A FORMULA FOR THE ORTHOGONAL PROJECTION

LEONARD TOMCZAK

Let V be an inner product space and $U \subseteq V$ a finite-dimensional subspace. In an introductory linear algebra course one defines the orthogonal projection pr_U of V onto U, and shows that it can be computed in terms of an orthogonal basis $u_1, \dots, u_n$ of U as follows:

$$\mathrm{pr}_U(v) = \sum_{i=1}^{n} \frac{\langle v, u_i \rangle}{\langle u_i, u_i \rangle} u_i.$$

It is straightforward to verify this satisfies this defining property of the orthogonal projection: $\mathrm{pr}_U(v) \in U$ and $v - \mathrm{pr}_U(v) \perp U$. What happens if we have a basis for U that is not necessarily orthogonal? How does the formula change? I recently saw the following formula in [Con85, Exercise I.4.5].[1]

Theorem. *Let $u_1, \dots, u_n$ be a basis for U. Let $v \in V$. The orthogonal projection $\mathrm{pr}_U(v)$ of v onto U is given by*

$$\mathrm{pr}_U(v) = \frac{-1}{\det(\langle u_i, u_j \rangle)_{i,j=1,\dots,n}} \det \begin{pmatrix} \langle u_1, u_1 \rangle & \cdots & \langle u_n, u_1 \rangle & \langle v, u_1 \rangle \\ \vdots & & \vdots & \vdots \\ \langle u_1, u_n \rangle & \cdots & \langle u_n, u_n \rangle & \langle v, u_n \rangle \\ u_1 & \dots & u_n & 0 \end{pmatrix}.$$

A word about the matrix: The last row contains vectors as entries, but that is fine, all the usual formulas (cofactor expansion, Leibniz formula, etc.) for computing the determinant will still work; basically since there is only exactly one row of all vectors. Formally, this could be interpreted as the determinant of an $(n+1) \times (n+1)$ matrix over the ring $\mathrm{Sym}\, U$, and since the rows are homogeneous of degrees $0, \dots, 0, 1$, the result will be homogeneous of degree 1, i.e. lie in $\mathrm{Sym}^1 U = U$. However, we will not need this interpretation.

Proof. Let u denote the right hand side in the formula. Expanding the determinant shows that u is a linear combination of $u_1, \dots, u_n$, hence it lies in U. It only remains to show that $v - u \perp U$. For this it suffices to verify $v - u \perp u_k$ for $k = 1, \dots, n$. We have:

$$\langle v - u, u_k \rangle = \frac{1}{\det(\langle u_i, u_j \rangle)} \left\langle \det(\langle u_i, u_j \rangle) v + \det \begin{pmatrix} \langle u_1, u_1 \rangle & \cdots & \langle u_n, u_1 \rangle & \langle v, u_k \rangle \\ \vdots & & \vdots & \vdots \\ \langle u_1, u_n \rangle & \cdots & \langle u_n, u_n \rangle & \langle v, u_n \rangle \\ u_1 & \dots & u_n & 0 \end{pmatrix}, u_1 \right\rangle$$

[1] The exercise is not quite correct, in the book's notation the indices in the determinant in the denominator should only go until $n - 1$.

$$= \frac{1}{\det(\langle u_i, u_j \rangle)} \left(\det(\langle u_i, u_j \rangle)\langle v, u_k \rangle + \det \begin{pmatrix} \langle u_1, u_1 \rangle & \cdots & \langle u_n, u_1 \rangle & \langle v, u_k \rangle \\ \vdots & & \vdots & \vdots \\ \langle u_1, u_n \rangle & \cdots & \langle u_n, u_n \rangle & \langle v, u_n \rangle \\ \langle u_1, u_k \rangle & \dots & \langle u_n, u_k \rangle & 0 \end{pmatrix} \right)$$

Now note that by cofactor expansion,

$$\det(\langle u_i, u_j \rangle)\langle v, u_k \rangle = \det \begin{pmatrix} \langle u_1, u_1 \rangle & \cdots & \langle u_n, u_1 \rangle & \langle v, u_1 \rangle \\ \vdots & & \vdots & \vdots \\ \langle u_1, u_n \rangle & \cdots & \langle u_n, u_n \rangle & \langle v, u_n \rangle \\ 0 & \dots & 0 & \langle v, u_k \rangle \end{pmatrix},$$

hence

$$\langle v - u, u_k \rangle = \frac{1}{\det(\langle u_i, u_j \rangle)} \det \begin{pmatrix} \langle u_1, u_1 \rangle & \cdots & \langle u_n, u_1 \rangle & \langle v, u_1 \rangle \\ \vdots & & \vdots & \vdots \\ \langle u_1, u_n \rangle & \cdots & \langle u_n, u_n \rangle & \langle v, u_n \rangle \\ \langle u_1, u_k \rangle & \dots & \langle u_n, u_k \rangle & \langle v, u_k \rangle \end{pmatrix} = 0,$$

since we have a repeated row.

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References

[Con85] J. B. Conway. *A Course in Functional Analysis*. Graduate Texts in Mathematics 96. 1985.

Department of Mathematics, Evans Hall, University of California, Berkeley, CA 94720, USA

Email address: leonard.tomczak@berkeley.edu