(\varpi)$. Since χ_1, χ_2 are unramified, these numbers determine the quasi-characters uniquely. To find the character ξ of $\mathcal{H}_K$ for this spherical representation, it suffices to know $\xi(T), \xi(R)$ which are given by:

Proposition 20.5. *Notation as above, we have $\pi(T)\phi_K = \lambda\phi_K$ and $\pi(R) = \mu\phi_K$ where*

$$\lambda = q^{1/2}(\alpha_1 + \alpha_2),$$
$$\mu = \alpha_1\alpha_2.$$

Proof. Evaluate both sides of $\pi(T)\phi_K = \lambda\phi_K$ at $I_2 \in G$ to get

$$\lambda = (\pi(T)\phi_K)(I_2) = \int_{K\begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}K} \phi_K(g)\,\mathrm{d}g.$$

Then split it up into left cosets mod K and use explicit Hermite normal form coset representatives to compute this. Same for $\pi(R)\phi_K$. $\square$

In fact the above two examples are exhaustive:

Theorem 20.6. *Let (V, π) be an irreducible admissible spherical representation of G. Then π is isomorphic to one of the two examples above.*

Proof. We make use of Theorem 20.4. Let ξ be the character of $\mathcal{H}_K$. Let $\lambda = \xi(T), \mu = \xi(R)$. Let α_1, α_2 be the roots of $X^2 - q^{1/2}\lambda X + \mu = 0$ and χ_1, χ_2 the unramified quasi-characters of $F^\times$ with $\chi_j(\varpi) = \alpha_j$. If $\mathcal{B}(\chi_1, \chi_2)$ is irreducible, it is spherical and the corresponding character of $\mathcal{H}_K$ is ξ by construction of α_1, α_2, the proposition and since $\mathcal{H}_K$ is generated by T, R, R^{-1}. Hence $(V, \pi) \cong \mathcal{B}(\chi_1, \chi_2)$. If $\mathcal{B}(\chi_1, \chi_2)$

is not irreducible, one argues similarly that π is isomorphic to the one-dimensional subrepresentation or quotient of $\mathcal{B}(\chi_1, \chi_2)$. $\square$

The following calculation is needed in the proof of Theorem 15.14.

Proposition 20.7 ([Bum97, Proposition 4.6.7])**.** *Let χ_1, χ_2 be unramified quasi-characters of $F^\times$ and $\alpha_i = \chi_i(\varpi)$. Let $\phi_{K,\chi}, \phi_{K,\phi'}$ denote the spherical vectors as defined above in $\mathcal{B}(\chi_1, \chi_2)$ and $\mathcal{B}(\chi_2, \chi_1)$. Let $M : \mathcal{B}(\chi_1, \chi_2) \to \mathcal{B}(\chi_2, \chi_1)$ be the intertwining operator defined in Section 15.3.2. Then*

$$M\phi_{K,\chi} = \frac{1 - q^{-1}\alpha_1\alpha_2^{-1}}{1 - \alpha_1\alpha_2^{-1}}\phi_{K,\chi'}.$$

Proof. $M\phi_{K,\chi}$ is a spherical vector in $\mathcal{B}(\chi_2, \chi_1)$, hence it is a multiple of $\phi_{K,\chi'}$. To compute the constant, we evaluate both sides at 1. For that we may assume $|\alpha_1| < |\alpha_2|$, so that the integral defining M converges. It is then a simple calculation using the geometric series to compute $(M\phi_{K,\chi})(1)$. $\square$

20.1. **Spherical Whittaker Function**

Let χ_1, χ_2 be unramified quasi-characters of $F^\times$ and $\alpha_i = \chi_i(\varpi)$. We consider the principal series representation $(V, \pi) = \mathcal{B}(\chi_1, \chi_2)$. It is spherical with spherical vector ϕ_K defined by $\phi_K(kb) = (\delta^{1/2}\chi)(k)$. We want to compute the spherical Whittaker function. First we define the Whittaker functional. For $f \in V$, let

$$\Lambda(f) = \int_F f\left(w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\psi(-x)\mathrm{d}x.$$

Here $w_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This converges absolutely if $|\alpha_1/\alpha_2| < 1$ by the same computation as in $(*)$. For general χ this can by analytically continued by

$$\Lambda(f) = \lim_{k\to\infty} \int_{\mathfrak{p}^{-k}} f\left(w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\psi(-x)\mathrm{d}x,$$

again using the same kind of argument as in $(*)$.

Let $W_0(g) = \Lambda(\pi(g)\phi_K)$. Then W_0 is the unique up to scaling spherical function in the Whittaker model of V.

We wish to compute $W_0(a_m)$ for $a_m = \begin{pmatrix} \varpi^m & 0 \\ 0 & 1 \end{pmatrix}$. We will assume that ψ has conductor $\mathfrak{p}^0$. Let $W = W_0$.

Note that knowing $W_0(a_m)$ is essentially the same as knowing W_0 because the matrices a_m form a set of representatives for $ZN\backslash G/K$.

Proposition 20.8. *$W(a_m) = 0$ for $m < 0$.*

Proof. For $x \in \mathcal{O}_F$ we have

$$W(a_m) = W\left(a_m \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = W\left(\begin{pmatrix} 1 & \varpi^m x \\ 0 & 1 \end{pmatrix} a_m\right) = \psi(\varpi^m x)W(a_m).$$

The first equality is because W is spherical. Now if $m < 0$ choose $x \in \mathcal{O}_F$ such that $\psi(\varpi^m x) \neq 1$ and we are done. $\square$

Proposition 20.9. $W(a_0) = 1 - q^{-1}\alpha_1\alpha_2^{-1}$.

Proof.
$$W(a_0) = \Lambda(\phi_K) = \lim_{k\to\infty}\int_{\mathfrak{p}^{-k}} \phi_K\left(w_0\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\psi(-x)\mathrm{d}x.$$
We have
$$\int_{\mathcal{O}_F} \phi_K\left(w_0\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\psi(-x)\mathrm{d}x = \int_{\mathcal{O}_F}\mathrm{d}x = 1.$$
Next:
$$\phi_K\left(w_0\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) = \phi_K\left(\begin{pmatrix}x^{-1} & -1\\ 0 & x\end{pmatrix}\begin{pmatrix}1 & 0\\ x^{-1} & 1\end{pmatrix}\right) = |x|^{-1}(\chi_1^{-1}\chi_2)(x)\phi_K\begin{pmatrix}1 & 0\\ x^{-1} & 1\end{pmatrix}$$
Therefore for $k \geq 1$:
$$\int_{\mathfrak{p}^{-k}-\mathfrak{p}^{-k+1}} \phi_K\left(w_0\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\psi(-x)\mathrm{d}x = q^{-k}\alpha_1^k\alpha_2^{-k}\int_{\mathfrak{p}^{-k}-\mathfrak{p}^{-k+1}}\psi(-x)\mathrm{d}x.$$
For $k = 1$, this last integral is -1 and for $k > 1$ it is 0, hence the claim. $\square$

Let $w_m = W(a_m)$. We will relate this to $(\pi(T)W)(a_m)$ where $T = T(\mathfrak{p})$. First recall that
$$K\begin{pmatrix}\varpi & 0\\ 0 & 1\end{pmatrix}K = \begin{pmatrix}1 & 0\\ 0 & \varpi\end{pmatrix}K \sqcup \coprod_{b \bmod \mathfrak{p}}\begin{pmatrix}\varpi & b\\ 0 & 1\end{pmatrix}K.$$
So for $m \geq 0$:
$$\begin{aligned}(\pi(T)W)(a_m) &= W\left(a_m\begin{pmatrix}1 & 0\\ 0 & \varpi\end{pmatrix}\right) + \sum_{b \bmod \mathfrak{p}} W\left(a_m\begin{pmatrix}\varpi & b\\ 0 & 1\end{pmatrix}\right)\\ &= W\left(a_{m-1}\begin{pmatrix}\varpi & 0\\ 0 & \varpi\end{pmatrix}\right) + \sum_{b \bmod \mathfrak{p}} W\left(\begin{pmatrix}1 & b\varpi^m\\ 0 & 1\end{pmatrix}a_{m+1}\right)\\ &= \chi_1(\varpi)\chi_2(\varpi)W(a_{m-1}) + \sum_{b \bmod \mathfrak{p}}\psi(b\varpi^m)W(a_{m+1})\\ &= \alpha_1\alpha_1 w_{m-1} + qw_{m+1}\end{aligned}$$
On the other hand from Proposition 20.5 we know that $(\pi(T)W)(a_m) = q^{1/2}(\alpha_1+\alpha_2)w_m$. Hence for $m \geq 0$ we get the recurrence relation
$$q^{1/2}(\alpha_1+\alpha_2)w_m = \alpha_1\alpha_2 w_{m-1} + qw_{m+1}$$
and $w_{-1} = 0, w_0 = 1 - q^{-1}\alpha_1\alpha_2^{-1}$. This is easily seen to have solution
$$w_m = q^{-m/2}\frac{\alpha_1^{m+1}-\alpha_2^{m+1}}{\alpha_1-\alpha_2}w_0.$$
We obtain:

Theorem 20.10. *The values of the spherical Whittaker function are given by*

$$W_0(a_m) = \begin{cases} 0 & \text{if } m < 0, \\ q^{-m/2} \frac{\alpha_1^{m+1} - \alpha_2^{m+1}}{\alpha_1 - \alpha_2} w_0 & \text{if } m \geq 0 \end{cases}$$

where $w_0 = W(I) = 1 - q^{-1}\alpha_1\alpha_2^{-1}$.

20.2. **Satake Isomorphism**

Define the *Satake map* (or *constant term map*)

$$\begin{aligned} S : \mathcal{H}_K &\longrightarrow C^\infty(T), \\ f &\longmapsto \left(a \mapsto \delta(a)^{1/2} \int_N f(an) \mathrm{d}n \right) \end{aligned}$$

Let $K_T = K \cap T$. Given an unramified quasi-character χ of T, obtain a character φ_χ of $\mathcal{H}(T, K_T)$ by integration against it. Then by pulling back via S, we get a character ξ_χ for the spherical Hecke algebra, defined by

$$\xi_\chi(f) = \varphi_\chi(Sf) = \int_T Sf(t)\chi(t)\mathrm{d}t.$$

Proposition 20.11. *Assume χ is such that the principal series representation $\mathcal{B}(\chi_1, \chi_2)$ is irreducible. Then $\xi_\chi(f)$ is exactly the character of $\mathcal{H}_K$ corresponding to the spherical representation $\mathcal{B}(\chi_1, \chi_2)$.*

Proof. Let φ_K the standard spherical vector in $\mathcal{B}(\chi_1, \chi_2)$ defined as before. We have to show that $\pi(f)\varphi_K = \xi_\chi(f)\varphi_K$ for $f \in \mathcal{H}_K$. It suffices to evaluate both sides at one, and show the results are equal. We compute

$$\begin{aligned} (\pi(f)\varphi_K)(1) &= \int_G \varphi(g) f(g) \mathrm{d}g \\ &= \int_B \varphi(g) f(g) \mathrm{d}g \\ &= \int_T \int_N \varphi(tn) f(tn) \mathrm{d}t \, \mathrm{d}n \\ &= \int_T \delta(t)^{1/2} \chi(n) \int_N f(tn) \mathrm{d}t \, \mathrm{d}n \\ &= \int_T S(f)(t)\chi(t)\mathrm{d}t \\ &= \xi_\chi(f). \end{aligned}$$

□

Theorem 20.12 ([Dei12, Theorem 8.2.3])**.** *The map S induces an isomorphism*

$$\mathcal{H}_K \xrightarrow{\simeq} \mathbb{C}[T/K_T]^W$$

Here W is the Weyl group. Note that $\mathbb{C}[T/K_T]^W = \mathbb{C}[X^{\pm 1}, Y^{\pm 1}]^{S_2}$, the ring of symmetric Laurent polynomials in two variables.

21. The Conductor of a Representation

For $\mathfrak{a}$ an ideal of $\mathcal{O}_F$ let $K_0(\mathfrak{a})$ be the group of matrices in K that are upper triangular modulo $\mathfrak{a}$. If g is a 2×2-matrix, denote by $d(g)$ its bottom right entry.

Theorem 21.1 ([Cas73, Theorem 1])**.** *Let (V,π) be an irreducible admissible representation of G with central quasi-character ω. There exists a largest ideal $\mathfrak{f} = \mathfrak{f}(\pi)$, called the* conductor *of π, such that*

$$W = \{v \in V \mid \pi(g)v = \omega(d(g))v \quad \forall g \in K_0(\mathfrak{f})\}$$

is non-zero. Moreover, $\dim W = 1$.

Note that if ω is trivial, then $\mathfrak{f} = \mathcal{O}_F$ if and only if π is unramified.

Proof sketch. □

TODO: What is difference with fixed vector for $K_1(\mathfrak{a})$?

22. Supercuspidal Representations

Let (V,π) be a smooth representation. Recall that V is quasi-cuspidal if the matrix coefficients are compactly supported modulo Z, and supercuspidal if it is quasi-cuspidal and admissible.

Proposition 22.1. *If π is quasi-cuspidal and irreducible, then π is admissible, i.e. supercuspidal.*

Proof. (Proof in [GH24, Proposition 8.3.4] and [BZ76]) Let K be an open compact subgroup of G. Fix a non-zero $v \in V^K$. Since V is irreducible, V^K is spanned by the elements $\pi(e_K)\pi(g)v$, $g \in G$. Let $(g_i)_{i\in I}$ be a collection in G such that the $v_i := \pi(e_K g_i)v$ form a basis for V^K. Let $f : V \to \mathbb{C}$ be the composition $V \to V^K \to \mathbb{C}$ where the last map sends every v_i to 1. Consider the matrix coefficient $\phi = \phi_{v,f}$. Then $\phi(g_i) = 1$. Since ϕ is compactly supported modulo Z there are finitely many $h_1, \dots, h_n \in G$ such that $\operatorname{supp}\phi = \bigcup_{i=1}^n h_i Z K$. Since $\phi(g_i) \neq 0$, we must have $g_i = h_j k z$ for some $j = 1, \dots, n$ and $k \in K, z \in Z$. Then $v_i = \pi(e_k g_i)v = \omega(z)\pi(e_k h_j)v$. So $\operatorname{span}\{v_i\} = \operatorname{span}\{\pi(e_K h_j)v\}_{j=1,\dots,n}$, hence V^K is finite-dimensional. □

Theorem 22.2 ([Cas+08, Proposition 5.4.2])**.** *Any supercuspidal representation of G is a countable direct sum of irreducible supercuspidal representations.*

Theorem 22.3. *Let (V,π) be an irreducible smooth representation. The following are equivalent:*

(1) There is a nonzero matrix coefficient in π that is compactly supported modulo the center.

(2) π is quasicuspidal.

(3) The space of the Kirillov model of π is $C_c^\infty(F^\times)$.

(4) $V_N = 0$.

(5) *V is not isomorphic to a subrepresentation of a principal series representation.*

(6) *The restriction of V to G^1 has compactly supported matrix coefficients.*

In this case π is admissible, hence supercuspidal.

Proof.

"(2) $\Rightarrow$ (1)" is trivial (recall we assume that irreducible representations are non-zero).

"(1) $\Rightarrow$ (2)" follows from Proposition 17.2 once we have shown that V is admissible. Indeed, this will be proven, and its proof only involves showing that (4) implies (2), and Proposition 22.1.

For "(4) $\Leftrightarrow$ (5)" use Frobenius reciprocity: We have $\mathrm{Hom}_G(\pi, \mathcal{B}(\chi_1, \chi_2)) = \mathrm{Hom}_B(\pi|_B, \chi \otimes \delta^{1/2}) = \mathrm{Hom}_T(\pi_N, \chi \otimes \delta^{1/2})$. This immediately gives "(4) $\Rightarrow$ (5)", for the other direction, one has to show that if $\pi_N \neq 0$, there is some quasi-character χ of T such that $\mathrm{Hom}_T(\pi_N, \chi \otimes \delta^{1/2}) \neq 0$. This can be seen quickly as follows. π_N is admissible, hence so is its contragredient. Any admissible representation of $(F^\times)^k$ has a one-dimensional invariant subspace (see [Bum97, Proposition 4.2.9]), hence there exists $0 \neq L \in \widehat{V_N}$ such that $L(\pi_N(t)v) = (\delta^{1/2}\chi)(t)v$ for some quasi-character χ of $T = F^\times \oplus F^\times$. Then $L \in \mathrm{Hom}_T(\pi_N, \chi \otimes \delta^{1/2})$. A different argument given in [BH06, Proposition 9.1] is to argue that V is finitely generated as a representation of G, hence V_N is finitely generated over T, and any finitely generated representation admits an irreducible quotient.

We have "(3) $\Leftrightarrow$ (4)" since by Theorem 14.2 in the Kirillov model the kernel of $V \to V_N$ is $C_c^\infty(F^\times)$.

"(2) $\Leftrightarrow$ (6)" holds since the matrix coefficients of $V|_{G^1}$ are the restrictions of matrix coefficients of V to G^1, $G^1/(Z \cap G^1)$ is of finite index in G/Z, and $Z \cap G^1$ is compact.

It remains to show "(4) $\Leftrightarrow$ (2)". Let $t = \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}$. Let T^+ be the set of all nonnegative powers of t. Then by the Cartan decomposition $T^+ \to ZK\backslash G/K$ is a bijection, i.e. T^+ is a set of double coset representatives.

"(4) $\Rightarrow$ (2)" Let $v \in V, \widetilde{v} \in \widehat{V}$. Since T^+ is a set of representatives for $ZK\backslash G/K$, we essentially have to show that $\phi_{v,\widetilde{v}}$ is nonzero at only finitely many elements in T^+. Let $N_1, N_2 \subseteq N$ be compact subgroups such that $\widetilde{v} \in \widehat{V}^{N_1}$ and $\pi(e_{N_2})v = 0$. The latter exists since by assumption $v \in V(N)$. For large enough n we have $t^n N_2 t^{-n} \subseteq N_1$, hence for such n we have

$$\phi_{v,\widetilde{v}}(t^n) = \langle \pi(t^n)v, \widetilde{v}\rangle = \langle \pi(t^n)v, \pi(e_{N_1})\widetilde{v}\rangle = \langle \pi(e_{N_1})\pi(t^n)v, \widetilde{v}\rangle = \langle \pi(e_{t^{-n}N_1t^n})v, \pi(t^{-n})\widetilde{v}\rangle = 0,$$

since $\pi(e_{t^{-n}N_1t^n})v = 0$ for $t^{-n}N_1t^n \supseteq N_2$.

"(2) $\Rightarrow$ (4)" Let $K_n = 1 + \mathfrak{p}^n M_{2\times 2}(\mathcal{O})$. Let $v \in V$ and choose n such that $v \in V^{K_n}$. For any $\widetilde{v} \in \widetilde{V}^{K_n}$, $\phi_{v,\widetilde{v}}$ is compactly supported modulo Z. Then $\phi_{v,\widetilde{v}}(t^a) = 0$ for all $a \in \mathbb{Z}$ large enough. Since $\widehat{V}^{K_n}$ is finite dimensional, there is a c such that $\phi_{v,\widetilde{v}}(t^a) = 0$ for all $a \geq c$ and $\widetilde{v} \in \widetilde{V}^{K_n}$. Since $\widehat{V}^{K_n} = (V^{K_n})^*$, we get $\pi(e_{K_n})\pi(t^a)v = 0$ for all $a \geq c$. Recall the notation from the proof of Theorem 11.4. We have $t^{-a}K_nt^a = K_{n-a}^+ K_n^0 K_{n+a}^-$. Therefore,

$$0 = \pi(e_{K_n})\pi(t^a)v = \pi(t^a)\pi(e_{t^{-a}K_nt^a})v = \pi(t^a)\pi(e_{K_{n-a}^+})\pi(e_{K_n^0})\pi(e_{K_{n+a}^-})v = \pi(t^a)\pi(e_{K_{n-a}^+})v.$$

Hence $0 = \pi(e_{K_{n-a}^+})v = \int_{x\in\mathfrak{p}^{n-a}} \pi\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} v\mathrm{d}x$, and we get $v \in V(N)$, which shows $V = V(N)$, so $V_N = 0$.

□

Proposition 22.4 ([JL70, Proposition 2.20]). *Let (V,π) be a supercuspidal. If the central quasi-character ω is unitary, then π is unitarizable. In particular, every supercuspidal representation is essentially square integrable.*

Proof. Immediate from Proposition 18.1. The last statement follows since any quasi-character can be twisted into a unitary character. □

22.1. **Construction of Supercuspidals**

One possible construction of supercuspidals is given in Section 25 using the Weil representation. This mimics the construction of cuspidal representations in the finite field case done in [Bum97]. In this section we construct supercuspidals by inflating cuspidal representations over the residue field.

Let k be the residue field $\mathcal{O}_F/\mathfrak{p}$. Let (V_0,π_0) be a cuspidal representation of $\mathrm{GL}_2(k)$. For the definition, see Section 27. Via the quotient map $K = \mathrm{GL}_2(\mathcal{O}) \to \mathrm{GL}_2(k)$ we lift π_0 to a representation of K. We also lift the central character of π_0 to $\mathcal{O}_F^\times$, and extend it to a unitary character ω_0 of $F^\times$. We then denote by (V_0,π_0) again the representation of KZ on V_0, where K acts via $K \to \mathrm{GL}_2(k)$ and Z acts via ω_0. Let $(V,\pi) = c\text{-}\mathrm{Ind}_{KZ}^G \pi_0$.

Theorem 22.5 ([Bum97, Theorem 4.8.1]). *(V,π) is an irreducible unitarizable supercuspidal representation of G.*

Proof. We proceed a little differently than Bump who does this essentially whith a bunch of Mackey theory. This proof takes the main ideas from [BH06, 11.4 Theorem].

Clearly π_0 is unitarizable as a representation over KZ, since V_0 is finite-dimensional and KZ is compact modulo Z. Hence V is also unitarizable by Theorem 9.15.

Let $K' = KZ$. Let $v \in V_0$. Then v gives rise to an element $f_v \in c\text{-}\mathrm{Ind}_{K'}^G V_0$, by letting $f_v(k) = \pi_0(k)v$ for $k \in K'$, and 0 otherwise. Similarly $\widetilde{v} \in \widehat{V_0}$ gives rise to an element $f_{\widetilde{v}} \in \mathrm{Ind}_{K'}^G \widehat{V_0}$. By Theorem 9.9, we have $\mathrm{Ind}_{K'}^G \widehat{V_0} \cong \widehat{c\text{-}\mathrm{Ind}_{K'}^G V_0}$. Inspecting the proof shows that the matrix element corresponding to the pair $v,\widetilde{v}$ is given by

$$\phi_{f_v,f_{\widetilde{v}}}(g) = \langle \pi(g)f_v, f_{\widetilde{v}}\rangle = \int_{K'\backslash G} \langle (\pi(g)f_v)(h), f_{\widetilde{v}}(h)\rangle \mathrm{d}\mu_{K'\backslash G}(h) = \int_{K'\backslash G} \langle f_v(hg), f_{\widetilde{v}}(h)\rangle \mathrm{d}\mu_{K'\backslash G}(h).^{13}$$

Now $f_{\widetilde{v}}(h) = 0$ if $h \notin K'$, and $f_v(hg) = 0$ for $h \notin K'g^{-1}$. Hence $\phi_{f_v,f_{\widetilde{v}}}(g) = 0$ if $K' \cap K'g^{-1} = \emptyset$, which happens iff $g \notin K'$. Hence $\operatorname{supp}\phi_{f_v,f_{\widetilde{v}}} \subseteq K' = KZ$ is compact mod Z. If we choose $v,\widetilde{v}$ such that $\widetilde{v}(v) \neq 0$, then $\phi_{f_v,f_{\widetilde{v}}}(1) \neq 0$. So we have constructed a matrix coefficient which is compactly supported modulo the center. It remains to show V is irreducible.

[13] K' and G are unimodular, and the quotient $K'\backslash G$ is discrete (since K' is open in G), hence these integrals are really sums over $K'\backslash G$.

Let $T : V_0 \to V|_{K'}$ be an intertwining operator of K' representations. We show that T must have image contained in the image of V_0 in $V|_{K'}$ in the above described way. So let $f \in T(V_0)$. We have to show that f is supported in K'. Suppose $f(g) \neq 0$ for some $g \in G - K'$. We have $\pi(gk) = (\pi(k)f)(g) = T(\pi_0(k)v)(g)$ for $k \in K'$, so we may replace g by gk by replacing f by $\pi(k)f$. Therefore we may assume that $g = \begin{pmatrix} \varpi^a & 0 \\ 0 & 1 \end{pmatrix}$ for some $a \geq 1$ (as these form representatives of the nonidentity cosets in $ZK\backslash G/K$). Let $n \in N \cap K = K_0^0 = \begin{pmatrix} 1 & \mathcal{O} \\ 0 & 1 \end{pmatrix}$ be arbitrary. Then we may write $n = g^{-1}mg$ for some $m \in K_1^+ = \begin{pmatrix} 1 & \mathfrak{p} \\ 0 & 1 \end{pmatrix}$. Then

$$(T(\pi_0(n)v))(g) = (\pi(n)f)(g) = (\pi(g^{-1}mg)f)(g) = f(mg) = \pi_0(m)(f(g)).$$

Since $m \equiv I_2 \bmod \mathfrak{p}$, we have $\pi_0(m)(f(g)) = f(g)$. Now somehow use that π_0 is a cuspidal representation of $\mathrm{GL}_2(k)$ (so can choose n such that $\pi_0(n)v \neq v$), but not sure how to continue. TODO

Assume we showed this, so that the range of T is in V_0. Then we get from (a version of) Frobenius reciprocity:

$$\mathrm{Hom}_G(V, V) \cong \mathrm{Hom}_{K'}(V_0, V) = \mathrm{Hom}_{K'}(V_0, V_0) = \mathbb{C}.$$

The last equality is by Schur's lemma since V_0 is irreducible. Since V is unitarizable it the follows that V irreducible.

□

23. Classification of Representations

Theorem 23.1. *Let (V, π) be a smooth irreducible representation of $G = \mathrm{GL}_2(F)$. Then π is admissible and it is isomorphic to exactly one of the following:*

- *a one dimensional representation of the form $\chi \circ \det$ for some quasi-character χ of $F^\times$;*
- *a principal series representation $\mathcal{B}(\chi_1, \chi_2)$ for quasi-characters χ_1, χ_2 of $\chi_1\chi_2^{-1} \neq |\cdot|^{-1}$ (which are uniquely determined up to order);*
- *a special representation $\chi \otimes \mathrm{St}$ for a quasi-character χ of $F^\times$*
- *a supercuspidal representation.*

Proof. If $V_N = 0$, then V is quasi-cuspidal and therefore admissible, hence supercuspidal by Theorem 22.3 and Proposition 22.1. Otherwise, again by Theorem 22.3, V is isomorphic to a subrepresentation of a principal series representation $\mathcal{B}(\chi_1, \chi_2)$ for some χ_1, χ_2, and the result follows again (principal series representations are admissible). □

See Section 19 for which of these representations are unitary.

24. L-Functions

Let (V, π) be an irreducible infinite-dimensional admissible representation of G with central quasi-character ω. [Bum97] and [JL70] define the zeta integrals in terms of the Kirillov model, while [BH06] and [GH11] define it in terms of matrix coefficients. We give both definitions and give proofs in both cases. (How to show directly they are equivalent??)

Definition (via matrix elements)**.** *Let $A = M_2(F)$ the space of 2×2 matrices over F. If $\Phi \in C_c^\infty(A), f \in \mathcal{C}(\pi)$, then*

$$Z(\Phi, f, s) = \int_G \Phi(g) f(g) \left|\det g\right|^{s+\frac{1}{2}} \,\mathrm{d}g$$

Definition (via the Kirillov model)**.** *Let $\mathcal{K} \subseteq C^\infty(F^\times)$ be the Kirillov model of V. For $\phi \in \mathcal{K}$, define*

$$Z_{\mathcal{K}}(\phi, s) = \int_{F^\times} \Phi(g) \left|\det g\right|^{s-\frac{1}{2}} \,\mathrm{d}g$$

Proposition 24.1. *There exists $s_0 \in \mathbb{R}$ such that the integral defining $Z(\Phi, f, s)$ (resp. $Z_{\mathcal{K}}(\phi, s)$) converges absolutely for $\operatorname{Re} s > s_0$.*

Proposition 24.2. *$Z(\Phi, f, s)$ (resp. $Z_{\mathcal{K}}(\phi, s)$) is a rational function in q^{-s} where q is the cardinality of the residue field of F.*

Theorem 24.3. *The functions Z have a "common denominator": There is a unique (up to scalar) function $L(\pi, s)$ such that*

$$\frac{Z(\Phi, f, s)}{L(\pi, s)} \quad (\textit{resp. } \frac{Z_{\mathcal{K}}(\phi, s)}{L(\pi, s)})$$

is an entire function which is constant 1 for some choice of Φ, f. The function $L(\pi, s)$ can be given as follows:

- *If $\pi \cong \pi(\chi_1, \chi_2)$, then $L(\pi, s) = L(\chi_1, s) L(\chi_2, s)$ where $L(\chi, s)$ is the L-function of χ as in Section 1, i.e.*
  $$L(\pi, s) = (1 - \alpha_1 q^{-s})^{-1} (1 - \alpha_2 q^{-s})^{-1},$$
  where $\alpha_i = \chi_i(\varpi)$ for some uniformizer ϖ if χ_i is unramified, and $\alpha_i = 0$ otherwise.
- *If $\pi \cong \sigma(\chi_1, \chi_2)$ with $\chi_1 \chi_2^{-1} = |\cdot|^{-1}$, then $L(\pi, s) = L(\chi_2, s)$.*
- *If π is supercuspidal, then $L(\pi, s) = 1$.*

Fix a nontrivial character ψ of F. Then there is an isomorphism $A \cong \widehat{A}$ given by $x \mapsto (y \mapsto \psi(\operatorname{Tr}(xy)))$. The Fourier transform $\widehat{\Phi}$ of $\Phi \in C_c^\infty(A)$ is then defined using the self-dual Haar measure on A with respect to this isomorphism. For a function f on G, $\check{f}$ denotes the function $g \mapsto f(g^{-1})$. The map $f \mapsto \check{f}$ is an isomorphism of vector spaces $\mathcal{C}(\pi) \to \mathcal{C}(\widetilde{\pi})$.

Theorem 24.4 (Functional Equation)**.** *There is a function $\gamma(\pi, \psi, s)$ such that $Z(\Phi, f, s)$ satisfies the functional equation*

$$Z(\widehat{\Phi}, \check{f}, 1 - s) = \gamma(\pi, \psi, s) Z(\Phi, f, s)$$

If we define

$$\varepsilon(\pi,\psi,s)=\gamma(\pi,\psi,s)\frac{L(\pi,s)}{L(\widetilde{\pi},1-s)},$$

then we have

$$\varepsilon(\pi,\psi,s)\varepsilon(\widetilde{\pi},\psi,1-s)=\omega(-1),$$

$$\frac{Z(\widetilde{\Phi},\check{f},1-s)}{L(\widetilde{\pi},1-s)}=\varepsilon(\pi,\psi,s)\frac{Z(\Phi,f,s)}{L(\pi,s)}.$$

Moreover, $\varepsilon(\pi,\psi,s)$ is of the form aq^{bs} for some constants $a\in\mathbb{C}^\times, b\in\mathbb{Z}$.

In the theorem $\widehat{\Phi}$ is the Fourier transform using the self-dual Haar measure (w.r.t. to a given nontrivial character ψ of F) of A.

24.1. **Whittaker Model Approach**

Let (V,π) be an irreducible admissible representation of G, admitting a Whittaker model $\mathcal{W}=\mathcal{W}(\pi,\psi)$.

Definition. *For $W\in\mathcal{W}$ define the Zeta integral*

$$Z(W,s)=\int_{F^\times}W\begin{pmatrix}y&0\\0&1\end{pmatrix}|y|^{s-\frac{1}{2}}\,\mathrm{d}^\times y.$$

Note that the functions of the form $y\mapsto W\begin{pmatrix}y&0\\0&1\end{pmatrix}$ are precisely the functions in the Kirillov model of π, so if ϕ is in the Kirillov model of π, we may also write

$$Z(\phi,s)=\int_{F^\times}\phi(y)\,|y|^{s-\frac{1}{2}}\,\mathrm{d}^\times y.$$

We can also generalize this (to $\mathrm{GL}_2\times\mathrm{GL}_1$ Zeta-functions) as follows: If χ is a quasi-character of $F^\times$, then let

$$Z(W,\chi,s)=\int_{F^\times}W\begin{pmatrix}y&0\\0&1\end{pmatrix}|y|^{s-\frac{1}{2}}\,\chi(y)\mathrm{d}^\times y.$$

Note however this doesn't really give something new, since the Whittaker model of $\chi\otimes\pi$ consists of functions of the form $\widetilde{W}(h)=\chi(h)W(h)$ (here $\chi(h)=\chi(\det h)$) where $W\in\mathcal{W}$.

Theorem 24.5 ([Bum97, Proposition 4.7.5])**.** *The integral defining $Z(W,s)$ converges absolutely for $\operatorname{Re}s\gg 0$ and has a meromorphic continuation to all s. There is a polynomial p such that $Z(W,s)=p(q^{-s})L(\pi,s)$. W can be chosen such that $p=1$.*

For the definition of $L(\pi,s)$ see Theorem 24.3.

Proof. Directly using the concrete description of the Kirillov model in Theorem 14.2. □

Proposition 24.6. *If π is unitarizable, the integral defining $Z(W,s)$ converges absolutely for $\operatorname{Re}s>\frac{1}{2}$.*

Proof. If $\pi \cong \mathcal{B}(\chi_1, \chi_2)$, then we know from Theorem 15.14 $|\chi_1| = |\cdot|^\sigma, |\chi_2| = |\cdot|^{-\sigma}$ for some σ with $-\frac{1}{2} < \sigma < \frac{1}{2}$ (the case where both χ_1, χ_2 are unitary is $\sigma = 0$). If $\chi_1 \neq \chi_2$, by Theorem 14.2, near 0 the integrand has the form $|\cdot|^s (C_1\chi_1 + C_2\chi_2)$. By the bound for σ this is easily seen to converge absolutely for $\operatorname{Re} s > \frac{1}{2}$. Similarly for the case $\chi_1 = \chi_2$ and the case $\pi \cong \sigma(\chi_1, \chi_2)$. Note that if π is supercuspidal, the integral converges for all $s \in \mathbb{C}$. $\square$

Theorem 24.7 (Local Functional Equation, [Bum97, Theorem 4.7.5])**.** *Let (V, π) be an admissible irreducible representation of G with Whittaker model $\mathcal{W}$ and central character ω. Let χ be a quasi-character of $F^\times$. There is a meromorphic function $\gamma(\pi, \chi, s, \psi)$ such that*

$$Z(\pi(w_1)W, \omega^{-1}\chi^{-1}, 1 - s) = \gamma(\pi, \chi, s, \psi)Z(W, \chi, s)$$

for all $W \in \mathcal{W}$, where $w_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

We need a lemma:

Lemma 24.8. *There are at most two values $s \in \mathbb{C}$ modulo $2\pi i/\log q$ such that $\operatorname{Hom}_{T_1}(V|_{T_1}, \chi |\cdot|^s)$ has dimension > 1.*

Proof. Explicitly, $\operatorname{Hom}_{T_1}(V|_{T_1}, \chi |\cdot|^s)$ consists of functionals $\Lambda : V \to \mathbb{C}$ satisfying $L\left(\pi \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} v\right) = \chi(y)\,|y|^s\,\Lambda(v)$ for all $y \in F^\times, v \in V$. Let Λ_1, Λ_2 be two such functionals. Their restrictions to $V(N) \cong C_c^\infty(F^\times)$ are linearly dependent by uniqueness of the twisted Haar measure. Hence, there are $a, b \in F$, not both 0, such that $\Lambda = a\Lambda_1 + b\Lambda_2$ factor through $V \to V_N = V/V(N)$. At most two characters of $F^\times \cong T_1$ occur in V_N, hence $\Lambda = 0$, unless s is one of the at most two values that makes $\chi |\cdot|^s$ occur in V_N. $\square$

Proof of Theorem 24.7. Fix s with $0 < \operatorname{Re} s < 1$. Consider the two functionals

$$\begin{aligned} \Lambda_1, \Lambda_2 &: \mathcal{W} \longrightarrow \mathbb{C}, \\ \Lambda_1(W) &= Z(W, \chi, s), \\ \Lambda_2(W) &= Z(\pi(w_1)W, \omega^{-1}\chi^{-1}, 1 - s). \end{aligned}$$

Then these functionals satisfy

$$\begin{aligned} \Lambda_1\left(\pi \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} v\right) &= \chi(y)^{-1}\,|y|^{-s+\frac{1}{2}}\,\Lambda_1(v), \\ \Lambda_2\left(\pi \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} v\right) &= \chi(y)^{-1}\,|y|^{-s+\frac{1}{2}}\,\Lambda_2(v). \end{aligned}$$

Therefore, for almost all s, by the lemma these functionals differ by a scalar, which we call $\gamma(\pi, \chi, s, \psi)$. $\square$

Proposition 24.9. *Some properties of the local gamma factor.*

(1) $\gamma(\pi, \chi, s, \psi) = \gamma(\chi \otimes \pi, 1, s, \psi)$.

(2) $\gamma(\pi,\chi,s,\psi)\gamma(\widetilde{\pi},\chi^{-1},1-s,\psi) = \omega(-1)$.

(3) $\gamma(\pi,\chi,s,\psi_a) = \chi(a)^2\omega(a)\,|a|^{2s-1}\,\gamma(\pi,\chi,s,\psi)$ *where* $\psi_a(x) = \psi(ax)$ *for* $a \in F^\times$.

Proof.

(1) If $W \in \mathcal{W}(\pi,\chi)$, then $\widetilde{W} = (\chi\circ\det)W \in \mathcal{W}(\chi\otimes\pi,\chi)$. The central character of $\chi\otimes\pi$ is $\omega\chi^2$. Therefore
$$Z(\pi(w_1)\widetilde{W},\omega^{-1}\chi^{-2},1-s) = \gamma(\chi\otimes\pi,1,s,\psi)Z(\widetilde{W},1,s).$$
But we also have
$$Z(\pi(w_1)\widetilde{W},\omega^{-1}\chi^{-2},1-s) = Z(\pi(w_1)W,\omega^{-1}\chi^{-1},1-s),$$
$$Z(\widetilde{W},1,s) = Z(W,\chi,s).$$
So the result follows.

(2) We have
$$Z(\pi(w_1)W,\omega^{-1}\chi^{-1},1-s) = \gamma(\pi,\chi,s,\psi)Z(W,\chi,s)$$
Now apply the functional equation with the two sides reversed. Then
$$Z(W,\chi,s) = \gamma(\pi,\omega^{-1}\chi^{-1},1-s,\psi)Z(\pi(w_1)^{-1}W,\omega^{-1}\chi^{-1},1-s).$$
Note that $\pi(w_1)^{-1} = \pi(-w_1) = \omega(-1)\pi(w_1)$. Hence, substituting the first equation into the second we get
$$\omega(-1)\gamma(\pi,\omega^{-1}\chi^{-1},1-s,\psi)\gamma(\pi,\chi,s,\psi) = 1.$$
The result then follows from $\omega^{-1}\otimes\pi \cong \widetilde{\pi}$, part (1) and $\omega(-1)^2 = 1$.

(3) If we replace ψ by ψ_a, then $W \in \mathcal{W}(\pi,\psi)$, will be replaced by $W' \in \mathcal{W}(\pi,\psi_a)$ where $W'(g) = W\left(\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} g\right)$. Then
$$Z(W',\chi,s) = |a|^{\frac{1}{2}-s}\,\chi(a)^{-1}Z(W,\chi,s),$$
$$Z(\pi(w_1)W',\omega^{-1}\chi^{-1},1-s) = |a|^{s-\frac{1}{2}}\,\omega(a)\chi(a)Z(\pi(w_1)W,\omega^{-1}\chi^{-1},1-s).$$
Hence,
$$\gamma(\pi,\chi,s,\psi_a) = |a|^{2s-1}\,\chi(a)^2\omega(a)\gamma(\pi,\chi,s,\psi).$$

$\square$

Theorem 24.10. *Assume π is unramified, so $\pi \cong \mathcal{B}(\chi_1,\chi_2)$ with unramified χ_1,χ_2. Let $\alpha_i = \chi_i(\varpi)$. Also assume that the conductor of ψ is $\mathcal{O}_F$. If W is the spherical Whittaker function with $W(1) = 1$, then*
$$Z(W,s) = L(\pi,s) = (1-\alpha_1 q^{-s})^{-1}(1-\alpha_2 q^{-s})^{-1}.$$

Proof. In the notation of Section 20.1 we have $W = W_0(1)^{-1}W_0$. By Theorem 20.10, $W(a_m) = q^{m/2}\frac{\alpha_1^{m+1}-\alpha_2^{m+1}}{\alpha_1-\alpha_2}$ for $m \geq 0$ and $W(a_m) = 0$ for $m < 0$, where $a_m = \begin{pmatrix} \varpi^m & 0 \\ 0 & 1\end{pmatrix}$. Since ψ has conductor

$\mathcal{O}_F$, that $\int_{\mathcal{O}_F} \mathrm{d}x = 1$ for the self-dual Haar measure, and so the multiplicative volume of $\mathcal{O}_F^\times$ is 1 by the standard normalization of multiplicative Haar measure. Now we calculate

$$\begin{aligned} Z(W,s) &= \int_{F^\times} W\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} |y|^{s-\frac{1}{2}}\, \mathrm{d}^\times y \\ &= \sum_{n\in\mathbb{Z}} q^{-n(s-\frac{1}{2})} \int_{\mathcal{O}_F^\times} W\begin{pmatrix} \varpi^n y & 0 \\ 0 & 1 \end{pmatrix} \mathrm{d}^\times y \\ &= \sum_{n\in\mathbb{Z}} q^{-n(s-\frac{1}{2})} \int_{\mathcal{O}_F^\times} W\begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix} \mathrm{d}^\times y \\ &= \sum_{n\geq 0} q^{-n(s-\frac{1}{2})} q^{-n/2} \frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2} \\ &= (\alpha_1 - \alpha_2)^{-1} \sum_{n\geq 0} q^{-ns} (\alpha_1^{n+1} - \alpha_2^{n+1}) \\ &= (\alpha_1 - \alpha_2)^{-1} (\alpha_1(1-\alpha_1 q^{-s})^{-1} - \alpha_2(1-\alpha_2 q^{-s})^{-1}) \end{aligned}$$

This is $L(\pi, s)$. □

We set $L(\pi, \chi, s) = L(\chi \otimes \pi, s)$.

Corollary 24.11. *Assumptions as in the theorem. If χ is an unramified quasi-character of $F^\times$, then*

$$Z(W, \chi, s) = L(\pi, \chi, s)$$

Proof. Indeed, $\widetilde{W}(g) = \chi(\deg g) W(g)$ defines the spherical Whittaker function with $\widetilde{W}(1) = 1$ for $\chi \otimes \pi$, so

$$Z(W, \chi, s) = Z(\widetilde{W}, s) = L(\chi \otimes \pi, s) = L(\pi, \chi, s).$$

□

Define the local Epsilon factor by

$$\varepsilon(\pi, \chi, s, \psi) = \gamma(\pi, \chi, s, \psi) \frac{L(\pi, \chi, s)}{L(\widetilde{\pi}, \chi^{-1}, 1-s)}.$$

Proposition 24.12. *Some properties of the local epsilon factor.*

(1) $\varepsilon(\pi, \chi, s, \psi) = \varepsilon(\chi \otimes \pi, 1, s, \psi)$.

(2) $\varepsilon(\pi, \chi, s, \psi)\varepsilon(\widetilde{\pi}, \chi^{-1}, 1-s, \psi) = \omega(-1)$.

(3) $\varepsilon(\pi, \chi, s, \psi_a) = \chi(a)^2 \omega(a) |a|^{2s-1} \varepsilon(\pi, \chi, s, \psi)$ *where* $\psi_a(x) = \psi(ax)$ *for* $a \in F^\times$.

(4) $\varepsilon(\pi, \chi, s, \psi)$ *is of the form* ab^s *for some constants* a, b.

(5) *If* π *and* χ *are unramified, and the conductor of* ψ *is* $\mathcal{O}_F$*, then* $\varepsilon(\pi, \chi, s, \psi) = 1$.

(6) *If* $\chi \otimes \pi$ *is not a ramified principal series or special representation, then* $\varepsilon(\pi, \chi, s, \psi) = \gamma(\pi, \chi, s, \psi)$.

Proof.

(1) Immediate from corresponding fact for the γ factors, see Proposition 24.9.

(2) Immediate from corresponding fact for the γ factors, see Proposition 24.9. The L-factors cancel out.

(3) Immediate from the way the γ-factors change, see Proposition 24.9.

(4) TODO.

(5) Let W be the spherical Whittaker function with $W(1) = 1$ for π. We know that $\widetilde{\pi} \cong \omega^{-1} \otimes \pi$, hence $\widetilde{W} = (\omega \circ \det)^{-1} W$ is the corresponding function for $\widetilde{\pi}$. Since W is spherical we have $\pi(w_1)W = W$. Then by the corollary:

$$L(\widetilde{\pi}, \chi^{-1}, 1-s) = Z(\widetilde{W}, \chi^{-1}, 1-s) = Z(W, \omega^{-1}\chi^{-1}, 1-s) = Z(\pi(w_1)W, \omega^{-1}\chi^{-1}, 1-s).$$

Hence

$$\varepsilon(\pi, \chi, s, \psi) = \gamma(\pi, \chi, s, \psi)\frac{L(\pi, \chi, s)}{L(\widetilde{\pi}, \chi^{-1}, 1-s)} = \frac{\gamma(\pi, \chi, s, \psi) Z(W, \chi, s)}{Z(\pi(w_1)W, \omega^{-1}\chi^{-1}, 1-s)} = 1$$

(6) Immediate, since in this case the L-factors are 1.

□

The γ factors determine the representation in the following sense.

Theorem 24.13 ([Bum97, Proposition 4.7.6])**.** *Let $(V_1, \pi_1), (V_2, \pi_2)$ be two irreducible admissible representations of G. Assume that π_1, π_2 have the same central quasi-character ω, and that*

$$\gamma(\pi_1, \chi, s, \psi) = \gamma(\pi_2, \chi, s, \psi)$$

for all quasi-characters[a] *χ of $F^\times$. Then $\pi_1 \cong \pi_2$.*

[a]Really we need to assume this for a class of quasi-characters χ such that we obtain every character of $\mathcal{O}^\times$ after restriction $\chi|_{\mathcal{O}^\times}$.

Proof. Assume that V_1, V_2 are in Kirillov form. Let $V_0 = V_1 \cap V_2$. Note that $C_c^\infty(F^\times) \subseteq V_0$ and $\pi_1(g)|_{V_0} = \pi_2(g)|_{V_0}$ for $g \in M$. Since π_1, π_2 have the same central character, this holds for all $g \in B$. We show that if $\phi \in V_0$, then $\pi_1(w_1)\phi = \phi_2(w_1)\phi$. Then $V_1 \cap V_2$ is stable under w_1, hence stable under G which is generated by w_1 and B, and therefore $V_1 = V = V_1$ and the actions of π_1, π_2 are the same. The idea to show $\pi_1(w_1)\phi = \phi_2(w_1)\phi$ is that we can "see" the action of w_1 using the local functional equation in the gamme factors. Let $\phi_i = \pi_i(w_1)\phi$. It suffices to show $\phi_1(1) = \phi_2(1)$: By applying this to $\phi' = \pi_1 \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \phi = \pi_2 \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \phi$, we get:

$$\phi_1(a) = \pi_1\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} w_1\right)\phi(1) = \pi_1(w_1)\phi'(1) = \pi_2(w_1)\phi'(1) = \pi_2\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} w_1\right)\phi(1) = \phi_2(a).$$

For a character χ of $F^\times$ and $n \in \mathbb{Z}$ we let

$$F_\chi(n) = \int_{|x|=q^{-n}} (\phi_1(x) - \phi_2(x))\chi(x) \mathrm{d}^\times x.$$

Let $f = (\phi_1 - \phi_2)|_{\mathcal{O}^\times}$. Note that $F_\chi(0)$ is the Fourier transform $\widehat{f}(\chi|_{\mathcal{O}^\times})$ on $\mathcal{O}^\times$. Since f is locally constant the compact space $\mathcal{O}^\times$, we have $\widehat{f}(\xi) = 0$ for all but a finite number of characters ξ of $\mathcal{O}^\times$.[14]. Therefore, the Fourier inversion formula on the compact group $\mathcal{O}^\times$ gives

$$f(1) = \sum_{\xi \in \widehat{\mathcal{O}^\times}} \widehat{f}(\xi).$$

We will show $\widehat{f}(\xi) = 0$. Fix $\xi \in \widehat{\mathcal{O}^\times}$. Let χ be any extension of ξ to $F^\times$. Then since $\gamma(\pi_1, \omega^{-1}\chi^{-1}, 1 - s, \psi) = \gamma(\pi_2, \omega^{-1}\chi^{-1}, 1 - s, \psi)$, the functional equation gives

$$Z(\phi_1, \chi, s) = Z(\phi_2, \chi, s).$$

But now note that, letting $t = q^{-s}$, we have

$$\sum_{n \in \mathbb{Z}} F_\chi(n) t^n = \int_{F^\times} (\phi_1(x) - \phi_2(x))\chi(x)\,|x|^s \,\mathrm{d}^\times x = Z(\phi_1, \chi, s + \frac{1}{2}) - Z(\phi_2, \chi, s + \frac{1}{2}),$$

for all s with $\operatorname{Re} s \gg 0$ (so that the integral converges absolutely), i.e. for all t with $|t|$ sufficiently small. This implies $F_\chi(n) = 0$ for all n, in particular for $n = 0$, we obtain $\widehat{f}(\xi) = F_\chi(0) = 0$ which is what we wanted to show. □

Theorem 24.14. *Let χ_1, χ_2 be quasi-characters of $F^\times$ such that $\pi = \mathcal{B}(\chi_1, \chi_2)$ is irreducible. Then for any quasi-character η of $F^\times$, we have*

$$\gamma(\pi, \eta, s, \psi) = \gamma(\eta\xi_1, s, \psi)\gamma(\eta\xi_2, s, \psi).$$

Here the gamma factors on the right are those from the GL_1 theory, Section 1. We give the proof at the end of 25.

25. Weil Representation

In this section we may for some parts allow F to be any local field of characteristic $\neq 2$, in particular it may be archimedean. As usual fix a non-trivial additive character ψ of F. Let (V, β) be a quadratic space over F, i.e. V is a finite-dimensional F-vector space and $\beta : V \to F$ is a non-degenerate quadratic form, which means the associated function $B : V \times V \to F$, defined by

$$B(u, v) = \frac{1}{2}(\beta(u + v) - \beta(u) - \beta(v)),$$

is a symmetric non-degenerate bilinear form. If $a \in F^\times$, then $a\beta$ is the quadratic form $(a\beta)(v) = a\beta(v)$. If $(V_1, \beta_1), (V_2, \beta_2)$ are quadratic spaces, we can consider their direct sum $(V_1, \beta_1) \oplus (V_2, \beta_2) = (V_1 \oplus V_2, \beta_1 \oplus \beta_2)$ where $\beta_1 \oplus \beta_2(v_1, v_2) = \beta_1(v_1) + \beta_2(v_2)$. If $a_1, \ldots, a_n \in F^\times$, we define $\mathrm{QF}(a_1, \ldots, a_n) = (F^n, \beta)$ where $\beta(x_1, \ldots, x_n) = \sum_{i=1}^n a_i x_i^2$. Note that $\mathrm{QF}(a_1, \ldots, a_n) = \bigoplus_{i=1}^n \mathrm{QF}(a_i)$. Every quadratic space is isomorphic to one of the form $\mathrm{QF}(a_1, \ldots, a_n)$. A quadratic space is *split* if it isomorphic to a direct sum of copies of hyperbolic planes $\mathrm{QF}(1, -1)$.

Of most importance to us will be the following cases:

[14] If $H \subseteq \mathcal{O}^\times$ is a finite index open subgroup such that f is constant on cosets of H, then $\widehat{f}(\xi) = 0$ for any ξ that is non-trivial on H.

(i) $V = L$ is a quadratic separable algebra over F, i.e. either L is a quadratic separable extension of F, or $L = F \oplus F$, and β is the norm form, given by $\beta(x) = x\sigma(x)$ where $\sigma : L \to L$ is the unique non-trivial F-automorphism.

(ii) If $L \neq \mathbb{C}$, V is the unique quaternion division algebra over F, and β the reduced norm form.

We would like to compute the Fourier transform of $F_\beta := \psi \circ \beta : V \to \mathbb{C}$. The problem is that this function is not integrable, the Fourier integral would be a kind of Fresnel integral. But we can make sense of this in a distributional sense.

$S(V)$ is the Schwartz space of V. The pairing $(u, v) \mapsto \psi(-2B(u, v))$ identifies V with $\widehat{V}$, its Pontryagin dual, hence if $f \in S(V)$, we define its Fourier transform $\widehat{f}$ by

$$\widehat{f}(\xi) = \int_V f(v)\psi(2B(\xi, v))\mathrm{d}v.$$

We normalize the Haar measure so that it becomes self-dual, i.e. the Fourier inversion formula

$$\widehat{\widehat{f}}(x) = f(-x)$$

holds. Note that the self-dual Haar measure depends on both ψ and β. We will denote the Fourier transform $\widehat{f}$ also by $\mathcal{F}f$. We now follow Bump (who follows [Wei65]), to define the Fourier transform of F_β by convolution with test functions. Note that another approach might be to instead view F_β as a distribution on V and thereby interpret and define its Fourier transform, which is done in [Cas].

Let $d = \dim V$.

Proposition 25.1 ([Bum97, Proposition 4.8.3]). *If $\Phi \in S(V)$, then $\Phi * F_\beta \in S(V)$. There is a number $\gamma(\beta) \in \mathbb{C}$ with $|\gamma(\beta)| = 1$ such that*

$$\mathcal{F}(\Phi * F_\beta) = \gamma(\beta)\mathcal{F}\Phi \cdot F_{-\beta}.$$

If $a \in F^\times$, then

$$\mathcal{F}(\Phi * F_{a\beta}) = |a|^{d/2}\gamma(a\beta)\mathcal{F}\Phi \cdot F_{-a^{-1}\beta}.$$

In other words, the first equation tells us that $\mathcal{F}F_\beta = \gamma(\beta)F_{-\beta}$ assuming we had defined $\mathcal{F}F_\beta$ (e.g. as a distribution). See also [Cas, Theorem 2.2].

Some properties of $\gamma(\beta)$:

Proposition 25.2.

(i) $\gamma(\beta_1 \oplus \beta_2) = \gamma(\beta_1)\gamma(\beta_2)$

(ii) $\gamma(-\beta) = \gamma(\beta)^{-1}$.

(iii) *If β is split, then* $\gamma(\beta) = 1$.

(iv) $\gamma(\beta)^8 = 1$.

Proof. $(i), (ii), (iii)$ are easy. For (iv) see [Cas]. ∎

Proposition 25.3 ([Bum97, Proposition 4.8.5]). *If F is nonarchimedean, then for every sufficiently large lattice $L \subseteq V$,[a] we have*

$$\gamma(\beta) = \int_L F_\beta(v)\mathrm{d}v.$$

[a] i.e. compact open subgroup of V

Recall that the Hilbert symbol is a certain symmetric bilinear map $F^\times/(F^\times)^2 \times F^\times/(F^\times)^2 \to \{\pm 1\}$. It can be defined in various equivalent ways:

(1) $(a, b) = 1$ if and only if $x^2 - ay^2 - bz^2 + abw^2 = 0$ has a solution $x, y, z, w \in F$, not all zero.

(2) $(a, b) = 1$ if and only if $\mathrm{Quat}(a, b)$ is split over F, i.e. $\mathrm{Quat}(a, b) \cong M_{2\times 2}(F)$.

(3) $\phi_K(a)b^{1/2} = (a, b)b^{1/2}$, where ϕ_K is the reciprocity map, see Theorem 4.1.

Theorem 25.4 ([Bum97, Theorem 4.8.4]). *Let $a, b \in F^\times$. Then*

$$(a, b) = \gamma(\mathrm{Quat}(a, b)) = \gamma(\mathrm{QF}(1, -a, -b, ab)).$$

Corollary 25.5. *Let $(V, \beta) = \mathrm{QF}(r_1, \dots, r_d)$ be a quadratic space with d even. If we let $\Delta = (-1)^{d/2}r_1r_2\cdots r_d$, then for any $a \in F^\times$, we have $\gamma(a\beta) = (\Delta, a)\gamma(\beta)$.*

Theorem 25.6. *Let (V, β) be a quadratic space. There is a projective representation r of $\mathrm{SL}_2(F)$ on $L^2(V)$ such that*

$$\left(r\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\Phi\right)(v) = \psi(x\beta(v))\Phi(v)$$
$$\left(r\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}\Phi\right)(v) = |a|^{d/2}\,\Phi(av)$$
$$r(w_1)\Phi = \widehat{\Phi},$$

where $x \in F, a \in F^\times$ and $w_1 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. This projective representation is unique up to scalar. The Schwartz space $S(V)$ is invariant.

Proof. See [Bum97, theorem 4.8.3]. The uniqueness holds because elements of the form indicated generate $\mathrm{SL}_2(F)$. ☐

Theorem 25.7 ([Bum97, theorem 4.8.3]). *Let (V, β) be a quadratic space of even dimension d. Then a scalar multiple r of the projective representation of $\mathrm{SL}_2(F)$ above is a genuine representation. The scalar is given as follows: Let $\chi : F^\times \to \{\pm 1\}$ be the quadratic character given by $\chi(a) = \frac{\gamma(a\beta)}{\gamma(\beta)} = (\Delta, a)$ (with the Δ from Corollary 25.5). Then the following defines a (continuous)* *representation r of $\mathrm{SL}_2(F)$ on $L^2(V)$, with $S(V)$ as an invariant subspace:*

$$\left(r\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\Phi\right)(v) = \psi(x\beta(v))\Phi(v),$$

$$\left(r\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}\Phi\right)(v) = |a|^{d/2}\chi(a)\Phi(av),$$

$$r(w_1)\Phi = \gamma(\beta)\widehat{\Phi}.$$

We now want to extend these representations to $G = \mathrm{GL}_2(F)$.

We now assume the case $\dim V = 2$. Up to rescaling (V,β) comes from an algebra E over F as above, so either E/F is a separable field extension, $E = F \oplus F$, or E is a nonsplit quaternion division algebra over F. In all the cases E is an algebra, and $\beta : E \to F$ is multiplicative. Hence, $F_+ := \beta(E^\times)$ is a subgroup of $F^\times$, and G_+, the subset of matrices $g \in \mathrm{GL}_2(F)$ with determinant $\det g \in F_+$, is a subgroup of $\mathrm{GL}_2(F)$. We have the following description of F_+:

- If $E = F \oplus F$, then $F_+ = F^\times$.
- If E/F is a quadratic separable extension, then $F_+ = N_{E/F}(E^\times)$ is the norm group of L in the sense of local class field theory.
- If E/F is a quaternion algebra, then $F_+ = F^\times$ if F is nonarchimedean, and $F_+ = \mathbb{R}_{>0}$ if $F = \mathbb{R}$, [Voi21, Lemma 13.4.9]

Note in every case G_+ has index at most 2 in G, and it contains the center.

Let $a \in F^\times$. How does the representation change if we replace ψ by ψ_a? Let r_{ψ_a} denote the represntation constructed with the character ψ_a. For $\Phi \in S(E)$ and $b \in E^\times$, we define $\lambda(b)\Phi(x) = \Phi(b^{-1}x), \rho(b)\Phi(x) = \Phi(xb)$.

Proposition 25.8 ([JL70, Lemma 1.4]). *We have*

$$r_{\psi_a}(g) = r_\psi\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} g \begin{pmatrix}a^{-1} & 0\\ 0 & 1\end{pmatrix}\right).$$

If $a = \beta(b)$ with $b \in E^\times$, then

$$r_{\psi_a}(g)\lambda(b^{-1}) = \lambda(b^{-1})r_\psi(g),$$

$$r_{\psi_a}(g)\rho(b) = \rho(b)r_\psi(g).$$

Note that if $\beta(b) = 1$, then $\lambda(b)$ and $\rho(b)$ commute with $r = r_\psi$.

We will first consider the case where E is a quadratic separable field extension of F. Let $E^1 := \ker\beta = \ker N_{E/F}$ be the norm 1 hyperplane. It is a compact subset of $E^\times$.

The character χ from before is now the nontrivial character of $F^\times/N_{E/F}(E^\times)$. Indeed, if $a = \beta(b) \in N_{E/F}(E^\times)$, then $(E, a\beta)$ and (E,β) are via $v \mapsto bv$ isomorphic quadratic spaces, hence $\gamma(a\beta) = \gamma(\beta)$, and therefore $\chi(a) = 1$. Alternatively, we have $E = K(\sqrt{a})$ for some $a \in F^\times$, and then $(E,\beta) \cong \mathrm{QF}(1,-a)$, so $\chi(b) = (a,b) = (b, E/K)$, by Corollary 25.5.

Let θ be a quasi-character of $E^\times$. Let $S(E,\theta)$ be the space of functions $\Phi \in S(E)$ satisfying

$$\Phi(xh) = \theta^{-1}(h)\Phi(x)$$

for all $x \in E$, $h \in E^1$.

Theorem 25.9.

(1) *$S(E,\theta)$ is an invariant subspace for r.*

(2) *The representation r of $\mathrm{SL}_2(F)$ on $S(E,\theta)$ can be extended to a representation r_θ of G_+, via*
$$\left(r_\theta \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \Phi\right)(x) = |h|_E^{1/2}\,\theta(h)\Phi(xh),$$
where $a = \beta(h), h \in E^\times, x \in E$.

(3) *The central quasi-character of r_θ is $\chi\theta$ on $Z(G_+) \cong F^\times$.*

Proof.

(1) By Proposition 25.8, r and $\rho(h)$ commute for $h \in E^1$.

(2) Let $H \subseteq G_+$ be the subgroup of matrices of the form $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ with $a \in F_+$. The formula clearly defines a continuous representation of H on $S(E)$. Note that $G_+ = \mathrm{SL}_2(F) \rtimes H$, so to show that this is compatible with the representation of $\mathrm{SL}_2(F)$, we need to check
$$\left(r\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) = r_\theta \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} r(g) r_\theta \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix},$$
this follows again from Proposition 25.8.

(3) Using $N_{E/F} a = a^2$, we have
$$\begin{aligned} r_\theta \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \Phi(x) &= r_\theta \left(\begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}\right)\Phi(x) \\ &= |a|^{-1}\chi(a) r_\theta \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}\Phi(a^{-1}x) = \chi(a)\theta(a)\Phi(x). \end{aligned}$$

□

We now get to $G = \mathrm{GL}_2(F)$. We let
$$\pi = \pi(\theta,\psi) := \mathrm{Ind}_{G_+}^G r_{\theta,\psi}$$
be the induction of r_θ to G. We finally got a representation of G! Note that both G, G_+ are unimodular, so we don't need to worry about modular quasi-characters in the induction. If ψ' is another non-trivial character of F, then $\pi(\theta,\psi) \cong \pi(\theta,\psi')$.

We say that θ is regular if it does not factor through $N_{E/F} : E^\times \to F^\times$.

Now assume that F is nonarchimedean.

Theorem 25.10 ([JL70, Theorem 4.6])**.** [a]

(1) *$r_{\theta,\psi}$ is an irreducible admissible representation of G_+.*

(2) *$\pi = \pi(\theta,\psi)$ is an irreducible admissible representation of G.*

(3) If θ is regular, then $\pi(\theta,\psi)$ is supercuspidal.

(4) If $\theta = \chi_0 \circ N_{E/F}$ for some quasi-character χ_0 of $F^\times$, then $\pi(\theta,\psi) \cong \mathcal{B}(\chi_0, \chi_0\chi)$.

[a] Note in [JL70], $\pi(\theta,\psi)$ denotes the representation $r_{\theta,\psi}$ of G_+.

In the last statement, χ is the character of $F^\times$ corresponding to E/F.

Proof. For admissibility, see [JL70] or [Bum97], I am too lazy for that right now. Consider the map $A : S(E,\theta) \to C^\infty(F_+)$, given by $A : \Phi \mapsto \varphi_\Phi$, where

$$\varphi_\Phi(a) = \theta(h)\, |h|_E^{1/2}\, \Phi(h)$$

where $a = N_{E/F}(h)$. A is injective. Conversely, define the map $B : C_c^\infty(F_+) \to S(E,\theta)$ by $B : \varphi \mapsto \Phi_\varphi$ where

$$\Phi_\varphi(h) = \theta(h)^{-1}\, |h|_E^{-1/2}\, \varphi(h).$$

Then $A \circ B = \mathrm{id}_{C_c^\infty(F_+)}$, so $V_+ = \operatorname{Ran} A \supseteq C_c^\infty(F')$. If θ is nontrivial on E^1 (i.e. is regular), then every $\Phi \in S(E,\theta)$ vanishes at 0, and therefore A maps into $C_c^\infty(F_+)$. Let $M_+ = G_+ \cap M$, i.e. it is the group of matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $a \in F_+, b \in F$. If $y = N_{E/F}(z)$ and $a = N_{E/F}(h)$, we have

$$\begin{aligned}
A\left(\pi\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\Phi\right)(a) &= \theta(h)\, |h|_E^{1/2} \left(\pi\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\Phi\right)(h) \\
&= |z|_E^{1/2}\, \theta(z)\theta(h)\, |h|_E^{1/2}\, \Phi(hz) \\
&= A(\Phi)(ay).
\end{aligned}$$

Similarly

$$\begin{aligned}
A\left(\pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\Phi\right)(a) &= \theta(h)\, |h|_E^{1/2} \left(\pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\Phi\right)(h) \\
&= \psi(xN_{E/F}(h))\theta(h)\, |h|_E^{1/2}\, \Phi(h) \\
&= \psi(xa)A(\Phi)(a).
\end{aligned}$$

Assume θ is nontrivial on E^1, i.e. θ is regular. Hence if we define a representation ξ_ψ^+ of M_+ on $C_c^\infty(F_+)$ by

$$\begin{aligned}
\pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\phi(x) &= \phi(ax), \\
\pi\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right)\phi(x) &= \psi(bx)\phi(x),
\end{aligned}$$

then $A : S(E,\theta) \to C^\infty(F_+)$ is a map of M_+ representations. Let ξ_ψ denote the representation of M on $C_c^\infty(F^\times)$.

Lemma. *ξ_ψ is the induction of ξ_ψ^+. In particular, ξ_ψ^+ is irreducible.*

Proof. For the first part if $\widetilde{\varphi} \in \operatorname{Ind}_{M_+}^M \xi_\psi^+$, then associate to it the function $\varphi \in C_c^\infty(F^\times)$ given by $\varphi(a) = \left(\widetilde{\varphi}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)(1)$. This gives an isomorphism $\operatorname{Ind}_{M_+}^M \xi_\psi^+ \to \xi_\psi$. The second part follows since by Proposition 12.2, ξ_ψ is irreducible. □

This shows that $r_{\theta,\psi}$ is irreducible. Since $M_+\backslash M \to G_+\backslash G$ is a bijection, we also have[15]

$$\pi(\theta,\psi)|_M = (\operatorname{Ind}_{G_+}^G r_{\theta,\psi})|_M = \operatorname{Ind}_{M_+}^M r_{\theta,\psi}|_{M_+} \cong \operatorname{Ind}_{M_+}^M \xi_\psi^+ \cong \xi_\psi.$$

Therefore $\pi(\theta,\psi)$ is irreducible. The isomorphism $\pi(\theta,\psi) \cong \xi_\psi$ then shows that the Kirillov model of $\pi(\theta,\psi)$ has space $C_c^\infty(F^\times)$, hence $\pi(\theta,\psi)$ is supercuspidal by Theorem 14.2.

The only thing left to do is consider the case when θ is not regular, so assume $\theta = \chi_0 \circ N_{E/F}$ for some quasi-character χ_0 of $F^\times$. Note that then $S(E,\theta)$ simply consists of the functions $\Phi \in S(E)$ that are invariant under translation by elements in E^1. Note under the map A, the elements in in $C_c^\infty(F')$ corresponds exactly to the elements in $S(E,\theta)$ vanishing at 0, hence $C_c^\infty(F_+)$ is of codimension 1 in $V_+ = \operatorname{Ran} A$. Since $C_c^\infty(F_+)$ is irreducible as an M_+-representation, to show irreducibility of $r_{\theta,\psi}$ it suffices to show that $A^{-1}(C_c^\infty(F_+))$ is not G_+-invariant (why? TODO why every nontrival invariant subspace contains $C_c^\infty(F_+)$...). Let $0 \neq \Phi \in S(E,\theta)$ be nonnegative, and $\Phi(0) = 0$. Then

$$(r_{\theta,\psi}(w_1)\Phi)(0) = \gamma(N_{E/F})\widehat{\Phi}(0) = \gamma(N_{E/F}) \int_E \Phi(x)\mathrm{d}x \neq 0,$$

so $A\Phi \in C_c^\infty(F_+)$, but $A(r_{\theta,\psi}(w_1)\Phi) \notin C_c^\infty(F_+)$. This shows that $r_{\theta,\psi}$ is irreducible. As above, we get

$$\pi(\theta,\psi)|_M = (\operatorname{Ind}_{G_+}^G r_{\theta,\psi})|_M = \operatorname{Ind}_{M_+}^M r_{\theta,\psi}|_{M_+} \cong \operatorname{Ind}_{M_+}^M V.$$

In the same way as in the lemma, we may view $\operatorname{Ind}_{M_+}^M V_+$ as a space of functions in $C^\infty(F^\times)$, concretely if $\widetilde{\varphi} \in \operatorname{Ind}_{M_+}^M V_+$, we associate to it the function $\varphi \in C^\infty(F^\times)$, given by $\varphi(a) = \left(\widetilde{\varphi}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)(1)$. Let $V \subseteq C^\infty(F^\times)$ be the space of these functions, so that $\pi(\theta,\psi)|_M \cong V$. $\xi_\psi^+ = C_c^\infty(F_+)$ is of codimension one in V_+, and so $\operatorname{Ind}_{M_+}^M \xi_\psi^+ \cong \xi_\psi$ is of codimension two in V. Since G_+ is open in G, the space V_+ embeds into V as a G_+-representation, and it is easily seen that V_+ generates V as a G-representation. Then [JL70] says that any nontrivial G-invariant subspace U of V must contain $C_c^\infty(F^\times)$ (why? TODO). Since $r_{\theta,\psi}$ is irreducible, U must contain V_+, hence $U = V$ since V_+ generates V over G.

Note that V is the Kirillov model of $\pi(\theta,\psi)$ and $C_c^\infty(F^\times)$ is of codimension 2 in V, hence $\pi(\theta,\psi)$ is a principal series representation. We only need to figure out which one. Let $L_+ : S(E,\theta) \to \mathbb{C}$ be the map given by $L_+(\Phi) = \Phi(0)$. Then it is easily checked that L_+ is a map $r_{\theta,\psi}|_{M_+} \to \delta^{1/2}(\chi_0 \boxtimes \chi_0\chi)$ of $B \cap G_+$ representations. Fix $\epsilon \in F^\times \setminus F_+$. Then we have a direct sum decomposition $\pi(\theta,\psi) = S(E,\theta) \oplus \pi(\theta,\psi)\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} S(E,\theta)$, and we may define $L : \pi(\theta,\psi) \to \mathbb{C}$ by

$$L(\Phi) = L_+(\Phi_1) + |\epsilon|^{1/2}\chi_0(\epsilon)L_+(\Phi_2)^{16}$$

[15] The modular functions of M, M_+ are not trivial, but they coincide, so again we don't worry about them in the induction.

[16] I believe [JL70] forgot the $|\epsilon|^{1/2}$ here.

where $\Phi = \Phi_1 + \pi \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \Phi_2 \in S(E,\theta) \oplus \pi \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} S(E,\theta)$. Then, after identification with V, L is trivial on $C_c^\infty(F^\times)$, and

$$L\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \Phi\right) = \left|\frac{a}{b}\right|^{1/2} \chi_0(a)\chi_0(b)\chi(b)L(\Phi).$$

Indeed, this holds for $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in T \cap G_+$ since it holds for L_+ then, and can be checked for $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}$, and these together generate T. Hence one of the characters of T occuring in the Jacquet module $\pi(\theta,\psi)_N$ is $\delta^{1/2}(\chi_0 \boxtimes \chi_0\chi)$. This implies the claim by Theorem 15.10. $\square$

Next we consider the case $E = F \oplus F$. Note that now $N_{E/F}(E^\times) = F^\times$, so χ is trivial. Let θ be a quasi-character of $E^\times$, so of the form $\theta_1 \boxtimes \theta_2$ with θ_1, θ_2 quasi-characters of $F^\times$. The reason this case is different than the previous one is that now the norm 1 hyperplane is not compact, so there are no non-zero functions in $S(E)$ that satisfy $\Phi(xh) = \theta^{-1}(h)\Phi(x)$ for all $x \in E, h \in E^1$.

Note that in this case there already is another natural action ρ of $\mathrm{SL}_2(F)$ on $L^2(E)$, given by

$$(\rho(g)\Phi)(v) = \Phi(vg),$$

where we view v as a row vector, and vg is the vector matrix product.

Proposition 25.11. *The two action r and ρ of $\mathrm{SL}_2(F)$ on $E = F \oplus F$ are isomorphic. More precisely, if $\Phi \in S(E)$, define the partial Fourier transform $\mathcal{F}_2\Phi \in S(E)$ by*

$$(\mathcal{F}_2\Phi)(v_1, v_2) = \int_F \Phi(v_1, u)\psi(uv_2)\mathrm{d}u,$$

in other words $\mathcal{F}_2\Phi$ is simply the one-dimensional Fourier transform with respect to the second variable, while keeping the first fixed. Then $\mathcal{F}_2 : S(E) \to S(E)$ is an isomorphism which extends to an isometry $L^2(E) \to L^2(E)$, and intertwines r, ρ:

$$\mathcal{F}_2 \circ r(g) = \rho(g) \circ \mathcal{F}_2.$$

Proof. Check $\mathcal{F}_2 \circ r(g) = \rho(g) \circ \mathcal{F}_2$ for the elements $g = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, and w_1. For w_1 use the Fourier inversion formula in the first component. $\square$

Now consider our quasi-character θ of $E^\times$. The action R of $\mathrm{SL}_2(F)$ on $L^2(E)$ is extended to an action of $\mathrm{GL}_2(F)$ via

$$(\rho(g)\Phi)(v) = |\det g|^{1/2}\, \theta_1(\det g)\Phi(vg).$$

We may then extend the action r of $\mathrm{SL}_2(F)$ on $L^2(E)$ similarly to $\mathrm{GL}_2(F)$, so that $\mathcal{F}_2$ remains an intertwining operator:

$$\left(r\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\Phi\right)(v_1, v_2) = |y|^{1/2}\, \theta_1(y)\Phi(yv_1, v_2).$$

Write $\theta_i = |\cdot|^{s_i}\, \xi_i$ where $s_1, s_2 \in \mathbb{C}$ and ξ_i are unitary characters of $F^\times$.

We will exhibit the principal series representation $V_{\theta_1,\theta_2} = \mathcal{B}(\theta_1, \theta_2)$ as a quotient of the representation R on $S(E)$. Define a map $\tau_{\theta_1,\theta_2} : S(E) \to \mathbb{C}$ via

$$\tau_{\theta_1,\theta_2}\Phi = \int_{F^\times} \Phi(0, y)(\theta_1\theta_2^{-1})(y)\,|y|\,\mathrm{d}^\times y.$$

Let $\chi = \theta_1\theta_2^{-1}$ and $\xi = \xi_1\xi_2^{-1}$. This integral converges absolutely if $\mathrm{Re}(s_1 - s_2 + 1) > 0$. It can be analytically continued to all s_1, s_2 except for the values where ξ is unramified and $\xi(\varpi)q^{-s_1+s_2-1} = 1$. One can check that

$$\tau_{\theta_1,\theta_2}\left(R\begin{pmatrix} y_1 & x \\ 0 & y_2 \end{pmatrix}\Phi\right) = \theta_1(y_1)\theta_2(y_2)\left|\frac{y_1}{y_2}\right|^{1/2}\tau_{\theta_1,\theta_2}\Phi,$$

hence the map $T_{\theta_1,\theta_2} : S(E) \to V_{\theta_1,\theta_2}$ defined by

$$(T_{\theta_1,\theta_2}\Phi)(g) = \tau_{\theta_1,\theta_2}(R(g)\Phi).$$

is well-defined, and an intertwining operator. The poles of τ_{θ_1,θ_2} correspond precisely to the case where V_{θ_1,θ_2} is reducible. Clearly, T_{θ_1,θ_2} is non-zero, so away from $\theta_1\theta_2^{-1} = |\cdot|^{-1}$, this shows that $\mathcal{B}(\theta_1, \theta_2)$ is a quotient of the representation R on $S(E)$.

To $\Phi \in S(E)$ associate the function $W_\Phi : G \to \mathbb{C}$, defined by

$$W_\Phi(g) = \int_{F^\times} \theta_1(t)\theta_2(t)^{-1}(r(g)\Phi)(t, t^{-1})\mathrm{d}^\times t.$$

The integral is absolutely convergent without any restriction since the integrand has compact support. Clearly,

$$W_\Phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \psi(x)W_\Phi(g)$$

Hence, we get a map $S(E) \to \mathcal{W}(\psi), \Phi \mapsto W_\Phi$. Also $W_{r(h)\Phi}(g) = W_\Phi(gh)$, so this is an intertwining map. Let $\mathcal{W}$ be the space of functions W_Φ with $\Phi \in S(E)$.

Theorem 25.12. *Let θ_1, θ_2 be quasi-characters of $F^\times$ such that $\theta_1\theta_2^{-1} \neq |\cdot|^{-1}$. Assume* $\mathrm{Re}(s_1 - s_2 + 1) > 0$*. Then the map $S(E) \to \mathcal{W}$, $\Phi \mapsto W_\Phi$, induces an isomorphism $V_{\theta_1,\theta_2} \cong \mathcal{W}$, hence $\mathcal{W}$ is the Whittaker model of $\mathcal{B}(\theta_1, \theta_2)$.*

Proof. TODO. □

Proof of Theorem 24.14. By analytic continuation, there is no loss in assuming that $\mathrm{Re}(s_1-s_2+1) > 0$. Let η be a quasi-character of $F^\times$. Let $\Phi \in S(E)$ be of the form $\Phi = \Phi_1 \otimes \Phi_2$ with $\Phi_1, \Phi_2 \in S(F)$. Let $W = W_\Phi$. Then

$$\begin{aligned} Z(W, \eta, s) &= \int_{F^\times} W\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\eta(y)\,|y|^{s-\frac{1}{2}}\,\mathrm{d}^\times y \\ &= \int_{F^\times}\int_{F^\times} \theta_1(t)\theta_2(t)^{-1}\left(r\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\Phi\right)(t, t^{-1})\mathrm{d}^\times t\,\eta(y)\,|y|^{s-\frac{1}{2}}\,\mathrm{d}^\times y \\ &= \int_{F^\times}\int_{F^\times} \theta_1(t)\theta_2(t)^{-1}\,|y|^{1/2}\,\theta_1(y)\Phi(yt, t^{-1})\mathrm{d}^\times t\,\eta(y)\,|y|^{s-\frac{1}{2}}\,\mathrm{d}^\times y \\ &= \int_{F^\times}\int_{F^\times} \theta_1(t)^{-1}\theta_2(t)\,|y|^{1/2}\,\theta_1(y)\Phi(yt^{-1}, t)\eta(y)\,|y|^{s-\frac{1}{2}}\,\mathrm{d}^\times y\,\mathrm{d}^\times t \end{aligned}$$

$$\overset{tu=y}{=} \int_{F^\times} \int_{F^\times} \theta_1(t)^{-1}\theta_2(t)\, |tu|^{1/2}\, \theta_1(tu)\Phi(u,t)\eta(tu)\, |tu|^{s-\frac{1}{2}}\, \mathrm{d}^\times u\, \mathrm{d}^\times t$$
$$= \int_{F^\times} \theta(u)\eta(u)\, |u|^s\, \Phi_1(u)\mathrm{d}^\times u \int_{F^\times} \theta_2(t)\eta(t)\, |t|^s\, \Phi_2(t)\mathrm{d}^\times t$$
$$= Z_{\mathrm{GL}_1}(\Phi_1, \theta_1\eta, s) Z_{\mathrm{GL}_1}(\Phi_2, \theta_2\eta, s)$$

Here the index GL_1 refers to the local zeta integrals from Section 1. Next note that $\pi(w_1)W = W_{r(w_1)\Phi} = W_{\widehat{\Phi}}$, and $\widehat{\Phi} = \widehat{\Phi}_2 \otimes \widehat{\Phi}_1$. The central quasi-character ω of $\mathcal{B}(\chi_1, \chi_2)$ is $\omega = \theta_1\theta_2$. The same calculation applied to $\widehat{\Phi}$ and the quasi-character $\omega^{-1}\eta^{-1}$ gives

$$Z(\pi(w_1)W, \eta^{-1}\omega^{-1}, s) = Z_{\mathrm{GL}_1}(\widehat{\Phi}_2, \theta_2^{-1}\eta^{-1}, s) Z_{\mathrm{GL}_1}(\widehat{\Phi}_1, \theta_1^{-1}\eta^{-1}, s).$$

□

26. Involution Method

Let G be a *tdlc* group. For $g \in G$ we define left and right translation by $\lambda(g)x = gx, \rho(g)x = xg^{-1}$ for $x, g \in G$. For $f \in C_c^\infty(G)$, we let $\lambda(g)f = f \circ \lambda(g)^{-1}$ and $\rho(g)f = f \circ \rho(g)^{-1}$. For $T \in \mathcal{D}(G) = (C_c^\infty(G))'$ we let $\lambda(g)T = T \circ \lambda(g)^{-1}$ and $\rho(g)T = T \circ \rho(g)^{-1}$.

Now let $G = \mathrm{GL}_2(F)$.

Let $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Define $\iota : G \to G$ by $\iota(g) = wg^T w$. It is an involution and induces involutions of $C_c^\infty(G)$ and $\mathcal{D}(G)$.

Let ψ be a nontrival character of F, viewed as a character of N.

Theorem 26.1 ([Bum97, Theorem 4.4.2])**.** *Suppose $\Delta \in \mathcal{D}(G)$ satisfies $\lambda(u)\Delta = \psi(u)^{-1}\Delta$ and $\rho(u)\Delta = \psi(u)\Delta$ for all $u \in N$. Then Δ is stable under ι.*

Proof. It suffices to prove the following: If $\Delta \in \mathcal{D}(G)$ satisfies $\lambda(u)\Delta = \psi(u)\Delta$ and $\rho(u)\Delta = \psi(u)^{-1}\Delta$ and $\iota(\Delta) = -\Delta$, then $\Delta = 0$. We call a distribution with these properties *invariant* for this proof.

Let $X = BwB$ be the open cell in the Bruhat decomposition. There is an exact sequence

$$0 \to \mathcal{D}(B) \to \mathcal{D}(G) \to \mathcal{D}(X) \to 0$$

We first show the image of Δ in $\mathcal{D}(X)$ is 0. So let $\Delta \in \mathcal{D}(X)$ be invariant, i.e. satisfy the above conditions. X is fibered over $Y = F^\times \times F^\times$ via $p : X \to Y$, where

$$p \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(c, \frac{ad - bc}{c}\right).$$

The fibers of p are the double cosets

$$N \begin{pmatrix} 0 & b_0 \\ c_0 & 0 \end{pmatrix} N$$

and are invariant under the action of $N \times N$ (left and right translation) and ι. It suffices to show that there are no non-zero invariant distributions on each fiber. There is a homeomorphism

$$N \times N \longrightarrow N \begin{pmatrix} 0 & b_0 \\ c_0 & 0 \end{pmatrix} N,$$

$$(u,v) \longmapsto u \begin{pmatrix} 0 & b_0 \\ c_0 & 0 \end{pmatrix} v^{-1}$$

This is $N \times N$-equivariant, where we let $N \times N$ on itself by left translations. Hence if we have an invariant distribution Δ on $N \begin{pmatrix} 0 & b_0 \\ c_0 & 0 \end{pmatrix} N$, we get an invariant distribution D on $N \times N$. But by uniqueness of the twisted Haar measure there is a scalar c such that

$$D(\varphi) = c \int_{N \times N} \psi(u)\psi(v)^{-1}\varphi(u,v)\mathrm{d}u\,\mathrm{d}v,$$

or

$$\Delta(\varphi) = c \int_{N \times N} \psi(u)\psi(v)\varphi\left(u \begin{pmatrix} 0 & b_0 \\ c_0 & 0 \end{pmatrix} v\right) \mathrm{d}u\,\mathrm{d}v.$$

Now note that

$$\begin{aligned}
\Delta(\iota(\varphi)) &= c \int_{N \times N} \psi(u)\psi(v)\varphi\left(wv^T \begin{pmatrix} 0 & b_0 \\ c_0 & 0 \end{pmatrix}^T u^T w\right) \mathrm{d}u\,\mathrm{d}v \\
&= c \int_{N \times N} \psi(u)\psi(v)\varphi\left(wv^T ww \begin{pmatrix} 0 & b_0 \\ c_0 & 0 \end{pmatrix}^T wwu^T w\right) \mathrm{d}u\,\mathrm{d}v \\
&= c \int_{N \times N} \psi(u)\psi(v)\varphi\left(u \begin{pmatrix} 0 & b_0 \\ c_0 & 0 \end{pmatrix} v\right) \mathrm{d}u\,\mathrm{d}v \\
&= \Delta(\varphi)
\end{aligned}$$

Hence $\Delta = 0$.

It remains to show that $\mathcal{D}(B)$ has no non-zero invariant distributions. This time we fiber over $Y = F^\times \times F^\times$, and the map $p : B \to Y$ is given by

$$p \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} (a,d)$$

Again the fibres of this map are stable under $N \times N$ and ι. There is a homeomorphism

$$\begin{aligned}
L : N &\longrightarrow p^{-1}(a,d), \\
u &\longmapsto u\delta,
\end{aligned}$$

where $\delta = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$. Note that this map preserves left multiplication, and $L(\rho(g)v) = \rho(\delta g \delta^{-1})L(v) = \psi(\delta g \delta^{-1})L(v)$ for $g \in N$. Hence, again by uniqueness of twisted Haar measure, there are c_1, c_2 such that

$$\begin{aligned}
\Delta\varphi &= c_1 \int_N \psi(n)\varphi(n\delta)\mathrm{d}n, \\
\Delta\varphi &= c_2 \int_N \psi(\delta^{-1}n\delta)\varphi(n\delta)\mathrm{d}n.
\end{aligned}$$

If $a \neq b$, these two alone already imply $c_1 = c_2 = 0$. Otherrwise, if $a = d$, one can check that these formulas define ι-invariant distributions, hence $\Delta = 0$ in all cases. $\square$

Theorem 26.2 ([Bum97, Theorem 4.2.3])**.** *If a distribution $\Delta \in \mathcal{D}(G)$ is invariant under conjugation, it is invariant under transposition.*

27. GL_2 over a finite field

In this section F will be a finite field with q elements. Let $G = \mathrm{GL}_2(F)$. Similarly, define w, N, T, B etc. as in the local field case. G has $q^2 - 1$ conjugacy classes, so we expect this many irreducible representations.

Definition. *Let (V, π) be an irreducible representation of G. π is called* cuspidal *if the restriction to N does not contain the trivial character.*

Since $V = V^N \oplus V(N)$, this is equivalent to $V(N) = V$, which corresponds to the characterization of supercuspidals in the local field case.

Let χ_1, χ_2 be characters of $F^\times$. As usual they induce a character of T, and by inflation one of B. We have the principal series representation $\mathcal{B}(\chi_1, \chi_2) := \mathrm{Ind}_B^G(\chi_1, \chi_2)$.

Theorem 27.1 ([BH06, 6.3 Proposition, Corollary 1])**.** *We have*

$$\mathrm{Hom}_G(\mathcal{B}(\chi_1, \chi_2), \mathcal{B}(\mu_1, \mu_2)) = \begin{cases} 2 & \text{if } \chi_1 = \chi_2 = \mu_1 = \mu_2, \\ 1 & \text{if } \chi = \mu \text{ or } \chi = \mu^w, \text{ but } \chi_1 \neq \chi_2, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $\mathcal{B}(\chi_1, \chi_2)$ is irreducible if and only if $\chi_1 \neq \chi_2$. If $\chi_1 = \chi_2$, then $\mathcal{B}(\chi_1, \chi_2)$ has length 2 with distinct composition factors.

Proposition 27.2. *An irreducible representation (V, π) is cuspidal if and only if it is not isomorphic to a subrepresentation of $\mathcal{B}(\chi_1, \chi_2)$ for some characters χ_1, χ_2 of $F^\times$.*

Proof. Frobenius reciprocity. □

Corollary 27.3. *There are $\frac{1}{2}(q^2 + q) - 1$ many irreducible noncuspidal representations of G up to isomorphism.*

Let E be a quadratic field extension of F. Let θ be a character of $E^\times$. As when discussing the Weil representation, we call θ *regular* if does not factor through $N_{E/F}$. Equivalently, $\theta^q \neq \theta$. By choosing a basis of E/F, we may identify $E^\times$ with a subgroup H_E of G. Let ψ be a non-trival character of N. We define θ_ψ on ZN by

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} n \mapsto \theta(a)\psi(n)$$

where $a \in F^\times, n \in N$.

Theorem 27.4 ([BH06, 6.4 Theorem])**.** *Let θ be a regular character of $E^\times$, and ψ a non-trivial character of N. Then there is a cuspidal irreducible representation π_θ of G with character*

$$\pi_\theta = \mathrm{Ind}_{ZN}^G \theta_\psi - \mathrm{Ind}_{H_E}^G \theta.$$

Moreover, $\dim \pi_\theta = q - 1$. If θ_1, θ_2 are both regular characters of $E^\times$, then $\pi_{\theta_1} \cong \pi_{\theta_2}$ if and only if $\theta_1 = \theta_2$ or $\theta_1 = \theta_2^q$.

Finally, every irreducible cupsidal representation G is obtained in this way.

Part 4. Global Theory

28. Classical Modular Forms

$\mathfrak{h}$ denotes the upper half plane. For a function $f : \mathfrak{h} \to \mathbb{C}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})^+$, $k \in \mathbb{Z}$ we define $f|_k[\gamma]$ by

$$f|_k[\gamma](z) = \det(\gamma)^{k/2}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \det(\gamma)^{k/2} j(\gamma, z)^{-k} f(\gamma z).$$[17]

Here $j(\gamma, z) = cz + d$.

Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Definition. *A* modular form *of weight k for Γ is a holomorphic function $f : \mathfrak{h} \to \mathbb{C}$ such that*

- *$f|_k[\gamma] = f$ for $\gamma \in \Gamma$, or explicitly $f(\gamma z) = (cz+d)^k f(z)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $z \in \mathfrak{h}$,*
- *f is holomorphic at the cusps of Γ.*

If in addition f vanishes at every cups, f is called cuspidal *or a* cusp form*. The space of modular forms (resp. cuspidal modular forms) of weight k for Γ is denoted $M_k(\Gamma)$ (resp. $S_k(\Gamma)$).*

A function f satisfying the first condition is holomorphic (resp. vanishes) at the cusps of Γ if $f|_k[\gamma](z)$ is bounded (resp. goes to 0) as $\operatorname{Im} z \to \infty$ for all $\gamma \in \mathrm{GL}_2(\mathbb{R})^+$.

Since $\Gamma_1(N)$ is a normal subgroup of $\Gamma_0(N)$, $\Gamma_0(N)$ acts on $M_k(\Gamma_1(N))$, and $S_k(\Gamma_1(N))$ is preserved under the action. Since by definition the action of $\Gamma_1(N) \subseteq \Gamma_0(N)$ is trivial on these spaces, the quotient $\Gamma_0(N)/\Gamma_1(N)$ acts on them. We have $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^\times$ via $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d$. If χ is a character of $(\mathbb{Z}/N\mathbb{Z})^\times$, then $M_k(N, \chi) := M_k(\Gamma_0(N), \chi)$ (resp. $S_k(N, \chi) := S_k(\Gamma_0(N), \chi)$) denotes the subspace of $M_k(\Gamma_1(N))$ (resp. $S_k(\Gamma_1(N))$) on which $(\mathbb{Z}/N\mathbb{Z})^\times$ acts via χ, explicitly it consists of those functions f satisfying $f|_k[\gamma] = \chi(\gamma) f$ for all $\gamma \in \Gamma_0(N)$, or

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. We have

$$M_k(\Gamma_1(N)) = \bigoplus_{\chi \in (\Gamma_0 \widehat{(N)/\Gamma_1}(N))} M_k(N, \chi), \quad S_k(\Gamma_1(N)) = \bigoplus_{\chi \in (\Gamma_0 \widehat{(N)/\Gamma_1}(N))} S_k(N, \chi).$$

Notice that $M_k(N, \chi) = 0$, unless $\chi(-1) = (-1)^k$. Also $M_k(N, \chi_0) = M(\Gamma_1(N))$ for the trivial character χ_0.

Fix k. Let $f, g : \mathfrak{h} \to \mathbb{C}$ be meromorphic functions. We define the differential form

$$\omega(f, g) = f(z)\overline{g(z)} y^k \frac{\mathrm{d}x \wedge \mathrm{d}y}{y^2}.$$

[17] In [DS05] and some other sources the exponent of $\det \gamma$ is $k - 1$ instead of $k/2$.

Proposition 28.1. *For $\alpha \in \mathrm{GL}_2(\mathbb{R})^+$, we have*
$$\alpha^*\omega(f,g) = \omega(f|_k[\alpha], g|_k[\alpha]).$$

Proof. Straightforward computation. □

No suppose that f, g are weight k modular forms for some (not necessarily the same) congruence subgroups such that at least one of them is a cusp form. Then f, g are modular forms for $\Gamma(N)$ for some N. By the proposition we have $\gamma^*\omega(f,g) = \omega(f|_k[\gamma], g|_k[\gamma]) = \omega(f,g))$ for $\gamma \in \Gamma(N)$, hence $\omega(f,g)$ descends to $\Gamma(N)\backslash\mathfrak{h}$. We define the *Petersson inner product* of f, g by
$$\langle f, g\rangle = \frac{1}{[\Gamma(1):\Gamma(N)]}\int_{\Gamma(N)\backslash\mathfrak{h}} \omega(f,g) = \frac{1}{[\Gamma(1):\Gamma(N)]}\int_{\Gamma(N)\backslash\mathfrak{h}} f(z)\overline{g(z)}y^k\,\frac{\mathrm{d}x\wedge\mathrm{d}y}{y^2},$$
This value is independent of the choice of N. It gives an inner product on $S_k(\Gamma)$.

Proposition 28.2. *Let $f, g \in M_k(\Gamma)$ such that at least one of f, g is cuspidal. Then for $\alpha \in \mathrm{GL}_2(\mathbb{Q})$, the slash operator is unitary in the following sense:*
$$\langle f, g\rangle = \langle f|_k[\alpha], g|_k[\alpha]\rangle.$$

Proof. $f|_k[\alpha], g|_k[\alpha]$ are modular for $\alpha^{-1}\Gamma\alpha\cap\Gamma$, so
$$\begin{aligned}\langle f|_k[\alpha], g|_k[\alpha]\rangle &= \frac{1}{[\Gamma(1):\alpha^{-1}\Gamma\alpha\cap\Gamma]}\int_{\alpha^{-1}\Gamma\alpha\cap\Gamma\backslash\mathfrak{h}} \omega(f|_k[\alpha], g|_k[\alpha])\\ &= \frac{1}{[\Gamma(1):\alpha^{-1}\Gamma\alpha\cap\Gamma]}\int_{\alpha^{-1}\Gamma\alpha\cap\Gamma\backslash\mathfrak{h}} \alpha^*\omega(f,g).\end{aligned}$$
The last equality is by Proposition 28.1. Now $x \mapsto \alpha x$ gives a diffeomorphism $\alpha^{-1}\Gamma\alpha\cap\Gamma\backslash\mathfrak{h} \to \Gamma\cap\alpha\Gamma\alpha^{-1}\backslash\mathfrak{h}$. Hence
$$\begin{aligned}\langle f|_k[\alpha], g|_k[\alpha]\rangle &= \frac{1}{[\Gamma(1):\alpha^{-1}\Gamma\alpha\cap\Gamma]}\int_{\alpha^{-1}\Gamma\alpha\cap\Gamma\backslash\mathfrak{h}} \alpha^*\omega(f,g)\\ &= \frac{1}{[\Gamma(1):\alpha^{-1}\Gamma\alpha\cap\Gamma]}\int_{\Gamma\cap\alpha\Gamma\alpha^{-1}\backslash\mathfrak{h}} \omega(f,g)\\ &= \frac{1}{[\Gamma(1):\Gamma\cap\alpha\Gamma\alpha^{-1}]}\int_{\Gamma\cap\alpha\Gamma\alpha^{-1}\backslash\mathfrak{h}} \omega(f,g)\\ &= \langle f, g\rangle.\end{aligned}$$
□

Let $f \in M_k(\Gamma_1(N))$ with q-expansion $f = \sum_{n=0}^{\infty} a_n q^n$. We define the L-series of f by
$$L(f,s) = \sum_{n=1}^{\infty} a_n n^{-s}.$$
If $f \in S_k(\Gamma_1(N))$, this is absolutely convergent for $s > 1 + \frac{k}{2}$.

28.1. **Fourier Expansions**

TODO

28.2. **Abstract Hecke Operators**

Let G be a group, Γ_1, Γ_2 commensurable subgroups such that $\Gamma_i \cap g\Gamma_i g^{-1}$ is of finite index in Γ_i for $i = 1, 2$ and all $g \in G$. Let us call (G, Γ_1, Γ_2) a Hecke triple[18] Γ_2 acts via right translation on the right cosets $\Gamma_1 g$. The stabilizer of $\Gamma_1 g$ is $\Gamma_2 \cap g^{-1}\Gamma_1 g$. The assumption implies that this has finite index in Γ_2. Hence the orbit of $\Gamma_1 g$, i.e. the double coset $\Gamma_1 g \Gamma_2$ is a finite union of right cosets $\Gamma_1 g_i$.

If R is a commutative ring, let $\mathcal{H}_R(G, \Gamma_1, \Gamma_2)$ denote the free R-module group with basis given by the double cosets $\Gamma_1 g \Gamma_2$, $g \in G$. Alternatively, $\mathcal{H}_R(G, \Gamma_1, \Gamma_2)$ is the set of functions $f : G \to R$ such that $f(g_1 g g_2) = f(g)$ for $g_1 \in \Gamma_1, g_2 \in \Gamma_2$ and $f(g) \neq 0$ only on finitely many double cosets. Note that $\mathcal{H}_R(G, \Gamma_1, \Gamma_2) = \mathcal{H}_{\mathbb{Z}}(G, \Gamma_1, \Gamma_2) \otimes_{\mathbb{Z}} R$.

Let M be a right G-module and denote by M^Γ the set of elements fixed by Γ. Then for $g \in G$ we define the operator $[\Gamma_1 g \Gamma_2]$ on M^{Γ_1} by

$$m|[\Gamma_1 g \Gamma_2] = \sum_{\gamma \in \Gamma_1 \backslash \Gamma_1 g \Gamma_2} m^\gamma$$

where the sum is over a set of representatives of the cosets. This is well-defined since the sum is finite and does not on the choice of representatives. We extend this to all of $\mathcal{H}(G, \Gamma_1, \Gamma_2)$ by linearity. Note that $[\Gamma_1 g \Gamma_2]$ maps M^{Γ_1} to M^{Γ_2}.

In the following we assume that $\Gamma_1 = \Gamma_2 =: \Gamma$ (though it can also be generalized to the two subgroups case). We equip $\mathcal{H}(G, \Gamma) := \mathcal{H}_{\mathbb{Z}}(G, \Gamma, \Gamma)$ that turns the action on M^Γ into an actual ring action. To do that we choose a "universal" M. Namely, let $M = \mathbb{Z}[\Gamma \backslash G]$. Then we have an isomorphism $\mathcal{H}(G, \Gamma) \cong M^\Gamma$ and the right action of $\mathcal{H}(G, \Gamma)$ on M^Γ defines a product on $\mathcal{H}(G, \Gamma)$. Explicitly, it is given as follows: If $g, h \in G$, write $\Gamma g \Gamma = \coprod_i \Gamma g_i, \Gamma h \Gamma = \coprod_\Gamma h_j$. Then let

$$\Gamma g \Gamma \cdot \Gamma h \Gamma = \sum_{k \in \Gamma \backslash G / \Gamma} c_k \Gamma k \Gamma$$

where $c_k = \#\{(i, j) \mid \Gamma g_i h_j = \Gamma k\}$. This definition is so that the action on M^Γ becomes an action as a ring, i.e. $m|[\Gamma g \Gamma \cdot \Gamma h \Gamma] = m|[\Gamma g \Gamma]|[\Gamma h \Gamma]$. If we view elements of $\mathcal{H}(G, \Gamma)$ as functions on G, then this product is the convolution product:

$$(f_1 * f_2)(g) = \sum_{h \in \Gamma \backslash G} f_1(gh^{-1}) f_2(h) = \sum_{h, k \in \Gamma \backslash G,\, \Gamma g = \Gamma hk} f_1(h) f_2(k).$$

The identity element is the identity coset Γ, or its indicator function $\mathbb{1}_\Gamma$. In the function interpretation the action of $f \in \mathcal{H}(G, \Gamma)$ on an element $m \in M^\Gamma$ is given by

$$m \cdot f = \sum_{g \in \Gamma \backslash G} f(g) m^g.$$

[18]Not sure if this a real term. If $\Gamma_1 = \Gamma_2 =: \Gamma$, then in the literature (G, Γ) is called a Hecke pair.

28.3. **Application to Modular Forms**

We take $G = \mathrm{GL}_2(\mathbb{Q})^+$.

Proposition 28.3. *If Γ is a finite index subgroup of $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$, then (G, Γ) is a Hecke pair.*

We will only apply this to congruence subgroups. Fix k and let

$$\mathcal{M}_k = \bigcup_{\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})} M_k(\Gamma), \quad \mathcal{S}_k = \bigcup_{\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})} S_k(\Gamma)$$

where the unions are taken over all congruence subgroups Γ. Then G acts on both these spaces on the right by $f \mapsto f|_k[\gamma] = f$ for $\gamma \in G$. Note that $M_k(\Gamma) = \mathcal{M}_k^\Gamma$ and $S_k(\Gamma) = \mathcal{S}_k^\Gamma$. Therefore we get an induced action of the Hecke algebra $\mathcal{H}(G, \Gamma)$ on $M_k(\Gamma)$ and $S_k(\Gamma)$, denoted $f \mapsto f|_k[\Gamma g \Gamma]$ for $f \in M_k(\Gamma)$ or $S_k(\Gamma)$, and $g \in G$.

If $g \in G$, we write T_g for the operator $f \mapsto f|_k[\Gamma g \Gamma]$. We write T_g on the right, i.e. $f|_k[\Gamma g \Gamma] = f|_k T_g = f|T_g$.

28.3.1. *Level* 1 *Case.* In this section fix $\Gamma(1) = \Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$. The first step is to find representatives for the double cosets $\Gamma(1) g \Gamma(1)$ for $g \in G = \mathrm{GL}_2(\mathbb{Q})^+$:

Proposition 28.4. *If $g \in G$, then there are unique $d_1, d_2 \in \mathbb{Q}_{>0}$ such that d_2/d_1 is a positive integer and*

$$\Gamma(1) g \Gamma(1) = \Gamma(1) \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \Gamma(1).$$

Corollary 28.5. *$\mathcal{H}(G, \Gamma(1))$ is commutative.*

Because of this we will also write $T_g f$ in place of $f|T_g$.

Corollary 28.6. *Any double coset $\Gamma(1) g \Gamma(1)$ has a common set of left and right coset representatives, i.e. there exist $g_1, \ldots, g_n$ such that $\Gamma(1) g \Gamma(1) = \coprod_{i=1}^n \Gamma(1) g_i = \coprod_{i=1}^n g_i \Gamma(1)$.*

Theorem 28.7. *Let $\alpha \in G$. Then the double coset operator T_α is self-adjoint on $S_k(\Gamma(1))$, i.e.*

$$\langle T_\alpha f, g \rangle = \langle f, T_\alpha g \rangle$$

for all $f, g \in S_k(\Gamma(1))$.

Proof. We may assume that $\alpha = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$ with d_1, d_2 positive rationals and $d_2/d_1 \in \mathbb{Z}$, see Corollary 28.6. Then $\alpha^{-1} = D^{-1} \begin{pmatrix} d_2 & 0 \\ 0 & d_1 \end{pmatrix} = D^{-1} w \alpha w$ where $D = d_1 d_2 I_2$, so $\Gamma(1) \alpha^{-1} \Gamma(1) = \Gamma(1) D \alpha \Gamma(1)$. By Corollary 28.6 there are $\alpha_1, \ldots, \alpha_n$ such that $\Gamma(1) \alpha \Gamma(1) = \coprod_{i=1}^n \Gamma(1) \alpha_i = \coprod_{i=1}^n \alpha_i \Gamma(1)$. Taking inverses gives

$$D^{-1} \Gamma(1) \alpha \Gamma(1) = \Gamma(1) \alpha^{-1} \Gamma(1) = \coprod_{i=1}^n \Gamma(1) \alpha_i^{-1}$$

Then by Proposition 28.2 we have

$$\langle T_\alpha f, g\rangle = \sum_{i=1}^{n} \langle f|_k[\alpha_i], g\rangle = \sum_{i=1}^{n} \langle f, g|_k[\alpha_i^{-1}]\rangle = \langle f, T_{D^{-1}} T_\alpha g\rangle = \langle f, T_\alpha g\rangle.$$

□

Corollary 28.8. *$S_k(\Gamma(1))$ decomposes into a direct sum of irreducible submodules under the action of $\mathcal{H}(G, \Gamma(1))$. On each such submodule the elements of $\mathcal{H}(G, \Gamma(1))$ act as scalars.*

Proof. Combine Theorem 28.7 with the fact that $\mathcal{H}(G, \Gamma(1))$ is commutative. □

Remark. One can also show that the Eisenstein series is an eigenfunction $\mathcal{H}(G, \Gamma(1))$, so the corollary also applies to $M_k(\Gamma(1))$.

We now single out a particular family of Hecke operators.

For $n \geq 1$, let Δ_n be the set of integer 2×2 matrices of determinant n.

We denote by T_n the operator on $M_k(\Gamma(1))$ or $S_k(\Gamma(1))$ given by $f \mapsto f|_k[\Delta_n]$. Note that

$$\Delta_n = \coprod_{d_1 | d_2,\, d_1 d_2 = n} \Gamma(1) \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \Gamma(1),$$

so this makes sense. To compute T_n we would like to find a set of representatives for the right cosets in Δ_n.

Proposition 28.9. *We have*

$$\Delta_n = \coprod_{\substack{a,b,d \in \mathbb{Z}_{\geq 0} \\ ad = n,\, 0 \leq b < d}} \Gamma(1) \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

For $n \geq 1$ let R_n be the element of $\mathcal{H}(G, \Gamma(1))$ corresponding to the double coset $\Gamma(1) \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix} \Gamma(1)$. Note that R_n acts trivially on $M_k(\Gamma(1))$ and $S_k(\Gamma(1))$.

Theorem 28.10. *In $\mathcal{H}(G, \Gamma(1))$ we have the relations*

(*i*) $R_n R_m = R_{nm}$ *for all* $n, m \geq 1$,

(*ii*) $T_m T_n = T_{mn}$ *for all coprime* $m, n \geq 1$,

(*iii*) $T_p T_{p^r} = T_{p^{r+1}} + p R_p T_{p^{r-1}}$ *for all primes* p *and* $r \geq 1$.

Proof. (i) is obvious. For (ii), we have

$$T_n = \sum_{g \in \Gamma(1) \backslash \Delta_n} \mathbb{1}_{\Gamma(1) g},$$

so

$$T_mT_n = \sum_{h\in\Gamma(1)\backslash\Delta_m, g\in\Gamma(1)\backslash\Delta_n} \mathbb{1}_{\Gamma(1)hg}$$

Note that as g runs through a set of representatives for $\Gamma(1)\backslash\Delta_n$, $g\mathbb{Z}^2$ runs through precisely the index n lattices in $\mathbb{Z}^2$. Since n, m are coprime any lattice of index nm has a unique sublattice of index n from which the equality $T_mT_n = T_{mn}$ follows.

The idea for (iii) is similar. We have

$$T_pT_{p^r} = \sum_{h\in\Gamma(1)\backslash\Delta_p, g\in\Gamma(1)\backslash\Delta_{p^r}} \mathbb{1}_{\Gamma(1)hg}.$$

Basically this corresponds to attaching to an index p^{r+1} subgroup Λ of $\mathbb{Z}^2$ an order p subgroup of $\mathbb{Z}^2/\Lambda$. We count how many there are depending on Λ. Suppose Λ is an index p^{r+1} subgroup of $\mathbb{Z}^2$. Then there are unique $0 \le a \le b$ with $a + b = r + 1$ and $\mathbb{Z}^2/\Lambda \cong \mathbb{Z}/p^a \oplus \mathbb{Z}/p^b$. If $a = 0$, there is exactly one order p subgroup, otherwise there are $p+1$ of them. Note that we have $a > 0$ if and only if $\Lambda \subseteq p\mathbb{Z}^2$. If $\Lambda = k\mathbb{Z}^2$, this is the case precisely when $k = pg$ for some $g \in \Delta_{p^{r-1}}$. Hence

$$\begin{aligned}
T_pT_{p^r} &= \sum_{h\in\Gamma(1)\backslash\Delta_p, g\in\Gamma(1)\backslash\Delta_{p^r}} \mathbb{1}_{\Gamma(1)hg} \\
&= \sum_{k\in\Gamma(1)\backslash(\Delta_{p^{r+1}} - p\Delta_{p^{r-1}})} \mathbb{1}_{\Gamma(1)k} + (p+1) \sum_{k\in\Gamma(1)\backslash\Delta_{p^{r-1}}} \mathbb{1}_{\Gamma(1)\begin{pmatrix} p & 0 \\ 0 & p\end{pmatrix}k} \\
&= \sum_{k\in\Gamma(1)\backslash\Delta_{p^{r+1}}} \mathbb{1}_{\Gamma(1)k} + p \sum_{k\in\Gamma(1)\backslash\Delta_{p^{r-1}}} \mathbb{1}_{\Gamma(1)\begin{pmatrix} p & 0 \\ 0 & p\end{pmatrix}k} \\
&= T_{p^{r+1}} + pR_pT_{p^{r-1}}.
\end{aligned}$$

Alternatively this could be done using the explicit coset representatives in Proposition 28.9 as is done in [Bum97, Proposition 4.6.4]. □

Corollary 28.11. *$\mathcal{H}(G, \Gamma(1))$ as a ring is generated by the operators T_p, R_p, R_p^{-1} for primes p.*

Proof. It is not difficult to see that it is generated by all the T_n, R_n, R_n^{-1}. Using (i) and (ii) in the theorem we can reduce to the case of prime powers, and then to primes using (iii). □

We can describe the action of T_m on the Fourier coefficients of a modular form explicitly:

Theorem 28.12. *Let $f \in M_k(\Gamma(1))$ and let $f = \sum_{n\ge0} a_nq^n$ its Fourier expansion. Then $T_mf = \sum_{n\ge0} b_nq^n$ where*

$$b_n = m^{1-\frac{k}{2}} \sum_{l|(n,m)} a_{nm/l^2} l^{k-1}$$

Proof. Straightforward computation using Proposition 28.9. □

Corollary 28.13. *If $f \in M_k(\Gamma(1))$, then*

(1) $a_0(T_m f) = m^{1-\frac{k}{2}}\sigma_{k-1}a_0(f)$,

(2) $a_1(T_m f) = m^{1-\frac{k}{2}}a_m(f)$.

Corollary 28.14. *Suppose $k > 0$ and $0 \neq f \in M_k(\Gamma(1))$ is a simultaneous eigenfunction for all T_n for $n \geq 0$. Normalize the eigenvalues so that $T_m f = m^{1-\frac{k}{2}}\lambda_m f$ for all $m \geq 1$. Let $a_n = a_n(f)$. Then $a_1 \neq 0$. If $a_1 = 1$, we call f* normalized. *In this case $\lambda_m = a_m$ for all $m \geq 1$ and moreover,*

(1) $a_{nm} = a_n a_m$ *for all coprime $n, m \geq 1$.*

(2) $a_p a_{p^r} = a_{p^{r+1}} + p^{k-1}a_{p^{r-1}}$ *for all primes p and $r \geq 1$.*

Proof. Suppose $a_1 = 0$. Then $n^{1-\frac{k}{2}}a_n(f) = a_1(T_n f) = n^{1-\frac{k}{2}}\lambda_n a_1 = 0$, so $a_n = 0$ for all $n \geq 1$, hence $0 \neq f$ is constant which is impossible since $k > 0$. Suppose f is normalized. Then the same computation shows that $\lambda_n = a_n$ for all $n \geq 1$. The other statements follow from Theorem 28.10. □

Corollary 28.15. *$S_k(\Gamma(1))$ is multiplicity free as a $\mathcal{H}(G, \Gamma(1))$-module, i.e. if two eigenforms have the same eigenvalues, they are scalar multiples of each other. Therefore, $S_k(\Gamma(1))$ has a* unique *(up to reordering) basis of normalized eigenforms.*

Proof. Suppose $f, g \in S_k(\Gamma(1))$ are two eigenforms with the same eigenvalues. We may assume they are normalized. Then by the previous corollary we have $a_n(f) = a_n(g)$ for all $n \geq 1$, hence $f = g$ as $a_0(f) = a_0(g) = 0$. □

28.3.2. *Higher Level Case.* Fix an integer $N \geq 1$. The notation here is a bit different from the previous section as I used different sources.

We introduce the *diamond* operator $\langle d_0 \rangle$ for integers d_0 coprime to N. Let

$$X = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \,\Big|\, d \equiv d_0 \bmod N \right\}$$

Let $\gamma \in X$. It is easy to see that we have

$$X = \Gamma_1(N)\gamma = \Gamma_1(N)\gamma\Gamma_1(N).$$

Hence X defines an element of $\mathcal{H}(G, \Gamma_1(N))$ and we have

$$f|_k[X] = f|_k[\gamma],$$

for $f \in M_k(\Gamma_1(N))$. We define $\langle d_0 \rangle f = f|_k[X]$. Note that this isn't really anything new, it is the action of $(\mathbb{Z}/N\mathbb{Z})^\times \cong \Gamma_0(N)/\Gamma_1(N)$ on $M_k(\Gamma_1(N))$.

Let $\Delta_0(N)$ be the set of integer matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{Z})$ such that $\det g > 0$, $c \equiv 0 \bmod N$ and $(a, N) = 1$. Note that $\Delta_0(N)$ is a subsemigroup of $\mathrm{GL}_2(\mathbb{Q})^+$. Even though it is not a group, we can still define $\mathcal{H}(\Delta_0(N), \Gamma_0(N))$ with the same definition as before. Then it is simple to check that this is a subring of $\mathcal{H}(\mathrm{GL}_2(\mathbb{Q})^+, \Gamma_0(N))$.

Proposition 28.16 ([Miy06, Lemma 4.5.2]). *For $\alpha \in \Delta_0(N)$ there exist unique positive integers l, m such that $l \mid m$, $(l, N) = 1$ and*

$$\Gamma_0(N)\alpha\Gamma_0(N) = \Gamma_0(N)\begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix}\Gamma_0(N).$$

Corollary 28.17. *$\mathcal{H}(\Delta_0(N), \Gamma_0(N))$ is commutative.*

Corollary 28.18. *Any double coset $\Gamma_0(N)g\Gamma_0(N)$ has a common set of left and right coset representatives, i.e. there exist $g_1, \dots, g_n$ such that $\Gamma_0(N)g\Gamma_0(N) = \coprod_{i=1}^n \Gamma_0(N)g_i = \coprod_{i=1}^n g_i\Gamma_0(N)$.*

As before if $g \in \Delta_0(N)$, we write T_g for the operator corresponding to the element $\Gamma_0(N)g\Gamma_0(N)$ of $\mathcal{H}(\Delta_0(N), \Gamma_0(N))$. A priori this is only defined on $M_k(\Gamma_0(N))$, but we can extend it to $M_k(N, \chi)$, where χ is a character of $(\mathbb{Z}/N\mathbb{Z})^\times \cong \Gamma_0(N)/\Gamma_1(N)$, as follows. First extend the character χ to $\Delta_0(N) \supseteq \Gamma_0(N)$ by $\chi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \overline{\chi(a)}$. Let $f \in M_k(N, \chi)$ and $g_1, \dots, g_n$ a set of right coset representatives for $\Gamma_0(N)$ in $\Gamma_0(N)g\Gamma_0(N)$. Then:

$$T_g f = \sum_{i=1}^n \overline{\chi(g_i)} f|_k[g_i]$$

It is easy to see that this is well defined, and this defines an action of $\mathcal{H}(\Delta_0(N), \Gamma_0(N))$ on $M_k(N, \chi)$.

As before we are interested in a particular class of operators. For $l \mid m$ and $(l, N) = 1$ let

$$T(l, m) = \Gamma_0(N)\begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix}\Gamma_0(N)$$

and set

$$T_n = T(n) = \sum_{\substack{g \in \Gamma_0(N)\backslash\Delta_0(N)/\Gamma_0(N) \\ \det g = n}} \Gamma_0 g \Gamma_0 = \sum_{\substack{l \mid m, lm = n \\ (l,N)=1}} T(l, m)$$

The second equality holds by Proposition 28.16.

In the notation of the previous chapter we have $T(n, n) = R_n$.

Theorem 28.19. *We have:*

(1) $T_m T_n = T_{mn}$ for all coprime $m, n \geq 1$.

(2) If p is a prime and $e \geq 1$, then $T_p T_{p^e} = \begin{cases} T_{p^{e+1}} + pT(p,p)T_{p^{e-1}} & \text{if } p \nmid N, \\ T_{p^{e+1}} & \text{if } p \mid N. \end{cases}$

Note that $T(p, p)$ acts by multiplication by $\chi(p)$ on $S_k(N, \chi)$. Therefore as operators acting on $S_k(N, \chi)$ we can also write

$$T_p T_{p^e} = T_{p^{e+1}} + p\chi(p)T_{p^{e-1}}.$$

Since we set $\chi(m) = 0$ for $(m, N) > 1$, this also holds for $p \mid N$.

Proof. See [Miy06, Lemma 4.5.7] and [Miy06, Lemma 4.5.8]. □

Corollary 28.20. *$\mathcal{H}(\Delta_0(N), \Gamma_0(N))$ as a ring is generated by the operators $T_p, T(p,p), T_q$ for primes $p \nmid N, q \mid N$.*

Note we don't need to include inverses here as opposed to the level 1 case here we only consider integral matrices (though one could extend everything suitably).

Theorem 28.21 ([Miy06, Theorem 4.5.4])**.** *If $l \mid m$ and $(lm, N) = 1$, the adjoint of $T(l,m)$ on $S_k(N,\chi)$ with respect to the Petersson inner product is given by $\overline{\chi(lm)}T(m,l)$. For $(n,N) = 1$, the adjoint of T_n is $\overline{\chi(n)}T_n$.*

Proof. Exactly like Theorem 28.7. □

Corollary 28.22. *$S_k(N,\chi)$ has a basis of common eigenfunctions for all the operators $T_n, T(l,m)$ where $(n,N) = (lm,N) = 1$ and $l \mid m$.*

Proof. By Theorem 28.19 and Theorem 28.21 these operators generate a commutative algebra of normal operators on $S_k(N,\chi)$ and are hence simultaneously (unitarily) diagonalizable. □

Proposition 28.23 ([Miy06, 4.5.25])**.** *We have*

$$T(l,m) = \coprod_{\substack{ad=lm, 0\le b<d \\ (a,b,d)=l, (a,N)=1}} \Gamma_0(N) \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

$$T_n = \coprod_{\substack{ad=n, 0\le b<d \\ (a,N)=1}} \Gamma_0(N) \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

Hence

$$T_n f = \sum_{\substack{ad=n, 0\le b<d \\ (a,N)=1}} \chi(a) f|_k[\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}] = \sum_{ad=n} \chi(a) \sum_{b=0}^{d-1} f|_k[\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}]$$

$$(T_n f)(z) = n^{k/2} \sum_{ad=n} \chi(a) d^{-k} \sum_{b=0}^{d-1} f\left(\frac{az+b}{d}\right)$$

We dropped the condition $(a,N) = 1$ in the sum since by convention $\chi(a) = 0$ if $(a,N) > 1$. Note that if all prime divisors of n also divide N, then

$$(T_n f)(z) = n^{-k/2} \sum_{b=0}^{n-1} f\left(\frac{z+b}{d}\right)$$

From this explicit description we see that under suitable conditions the Hecke operators are compatible for different N:

Proposition 28.24 ([Miy06, 4.5.10])**.** *Suppose M is a multiple of N. Then we get an induced character χ on $(\mathbb{Z}/M\mathbb{Z})^\times$. Suppose that either $(n, M) = 1$ or that all primes dividing n also divide N. Then the Hecke operator T_n^M on $M_k(M, \chi)$ restricts to the Hecke operator T_n^N on $M_k(N, \chi) \subseteq M_k(N, \chi)$.*

As before we describe the action of T_m on the Fourier coefficients.

Theorem 28.25. *Let $f \in M_k(N, \chi)$ and write $f = \sum_{n\geq 0} a_n q^n$. Then $T_m f = \sum_{n\geq 0} b_n q^n$ where*

$$b_n = m^{1-\frac{k}{2}} \sum_{l|(n,m)} a_{mn/l^2} \chi(l) l^{k-1}.$$

Proof. Straightforward computation using Proposition 28.23. □

Corollary 28.26. *If $f \in M_k(N, \chi)$, then*

(1) $a_0(T_m f) = m^{1-\frac{k}{2}} \left(\sum_{l|m} \chi(l) l^{k-1}\right) a_0(f)$,

(2) $a_1(T_m f) = m^{1-\frac{k}{2}} a_m(f)$.

Proposition 28.27. *Suppose $f \in M_k(N, \chi)$ is an eigenfunction of T_p for primes p in a set M. Normalize the eigenvalues such that $T_p f = p^{1-\frac{k}{2}} \lambda_p f$. Suppose m is an integer only divisible by primes in M. Then f is an eigenfunction of T_m and if we write $T_m f = m^{1-\frac{k}{2}} \lambda_m f$, then $a_m(f) = \lambda_m a_1(f)$ and*

$$L(f, s) = \prod_{p\in M} (1 - \lambda_p p^{-s} + \chi(p) p^{k-1-2s})^{-1} \sum_n{}' a_n(f) n^{-s}$$

where the sum runs over all integers n not divisible by any prime in M.

Proof. If m only has prime factors in M, then T_m is a polynomial in the T_p with $p \in M$ by Theorem 28.19, so f is an eigenfunction of T_m. By the corollary we have

$$m^{1-\frac{k}{2}} \lambda_m a_1(f) = a_1(m^{1-\frac{k}{2}} \lambda_m f) = a_1(T_m f) = m^{1-\frac{k}{2}} a_m(f),$$

hence $a_m(f) = \lambda_m a_1(f)$. More generally, Theorem 28.25 shows that for coprime n, m we have

$$\lambda_m a_n(f) = a_{nm}(f),$$

and therefore

$$L(f, s) = \left(\sum_m{}'' \lambda_m m^{-s}\right) \left(\sum_n{}' a_n(f) n^{-s}\right),$$

where $\sum''$ is taken over all integers m only divisible by primes in M. Now Theorem 28.19 implies that $\lambda_{nm} = \lambda_n \lambda_m$ for coprimes n, m that are only divisible by primes in M. Hence

$$\sum_m{}'' \lambda_m m^{-s} = \prod_{p\in M} \left(\sum_{j=0}^{\infty} \lambda_{p^j} p^{-js}\right).$$

By the same theorem we also have

$$\lambda_p \lambda_{p^e} = \lambda_{p^{e+1}} + p^{1-k}\chi(p)\lambda_{p^{e-1}},$$

which implies

$$\sum_{j=0}^{\infty} \lambda_{p^j} p^{-js} = (1 - \lambda_p p^{-s} + \chi(p)p^{k-1-2s})^{-1}.$$

□

We now define old- and newforms. Let $\alpha = \alpha_d = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$. Note first that

$$\alpha^{-1}\Gamma_0(N)\alpha \supseteq \Gamma_0(dN)$$
$$\alpha^{-1}\Gamma_1(N)\alpha \supseteq \Gamma_1(dN)$$

Therefore, if $f \in M_k(\Gamma_1(N))$, then $f|_k[\alpha] \in M_k(\Gamma_1(dN))$, so $f(dz) = d^{-k/2} f|_k[\alpha](z) \in M_k(\Gamma_1(dN))$. Also $\chi(\alpha g \alpha^{-1}) = \chi(g)$ for $g \in \Gamma_0(dN)$, hence if $f \in M_k(N,\chi)$, then $f|_k[\alpha] \in M_k(dN,\chi)$. Of course this also preserves the cuspidal subspace. If M is a divisor of N, define the map

$$\begin{aligned} i_M : M_k(\Gamma_1(N/M))^2 &\longrightarrow M_k(\Gamma_1(N)) \\ (f,g) &\longmapsto f + g|_k[\alpha_M] \end{aligned}$$

The space $M_k(\Gamma_1(N))^{\text{old}}$ of *oldforms* at level N is the subspace of $M_k(\Gamma_1(N))$ spanned by all the images of the i_M for $M > 1, M \mid N$, i.e.

$$M_k(\Gamma_1(N))^{\text{old}} = \sum_{M>1, M \mid N} i_M(M_k(\Gamma_1(N/M))^2)$$

We also set

$$M_k(N,\chi)^{\text{old}} = M_k(N,\chi) \cap M_k(\Gamma_1(N))^{\text{old}}$$
$$M_k(\Gamma_1(N))^{\text{old}} = S_k(\Gamma_1(N)) \cap M_k(\Gamma_1(N))^{\text{old}}$$
$$S_k(N,\chi)^{\text{old}} = S_k(N,\chi) \cap M_k(\Gamma_1(N))^{\text{old}}$$

28.4. **Some Examples**

We consider the level $N = 1$ case here. Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. For $w, z \in \mathfrak{h}$ we write $w \sim z$, and say that w, z are $\Gamma(1)$-*equivalent*, if $w = gz$ for some $g \in \Gamma(1)$.

Proposition 28.28. $\mathrm{SL}_2(\mathbb{Z})$ *is generated by* S, T.

Let

$$\mathcal{F} = \{z \in \mathfrak{h} \mid \operatorname{Re}\tau \in [-\frac{1}{2}, \frac{1}{2}), |z| \geq 1, (|z| = 1 \implies \operatorname{Re} \leq 0)\}.$$

Proposition 28.29. *Every* $\tau \in \mathfrak{h}$ *is* $\Gamma(1)$-*equivalent to a unique element in* $\mathcal{F}$, *i.e. the map* $\mathcal{F} \to \Gamma(1)\backslash\mathfrak{h}$ *is bijective.*

Corollary 28.30. $\Gamma(1)\backslash\mathfrak{h}$ *has volume* $\frac{\pi}{3}$.

Proof. We have

$$\begin{aligned}\operatorname{vol}(\Gamma(1)\backslash\mathfrak{h}) &= \operatorname{vol}(\mathcal{F})\\ &= \int_{\mathcal{F}} \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}\\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{y=\sqrt{1-x^2}}^{\infty} \frac{1}{y^2}\mathrm{dydx}\\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{y=\sqrt{1-x^2}}^{\infty} \frac{1}{y^2}\mathrm{dydx}\\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\mathrm{dx}\\ &= 2\arcsin(1/2) = \frac{\pi}{3}.\end{aligned}$$

□

For $\tau \in \mathfrak{h}$ let

$$e_\tau = \#\operatorname{Stab}_{\mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}} \tau = \begin{cases} 1 & \text{if } \tau \not\sim \rho, i,\\ 2 & \text{if } \tau \sim i,\\ 3 & \text{if } \tau \sim \rho.\end{cases}$$

Here $\rho = e^{2\pi i/3}$.

Proposition 28.31 (Valence Formula)**.** *Let* $f : \mathfrak{h} \to \mathbb{C}$ *be a meromorphic function such that* $f|_k[g] = f$ *for* $g \in \mathrm{SL}_2(\mathbb{Z})$ *assume* f *is meromorphic at* ∞*. Then*

$$v_\infty(f) + \sum_{\tau \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}} \frac{1}{e_\tau} v_\tau(f) = \frac{k}{12}.$$

For $k \geq 3$ let

$$G_k(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\backslash\{0\}} (m\tau+n)^{-k} = \sum_{\lambda\in\mathbb{Z}\oplus\tau\mathbb{Z}} \lambda^{-k}.$$

This converges absolutely and defines a modular form in $M_k(\Gamma(1))$. Normalize G_k by

$$E_k := \frac{1}{2\zeta(k)} G_k.$$

It follows easily from the valence formula that G_4 (resp. G_6) has no zeros, except a simple one at $\tau = \rho$ (resp. $\tau = i$).

Proposition 28.32. *The Fourier expansion of G_k is*

$$G_k(\tau) = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n.$$

The E_k have rational Fourier coefficients, E_4, E_6 in fact integral ones. More precisely, for even k:

$$E_k = 1 + \frac{(2\pi i)^k}{\zeta(k)(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n = 1 + \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$

For $k = 4, 6$ in particular:

$$E_4 = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n$$

$$E_6 = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n$$

Let

$$\Delta = \frac{E_4^3 - E_6^2}{1728}.$$

Note $\Delta \in S_{12}(\Gamma(1))$. It is not too difficult to see from the above formulas for E_4, E_6 that $\Delta \in q + q^2\mathbb{Z}[\![q]\!]$. Δ has a zero at ∞ and is holomorphic in $\mathfrak{h}$, hence by Proposition 28.31, the order of the zero at ∞ is 1 and Δ is non-vanishing on $\mathfrak{h}$. We immediately get from this:

Proposition 28.33. *Multiplication by Δ induces an isomorphism $M_k(\Gamma(1)) \cong S_{k+12}(\Gamma(1))$.*

Similar considerations with the valence formula give that $M_k(\Gamma(1)) = \mathbb{C}E_k$ for $4 \le k \le 10$, $M_0(\Gamma(1)) = \mathbb{C}$ and $M_k(\Gamma(1)) = 0$ for $k < 0$ or $k = 2$.

Corollary 28.34. *Let $k \ge 0$ be even. Then*

$$\dim M_k(\Gamma(1)) = \begin{cases} \lfloor \frac{k}{12} \rfloor & \text{if } k \equiv 2 \pmod{12}, \\ \lfloor \frac{k}{12} \rfloor + 1 & \text{if } k \not\equiv 2 \pmod{12}, \end{cases}$$

Corollary 28.35. *We have the following relations:*

$$E_4^2 = E_8, \quad E_4E_6 = E_{10}.$$

Proposition 28.36. *The algebra homomorphism $\Phi : \mathbb{C}[x,y] \to \mathcal{M}(\Gamma(1)) = \bigoplus_{k=0}^{\infty} M_k(\Gamma(1))$, sending E_4, E_5 to x, y respectively, is an isomorphism. If $f \in \mathcal{M}(\Gamma(1))$ has integral Fourier coefficients, then $f = p(M_4, M_6)$ for some polynomial $p \in \mathbb{Z}[x,y]$.*

Proof. Suppose by induction $\bigoplus_{k=0}^{K} M_k(\Gamma(1)) \subseteq \operatorname{Im}\Phi$. By the preceeding corollary we may assume $K \ge 10$. If $f \in M_{K+2}(\Gamma(1))$, then either $K+2 = 4k+2$ or $K+2 = 4k$ for some k, accordingly consider $g = f - cE_4^{k-1}E_6$ or $g = f - cE_4^k$ where $c = f(\infty)$. Then $\frac{g}{\Delta} \in S_{K-10}(\Gamma(1)) \subseteq \operatorname{Im}\Phi$, hence g and therefore also $f \in \operatorname{Im}\Phi$. To see that Φ is injective, assume that E_4, E_6 are algebraically dependent. By homogeneity considerations, a non-trivial algebraic dependence relation can be chosen to be homogeneous. If in such a relation we have a term E_4^r, then we may write $E_4^r + E_6 p(E_4, E_6) = 0$ for some polynomial p. But then evaluation at $\tau = i$ to get $E_4(i) = 0$, a contradiction. Hence no pure power of E_4 occurs and we can cancel a factor of E_6 to get an equation of smaller degree. Similarly if a pure power of E_6 occurs.

The last statement follows basically by inspecting the way we proved the surjectivity of Φ, noting that $\Delta^{-1} \in q^{-1}\mathbb{Z}[\![q]\!]$. $\square$

We also introduce the j-function:

$$j = \frac{E_4^3}{\Delta}$$

j is a modular function of weight 0, with a simple pole at ∞.

Proposition 28.37. *j induces a bijection $\Gamma(1)\backslash\mathfrak{h} \to \mathbb{C}$. The field of modular functions of weight 0, level 1, is the function field $\mathbb{C}(j)$.*

The first part follows essentially from

$$-1 + \sum_{\tau \in \Gamma(1)\backslash\mathfrak{h}} e_\tau^{-1} v_\tau(j - z) = 0,$$

for $z \in \mathbb{C}$, which shows that $v_\tau(j - z) > 0$ for exactly one $\tau \in \mathfrak{h}$ up to $\Gamma(1)$-equivalence (using $e_\tau \in \{1, 2, 3\}$). Note that in particular if $v_\tau(j - z) > 1$, then $e_\tau > 1$. Indeed, for $\tau = \rho$ we have $j(\rho) = 0$ and $v_\rho(j) = v_\rho(E_4^3) = 3$. For $\tau = i$ we have $j(i) = 1728$ and $v_i(\rho - 1728) = v_i(E_6^2) = 2$. In particular we see that j induces a covering map

$$\mathfrak{h} \setminus (\Gamma(1)i \cup \Gamma(1)\rho) = \mathfrak{h} \setminus j^{-1}(\{i, \rho\}) \longrightarrow \mathbb{C} \setminus \{0, 1728\}.$$

This easily implies Picard's theorem:

Theorem 28.38 ([Apo90, Theorem 2.10])**.** *If f is an entire function omitting at least 2 values, then f is constant.*

Proof. After rescaling we may assume that f omits the values 0 and 1728. Since $\mathbb{C}$ is simply connected (and pathwise connected, locally pathwise connected, whatever we need), there is a lifting $\widetilde{f} : \mathbb{C} \to \mathfrak{h} \setminus j^{-1}(\{i, \rho\})$, i.e. a map $\widetilde{f}$ such that

$$\begin{tikzcd} & \mathbb{C} \arrow[dl, dashed, "\exists \widetilde{f}"'] \arrow[d, "f"] \\ \mathfrak{h} \setminus j^{-1}(\{i, \rho\}) \arrow[r, "j"] & \mathbb{C} \setminus \{0, 1728\} \end{tikzcd}$$

commutes. But any map $\mathbb{C} \to \mathfrak{h}$ is constant by Liouville's theorem (compose it with a Möbius transformation to get $\mathfrak{h} \cong \mathbb{D}$). $\square$

Note that even though the series defining E_2 doesn't converge absoluteley, we may still consider the q-expansion

$$E_2 = 1 + \frac{4}{B_2} \sum_{n=1}^{\infty} \sigma_1(n) q^n = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n.$$

This defines a holomorphic function in the upper half plane, and clearly $E_2|_2[T] = E_2$. But $E_2|_2[S]$ cannot be E_2, since otherwise $E_2 \in M_2(\Gamma(1)) = 0$. It turns out that E_2 satisfies the following transformation law under S:

Theorem 28.39. *E_2 satisfies:*

$$E_2|_2[S](z) = E_2(z) + \frac{6}{\pi i z},$$

or equivalently

$$E_2\left(-\frac{1}{z}\right) = z^2 E_2(z) + \frac{6z}{\pi i}.$$

We introduce the Dirichlet η-function by

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) = e^{2\pi i\tau/24}\prod_{n=1}^{\infty}(1-e^{2\pi i n\tau}).$$

Theorem 28.40. *η satisfies*

$$\eta\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{i}}\eta(z),$$

for all $z \in \mathfrak{h}$.

Note that z/i only takes on values in the right half plane, so we have a well-defined branch of the squareroot function determined by $\sqrt{1} = 1$.

Theorem 28.39 and Theorem 28.40 are essentially equivalent. Indeed, it is easy to see that $\frac{\mathrm{d}}{\mathrm{d}\tau}\log\eta(z) = \frac{\eta'(z)}{\eta(z)} = \frac{\pi i}{2}E_2(z)$. Hence assuming Theorem 28.39,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\log f\left(-\frac{1}{z}\right) &= z^{-2}\left(\frac{\mathrm{d}}{\mathrm{d}z}\log\eta\right)\left(-\frac{1}{z}\right)\\ &= \frac{\pi i}{12}E_2\left(-\frac{1}{z}\right) = \frac{\pi i}{12}E_2(z) + \frac{1}{2z}\\ &= \left(\frac{\mathrm{d}}{\mathrm{d}z}\log\eta\right)(z) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\log z.\end{aligned}$$

We get $\log\eta\left(-\frac{1}{z}\right) = \log\eta(z) + \frac{1}{2}\log z + C$ for some constant C, or

$$\eta\left(-\frac{1}{z}\right) = z^{1/2}\eta(z)c,$$

for come constant c. Evaluating at $\tau = i$ gives $c = i^{-1/2}$. Conversely, assuming Theorem 28.40, Theorem 28.39 follows by taking logarithmic derivatives.

Corollary 28.41. *We have $\eta^{24} = \Delta$.*

Proof. Let $f = \eta^{24}$. Clearly, $f|_{12}[T] = f$. It follows from Theorem 28.40 that also $f|_{12}[S] = f$. From the definition it is clear that f vanishes at ∞, so this shows $f \in S_{12}(\Gamma(1))$. Since this space is one-dimensional, we must have $f = \Delta$ by examining their leading term. $\square$

It remains to show Theorem 28.39.

Proof 1 of Theorem 28.39. Proof adapted from [Miy06]. Let $f = \frac{1-E_2}{24}$. Let

$$\Lambda(f,s) = \int_0^\infty f(iy) y^s \mathrm{d}^\times y$$

be the Mellin transform of $f(i\cdot)$. Explicitly,

$$\Lambda(f,s) = \sum_{n=1}^\infty \sigma_1(n) \int_0^\infty e^{-2\pi n y} y^s \mathrm{d}^\times y = \Gamma(s)(2\pi)^{-s} \sum_{n=1}^\infty \sigma_1(n) n^{-s} = \Gamma(s)(2\pi)^{-s}\zeta(s)\zeta(s-1)$$

Let $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. Then

$$\Lambda(f,s) = \frac{s-1}{4\pi}\Lambda(s)\Lambda(s-1),$$

and from the functional equatino $\Lambda(s) = \Lambda(1-s)$ for the Zeta function we get

$$\Lambda(f,s) = -\Lambda(f, 2-s).$$

Now use the Mellin inversion formula. □

Proof 2 of Theorem 28.39. Proof from [Ser73]. TODO □

Proof 3 of Theorem 28.39. Proof using the first Kronecker limit formula. Let

$$E(z,s) = \pi^{-s}\Gamma(s)\frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2 - \{(0,0)\}} \frac{(\operatorname{Im} z)^s}{|mz+n|^{2s}}$$

be the *non-holomorphic Eisenstein series.* (TODO do some more about this, convergence etc. in a separate section) Note that

$$E(z,s) = \pi^{-s}\Gamma(s)\zeta(2s) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma(1)} (\operatorname{Im}\gamma z)^s.$$

Clearly, $E(gz,s) = E(z,s)$ for $g \in \Gamma(1)$. Let $\xi(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$ be the completed Zeta function ($= L(|\cdot|^s)$ in the notation of Section 3.5).

Theorem 28.42. *$E(z,s)$ has the Fourier expansion $E(z,s) = \sum_{n=-\infty}^\infty a_n(y,s)e^{2\pi i n x}$ where $z = x+iy$ and*

$$a_n(y,s) = \begin{cases} \xi(2s)y^s + \xi(2(1-s))y^{1-s} & \text{if } n = 0, \\ 2\sqrt{y}\,|n|^{s-\frac{1}{2}}\,\sigma_{1-2s}(|n|)K_{s-\frac{1}{2}}(2\pi\,|n|\,y) & \text{if } n \neq 0. \end{cases}$$

Here K denotes the K-Bessel function, defined by

$$K_s(y) = \frac{1}{2}\int_0^\infty e^{-y(t+t^{-1})/2} t^s \frac{\mathrm{d}t}{t}.$$

Proof. This is a relatively simple computation using the definition of $E(z,s)$ and the functional equation of ξ. □

Theorem 28.43. *$E(z,s)$ satisfies the functional equation $E(z,s) = E(z,1-s)$.*

Proof. This can be proven either directly using two-dimensional theta functions, or by proving that the Fourier coefficients satisfy $a_n(y,s) = a_n(y,1-s)$. □

$E(z,s)$ has a simple pole at $s=1$ coming from the corresponding pole of a_0 there.

Proposition 28.44. *For $z = x+iy$ we have*

$$\lim_{s\to 1}(E(z,s) - a_0(y,s)) = -2\log|\eta(z)| - \frac{\pi y}{6}.$$

Proof. One can prove that $K_{1/2}(y) = \sqrt{\frac{\pi}{2y}}e^{-y}$. We have

$$\lim_{s\to 1}(E(z,s) - a_0(y,s)) = \lim_{s\to 1}\sum_{n\neq 0} 2\sqrt{y}\,|n|^{s-\frac{1}{2}}\,\sigma_{1-2s}(|n|)K_{s-\frac{1}{2}}(2\pi\,|n|\,y)e^{2\pi inx}$$

Due to the locally uniform exponential decay of $K_{s-\frac{1}{2}}(2\pi\,|n|\,y)$ (in k), we may interchange limit and sum to get

$$\lim_{s\to 1}(E(z,s) - a_0(y,s)) = \sum_{n\neq 0} 2\sqrt{y}\,|n|^{\frac{1}{2}}\,\sigma_{-1}(|n|)K_{\frac{1}{2}}(2\pi\,|n|\,y)e^{2\pi inx}.$$

One can prove that $K_{1/2}(y) = \sqrt{\frac{\pi}{2y}}e^{-y}$, hence we get

$$\begin{aligned}
\lim_{s\to 1}(E(z,s) - a_0(y,s)) &= 2\sqrt{y}\sum_{n\neq 0}\sigma_{-1}(|n|)\,|n|^{\frac{1}{2}}\sqrt{\frac{\pi}{4\pi\,|n|\,y}}e^{-2\pi|n|y}e^{2\pi inx}\\
&= \sum_{n\neq 0}\sigma_{-1}(|n|)e^{-2\pi|n|y}e^{2\pi inx}\\
&= \sum_{n=1}^{\infty}\sigma_{-1}(n)(q^n + \overline{q}^n)\\
&= \sum_{n=1}^{\infty}\sum_{d=1}^{\infty}\frac{q^{nd} + \overline{q}^{nd}}{d}\\
&= -2\log\prod_{n=1}^{\infty}|1-q^n|\\
&= -2\log\left|\frac{\eta(s)}{q^{1/24}}\right| = -2\log|\eta(s)| - \frac{\pi y}{6}.
\end{aligned}$$

□

We can now finish the proof. We examine how the left and the right hand side in Proposition 28.44 change under $z \mapsto -\frac{1}{z}$. Note that under $z \mapsto -\frac{1}{z}$, y becomes $\frac{y}{|z|^2}$. Letting $f(z)$ denote the left side, we have

$$\begin{aligned} f(z) - f\left(-\frac{1}{z}\right) &= \lim_{s\to 1}\left(E(z,s) - a_0(y,s) - E\left(-\frac{1}{z}, s\right) + a_0\left(\frac{y}{|z|^2}, s\right)\right) \\ &= \lim_{s\to 1}\left(-a_0(y,s) + a_0\left(\frac{y}{|z|^2}, s\right)\right) \\ &= \lim_{s\to 1}\left(-\xi(2s)y^s + \xi(2s)y^s\,|z|^{-2s} - \xi(2(1-s))y^{1-s} + \xi(2(1-s))y^{1-s}\,|z|^{2s-2}\right) \\ &= \xi(2)y(|z|^{-2} - 1) + y\lim_{s\to 1}\left(\xi(2(1-s))(|z|^{2s-2} - 1)\right) \end{aligned}$$

For the first term use $\zeta(2) = \frac{\pi^2}{6}$ and for the second that ξ has a simple pole with residue -1 at 0, to get

$$f(z) - f\left(-\frac{1}{z}\right) = \frac{\pi}{6}y(|z|^{-2} - 1) + \log|z|\,.$$

Now if g denotes the right hand side in Proposition 28.44, we have

$$g(z) - g\left(-\frac{1}{z}\right) 2\log\left|\frac{\eta\left(-\frac{1}{z}\right)}{\eta(z)}\right| - \frac{\pi y}{6} + \frac{\pi y}{6\,|z|^2}$$

Hence, using $g = f$, we obtain

$$\left|\frac{\eta\left(-\frac{1}{z}\right)}{\eta(z)}\right| = \left|z^{1/2}\right|.$$

This easily implies $\eta\left(-\frac{1}{z}\right) = cz^{1/2}\eta(z)$ for some c with $|c| = 1$, and evaluating at $z = i$ gives $c = i^{-1/2}$, which shows Theorem 28.40. □

Proof 4 of Theorem 28.39. Here we prove Theorem 28.40 directly instead. This is the proof in [Bum97]. We assume Euler's pentagonal number theorem:

$$\prod_{n=1}^{\infty}(1 - q^n) = \sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{n(3n-1)}{2}},$$

which follows from the Jacobi triple product formula. Then we may write

$$\eta(\tau) = \sum_{n\in\mathbb{Z}}(-1)^n q^{(6n+1)^2/24} = \sum_{n=1}^{\infty}\chi(n)q^{n^2/24},$$

where χ is the unique primitive quadratic Dirichlet character mod 12. This is a kind of twisted Theta function and a twisted version of the Poisson summation formula can then be used to show

$$\eta(-1/\tau) = (\tau/i)^{1/2}\eta(\tau).$$

□

29. Classical Automorphic Forms

Let $G = \mathrm{GL}_2(\mathbb{R})^+$.

29.1. **Some Differential Operators**

Let $k \in \mathbb{Z}$. If $z \in \mathfrak{h}$, we write $z = x + iy$ with $x, y \in \mathbb{R}$. On smooth functions on the upper half plane we define the following operators

$$\begin{aligned}
\Delta_k &= -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\frac{\partial}{\partial x} = -y^2\Delta_e + iky\frac{\partial}{\partial x},\\
R_k &= iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{k}{2} = (z - \overline{z})\frac{\partial}{\partial z} + \frac{k}{2},\\
L_k &= -iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{k}{2} = -(z - \overline{z})\frac{\partial}{\partial \overline{z}} - \frac{k}{2}.
\end{aligned}$$

Δ_k is the weight k Laplacian. R_k (resp. L_k) is the Maass raising (resp. lowering) operator.

Let $G = \mathrm{GL}_2(\mathbb{R})^+$. For $g \in G$, there are unique $z \in Z(\mathbb{R}), b \in \mathrm{SL}_2(\mathbb{R}) \cap B(\mathbb{R})$, $k \in \mathrm{SO}(2)$ such that $g = zbk$. Write

$$\begin{aligned}
z &= \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix},\\
b &= \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix},\\
k = k_\theta &:= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.
\end{aligned}$$

Here $u, y \in \mathbb{R}_{>0}, x \in \mathbb{R}, \theta \in \mathbb{R}/2\pi\mathbb{Z}$ are uniquely determined.

We take u, x, y, θ as coordinates on G. Then Haar measure is (up to scaling) given by

$$\mathrm{d}g = \frac{\mathrm{d}u}{u}\frac{\mathrm{d}x\mathrm{d}y}{y^2}\mathrm{d}\theta$$

We define differential operators on G by

$$\begin{aligned}
\Delta &= -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + y\frac{\partial^2}{\partial x \partial \theta},\\
R &= e^{2i\theta}\left(iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2i}\frac{\partial}{\partial \theta}\right),\\
L &= e^{-2i\theta}\left(-iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{1}{2i}\frac{\partial}{\partial \theta}\right).
\end{aligned}$$

30. Generalities on Adele groups

From now on write $G = \mathrm{GL}_2$ for convenience.

Let F be a number field. $\mathbb{A}$ denotes its Adele ring. Recall some notation from Section 2. If v is a place of F, we let $K_v = O(2)$, $U(2)$ or $\mathrm{GL}_2(\mathcal{O}_v)$ depending on whether v is real, complex or p-adic. We set $K = \prod_v K_v$ and $K_\infty = \prod_{v|\infty} K_v, K^\infty = \prod_{v\nmid\infty} K_v = \mathrm{GL}_2(\widehat{\mathcal{O}}_F)$. We also let $\mathfrak{g}_\infty$ be the Lie algebra of $\mathrm{GL}_2(F_\infty)$, equivalently $\mathfrak{gl}_\infty = \prod_{v|\infty} \mathfrak{gl}_2(F_v)$. We view $\mathbb{A}^\times$ embedded as the diagonal in $\mathrm{GL}_2(\mathbb{A})$, under this identification, it is the center of $\mathrm{GL}_2(\mathbb{A})$.

We let $C_c^\infty(\mathrm{GL}_2(\mathbb{A}))$ denote the space spanned by functions of the form $\phi = \bigotimes_v \phi_v$ where $\phi_v \in C_c^\infty(\mathrm{GL}_2(F_v))$ and $\phi_v = \mathbb{1}_{\mathrm{GL}_2(\mathcal{O}_v)}$ for almost all v.

Let $\mathbb{R}_{>0}$ be embedded diagonally in F_∞. Let $Z_{\mathbb{R}_{>0}}$ be the subgroup of $G(\mathbb{A})$ of matrices of the form cI_2 with $c \in \mathbb{R}_{>0}$. Let $G(\mathbb{A})^1$ be the subgroup of matrices g with $|\det g| = 1$. We have

$$Z_{\mathbb{R}_{>0}} G(F)\backslash G(\mathbb{A}) \cong G(F)\backslash G(\mathbb{A})^1.$$

In [GH24], $Z_{\mathbb{R}_{>0}} G(F)\backslash G(\mathbb{A})$ is called the *adelic quotient* of G, and denoted $[G]$. Note since $-I_2 \in G(F)$ we have $Z_{\mathbb{R}} G(F) = Z_{\mathbb{R}_{>0}} G(F)$. We also denote by $G(\mathbb{R})^+$ the subgroup of matrices in $G(\mathbb{R})$ with positive determinant.

In the case $F = \mathbb{Q}$ we define the following groups.

Let $N \geq 1$ be an integer. We define compact subgroups

$$\begin{aligned}
K_0(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) \middle| c \equiv 0 \bmod N \right\}, \\
K_1(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) \middle| c \equiv 0 \bmod N, d \equiv 1 \bmod N \right\}, \\
K(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) \middle| b, c \equiv 0 \bmod N, a, d \equiv 1 \bmod N \right\}.
\end{aligned}$$

We can write $K_0(N)$ (resp. $K_1(N)$, $K(N)$) as the product of the subgroup of matrices in $\mathrm{GL}_2(\mathbb{Z}_p)$ that are upper triangular (resp. upper triangular unimodular, the identity) mod N over all primes p. We have

$$\begin{aligned}
K_0(N) \cap \mathrm{GL}_2(\mathbb{Q})_+ &= \Gamma_0(N), \\
K_1(N) \cap \mathrm{GL}_2(\mathbb{Q})_+ &= \Gamma_1(N), \\
K(N) \cap \mathrm{GL}_2(\mathbb{Q})_+ &= \Gamma(N).
\end{aligned}$$

Let $\mathfrak{m}$ be the cycle $\mathfrak{m} = (\infty)(N)$ of $\mathbb{Q}$. Let

$$U(N) = W_{\mathfrak{m}} = \mathbb{R}_+^\times \times \prod_{p<\infty} W_p(\mathfrak{m}) = \mathbb{R}_+^\times \times \{x \in \widehat{\mathbb{Z}}^\times \mid x \equiv 1 \bmod N\}$$

Then by Section 2.1 we have

$$\mathbb{A}^\times / \mathbb{Q}^\times U(N) \cong C_{\mathfrak{m}} = (\mathbb{Z}/N\mathbb{Z})^\times$$

If χ is a character of $(\mathbb{Z}/N\mathbb{Z})^\times$, we get an induced character ω of $\mathbb{A}^\times$ trivial on $\mathbb{Q}^\times$ via this isomorphism, see also Proposition 2.4. We have $\omega(p_p) = \chi(p)$ for any prime $p \nmid N$ where p_p denotes the idele that is p in the p-adic place and 1 at all other places. We then define a character λ of $K_0(N)$ via

$$\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \omega(d_N) = \prod_{p|N} \omega_p(d_p),$$

where d_N denotes the projection of $d \in \mathbb{A}^\times$ onto $\prod_{p|N} \mathbb{Q}_p$.

30.1. **Strong Approximation and Finiteness**

Let F be a number field.

Theorem 30.1. $[G] = Z_{\mathbb{R}} G(F) \backslash G(\mathbb{A}) \cong G(F) \backslash G(\mathbb{A})^1$ *has finite measure.*

TODO reference.

Theorem 30.2. $\mathrm{SL}_2(F_\infty)\,\mathrm{SL}_2(F)$ *is dense in* $\mathrm{SL}_2(\mathbb{A})$.

Proof. The idea is to apply strong approximation to the additive group $\mathbb{A}$ and then use that the nilpotent radicals in $\mathrm{SL}_2(\mathbb{A})$ (which are isomorphic to $\mathbb{A}$) generate $\mathrm{SL}_2(\mathbb{A})$, see [Hum80, 14.3]. □

Corollary 30.3. $\mathrm{SL}_2(F)$ *is dense in* $\mathrm{SL}_2(\mathbb{A}_{\mathrm{f}})$.

Theorem 30.4. *Let K_0 be an open compact subgroup of $G(\mathbb{A}_{\mathrm{f}})$. Assume that the image of K_0 in $\mathbb{A}_{\mathrm{f}}^\times$ under the determinant map is $\widehat{\mathcal{O}_F}^\times = \prod_{v\nmid\infty} \mathcal{O}_v^\times$. Then the cardinality of*

$$G(F)G(F_\infty)\backslash G(\mathbb{A})/K_0$$

is the the class number of F.

Proof. First note that we have a bijection

$$G(F)\backslash G(\mathbb{A}_{\mathrm{f}})/K_0 \xrightarrow{\sim} G(F)G(F_\infty)\backslash G(\mathbb{A})/K_0,$$

induced by the inclusion $G(\mathbb{A}_{\mathrm{f}}) \to G(\mathbb{A})$. Then consider the map

$$G(F)\backslash G(\mathbb{A}_{\mathrm{f}})/K_0 \xrightarrow{\det} F^\times\backslash \mathbb{A}_{\mathrm{f}}/\widehat{\mathcal{O}_F}^\times.$$

It is clearly surjective, and the assumption on K_0 together with Corollary 30.3 shows it is injective. □

Let $F = \mathbb{Q}$.

Theorem 30.5 ([Bum97, Proposition 3.3.1])**.** *We have $G(\mathbb{A}) = G(\mathbb{Q})G(\mathbb{R})^+K_0(N)$ and the inclusion $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{GL}_2(\mathbb{A})$ induces a bijection*

$$\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{R}) \cong \mathbb{A}^\times G(\mathbb{Q})\backslash G(\mathbb{A})/K_0(N).$$

Proof. TODO □

31. Siegel Sets and Reduction Theory

There are slightly different conventions for the definition of a Siegel set. We choose the following. Let $\omega \subseteq \mathbb{A}$ be a compact subset. For a scalar $t > 0$ we define the *Siegel set* $\mathfrak{S}(\omega, t) \subseteq G(\mathbb{A})$ to be the set consisting of the matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} k$$

where $x \in \omega$, $m_1, m_2 \in \mathbb{A}^\times$ such that $|m_1/m_2| \geq t$, and $k \in K = K_\infty G(\widehat{O})$.

Theorem 31.1 ([Gar18, Corollary 2.2.8])**.** *Let $\omega \subseteq \mathbb{A}$ be compact such that $\mathbb{A} = F + \omega$. Then there is a $t > 0$ such that*

$$G(F)\mathfrak{S}(\omega, t) = G(\mathbb{A}).$$

Bump defines in the case of $F = \mathbb{Q}$ the following Siegel sets: For $c, d > 0$ define $\mathcal{G}_{c,d}$ to be the set of adeles of the form $(g_v)_v$ where g_∞ is of the form

$$\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} k_\infty$$

where $z \in \mathbb{R}^\times, c \leq y, 0 \leq x \leq d, k_\infty K_\infty$, and the finite places are in K_v. Let $\omega = [0, d] \times \widehat{\mathbb{Z}} \subseteq \mathbb{A}_\mathbb{Q}$. Then by Theorem 30.4 it is easy to see that $G(\mathbb{Q})\mathfrak{S}(\omega, c) = G(\mathbb{Q})\mathcal{G}_{c,d}$, so this isn't that different.

Proposition 31.2. *For $d \geq 1$ and $c \leq \frac{\sqrt{3}}{2}$ we have*

$$G(\mathbb{Q})\mathcal{G}_{c,d} = G(\mathbb{A}_\mathbb{Q})$$

Proof. Let $G(\mathbb{Q})g \in G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})$. By Theorem 30.4, we may assume that $g \in G(\mathbb{R}) \times G(\widehat{\mathbb{Z}})$. We may also assume that $\det g_\infty > 0$. Let $\tau = g_\infty \cdot i \in \mathfrak{h}$. By Proposition 28.29 there is a (unique) $z \in \mathcal{F} + \frac{1}{2}$ such that $w = \gamma\tau$ for some $\gamma \in \mathrm{SL}_2(\mathbb{Z}) \subseteq G(\mathbb{Q}) \cap G(\widehat{Z})$. Then writing $w = x+iy$, we have $0 \leq x \leq 1 \leq d$ and $y \geq \frac{\sqrt{3}}{2} \geq c$. Then note $w = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} i$, so $\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} i = \gamma g_\infty i$, hence $\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} = \gamma g_\infty \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} k_\infty$ for some $z > 0$, and $k_\infty \in \mathrm{SO}(2) \subseteq K_\infty$. We get that

$$g_\infty = \gamma^{-1} \begin{pmatrix} z^{-1} & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} k_\infty^{-1},$$

is of the desired form. □

32. Definition of Automorphic Forms and Representations

For $g \in G(\mathbb{A})$ we define $\|g\| = \prod_v \|g_v\|_v$ where v runs over all places of F and

$$\|g_v\|_v = \max\{|(g_v)_{ij}|_v\} \cup \{|\det g_v|_v^{-1}\}.$$

Definition. *Let $\varphi : G(\mathbb{A}) \to \mathbb{C}$ be a function. φ is called*

- ***smooth*** *if for any $g \in G(\mathbb{A})$ there is a neighborhood U of g and a smooth function f on $G(F_\infty)$ such that $\varphi(h) = f(h_\infty)$ for $h \in U$ (so basically φ is locally constant on the finite places and smooth in the usual sense on the infinite places);*
- ***K-finite*** *if the right translates of φ under K generate a finite-dimensional subspace;*
- ***$\mathcal{Z}$-finite*** *if it is smooth and $\{D\varphi : D \in Z(\mathcal{U}(\mathfrak{gl}_2(F_v)))\}$ generates a finite dimensional vector space for every infinite place v.*
- *of **moderate growth** if there are constants C, N such that $|\varphi(g)| < C \|g\|^N$ for all $g \in G(\mathbb{A})$.*

Definition. *An* automorphic form *is a function* $\varphi : G(\mathbb{A}) \to \mathbb{C}$ *such that*

- $\varphi(\gamma g) = \varphi(g)$ *for* $\gamma \in G(F), g \in G(\mathbb{A})$,
- φ *is smooth,*
- φ *is* K*-finite,*
- φ *is* $\mathcal{Z}$*-finite,*
- φ *is of moderate growth.*

Let ω *by a quasi-character of* $\mathbb{A}^\times/F^\times$. *If* φ *additionally satisfies* $\varphi(zg) = \omega(z)\varphi(g)$ *for* $z \in \mathbb{A}^\times, g \in G(\mathbb{A})$*, then* φ *is called an* automorphic form with central quasi-character ω. *The space of automorphic forms (resp. automorphic forms with quasi-character* ω*) is denoted* $\mathcal{A}(G(F)\backslash G(\mathbb{A}))$ *(resp.* $\mathcal{A}(G(F)\backslash G(\mathbb{A}), \omega)$*).*

Definition. *By an algebraic* representation *of* $G(\mathbb{A})$ *we mean a vector space* V *equipped with compatible structures of a* $(\mathfrak{g}_\infty, K_\infty)$*-module and a* $G(\mathbb{A}_\mathrm{f})$*-representation (in the usual sense).*

Definition. *Let* V *be a representation of* $G(\mathbb{A})$ *in this sense.* V *is called*

- ***smooth****, if it is smooth as a* $G(\mathbb{A}_\mathrm{f})$*-representation, i.e. any* $v \in V$ *is fixed by some open compact subgroup of* $G(\mathbb{A}_\mathrm{f})$*;*
- ***admissible****, if it is smooth, every vector is* K*-finite and for any irreducible representation* ρ *of* K*, the* ρ*-isotypic component* $V(\rho)$ *of* V *is finite-dimensional.*

$\mathcal{A}(G(F)\backslash G(\mathbb{A}), \omega)$ becomes a smooth (but not admissible) representation in this sense where $G(\mathbb{A}_\mathrm{f})$ and K_∞ act via right translation, and $\mathfrak{g}_\infty$ via differentiation.

Definition. *An irreducible representation* π *of* $G(\mathbb{A})$ *is called a* constituent *of a representation* W*, if there are invariant subspaces* $U \subseteq V$ *of* W *such that* $\pi \cong V/U$*, i.e. if* π *is a subquotient of* W*.*

An representation of $G(\mathbb{A})$ *is* automorphic *if it is a constituent of* $\mathcal{A}(G(F)\backslash G(\mathbb{A}), \omega)$ *for some* ω*, i.e. if it is isomorphic to an irreducible subquotient of* $\mathcal{A}(G(F)\backslash G(\mathbb{A}), \omega)$*.*

We can also define L^2-automorphic forms and representations:

Definition. *A* unitary Hilbert space representation *of a topological group* G *is a Hilbert space* V *with a homomorphism* $\pi : G \to \mathrm{Aut}(V)$ *such that* $\pi(g)$ *is unitary and for every* $v \in V$*, the map* $g \mapsto \pi(g)v$ *is continuous (i.e.* π *is continuous for the strong operator topology on* $\mathcal{B}(H)$*).*

Definition. *A unitary Hilbert space representation* (V, π) *of a locally compact group* G *is admissible if for some compact subgroup* K *every irreducible representation of* K *occurs with finite multiplicity in* V*.*

If the condition holds for some K, it holds for all compact $K' \supseteq K$, see [Dei12, Lemma 7.5.22].

Definition. *Assume that ω is* unitary. *We let $L^2(G(F)\backslash G(\mathbb{A}),\omega)$ denote the space of measurable functions $\varphi : G(\mathbb{A}) \to \mathbb{C}$ such that*

- $\varphi(zg) = \omega(z)\varphi(g)$ *for $z \in \mathbb{A}^\times$ and a.e. $g \in G(\mathbb{A})$,*
- $\varphi(\gamma g) = \varphi(g)$ *for $\gamma \in G(F)$ and a.e. $g \in G(\mathbb{A})$,*
- $\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} |\varphi(g)|^2 \,\mathrm{d}g < \infty$*, i.e. φ is square integrable modulo the center.*

Note that unlike on $\mathcal{A}(G(F)\backslash G(\mathbb{A}),\omega)$ the full group $G(\mathbb{A})$ acts via right translation on $L^2(G(F)\backslash G(\mathbb{A}),\omega)$. Also note that if ω is the trivial character, then $L^2(G(F)\backslash G(\mathbb{A}),\omega) = L^2(\mathbb{A}^\times G(F)\backslash G(\mathbb{A}))$.

Note that if $\phi \in C_c^\infty(G(\mathbb{A}))$ (or more generally $L^1(G(\mathbb{A}))$ should work), then ϕ acts on $L^2(G(F)\backslash G(\mathbb{A}),\omega)$ via

$$\pi(\phi)f = \int_{G(\mathbb{A})} \phi(h)f(gh)\mathrm{d}h$$

By unfolding we can also write this as $\pi(\phi)f = \int_{Z(\mathbb{A})\backslash G(\mathbb{A})} \phi_\omega(h)f(gh)\mathrm{d}h$, where

$$\phi_\omega(g) = \int_{\mathbb{A}^\times} \phi(zg)\omega(z)\mathrm{d}z.$$

These are the definitions in [Bum97] and [CKM04]. In [GH24] and [Dei12], they instead look at $L^2([G]) = L^2(Z_\mathbb{R} G(F)\backslash G(\mathbb{A}))$. The relationship is as follows. If ω is a character of $G(F)\backslash G(\mathbb{A})$. Then $\omega \otimes (|\cdot|^s \circ \det)$ is trivial on $Z_\mathbb{R}$ for some s. Assume ω is trivial on $Z_\mathbb{R}$. Then $L^2(G(F)\backslash G(\mathbb{A}),\omega) \subseteq L^2([G])$, and in fact:

Proposition 32.1 ([GH24, Lemma D.2.1])**.** *We have*

$$L^2([G]) \cong \widehat{\bigoplus_\omega} L^2(G(F)\backslash G(\mathbb{A}),\omega)$$

where the sum is over the characters of $Z_\mathbb{R} Z(F)\backslash Z(\mathbb{A})$.

Proof. $Z_\mathbb{R} Z(F)\backslash Z(\mathbb{A}) = \mathbb{R}_{>0}F^\times\backslash\mathbb{A}^\times$ is compact. □

Definition. *Let $\varphi \in \mathcal{A}(G(F)\backslash G(\mathbb{A}),\omega)$ (resp. $\varphi \in L^2(G(F)\backslash G(\mathbb{A}),\omega)$). φ is called* cuspidal *if*

$$\int_{F\backslash\mathbb{A}} \varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) \mathrm{d}x = 0$$

for every $g \in G(\mathbb{A})$ (resp. a.e. $g \in G(\mathbb{A})$). The subspace of cuspidal forms is denoted by $\mathcal{A}_0(G(F)\backslash G(\mathcal{A}),\omega)$ resp. $L^2_0(G(F)\backslash G(\mathbb{A}),\omega)$.

Definition. *An* automorphic cuspidal *representation of $G(\mathbb{A})$ is an irreducible subrepresentation of $\mathcal{A}_0(G(F)\backslash G(\mathbb{A}),\omega)$.*

Depending whether we want to condider functions (or representations) in $\mathcal{A}(G(F)\backslash G(\mathbb{A}),\omega)$ or $L^2(G(F)\backslash G(\mathbb{A}),\omega)$ we might call them *algebraic* or L^2 (note this is made up, and not standard terminology).

Given a unitary representation (V, π) of G, we let V_{fin} be the space of K-finite vectors, i.e. the space of vectors whose K-orbit spans a finite-dimensional subspace.

33. The Hecke Algebra

We define the global Hecke algebra $\mathcal{H}$ as the restricted tensor product

$$\mathcal{H} = \bigotimes_v{}' \mathcal{H}_v$$

of certain local Hecke algebras $\mathcal{H}_v$ which we have to define next.

- v nonarchimedean. In this case the Hecke algebra $\mathcal{H}_v$ is as defined in Section 9.1. We $\mathcal{H}_v = C_c^\infty(\mathrm{GL}_2(F_v))$ with convolution as the product. For any open compact subgroup $K \subseteq \mathrm{GL}_2(F_v)$ there is a fundamental idempotent $\xi_K = \frac{1}{\mathrm{vol}(K)} \mathbb{1}_K$. Of particular importance is the case $\xi_v^\circ := \xi_{\mathrm{GL}_2(\mathcal{O}_v)}$.
- v archimedean. Let $K = \mathrm{SO}(2)$ or $\mathrm{U}(2)$, depending on v real or complex, be the standard maximal connected compact subgroup of $\mathrm{GL}_2(F_v)$. $\mathcal{H}_v$ is the algebra of compactly supported distributions on $\mathrm{GL}_2(F_v)$ that have their support contained in K and are K-finite under left and right translation. The operation is convolution. There are two important classes of such distributions: First we have $\mathcal{H}_K$, the space of smooth functions on K. Secondly any $D \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ defines a differential operator $f \mapsto Df(1)$ on $C_c^\infty(\mathrm{GL}_2(F_v))$, supported at the identity. $\mathcal{U}(\mathfrak{k}_\mathbb{C})$ acts on $\mathcal{H}_K$, hence we obtain a homomorphism $\mathcal{H}_K \otimes_{\mathcal{U}(\mathfrak{k}_\mathbb{C})} \mathcal{U}(\mathfrak{g}_\mathbb{C}) \to \mathcal{H}_v$ which turns out to be an isomorphism ([KV16, Corollary 1.71]). For any irreducible representation σ of K there is a fundamental idempotent ξ_σ defined by $\xi_\sigma = \mathrm{vol}(K)^{-1} \dim(\sigma)^{-1} \operatorname{Tr} \sigma(k^{-1})$.

  In [JL70] the archimedean Hecke algebra is defined slightly different: TODO

Now the global Hecke algebra is the restricted tensor product of the local $\mathcal{H}_v$ with respect to the vectors ξ_v° for nonarchimedean v, i.e. it is spanned by tensors of the form $\otimes_v f_v$ with $f_v = \xi_v^\circ$ for almost all v. It is an idempotented algebra, with a collection of fundamental idempotents being given by $\otimes_v \xi_v$ where ξ_v is a fundamental idempotent of $\mathcal{H}_v$ and $\xi_v = \xi_v^\circ$ for almost all v.

A module M of $\mathcal{H}$ is *admissible* if ξM is finite-dimensional for every fundamental idempotent ξ.

Proposition 33.1. *There is a bijection between irreducible admissible algebraic representations of* $\mathrm{GL}_2(\mathbb{A})$ *and simple admissible modules for* $\mathcal{H}$.

34. Tensor Product Theorem

Algebraic Version:

Theorem 34.1 ([Bum97, Theorem 3.3.3])**.** *Let* (V, π) *be an irreducible algebraic admissible representation of* $\mathrm{GL}_2(\mathbb{A})$ *(or equivalently a module for the global Hecke algebra). Then for every place* v *of* F *there exists an irreducible admissible representation* (V_v, π_v) *of* $\mathcal{H}_v$,[a] *with a* K_v*-fixed* *vector ξ_v° for almost places v, such that*

$$\pi \cong \bigotimes_v{}' \pi_v$$

where the dashed tensor product denotes restricted tensor product with respect to the chosen K_v-fixed vectors.

[a] i.e. an admissible $(\mathfrak{g}_v, K_v)$-module for archimedean places and an admissible $\mathrm{GL}_2(F_v)$-module for nonarchimedean places.

Hilbert space version:

Theorem 34.2 ([Dei12, Theorem 7.5.23]). *Let π be an irreducible admissible unitary Hilbert space representation of $\mathrm{GL}_2(\mathbb{A})$. Then for every place v there is an irreducible admissible unitary representation π_v of $\mathrm{GL}_2(F_v)$, such that almost all π_v are unramified (i.e. have $\mathrm{GL}_2(\mathcal{O}_v)$-fixed vectors) and $\pi \cong \widehat{\bigotimes}_v{}' \pi_v$. Here the restricted Tensor product is taken with respect to a fixed choice of normalized spherical vectors at the unramified places.*

35. Discreteness of the Cuspidal Spectrum

Lemma 35.1 ([DE09, Lemma 9.2.7]). *Let G be a locally compact group and (V, π) a unitary Hilbert space representation of G. Suppose there is a Dirac net $(f_j)_j \subseteq L^1(G)$ such that $\pi(f_j)$ is self-adjoint and compact. Then V is a direct sum of irreducible representations with finite multiplicities.*

Note that if G is unimodular, then $\pi(f_j)$ is self-adjoint if $f_j = \overline{f_j(g^{-1})}$.

Now consider $G = \mathrm{GL}_2$ and the representation π of $G(\mathbb{A})$ on $L_0^2(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}), \omega)$.

Theorem 35.2 ([Bum97, Proposition 3.3.3 (a)]). *Let $\phi \in C_c^\infty(G(\mathbb{A}))$. Then there is a constant $C > 0$ such that $\|\pi(\phi)f\|_{G(\mathbb{A}),\infty} \leq C\,\|f\|_2$ for all $f \in L_0^2(G(F)\backslash G(\mathbb{A}), \omega)$.*

Proof. If we let $\phi_\omega(h) = \int_{Z(\mathbb{A})} \phi(zh)\omega(z)\mathrm{d}z$, we have

$$\begin{aligned}
(\pi(\phi))f(g) &= \int_{Z(\mathbb{A})\backslash G(\mathbb{A})} \phi_\omega(h) f(gh)\mathrm{d}h \\
&= \int_{Z(\mathbb{A})\backslash G(\mathbb{A})} \phi_\omega(g^{-1}h) f(h)\mathrm{d}h \\
&= \int_{N(F)Z(\mathbb{A})\backslash G(\mathbb{A})} \sum_{\gamma\in N(F)} \phi_\omega(g^{-1}\gamma h) f(\gamma h)\mathrm{d}h
\end{aligned}$$

Since $N(F) \subseteq G(F)$, we have $f(\gamma h) = f(h)$, hence this is

$$\int_{N(F)Z(\mathbb{A})\backslash G(\mathbb{A})} K(g,h) f(h)\mathrm{d}h,$$

where

$$K(g,h) = \sum_{\gamma \in N(F)} \phi_\omega(g^{-1}\gamma h).$$

Define[19]

$$K_0(g,h) = \int_{\mathbb{A}} \phi_\omega \left(g^{-1} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} h \right) \mathrm{d}x = \int_{\mathbb{A}/F} K\left(g, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} h \right) \mathrm{d}x$$

Then

$$\begin{aligned} \int_{N(F)Z(\mathbb{A})\backslash G(\mathbb{A})} K_0(g,h) f(h) \mathrm{d}h &= \int_{N(F)Z(\mathbb{A})\backslash G(\mathbb{A})} \int_{\mathbb{A}/F} K\left(g, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} h\right) \mathrm{d}x f(h) \mathrm{d}h \\ &= \int_{N(F)Z(\mathbb{A})\backslash G(\mathbb{A})} K(g^{-1}h) \int_{\mathbb{A}/F} f\left(\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} h \right) \mathrm{d}x \, \mathrm{d}h \\ &= 0 \end{aligned}$$

since f is cuspidal. Hence we may write

$$(\pi(\phi)f)(g) = \int_{N(F)Z(\mathbb{A})\backslash G(\mathbb{A})} K'(g,h) f(h) \mathrm{d}h,$$

where

$$K'(g,h) = K(g,h) - K_0(g,h)$$

Let $\Phi_{g,h} : \mathbb{A} \to \mathbb{C}$ be the compactly supported continuous function defined by

$$\Phi_{g,h}(x) = \phi_\omega \left(g^{-1} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} h \right).$$

Then by definition

$$K(g,h) = \sum_{\xi \in F} \Phi_{g,h}(\xi), \quad K_0(g,h) = \widehat{\Phi}_{g,h}(0).$$

Hence, the Poisson summation formula gives

$$K(g,h) = \sum_{\xi \in F} \widehat{\Phi}_{g,h}(\xi),$$

so

$$K'(g,h) = \sum_{\xi \in F^\times} \widehat{\Phi}_{g,h}(\xi),$$

Now we want to bound $\widehat{\Phi}_{g,h}(x)$. Assume $F = \mathbb{Q}$ for simplicity. Let $g \in \mathcal{G}_{c,d}$, so that

$$g = \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} k_g,$$

where $\eta \in \mathbb{R}^\times$, and $0 \leq x \leq c, y \geq d$ and $k_g \in K$. We can write (e.g. Adelic Iwasawa decomposition)

$$h = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix} \begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix} k_h,$$

where $\zeta, v \in \mathbb{A}^\times, u \in \mathbb{A}$ and $k_h \in K$.

[19]Bump takes the first integral over $\mathbb{A}/F$, but this seems weird to me, and isn't compatible with $\widehat{\Phi}_{g,h}(0) = K_0(g,h)$, or the proof in the classical theory.

Then we have

$$\begin{aligned}
\widehat{\Phi}_{g,h}(\xi) &= \int_{\mathbb{A}} \Phi_{g,h}(x)\psi(-\xi t)\mathrm{d}t \\
&= \int_{\mathbb{A}} \phi_\omega \left(k_g^{-1} \begin{pmatrix} \eta^{-1}\zeta & 0 \\ 0 & \eta^{-1}\zeta \end{pmatrix} \begin{pmatrix} y^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t-x+u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} k_h \right) \psi(-\xi t)\mathrm{d}t \\
&= \omega(\zeta^{-1}\eta) \int_{\mathbb{A}} \phi_\omega \left(k_g^{-1} \begin{pmatrix} y^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t-x+u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} k_h \right) \psi(-\xi t)\mathrm{d}t \\
&= \omega(\zeta^{-1}\eta)\psi(\xi(x-u)) \int_{\mathbb{A}} \phi_\omega \left(k_g^{-1} \begin{pmatrix} y^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} k_h \right) \psi(-\xi t)\mathrm{d}t \\
&= \omega(\zeta^{-1}\eta)\psi(\xi(x-u)) \int_{\mathbb{A}} F_{k_g,k_h,y^{-1}v}(y^{-1}t)\psi(-\xi t)\mathrm{d}t \\
&= \omega(\zeta^{-1}\eta)\psi(\xi(x-u))\, |y|\, \widehat{F}_{k_g,k_h,y^{-1}v}(\xi y),
\end{aligned}$$

where

$$F_{k_g,k_h,y}(t) = \phi_\omega \left(k_g^{-1} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} k_h \right).$$

Hence we get

$$|K'(g,h)| = \Big| \sum_{\xi \in F^\times} \widehat{\Phi}_{g,h}(\xi) \Big| \le |y| \sum_{\xi \in F^\times} \left| \widehat{F}_{k_g,k_h,y^{-1}v}(\xi y) \right|$$

Since $K(\operatorname{supp}\phi)K$ is compact, so is $K(\operatorname{supp}\phi)K \cap B(\mathbb{A})$, so there is a compact subset Ω of $\mathbb{A}^\times$ such that $F_{k_g,k_h,y}(t) = 0$ for $y \notin \Omega$. So $F_{k_g,k_h,y^{-1}v}(y)$ is a Schwartz function, as a function of y which vanishes, unless $(k_g, k_h, y^{-1}h)$ lies in the compact set $K \times K \times \Omega$. Hence its Fourier transform is rapidly decreasing and we can get for any $N > 0$ an estimate of the form (TODO Fill in some details here, see [Gar18, 7])

$$|K'(g,h)| \le C_N \, |y|^{-N}.$$

Hence,

$$\begin{aligned}
|(\pi(\phi))f(g)| &\le \int_{N(\mathbb{Q})Z(\mathbb{A})\backslash G(\mathbb{A})} |K'(g,h)|\, |f(h)| \,\mathrm{d}h \\
&\le C_N \, |y|^{-N} \int_{\mathbb{A}/\mathbb{Q}} \int_{y^{-1}v \in \Omega} \int_K \left| f\left(\begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix} k_h \right) \right| \mathrm{d}k_h \, \mathrm{d}^\times v \, \mathrm{d}u \\
&\ll C \, |y|^{-N} \, \|f\|_{L^1(G(\mathbb{Q})\backslash G(\mathbb{A}),\omega)} \le C' y^{-N} \, \|f\|_{L^2(G(\mathbb{Q})\backslash G(\mathbb{A}),\omega)}
\end{aligned}$$

□

Proposition 35.3. *$\pi(\phi)$ is a compact operator on $L^2_0(G(F) \setminus G(\mathbb{A}), \omega)$. In fact, it is Hilbert-Schmidt.*

Proof.

- Proof in [Bum97]. This only shows that $\pi(\phi)$ is compact. By the proposition the image D of the unit ball in $L^2_0(G(\mathbb{F}) \setminus G(\mathbb{A}), \omega)$ under $\pi(\phi)$ is a bounded set (with respect to the uniform norm) of continuous functions. If we can show that X is equicontinuous, then X is precompact by Arzela-Ascoli, and hence it will also be precompact in $L^2_0(G(F)\backslash G(\mathbb{A}), \omega)$ (since $\mathbb{A}^\times G(F)\backslash G(\mathbb{A})$ has finite measure). For equicontinuity the idea is that at the finite places, we have $\phi(gk) = \phi(g)$ for k in some open subgroup of $G(\mathbb{A}_\mathrm{f})$, while at the infinite places we apply the bound from (a) to the derivatives $X\pi(\phi)f$ for $X \in \mathfrak{g}_\infty$.

- Proof in [GH24, Lemma 9.3.3] (adapted to work with central character). Let $x \in G(\mathbb{A})$. Since $\pi(\phi)f$ is a continuous function, we we have a functional $f \mapsto \pi(\phi)f(x)$ which is continuous by the Proposition. Hence there exists $K_{\pi(\phi),x} \in V := L^2_0(G(F)\backslash G(\mathbb{A}), \omega)$ such that

$$\pi(\phi)f(x) = \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} K_{\pi(\phi),x}(y)f(y)\mathrm{d}y,$$

for all $f \in V$. Also $\left\|K_{\pi(\phi),x}\right\|_2$ is the norm of the functional which is bounded by C as in the lemma, which is independent of x. Let $K(x,y) = K_{\pi(\phi),x}(y)$. Then

$$\begin{aligned}\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} |K(x,y)|^2 \,\mathrm{d}y\,\mathrm{d}x &\leq \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \left\|K_{\pi(\phi),x}\right\|^2 \mathrm{d}x \\ &\leq C^2 \int_{\mathbb{Z}(\mathbb{A})G(F)\backslash G(\mathbb{A})} \mathrm{d}x\end{aligned}$$

So $K(x,y)$ is L^2 and is an integral kernel for $\pi(\phi)$, so $\pi(\phi)$ is Hilbert-Schmidt.

□

Corollary 35.4. $\pi(\phi)$ *a trace class operator on* $L^2_0(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}), \omega)$.

Proof. By the Dixmier-Malliavin lemma any $\phi \in C_c^\infty(\mathrm{GL}_2(\mathbb{A}))$ is a finite sum of convolutions $\phi = \sum_i \phi_{i1} * \phi_{i2}$ where $\phi_{i1}, \phi_{i2} \in C_c^\infty(\mathrm{GL}_2(\mathbb{A}))$. By the proposition $\pi(\phi_{i1}), \pi(\phi_{i2})$ are Hilbert-Schmidt, hence their product and then the sum over i is trace class. □

Theorem 35.5 ([Bum97, Theorem 3.3.2])**.** $L^2_0(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}), \omega)$ *decomposes into a (Hilbert) direct sum of irreducible subspaces for* $\mathrm{GL}_2(\mathbb{A})$ *each occuring with finite multiplicities.*

Proof. Combine Lemma 35.1 with Theorem 35.2. □

36. Going from Unitary to Algebraic Representations

Theorem 36.1 ([GH24, 6.6.2])**.** *Let* (V, π) *be an irreducible unitary representation of* $G(\mathbb{A})$*. Then* V_{fin} *is admissible and irreducible. If* $V = V_\infty \widehat{\otimes} V^\infty$ *where* V_∞, V^∞ *are irreducible unitary representations of* $G(F_\infty)$ *and* $G(\mathbb{A}^\infty)$ *respectively*[a]*, then* $V_{\mathrm{fin}} = V_{\infty\mathrm{fin}} \otimes V^\infty_{\mathrm{fin}}$*. If* (W, π) *is another irreducible unitary representation of* $G(\mathbb{A})$ *such that* $V_{\mathrm{fin}} \cong W_{\mathrm{fin}}$*, then* V, W *are unitarily equivalent.*

[a] These always exist and are uniquely determined, see [GH24, Theorem 6.6.1]

Proof. TODO □

Theorem 36.2 ([Bum97, Theorem 3.3.4], [GH24, Theorem 6.6.4]). *Let (V,π) an irreducible unitary subreprepresentation of $L^2_0(\mathrm{GL}_2(F)\setminus\mathrm{GL}_2(\mathbb{A}),\omega)$. Then V_{fin} is an admissible algebraic representation of $\mathrm{GL}_2(\mathbb{A})$.*

Proof. Proof in [GH24] uses previous result. Here is proof in [Bum97]. TODO □

Theorem 36.3 ([GH24, Theorem 6.5.3]). [a] *$\mathcal{A}_0(\mathrm{GL}_2(F)\setminus\mathrm{GL}_2(\mathbb{A}),\omega)$ is a dense subspace of $L^2_0(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}),\omega)$. We have*

$$\mathcal{A}_0(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}),\omega) = L^2_0(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}),\omega)_{\mathrm{fin}}$$

If (V,π) is an irreducible subrepresentation then V_{fin} is a cuspidal algebraic automorphic representation. and

$$\mathcal{A}_0(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}),\omega)(\pi|_{V_{\mathrm{fin}}}) = L^2_0(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}),\omega)(\pi)_{\mathrm{fin}}.$$

The multiplicity of $\pi|_{V_{\mathrm{fin}}}$ in $\mathcal{A}_0(\mathrm{GL}_2(F)\setminus\mathrm{GL}_2(\mathbb{A}),\omega)$ is the same as that of π in $L^2_0(\mathrm{GL}_2(F)\setminus\mathrm{GL}_2(\mathbb{A}),\omega)$. We have

$$\mathcal{A}_0(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}),\omega) = \bigoplus_{\pi}\mathcal{A}_0(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}),\omega)(\pi),$$

the sum ranging over isomorphism classes of cuspidal automorphic representations.

[a] In [GH24] everything is formulated on $[G]$ without the ω, but that doesn't really change a lot, see their Appendix D.

Proof. By Theorem 36.4, $\mathcal{A}_0(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}),\omega) \subseteq L^2_0(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}),\omega)$. TODO □

Theorem 36.4. *Let $\phi \in \mathcal{A}_0(G(F)\backslash G(\mathbb{A}),\omega)$. Then ϕ is bounded.*

37. Adelization of Classical Modular Forms

In this section $F=\mathbb{Q}$. Let $N\geq 1$. Fix a character χ of $(\mathbb{Z}/N\mathbb{Z}^\times)$. We describe how to get adelic automorphic forms from cusp forms in $S_k(N,\chi)$. As in Section 30 we get an induced idelic character $\omega:\mathbb{A}^\times/\mathbb{Q}^\times\to\mathbb{C}^\times$ and a character λ of $K_0(N)$.

We want to lift modular forms to automorphic forms on $\mathrm{GL}_2(\mathbb{A})$. We first lift them to functions on $\mathrm{GL}_2(\mathbb{R})_+$. It seems there are a lot of different conventions on how to do this. Let $f:\mathfrak{h}\to\mathbb{C}$, we define a function $F=F_f:\mathrm{GL}_2(\mathbb{R})_+\to\mathbb{C}$. Fix an integer k.

- [Bum97], [GH11]. We define a new action of $\mathrm{GL}_2(\mathbb{R})^+$ on functions on $\mathfrak{h}$ by

$$\begin{aligned}(f|_k\gamma)(z) &= j(\gamma,\overline{z})^k\,|j(\gamma,z)|^{-k}\,f(\gamma z) = \left(\frac{c\overline{z}+d}{|cz+d|}\right)^k f\left(\frac{az+b}{bz+d}\right)\\ &= j(\gamma,z)^{-k}\,|j(\gamma,z)|^k\,f(\gamma z) = \left(\frac{cz+d}{|cz+d|}\right)^{-k} f\left(\frac{az+b}{bz+d}\right)\end{aligned}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})^+$. If we write $|_k^o$ for the slash operator defined in Section 28, then we have

$$y^{k/2}(f|_k\gamma)(z) = ((y^{-k/2}f)|_k^o\gamma)(z).$$

Then we let

$$F(g) = (f|_k g)(i) = \left(\frac{-ci+d}{|ci+d|}\right)^k f(g\cdot i) = ((y^{-k/2}f)|_k^o[g])(i).$$

- [CKM04], [Gel16]. Define F by
$$F(g) = (\det g)^{-k/2} j(g,i)^k f(g\cdot i).$$

We will go with the notation in [Bum97] for the moment. Let $f \in M_k(N,\chi)$ for some integer $N \geq 1$ and Dirichlet character χ. Let $F = F_{y^{k/2}f}$. Then for $\gamma \in \Gamma_0(N), g \in \mathrm{GL}_2(\mathbb{R})^+$ we have

$$F(\gamma g) = (f|_k^o[\gamma g])(i) = (\chi(\gamma) f|_k^o[g])(i) = \chi(\gamma)F(g).$$

By Theorem 30.5 we have $\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{Q})\,\mathrm{GL}_2(\mathbb{R})^+ K_0(N)$. Write $g \in \mathrm{GL}_2(\mathbb{A})$ as $g = \gamma g_\infty k_0$ with $\gamma \in \mathrm{GL}_2(\mathbb{Q}), g_\infty \in \mathrm{GL}_2(\mathbb{R})^+, k_0 \in K_0(N)$. Then define

$$\phi_f(g) = F(g_\infty)\lambda(k_0).$$

Here λ is as in Section 30.

We need to check that this is independent of the decomposition $g = \gamma g_\infty k_0$. This amounts to the following: If $g'_\infty = \gamma g_\infty k_0$ with $\gamma \in \mathrm{GL}_2(\mathbb{Q}), g_\infty, g'_\infty \in \mathrm{GL}_2(\mathbb{R})^+, k_0 \in K_0(N)$, then

$$F(g'_\infty) = F(g_\infty)\lambda(k_0).$$

Comparing infinite and finite part gives $g'_\infty = \gamma_\infty g_\infty$ and $\gamma_{\mathrm{f}} k_0 = 1$. This shows that $\gamma_\infty \in \Gamma_0(N)$, so

$$F(g'_\infty) = F(g_\infty)\chi(\gamma_\infty),$$

and we have to show $\chi(\gamma_\infty) = \lambda(k_0)$. For a matrix A denote by $d(A)$ the bottom right entry. Then

$$\begin{aligned}\lambda(k_0) &= \lambda(k_0^{-1})^{-1} \\ &= \lambda((1,\gamma,\gamma,\gamma,\dots))^{-1} \\ &= \prod_{p|N} \omega_p(d(\gamma))^{-1}\end{aligned}$$

Since $\omega(d(\gamma)) = 1$, this is

$$= \prod_{p\nmid N} \omega_p(d(\gamma)) = \chi(d(\gamma)) = \chi(\gamma).$$

Hence we get a well defined function $\phi_f : G(\mathbb{A}) \to \mathbb{C}$ in this way.

Proposition 37.1. *If $f \in M_k(N,\chi)$, then $\phi_f \in \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}),\omega)$. If f is cuspidal, then so is ϕ_f.*

Proof. Denote $\phi = \phi_f$.

- $\phi(\gamma g) = \phi(g)$ for $\gamma \in G(\mathbb{Q}), g \in G(\mathbb{A})$. This is by definition.

- ϕ is smooth. Let $g \in G(\mathbb{A})$. Let $K \subseteq K_0(N)$ be a small enough open subgroup such that λ is trivial on K'. Let $U = gG(\mathbb{R})^+K$. Then U is an open neighborhood of g. Let $gh \in U$ and write $g = \gamma g_\infty k_0, h = h_\infty k$ with $\gamma \in G(\mathbb{Q}), g_\infty, h_\infty \in G(\mathbb{R})^+, k_0 \in K_0(N), k \in K'$. Then, using that h_∞ and k_0 commute since they are supported in different places, we have

  $$\phi(gh) = \phi(\gamma g_\infty k_0 h_\infty k) = \phi(\gamma g_\infty h_\infty, k_0 k) = F(g_\infty h_\infty)\lambda(k_0 k) = F((gh)_\infty)\lambda(k_0).$$

  As F is a smooth function on $G(\mathbb{R})^+$, this shows that ϕ is smooth in the sense of Section 32.

- ϕ is K-finite. Let $K' = \mathrm{SO}(2)K_0(N)$. Since K' is of finite index in K, it suffices to prove that ϕ is K'-finite. For $k = k_\infty k_\mathrm{f} \in K'$ we have

  $$\phi(gk) = \phi(\gamma g_\infty k_\infty k_0 k_\mathrm{f}) = F(g_\infty k_\infty)\lambda(k_0 k_\mathrm{f})$$

  Now note that if we let $\widetilde{f} = y^{k/2} f$, then $F(g_\infty k_\infty) = \widetilde{f}|_k[g_\infty k_\infty](i) = j(k_\infty, i)^{-k}\widetilde{f}|_k[g_\infty](i)$ since $k_\infty i = i$ (note we use the letter k in two different ways). Hence $\phi(gk) = j(k_\infty, i)^{-k}\lambda(k_\mathrm{f})\phi(g)$, and ϕ is K'-finite.

- ϕ is $\mathcal{Z}$-finite. TODO (basically $\mathcal{Z}$ generated by $Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the Casimir element which acts as the Laplacian. We have $D_Z = 0$ and ϕ is an eigenfunction of the Laplacian.)

- ϕ is of moderate growth. Somehow need to relate the growth of f with the adlic norm. See [GH11, p. 122] for details.

- $\phi(zg) = \omega(z)\phi(g)$ for $z \in \mathbb{A}^\times, g \in \mathrm{GL}_2(\mathbb{A})$. Write $z = rz_\infty z_\mathrm{f}$ with $r \in \mathbb{Q}^\times, z_\infty \in \mathbb{R}_{>0}, z_\mathrm{f} \in K_0(N)$. Then

  $$\phi(zg) = \phi((r\gamma)(z_\infty g_\infty)(z_\mathrm{f} k_0)) = F(z_\infty g_\infty)\lambda(z_\mathrm{f} k_0) = \phi(g)\lambda(z_\mathrm{f}).$$

  Now

  $$\lambda(z_\mathrm{f}) = \prod_{p \mid N} \omega_p(z_\mathrm{f}) = \omega(z_\mathrm{f}) = \omega(z)$$

Finally we show that if f is cuspidal, so is ϕ. For simplicity assume $N = 1$. Let $g \in \mathrm{GL}_2(\mathbb{A})$. By taking the Iwasawa decomposition at every place, we may write $g = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} k_0$ with $u \in \mathbb{A}, y, r \in \mathbb{A}^\times$ and $k_0 \in K$. Then

$$\int_{\mathbb{Q}\backslash\mathbb{A}} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \mathrm{d}x = \int_{\mathbb{Q}\backslash\mathbb{A}} \phi\left(\begin{pmatrix} 1 & x+u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} k_0\right) \mathrm{d}x = \int_{\mathbb{Q}\backslash\mathbb{A}} \phi\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \mathrm{d}x.$$

In the last step we substituted $x \mapsto x - u$ and used automorphy of ϕ. Let $t \in \mathbb{Q}^\times$ be such that $(t^{-1}y)_p \in K_p$ for all finite primes p. Then we have

$$\begin{aligned} \int_{\mathbb{Q}\backslash\mathbb{A}} \phi\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \mathrm{d}x &= \int_{\mathbb{Q}\backslash\mathbb{A}} \phi\left(\begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) \mathrm{d}x \\ &= \int_{\mathbb{Q}\backslash\mathbb{A}} \phi\begin{pmatrix} t^{-1}y & t^{-1}x \\ 0 & 1 \end{pmatrix} \mathrm{d}x \\ &= \int_{\mathbb{Q}\backslash\mathbb{A}} \phi\begin{pmatrix} t^{-1}y & x \\ 0 & 1 \end{pmatrix} \mathrm{d}x \end{aligned}$$

$$= \int_{[0,1]\times\widehat{\mathbb{Z}}} \phi\left(\begin{pmatrix}(t^{-1}y)_\infty & x_\infty \\ 0 & 1\end{pmatrix}\begin{pmatrix}(t^{-1}y)^\infty & x^\infty \\ 0 & 1\end{pmatrix}\right) \mathrm{d}x$$
$$= \int_{[0,1]\times\widehat{\mathbb{Z}}} \phi\begin{pmatrix}(t^{-1}y)_\infty & x_\infty \\ 0 & 1\end{pmatrix} \mathrm{d}x$$
$$= \int_0^1 \phi\begin{pmatrix}(t^{-1}y)_\infty & x_\infty \\ 0 & 1\end{pmatrix} \mathrm{d}x_\infty$$
$$= \int_0^1 (t^{-1}y)_\infty^{k/2} f((t^{-1}y)_\infty i + x_\infty)\mathrm{d}x_\infty.$$

This last integral is 0 by cuspidality of f. □

Theorem 37.2. *Let $f \in S_k(N,\chi)$ and assume that f is an eigenfunction for the Hecke operators T_p for $p \nmid N$. Then φ_f lies in a unique irreducible constituent of $\mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}),\omega)$.*

Proof. We will give the proof in the next section using the multiplicity one theorem. □

38. Whittaker Models, Fourier Expansions and Multiplicity One

Fix a nontrivial character ψ of $\mathbb{A}/F$. It gives a character of $N(\mathbb{A})$. As in the local case we denote by $\mathcal{W} = \mathcal{W}(\psi)$ the space of "nice" functions $W : G(\mathbb{A}) \to \mathbb{C}$ satisfying

$$W(ng) = \psi(n)W(g)$$

for all $n \in N(\mathbb{A}), g \in G(\mathbb{A})$. Here "nice" means:

- W is smooth.
- W is K-finite.
- W is $\mathcal{Z}$-finite.
- W is of moderate growth.

Then $\mathcal{W}$ is an algebraic representation of $G(\mathbb{A})$.

Let (V,π) be an algebraic irreducible admissible representation of $G(\mathbb{A})$.

Definition. *A Whittaker model for V is a subspace $\mathcal{W} = \mathcal{W}(\pi,\psi) \subseteq \mathcal{W}(\psi)$ closed under the action of $G(\mathbb{A})$, together with an isomorphism $V \to \mathcal{W}, v \mapsto W_v$ of $G(\mathbb{A})$-representations.*

Theorem 38.1 ([Bum97, Theorem 3.5.4])**.** *π has a Whittaker model $\mathcal{W}$ if and only if each π_v has a Whittaker model $\mathcal{W}_v$. In this case $\mathcal{W}$ is unique and consists of finite linear combinations of functions of the form $\bigotimes_v W_v$ where $W_v \in \mathcal{W}_v$ and $W_v = W_v^\circ$ for almost all v.*

The proof in [Bum97] shows that:

Proposition 38.2. *If π has a Whittaker model $\mathcal{W}$, then any $W \in \mathcal{W}$ is rapidly decreasing.*

Let $\varphi \in \mathcal{A}(G(F)\backslash G(\mathbb{A}),\omega)$. For fixed $g \in G(\mathbb{A})$ consider $f : \mathbb{A} \to \mathbb{C}$, defined by

$$f(x) = \varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right).$$

Since φ is invariant under left translation by $G(F)$, f is periodic, i.e. $f(x+\alpha) = f(x)$ for $x \in \mathbb{A}, \alpha \in F$. F is discrete in $\mathbb{A}$ and $F \cong \widehat{A/F}$ via $\xi \mapsto \psi_\xi$, see the beginning of Section 3.1. Therefore f has a Fourier expansion

$$f(x) = \sum_{\xi \in F} \widehat{f}(\xi)\psi(\xi x)$$

where

$$\widehat{f}(\xi) = \int_{\mathbb{A}/F} f(x)\psi(-\xi x)\mathrm{d}x = \int_{\mathbb{A}/F} \varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right)\psi(-\xi x)\mathrm{d}x.$$

f is smooth since φ is. This implies that the Fourier series converges absolutely. Now assume that f is a cusp form. Then by definition $\widehat{f}(0) = 0$, so we can assume $\xi \neq 0$. In this case we can change variables using $|\xi| = 1$ and get

$$\begin{aligned}
\widehat{f}(\xi) &= \int_{\mathbb{A}/F} \varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right)\psi(-\xi x)\mathrm{d}x\\
&= \int_{\mathbb{A}/F} \varphi\left(\begin{pmatrix}1 & \xi^{-1}x\\ 0 & 1\end{pmatrix} g\right)\psi(-x)\mathrm{d}x\\
&= \int_{\mathbb{A}/F} \varphi\left(\begin{pmatrix}\xi^{-1} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix} g\right)\psi(-x)\mathrm{d}x\\
&= \int_{\mathbb{A}/F} \varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix} g\right)\psi(-x)\mathrm{d}x
\end{aligned}$$

If we let

$$W_\varphi(g) = \int_{\mathbb{A}/F} \varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right)\psi(-x)\mathrm{d}x,$$

then we see that

$$\varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) = \sum_{\xi \in F^\times} W_\varphi\left(\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix} g\right)\psi(\xi x).$$

We may substitute $x = 0$ to get

$$\varphi(g) = \sum_{\xi \in F^\times} W_\varphi\left(\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix} g\right).$$

We set $\widehat{\varphi}_\xi(g) = W_\varphi\left(\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix} g\right)$.

The important property of the functions W_φ is that they are Whittaker functions:

Theorem 38.3. *Let (V,π) be an algebraic irreducible cuspidal representation of $G(\mathbb{A})$. Then the map $V \to \mathcal{W}(\psi)$ given by $\varphi \mapsto W_\varphi$ is a Whittaker model of π.*

Proof. We have to prove that $W = W_\varphi$ is indeed a Whittaker function, that $\varphi \mapsto W_\varphi$ is injective and equivariant for the action of $G(\mathbb{A})$.

- $W(ng) = \psi(n)W(g)$ is immediate from a change of variables in the integral defining W.
- W smooth follows from φ being smooth.
- W is K-finite because φ is.
- W is $\mathcal{Z}$-finite because φ is.
- W is of moderate growth because φ is.

Therefore $W_\varphi \in \mathcal{W}(\psi)$. That the map is equivariant for the action is easy. If $W_\varphi = 0$, then get $\varphi = 0$ from the Fourier expansion $\varphi(g) = \sum_{\xi \in F^\times} W_\varphi \left(\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} g \right)$. □

As a consequence we get:

Theorem 38.4 (Multiplicity One). *Let (V, π) be an irreducible admissible representation of $G(\mathbb{A})$. Then its multiplicity in $\mathcal{A}_0(G(F)\backslash G(\mathbb{A}), \omega)$ is at most one.*

Proof. Suppose there are two subrepresentations $V_1, V_2 \subseteq \mathcal{A}_0(G(F)\backslash G(\mathbb{A}), \omega)$, both isomorphic to V. By Theorem 38.1, V_1, V_2 have the same Whittaker model $\mathcal{W}$. But we can reconstruct a subrepresentation of $\mathcal{A}_0$ from the Whittaker model via the Fourier expansion: If $W \in \mathcal{W}$, then the corresponding element in $\mathcal{A}_0$ is $g \mapsto \sum_{\xi \in F^\times} W \left(\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} g \right)$. □

Theorem 38.5 (Strong Multiplicity One). *Let π, π' be two algebraic cuspidal automorphic representations with the same central character. If $\pi_v \cong \pi'_v$ for all archimedean and all but finitely many non-archimedean places v, then $\pi \cong \pi'$.*

Proof. Consider their Whittaker models $\mathcal{W} = \bigotimes_v \mathcal{W}_v, \mathcal{W}' = \bigotimes_v \mathcal{W}'_v$. The goal is to construct particular Whittaker functions $W \in \mathcal{W}, \mathcal{W}'$ such that their corresponding cusp forms are the same. Then $V \cap V' \neq 0$, and the result follows. Let S be a finite set of finite places such that $\pi_v \cong \pi'_v$ for all $v \notin S$. For all $v \notin S$ choose any $W_v = W'_v \in \mathcal{W}_v = \mathcal{W}'_v$, with the restriction that for almost all v, $W_v = W'_v$ is the spherical element with $W_v(1) = 1$. For $v \in S$ we can at least choose non-zero $W_v \in \mathcal{W}_v, W'_v \in \mathcal{W}'_v$ such that

$$W_v \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} = W'_v \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$$

for all $y \in F_v^\times$. This is possible since the Kirillov models of π_v, π'_v contain $C_c^\infty(F_v^\times)$, Theorem 14.2. Then let $W = \bigotimes_v W_v, W' = \bigotimes_v W'_v$. Let ϕ be the element of V corresponding to $W \in \mathcal{W}$, i.e. ϕ is defined by

$$\phi(g) = \sum_{\xi \in F^\times} W \left(\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} g \right).$$

Similarly define ϕ'. There is some open compact subgroup $K_0 \subseteq G(\mathbb{A}_\mathrm{f})$ such that W, W' are right invariant under K_0. Also being automorphic forms, ϕ, ϕ' are left invariant under $G(F)$. Also since

π, π' have the same central character, they transform in the same way under $Z(\mathbb{A})$. Hence, by the choice of W, W' it follows that $\phi(g) = \phi'(g)$ whenever g is of the form

$$z\gamma \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} g_\infty k_0$$

with $z \in Z(\mathbb{A}), \gamma \in G(F), y \in \mathbb{A}^\times, g_\infty \in G(F_\infty), k_0 \in K_0$. The proof of Theorem 30.4 shows that every element in $G(\mathbb{A})$ is of this form, hence $\phi = \phi'$. (Do we really need the z here? Bump doesn't mention it, but is the result true without the assumption that they have the same central character? TODO) □

Proof of Theorem 37.2. $\mathcal{A}_0(G(F)\backslash G(\mathbb{A}), \omega)$ decomposes into a direct sum of irreducible invariant subspaces. Let (V, π) be an irreducible invariant subspace such that $\varphi = \varphi_f$ has non-zero projection onto it. Write $V = \bigotimes_p V_p$. By Theorem 39.1, for all $p \nmid N$, the eigenvalues of $\mathcal{H}_p$ on V_p are determined by f and χ. In particular, by Theorem 20.4 V_p is independent of V. By the strong multiplicity one theorem (what about the infinite places?? TODO, must show that V_∞ is the weight k discrete series), V is uniqueley determined. Hence $\varphi \in V$. □

38.1. **Comparison with the Classical Fourier Expansion**

Let $f \in S_k(N, \chi)$. As describe in Section 37 we get a corresponding form $\phi = \phi_f \in \mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$ where ω is the adelic lift of χ. We describe the relation of the q-expansion of f with the Fourier expansion of ϕ in terms of Whittaker functions.

For simplicity first assume $f \in S_k(\Gamma(1))$. Write $f = \sum_{n=1}^\infty a_n q^n$. Let us compute $\widehat{\phi}_\xi(g)$ for $g = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$ with $y \in \mathbb{R}_{>0}$. We have

$$\begin{aligned}
\widehat{\phi}_\xi(g) = W_\phi\left(\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} g\right) &= \int_{A/F} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right)\psi(-\xi x)\mathrm{d}x \\
&= \int_{A/F} \phi\left(\begin{pmatrix} y & x_\infty \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x^\infty \\ 0 & 1 \end{pmatrix}\right)\psi(-\xi x)\mathrm{d}x \\
&= \int_{A/F} \phi\begin{pmatrix} y & x_\infty \\ 0 & 1 \end{pmatrix}\psi(-\xi x)\mathrm{d}x \\
&= \int_0^1 \phi\begin{pmatrix} y & x_\infty \\ 0 & 1 \end{pmatrix}\psi_\infty(-\xi x_\infty)\mathrm{d}x_\infty \int_{\prod_p \mathbb{Z}_p} \psi^\infty(-\xi x^\infty)\mathrm{d}x^\infty \\
&= y^{k/2}\int_0^1 \phi(x+iy)e^{-2\pi i\xi x}\mathrm{d}x \int_{\prod_p \mathbb{Z}_p} \psi^\infty(-\xi x)\mathrm{d}x
\end{aligned}$$

Now note that $x \mapsto \psi(-\xi x)$ is trivial on $\prod_p \mathbb{Z}_p$ if and only if when $\xi \in \mathbb{Z}$. Since the measure of $\prod_p \mathbb{Z}_p$ is 1, this gives

$$\widehat{\phi}_\xi\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} = \begin{cases} y^{k/2}a_n e^{-2\pi n y} & \text{if } \xi = n \in \mathbb{Z}, \\ 0 & \text{if } \xi \notin \mathbb{Z}. \end{cases}$$

Note the factor of $y^{k/2}$ makes sense since in the definition of the adelic lift of f we technically built this in.

39. Hecke Operators

Let v be a place of F. Suppose $f \in C_c^\infty(G(F_v))$ and $\phi \in \mathcal{A}(G(F)\backslash G(\mathbb{A}), \omega)$, then f acts on ϕ via

$$\pi(f)\phi(g) = \int_{G(F_v)} f(h)\phi(gh)\lambda^{-1}(h)\mathrm{d}h.$$

Here we view $h \in G(F_v)$ as an element of $G(\mathbb{A})$ via the usual inclusion $F_v \hookrightarrow \mathbb{A}$.

Specialize to $F = \mathbb{Q}$. Take for f the characteristic function of $K_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} K_p$. We denote it by T_p. Let $\varphi = \varphi_f$ be the automorphic form corresponding to a modular form $f \in M(N, \chi)$. Assume $p \nmid N$. Then $K_0(N)_p = K_p = G(\mathbb{Z}_p)$. Then we have (note that λ is actually trivial on K_p)

$$\begin{aligned}
\pi(T_p)\varphi(g) &= \sum_{A \in K_p \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) K_p / K_p} \int_{G(\mathbb{Z}_p)} \varphi(gAh)\lambda^{-1}(h)\mathrm{d}h \\
&= \sum_{A \in K_p \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) K_p / K_p} \int_{G(\mathbb{Z}_p)} \varphi(gA)\mathrm{d}h \\
&= \sum_{A \in K_p \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) K_p / K_p} \varphi(gA).
\end{aligned}$$

Recall from Section 20 the explicit coset representatives for T_p (thought we won't need them). Then:

$$\pi(T_p)\varphi(g) = \varphi\left(gi_p\left[\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\right]\right) + \sum_{b \bmod p} \varphi\left(gi_p\left[\begin{pmatrix} p & b \\ 0 & 1 \end{pmatrix}\right]\right)$$

Here i_p denotes the inclusion $G(\mathbb{Q}_p) \hookrightarrow G(\mathbb{A})$. Let A denote one of these coset representatives. Write $g = \gamma g_\infty k_0$. There is another coset representative B such that $k_0 A = Bk_0'$ for some $k_0' \in K_0(N)$. Then

$$gi_p(A) = \gamma g_\infty Bk_0' = (\gamma i_{\mathbb{Q}}(B))(i_\infty(B)^{-1}g_\infty)(i_{\mathrm{f}}(B)^{-1}i_p(B)k_0')$$

and so

$$\varphi(gi_p(A)) = F(i_\infty(B)^{-1}g_\infty)\lambda(i_{\mathrm{f}}(B)^{-1}i_p(B)k_0') = F(i_\infty(B)^{-1}g_\infty)\lambda(i_{\mathrm{f}}(B))^{-1}\lambda(k_0).$$

The last equality holds becaose λ only depends on the places dividing N. Then[20]

$$\varphi(gi_p(A)) = (f|_k^o[i_\infty(B)^{-1}g_\infty])(i)\chi(B)\lambda(k_0)$$

Then by definition of the classical Hecke operators T_p we have

$$\begin{aligned}
\pi(T_p)\varphi(g) = \sum_A \varphi(gi_p(A)) &= \sum_B (f|_k^o[i_\infty(B)^{-1}g_\infty])(i)\chi(B)\lambda(k_0) \\
&= \left(\sum_B \chi(B)(f|_k^o[i_\infty(B)^{-1}])\right)|_k^o[g_\infty](i)\lambda(k_0) \\
&= (T_pf)|_k^o[g_\infty](i)\lambda(k_0) \\
&= \varphi_{T_pf}(g)
\end{aligned}$$

[20]For a matrix $\gamma \in \Gamma_0(N)$ we have $\lambda(k_0) = \chi(\gamma)$ where $k_0 = \gamma_{\mathrm{f}}^{-1}$, see the computations before Proposition 37.1.

We also consider the action of the Hecke operator R_p corresponding to the coset $K_p \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$. The same calculation as above shows that

$$\pi(R_p)\varphi(g) = \chi(p)\varphi(g).$$

This whole discussion gives:

Theorem 39.1. *If $f \in M_k(N, \varphi)$ is an eigenfunction of T_p with $p \nmid N$, then φ_f is an eigenfunction for the local Hecke algebra $\mathcal{H}_p$. The eigenvalues are determined by χ and the eigenvalue of T_p on f.*

40. L-Functions and Functional Equation

As in the local case there are different approaches: Either via Whittaker models or via matrix coefficients.

We go the route with Whittaker models.

Let π be an irreducible algebraic admissible representation of $G(\mathbb{A})$, admitting a Whittaker model. Factor $\pi \cong \bigotimes_v \pi_v$ and let $\mathcal{W}_v$ be the local Whittaker model of π_v, so that $\mathcal{W} = \bigotimes_v \mathcal{W}_v$ is the Whittaker model of π.

Definition. *For $W \in \mathcal{W}$ and χ a quasi-character of $\mathbb{A}^\times/F^\times$, we define the* global Zeta integral *by*

$$Z(W, \chi, s) = \int_{\mathbb{A}^\times} W \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} |y|^{s-\frac{1}{2}} \chi(y) \mathrm{d}^\times y$$

The L-function *of π is defined as*

$$L(\pi, \chi, s) = \prod_v L_v(\pi_v, \chi_v, s),$$

where $L_v(\pi_v, \chi_v, s)$ is the local L-factor at v, defined in Section 24 for nonarchimedean v, and in TODO for archimedean v.

Lastly, we define the global Epsilon factor *by*

$$\varepsilon(\pi, \chi, s) = \prod_v \varepsilon_v(\pi_v, \chi_v, s, \psi_v).$$

We set $Z(W, s) = Z(W, 1, s)$, $L(\pi, s) = L(\pi, 1, s)$ and $\varepsilon(\pi, s) = \varepsilon(\pi, 1, s)$.

Note that there is no reason why the product defining the L-function should converge.

Since almost all π_v and χ_v are unramified, and the conductor of ψ_v is $\mathcal{O}_{F_v}$ for almost all v, we have $\varepsilon_v(\pi_v, \chi_v, s, \psi_v) = 1$ for almost all v, see Proposition 24.12. Also the same proposition implies that ε is independent of the choice of ψ, hence we dropped it from the notation.

Theorem 40.1 ([JL70, Theorem 11.1])**.** *Suppose π is a constituent of $\mathcal{A}(G(F)\backslash G(\mathbb{A}), \omega)$. The products defining $L(\pi, s)$ and $L(\widetilde{\pi}, s)$ converge absolutely for* $\mathrm{Re}(s)$ *large enough. They can be meromorphically continued to the whole complex plane with only finitely many poles, and are* *entire if π is cuspidal. They satisfy the functional equation*

$$L(\pi, s) = \varepsilon(\pi, s) L(\widetilde{\pi}, 1-s).$$

The L function is bounded in vertical strips.

We will prove this in the next section case if π is cuspidal.

40.1. L-Functions of Automorphic Forms

Let (V, π) be an algebraic automorphic cuspidal representation of $G(\mathbb{A})$. Let $\varphi \in V$.

Lemma 40.2. $\varphi \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$ *is rapidly decreasing as $|y| \to \infty$ or $|y| \to 0$.*

Proof. [GH11, Proposition 8.9.2]. □

Because of this the integral

$$Z(\varphi, s) = \int_{\mathbb{A}^\times / F^\times} \varphi \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} |y|^{s - \frac{1}{2}} \, \mathrm{d}^\times y$$

converges for all s.

More generally if χ is a character of $\mathbb{A}^\times / F^\times$, then we can consider the twisted Zeta function (or $\mathrm{GL}_2 \times \mathrm{GL}_1$ L-function)

$$Z(\varphi, \chi, s) = \int_{\mathbb{A}^\times / F^\times} \varphi \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} |y|^{s - \frac{1}{2}} \, \chi(y) \mathrm{d}^\times y.$$

Theorem 40.3 (Functional Equation)**.** *For all $\varphi \in V, s \in \mathbb{C}$, Hecke characters χ we have*

$$Z(\varphi, \chi, s) = Z(\pi(w_1)\varphi, \omega^{-1}\chi^{-1}, 1-s)$$

where ω is the central character of V and $w_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Proof.

$$\begin{aligned}
Z(\varphi, \chi, s) &= \int_{\mathbb{A}^\times / F^\times} \varphi \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} |y|^{s - \frac{1}{2}} \, \chi(y) \mathrm{d}^\times y \\
&= \int_{\mathbb{A}^\times / F^\times} \varphi \left(w_1 \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) |y|^{s - \frac{1}{2}} \, \chi(y) \mathrm{d}^\times y \\
&= \int_{\mathbb{A}^\times / F^\times} \varphi \left(\begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} w_1 \right) |y|^{s - \frac{1}{2}} \, \chi(y) \mathrm{d}^\times y \\
&= \int_{\mathbb{A}^\times / F^\times} \varphi \left(\begin{pmatrix} 1 & 0 \\ 0 & y^{-1} \end{pmatrix} w_1 \right) |y|^{-s + \frac{1}{2}} \, \chi(y^{-1}) \mathrm{d}^\times y \\
&= \int_{\mathbb{A}^\times / F^\times} \pi \left(\begin{pmatrix} y^{-1} & 0 \\ 0 & y^{-1} \end{pmatrix} w_1 \right) \varphi \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} |y|^{-s + \frac{1}{2}} \, \chi(y^{-1}) \mathrm{d}^\times y
\end{aligned}$$

$$= \int_{\mathbb{A}^\times/F^\times} \pi(w_1)\varphi\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix} |y|^{-s+\frac{1}{2}} (\omega^{-1}\chi^{-1})(y)\mathrm{d}^\times y$$
$$= Z(\pi(w_1)\varphi, \omega^{-1}\chi^{-1}, 1-s)$$

□

These Zeta integrals are related to those defined in terms of the Whittaker model defined previously as follows: Write $\varphi\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix} = \sum_{\xi\in F^\times} W_\varphi\left(\begin{pmatrix} \xi & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix}\right)$. Then by unfolding we have

$$Z(\varphi, s) = \int_{\mathbb{A}^\times} W_\varphi\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix} |y|^{s-\frac{1}{2}}\, \mathrm{d}^\times y,$$

which is the Zeta integral as defined before. This is valid as long as this is absolutely convergent. Let $\pi \cong \bigotimes_v \pi_v$, and assume that φ corresponds to $\otimes_v \varphi_v$. Let W_v be the local Whittaker function corresponding to φ_v. Then we have $W(g) = \prod_v W_v(g_v)$. Then we have

$$\int_{\mathbb{A}^\times} W_\varphi\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix} |y|^{s-\frac{1}{2}}\, \mathrm{d}^\times y = \prod_v \int_{F_v^\times} W_v\begin{pmatrix} y_v & 0 \\ 0 & 1\end{pmatrix} |y_v|^{s-\frac{1}{2}}\, \mathrm{d}^\times y = \prod_v Z_v(W_v, s)$$

We can use this to determine when the integral above is absolutely convergent. By Proposition 24.6 and its archimedean analog TODO, the local integral $\int_{F_v^\times} W_v\begin{pmatrix} y_v & 0 \\ 0 & 1\end{pmatrix} |y_v|^{s-\frac{1}{2}}\, \mathrm{d}^\times y$ converges for Re $> \frac{1}{2}$. By Theorem 24.10, for almost all v we have

$$Z_v(W_v, s) = \frac{1}{(1-\alpha_1 q_v^{-s})(1-\alpha_2 q_v^{-s})},$$

where $\alpha_i = \chi_i(\varpi_v)$ where $\pi_v \cong \mathcal{B}(\chi_1, \chi_2)$. Then by Theorem 15.14 we have $|\alpha_i| < q_v^{1/2}$. This easily implies that the products $\prod_v Z_v(W_v, s)$ and $\prod_v L_v(W_v, s)$ converge for $\operatorname{Re} s > \frac{3}{2}$.

If χ is a quasi-character of $\mathbb{A}^\times/F^\times$, then

$$Z(\varphi, \chi, s) = \int_{\mathbb{A}^\times/F^\times} \varphi\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix} |y|^{s-\frac{1}{2}}\, \chi(y)\mathrm{d}^\times y = \int_{\mathbb{A}^\times} W_\varphi\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix} |y|^{s-\frac{1}{2}}\, \chi(y)\mathrm{d}^\times y = \prod_v Z_v(W_v, \chi_v, s).$$

Let S be a finite set of primes, containing the infinite places, the places where π or χ is ramified and those where ψ_v has conductor $\neq \mathcal{O}_F$. Then for $v \notin S$ we have $L_v(\pi_v, \chi, s) = Z_v(W_v, \chi, s)$. Then where the infinite product converges we have

$$Z(\varphi, \chi, s) = L(\pi, \chi, s)\prod_{v\in S}\frac{Z_v(W_v, \chi, s)}{L_v(\pi_v, \chi, s)}.$$

At the finitely many places $v \in S$ we may choose W_v such that $\frac{Z_v(W_v,\chi,s)}{L_v(\pi_v,\chi,s)}$ is 1, hence $L(\pi, \chi, s)$ admits an analytic continuation to an entire function. For $\operatorname{Re} s \ll 0$, and hence for all by analytic continuation, we have

$$Z(\pi(w_1)W, \omega^{-1}\chi^{-1}, 1-s) = L(\pi, \omega^{-1}\chi^{-1}, 1-s)\prod_{v\in S}\frac{Z_v(\pi_v(w_1)W_v, \omega^{-1}\chi^{-1}, 1-s)}{L_v(\pi_v, \omega^{-s}\chi^{-1}, 1-s)}$$

Then by Theorem 40.3, we get

$$L(\pi,\omega^{-1}\chi^{-1},1-s)\prod_{v\in S}\frac{Z_v(\pi_v(w_1)W_v,\omega^{-1}\chi^{-1},1-s)}{L_v(\pi_v,\omega^{-s}\chi^{-1},1-s)}=L(\pi,\chi,s)\prod_{v\in S}\frac{Z_v(W_v,\chi,s)}{L_v(\pi_v,\chi,s)},$$

or after reordering terms

$$L(\pi,\omega^{-1}\chi^{-1},1-s)\prod_{v\in S}\frac{Z_v(\pi_v(w_1)W_v,\omega^{-1}\chi^{-1},1-s)}{L_v(\pi_v,\omega^{-1}\chi^{-1},1-s)}\frac{L_v(\pi_v,\chi,s)}{Z_v(W_v,\chi,s)}=L(\pi,\chi,s),$$

The term in the product is $\varepsilon_v(\pi_v,\chi_v,s,\psi_v)$. Since $\varepsilon_v=1$ for all $v\notin S$, we get

$$L(\pi,\omega^{-1}\chi^{-1},1-s)\varepsilon(\pi,\chi,s)=L(\pi,\chi,s).$$

This proves Theorem 40.1, noting $\widetilde{\pi}\cong\omega^{-1}\pi$.

Appendix A. Haar Measures and Modular Quasi-characters

In the following G is a locally compact group. Let μ denote a left Haar measure on G.

Definition. *The* modular function, *or* modular quasi-character, *of G is the function $\delta_G : G \to \mathbb{R}_{>0}$ such that*

$$\int_G f(gh)\mathrm{d}\mu(g) = \delta(h) \int_G f(g)\mathrm{d}\mu(g) \tag{$*$}$$

for any $f \in L^1(G)$ (or $C_c(G)$ is sufficient).

We might also write this as $\mathrm{d}(gh) = \delta(h)^{-1}\mathrm{d}g$. Equivalently, for any measurable set $A \subseteq G$ and $h \in G$ we have

$$\mu(Ah) = \delta(h)^{-1}\mu(A).$$

We will usually just write $\mathrm{d}g = \mathrm{d}\mu(g)$.

Proposition A.1. *$\delta_G \to \mathbb{R}_{>0}$ is a continuous homomorphism.*

Proof. The homomorphism property is immediate from the definition. For continuity fix a function $f \in L^1(G)$ with $\int f \neq 0$. The map $G \to L^1(G), h \mapsto R_h f$ where $R.f(g) = f(gh)$, is continuous. Hence $\delta = (\int_G f(g)\mathrm{d}g)^{-1} \int_G (R_- f)(g)\mathrm{d}g$ is continuous. □

Proposition A.2. *A right Haar measure on G is given by $\mathrm{d}_r g = \delta(g)\mathrm{d}g$, i.e. a right Haar integral is*

$$f \mapsto \int_G f(g)\delta(g)\mathrm{d}g.$$

Proof. Immediate from $(*)$. □

Proposition A.3. *We have $\int_G f(g^{-1})\mathrm{d}g = \int_G f(g)\delta(g)\mathrm{d}g$.*

Proof. $f \mapsto \int_G f(g^{-1})\mathrm{d}g$ is a right invariant Haar integral, hence it coincides with $\int_G f(g)\delta(g)\mathrm{d}g$ up to a scalar by the previous proposition. To see that the constant is 1, test with the characteristic functions of a symmetric neighborhoods of e of finite positive measure such that δ is close to 1 on them. □

Remark. In various books there are different conventions:

- In [BH06], [Fol15], [DE09], [BZ76], their δ, Δ, Δ is our δ^{-1}.
- In [GH24], [Bum97], [Car79], their δ agrees with our δ.

Example. Let F be a locally compact field. $\mathrm{GL}_2(F)$ is unimodular. The standard Borel subgroup B is not. Its modular function is given by

$$\delta_B\begin{pmatrix} y_1 & x \\ 0 & y_2 \end{pmatrix} = \left|\frac{y_1}{y_2}\right|.$$

Theorem A.4 ([Bum97, Proposition 2.1.5])**.** *Let G be unimodular, P, K closed subgroups with $P \cap K$ compact and $G = PK$. Then a Haar measure on G is given by*

$$\int_G f(g)\mathrm{d}g = \int_P \int_K f(pk)\mathrm{d}_l p\, \mathrm{d}_r k.$$

For the next part we follow [Bou04, VII § 2] (note Bourbaki does the things on the right).

Let G be locally compact and H a closed subgroup. Let $\chi : H \to \mathbb{C}^\times$ be a quasi-character. We denote by $C(H\backslash G, \chi)$ the space of continuous functions $f : G \to \mathbb{C}$ such that $f(hg) = \chi(h)f(g)$ for all $h \in H, g \in G$. $C_c(H\backslash G, \chi)$ denotes the subspace of functions that are compactly supported mod H. If G is *tdlc* we also consider $C_c^\infty(H\backslash G, \chi)$, the subspace of $C_c(H\backslash G, \chi)$ consisting of locally constant functions.

Given $f \in C_c(G)$, let $P^\chi(f) : G \to \mathbb{C}$ be defined by

$$(P^\chi f)(x) = \int_H f(\xi x)\chi(\xi)^{-1}\delta_H(\xi)\mathrm{d}_H\xi,$$

where $d_H\xi$ is a fixed left Haar measure on H, so that $\delta_H(\xi)\mathrm{d}_H\xi$ is a right Haar measure.

For $f : G \to \mathbb{C}$ and $g \in G$ let $R_g f$ and $L_g f$ denote the functions $(R_g f)(h) = f(hg)$, $(L_g f)(h) = f(gh)$.

Proposition A.5. *P^χ maps $C_c(G)$ to $C_c(H\backslash G, \chi)$. We have $P^\chi(L_y f) = \chi(y)\delta_H(y)^{-1}P^\chi(f)$ for $y \in H$, and $P^\chi(R_g f) = R_g(P^\chi f)$ for $g \in G$. The map $P^\chi : C_c(G) \to C_c(H\backslash G, \chi)$ is surjective.*

Proof. Clear except for the surjectivity. For the latter let $g \in C_c(H\backslash G, \chi)$. Fix a compact set $K \subseteq G$ such that $\operatorname{supp} g \subseteq HK$. Let $\phi \in C_c(G)$ be a function only taking on nonnegative real values and such that $\phi \equiv 1$ on K. Define $f : G \to \mathbb{C}$ by $f = g\phi/P^1(\phi)$ (here P^1 is the map P^χ for the trivial character) on HK and $f = 0$ elsewhere. Then $f \in C_c(G)$ and $P^\chi(f) = g$. □

If χ only takes on values in $\mathbb{R}_{>0}$ the proof shows that $P^\chi : C_c(G)^+ \to C_c(H\backslash G, \chi)^+$ is surjective, where the superscript $+$ indicates the subspace of functions that only take on nonnegative real values.

If G is *tdlc*, then P^χ restricts to a surjective map $C_c^\infty(G) \to C_c^\infty(H\backslash G, \chi)$ [BH06, 3.4].

Theorem A.6. *Let μ be a regular Borel measure on G (not necessarily Haar). Let $\chi : H \to \mathbb{C}^\times$ be a quasi-character. The following are equivalent:*

(1) There is a relatively bounded[a] *functional $I : C_c(H\backslash G, \chi) \to \mathbb{C}$ such that $I(P^\chi f) = \int_G f\mathrm{d}\mu$ for every $f \in C_c(G)$.*

(2) $\mathrm{d}\mu(\xi g) = \chi(\xi)^{-1}\delta_H(\xi)\mathrm{d}\mu(g)$ *for $\xi \in H$.*

[a]This means that for every $f \in C_c(H\backslash G, \chi)^+$, I is bounded on the set of $g \in C_c(H\backslash G, \chi)$ satisfying $|g| \leq f$.

Proof. "(1) ⇒ (2)" For $f \in C_c(G)$ we have

$$\int_G f(g)\mathrm{d}\mu(\xi g) = I(P^\chi(L_{\xi^{-1}}f)) = \chi(\xi)^{-1}\delta_H(\xi)I(P^\chi(f)) = \chi(\xi)^{-1}\delta_H(\xi)\int_G f(g)\mathrm{d}\mu(g).$$

For "(2) ⇒ (1)″ one has to show that if $P^\chi f = 0$, then $\int_G f\mathrm{d}\mu = 0$. Then $f \mapsto \int_G f\mathrm{d}\mu$ factors through P^χ and the claim follows. This isn't difficult, but see Bourbaki... □

In particular, if μ is a left Haar measure, a χ-twisted invariant measure on $H\backslash G$ exists if and only if $\delta_G|_H^{-1}\delta_H = \chi$. In the following assume this. Then we use the notation (from [BH06], [BZ76]):

$$I(f) = \int_{H\backslash G} f(g)\mathrm{d}_{H\backslash G}(g)$$

where $f \in C_c(H\backslash G, \delta_G|_H^{-1}\delta_H)$.

Note that if $f \in C_c(H\backslash G, \delta_G|_H^{-1}\delta_H)$ and $f = P^\chi f'$, then

$$I(R_g f) = I(R_g P^\chi f') = I(P^\chi(R_g f')) = \int_G f'(hg)\mathrm{d}h = \int_G f'(h)\mathrm{d}h = I(f).$$

(Recall we are using a *right* Haar measure on G here.) Note that if $C_c(H\backslash G, \delta_G|_H^{-1}\delta_H)^+$, then by the remark after the proof of Proposition A.5 there is a $f' \in C_c(G)^+$ such that $P^\chi f' = f$. Then

$$I(f) = I(P^\chi f') = \int_G f'(g)\mathrm{d}\mu(g) \geq 0.$$

So I is positive. We also see that if additionally $I(f) = 0$, then $f = 0$.

Proposition A.7. *If $J : C_c(H\backslash G, \delta_G|_H^{-1}\delta_H) \to \mathbb{C}$ is another positive functional invariant under right G-translations, then $J = cI$ for some (nonnegative) constant c.*

Proof. Pull back via P^χ to $C_c(G)$ and use uniqueness of Haar measure. □

This allows for a concrete way of computing $I(f)$. Assume that G is unimodular (not sure if this is really necessary, maybe can go without by inserting modular function?). Suppose $G = HK$ for some subgroup K such that $H \cap K$ is compact. Define $J : C_c(H\backslash G, \delta_G|_H^{-1}\delta_H) \to \mathbb{C}$ by

$$J(f) = \int_K f(k)\mathrm{d}_r k$$

Proposition A.8. *After rescaling we have $J = I$.*

Proof. By Theorem A.4

$$f \mapsto \int_H \int_K f(hk)\mathrm{d}_l h\mathrm{d}_r k$$

is a (both left and right) Haar integral on G, hence coincides with μ after rescaling. Then note that

$$\int_H \int_K f(hk)\mathrm{d}_l h \mathrm{d}_r k = \int_K (P^{\delta_H} f)(k)\mathrm{d}_r k = J(P^{\delta_H} f).$$

Since $P^{\delta_H} : C_c(G) \to C_c(H\backslash G, \delta_H)$ is surjective, this shows that J is right G-invariant and we are done by the previous proposition. □

For example, if F is a nonarchimedean local field we might apply this in the situation $G = \mathrm{GL}_2(F)$, $H = B(F)$, $K = \mathrm{GL}_2(\mathcal{O}_F)$.

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