Differential Geometry

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1 Manifolds and smooth maps

1.1 Manifolds

Definition. *A* topological n-manifold *is a topological space X such that for all $p \in X$ there exists an open neighborhood U of p, an open set $V \subseteq \mathbb{R}^n$ and a homeomorphism $\varphi : U \to V$. We also require that X is Hausdorff and second countable.*

Remark. One can show that for spaces locally homeomorphic to $\mathbb{R}^n$ the condition "Hausdorff + second countable" is equivalent to "metrisable + has countably many components"

Maps φ as in the definition are called *charts* for X. A collection of charts whose domains cover X is called an *atlas* for X.

If $\varphi_\alpha : U_\alpha \to V_\alpha, \varphi_\beta : U_\beta \to V_\beta$ are overlapping charts, then

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is the *transition map.* It expresses the φ_β coordinates as functions of the φ_α coordinates.

Definition. *Given an atlas $\mathbb{A} = \{\varphi_\alpha : U_\alpha \to V_\alpha \mid \alpha \in \mathcal{A}\}$ on X and a fucntion $f : W \to \mathbb{R}$ on an open $W \subseteq X$, say f is* smooth *with repsect to $\mathbb{A}$ if for all α the map*

$$f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap W) \to \mathbb{R}$$

is smooth. The atlas is smooth *if every transition map $\varphi_\beta \circ \varphi_\alpha^{-1}$ is smooth. Two smooth atlases are* equivalent *if their union is smooth.*

Here smooth means C^∞.

Definition. *A* smooth structure *on a topological n-manifold X is an equivalence class of smooth atlases. A* smooth n-manifold *is a topological n-manifold together with the choice of a smooth structure.*

Note: Being a topological manifold is a property of a space, but for a differentiable manifold one needs to choose additional structure, i.e. the smooth structure. There are topological manifolds with different smooth structures (not even diffeomorphic).

Example. The n-sphere S^n has the underlying set

$$\{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\} \subseteq \mathbb{R}^{n+1},$$

endowed with the subspace topology. We define an atlas on S^n. Let $U_\pm = S^n \backslash \{(0,\dots,0,\pm 1)\}$. Define $\varphi_\pm : U_\pm \xrightarrow{\simeq} \mathbb{R}^n$ by

$$\varphi_\pm(x_1,\dots,x_{n+1}) = \frac{1}{1 \mp x_{n+1}}(x_1,\dots,x_n).$$

This defines a smooth structure on S^n.

1.2 Tangent spaces

Fix an n-manifold and point $p \in X$.

Definition. *A* curve based at p *is a smooth map of the form $\gamma : I \to X$ sending $0 \mapsto p$, where $I \subseteq \mathbb{R}$ is an open neighborhood of 0. Say two curves γ_1, γ_2 (based at p)* agree to first order *if there exists a chart φ around p such that*

$$(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0)$$

We write π_p^φ for the map $\gamma \mapsto (\varphi \circ \gamma)'(0)$.

Lemma 1.1. *If γ_1, γ_2 satisfy $\pi_p^\varphi(\gamma_1) = \pi_p^\varphi(\gamma_2)$, then for all charts ψ around p we have*

$$\pi_p^\psi(\gamma_1) = \pi_p^\psi(\gamma_2)$$

Proof. Clear, using chain rule and inserting transition maps. □

Corollary 1.2. *Agreement to first order is an equivalence relation.*

Definition. *The* tangent space to X at p, *denoted T_pX, is*

$$\{\textit{curves based at } p\}/\textit{agreement to first order.}$$

Proposition 1.3. *T_pX is naturally an n-dimensional $\mathbb{R}$-vector space.*

Proof. Given a chart φ about p, the map $\pi_p^\varphi : \{\text{curves at } p\} \to \mathbb{R}^n$ factors through T_pX and thus induces an injection $T_pX \hookrightarrow \mathbb{R}^n$. It is easily seen to be surjective. This bijection defines a vector space structure on T_pX. Different charts give rise to different bijections but they are related by a linear automorphism of $\mathbb{R}^n$, hence they all induce the same vector space structure on T_pM. □

Definition. *Given a chart φ about p with coordinates $x_1,\dots,x_n$, define $\frac{\partial}{\partial x_i} \in T_pX$ to be $(\pi_p^\varphi)^{-1}(e_i)$ where $e_i \in \mathbb{R}^n$ is the i-th standard basis vector. We will often abbreviate this by ∂_{x_i} or ∂_i.*

Warning. ∂_{x_i} depends on the choice of the whole coordinate system $x_1,\dots,x_n$, not just x_i.

Lemma 1.4. *On chart overlaps we have*

$$\frac{\partial}{\partial y_i} = \sum_{j=1}^{n} \frac{\partial x_j}{\partial y_i}\Big|_p \frac{\partial}{\partial x_j}$$

Here $\frac{\partial x_j}{\partial y_i}|_p := \frac{\partial}{\partial y_i} x_j := \varphi_j(\psi^{-1}(\psi(p) + te_i))'(0)$ *where* φ, ψ *are charts inducing the local coordinates* x_j, y_i.

Proof. We have

$$\begin{aligned}
\frac{\partial}{\partial y_i} &= (\pi_p^{\psi})^{-1}(e_i) \\
&= (\pi_p^{\varphi})^{-1}(\pi_p^{\varphi} \circ (\pi_p^{\psi})^{-1})(e_i) \\
&= (\pi_p^{\varphi})^{-1}(t \mapsto \varphi(\psi^{-1}(p) + te_i))'(0) \\
&= (\pi_p^{\varphi})^{-1} \left(\sum_{j=0}^{n} (t \mapsto \varphi_j(\psi^{-1}(p) + te_i))'(0) e_j \right) \\
&= (\pi_p^{\varphi})^{-1} \left(\sum_{j=0}^{n} \frac{\partial x_j}{\partial y_i}\Big|_p e_j \right) \\
&= \sum_{j=0}^{n} \frac{\partial x_j}{\partial y_i}\Big|_p \frac{\partial}{\partial y_j}.
\end{aligned}$$

□

1.3 Derivatives

Fix manifolds X, Y and a smooth map $F : X \to Y$.

Definition. *The* derivative *of* F *at* $p \in X$ *is the map* $D_pF : T_pX \to T_{F(p)}Y$ *defined by* $[\gamma] \mapsto [F \circ \gamma]$. *Sometimes we will write it as* F_*,*called pushforward by* F *on tangent vectors.*

Lemma 1.5. *The map* D_pF *is well-defined and linear.*

Proof. Let φ, ψ be charts about p, resp. $F(p)$. We have $\pi^{\psi}_{F(p)}(F \circ \gamma) = (\psi \circ F \circ \gamma)'(0) = ((\psi \circ F \circ \varphi^{-1}) \circ (\varphi \circ \gamma))'(0) = T\pi_p^{\varphi}(\gamma)$ where T is the derivative of $\psi \circ F \circ \varphi^{-1}$ at $\varphi(p)$.

$$\begin{array}{ccc}
T_pX & \xrightarrow{D_pF} & T_{F(p)}X \\
\downarrow{\scriptstyle \pi_p^{\varphi}} & & \downarrow{\scriptstyle \pi^{\psi}_{F(p)}} \\
\mathbb{R}^{\dim X} & \xrightarrow{T} & \mathbb{R}^{\dim Y}
\end{array}$$

□

Suppose that $\{x_i\}, \{y_j\}$ are coordinates associated to φ resp. ψ. Then $\psi \circ F \circ \varphi^{-1}$ expresses the y_j as functions of the x_i via F, so $T = \frac{\partial y_j}{\partial x_i}\Big|_p$ and

$$D_pF(\partial_{x_i}) = D_F((\pi_p^{\varphi})^{-1}(e_i)) = (\pi_{F(p)}^{\psi})^{-1}(Te_i) = \sum_j (\pi_{F(p)}^{\psi})^{-1}\Big(\frac{\partial y_j}{\partial x_i}\Big|_p e_j\Big).$$

Remarks.

(i) For $X = \mathbb{R}^n, Y = \mathbb{R}^m$, the new notion of derivative coincides with the standard one from multivariable calculus.

(ii) Given $f : X \to \mathbb{R}$, we have $D_pf(\partial_{x_i}) = \frac{\partial f}{\partial x_i}\Big|_p$.

(iii) We can write

$$[\gamma] = D_0\gamma(\partial_t)$$

where t is the parameter of γ.

Lemma 1.6 (Chain Rule)**.** *Suppose we have smooth maps*

$$X \xrightarrow{F} Y \xrightarrow{G} Z$$

Then $D_p(G \circ F) = D_{F(p)}G \circ D_pF$.

Proof. By definition. □

1.4 Immersions, Submersions and local Diffeomorphisms

Definition. *A smooth map $F : X \to Y$ is a* immersion/submersion/local diffeomorphism *(at $p \in X$) if D_qF is injective/surjective/bijective for all $q \in X$ (resp. only at $q = p$).*

Say p is a regular point *of F if D_pF is surjective (i.e. F is a submersion at p) and a* critical point *otherwise. Say that $q \in Y$ is a* regular value *if all $p \in F^{-1}(q)$ are regular and otherwise* critical value*.*

Lemma 1.7. *F is a local diffeomeorphism at p (in the sense of the definition above) iff there are open neighborhoods U of p, V of $F(p)$ such that $F|_U : U \to V$ is a diffeomorphism.*

Proof. "⇐" is obvious from the chain rule.

"⇒" Pick charts $\varphi : A \to B, \psi : C \to D$ about $p, F(p)$. By shrinking φ, WLOG $F(A) \subseteq C$. Now consider $\psi \circ F \circ \varphi^{-1} : B \to D$. This is a sooth map between open subsets of Euclidean space with invertible derivative at $\varphi(p)$. By the inverse function theorem there exist open neighborhoods $B' \subseteq B$ of $\varphi(p)$ and $D' \subseteq D$ of $\psi(F(p))$ such that $(\psi \circ F \circ \varphi^{-1})|_{B'} : B' \to D'$ has a smooth inverse H. Now set $U = \varphi^{-B'}, V = \psi^{-1}(D')$. Then $F|_U : U \to V$ has smooth inverse $\varphi^{-1} \circ H \circ \psi$. □

Notice: We could have chosen $\psi \circ F$ as φ. W.r.t. the charts $\psi \circ F, \psi$ the map F looks like the identity in local coordinates.

Proposition 1.8. *Suppose F is an immersion at p. Given local coordinates $x_1, \ldots, x_n$ on X about p, there exist local coordinates $y_1, \ldots, y_m$ on Y about $F(p)$ w.r.t. which F looks like $\mathbb{R}^n = \mathbb{R}^n \times 0 \hookrightarrow \mathbb{R}^n \times \mathbb{R}^{m-n} = \mathbb{R}^m$. Similarly, if F is a submersion of p, then given coordinates y about $F(p)$ there exist coordinates x about p w.r.t. which F looks like $\mathbb{R}^n \to \mathbb{R}^n/(0 \times \mathbb{R}^{n-m}) = \mathbb{R}^m$.*

Proof. Exercise. □

1.5 Submanifolds

Fix an n-manifold X.

Definition. *A subset $Z \subseteq X$ is a* submanifold of codimension k *if for all $p \in Z$ there exist local coordinates $x_1, \ldots, x_n$ on X about p such that Z is given locally by $x_1 = \cdots = x_k = 0$. (Formally: on the domain of the chart $Z = \{x_1 = \cdots = x_k = 0\}$).*

Z is a properly embedded submanifold *if the same holds for all $p \in X$ (not merely $p \in Z$).*

E.g. $0 \times \mathbb{R} \subseteq \mathbb{R}^2$ is a properly embedded submanifold. $0 \times \mathbb{R}^* \subseteq \mathbb{R}^2$ is a submanifold but not properly embedded.

Given a codimension k submanifold $Z \subseteq X$,

- Equip Z with the subspace topology (automatically Hausdorff and 2nd-countable because X is)
- For $p \in Z$ can choose local coordinates $x_1, \ldots, x_n$ on X as in the definition. Then $x_{k+1}, \ldots, x_n$ define a chart on Z about p.
- Transition functions are smooth since original transition functions on X were smooth.

Proposition 1.9. *A codimension k submanifold $Z \subseteq X$ is naturally an $(n-k)$-manifold. The inclusion map $\iota : Z \to X$ is a smooth immersion and a homeomorphism onto its image. Composition with ι gives a bijection*

$$\{\textit{smooth maps to } Z\} \xrightarrow{\iota \circ -} \left\{ \begin{smallmatrix} \textit{smooth maps to } X \\ \textit{with image} \subseteq Z \end{smallmatrix} \right\}.$$

Definition. *A map $F : Y \to X$ is an* embedding *if it is a smooth immersion and a homeomorphism onto its image.*

Lemma 1.10. *The image of an embedding $F : Y \to X$ is a submanifold of X and F induces a diffeomorphism from Y to that submanifold.*

So: Submanifolds $\leftrightarrow$ images of embeddings.

Example. The inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$ is an embedding. So S^n is a submanifold of $\mathbb{R}^{n+1}$ and the induced manifold structure agrees with the one we already defined.

Proposition 1.11. *If $F : X \to Y$ is a smooth map, and $q \in Y$ is a regular value, then $F^{-1}(q)$ is a submanifold of X of codimension $\dim Y$.*

Proof. Take a point $p \in F^{-1}(q)$. Since q is a regular value, F is a submersion at p, so there exist local coordinates x_i on X about p and y_j on Y about q such that $y \circ F = (x_1, \dots, x_m)$ where $m = \dim Y$. By translating the y-coords we may assume that $y(q) = 0$. Let U be an open neighborhood of p on which the x-coordinates and $y \circ F$ are defined. On U we have $U \cap F^{-1}(q) = U \cap (y \circ F)^{-1}(0) = \{x \in U : x_1 = \cdots = x_m = 0\}$, so the x_i give the required chart about p. $\square$

Example. Consider $F : \mathbb{R}^{n+1} \to \mathbb{R}, y \mapsto \|y\|^2$. This is smooth and $DF|_y = (2y_1 \dots 2y_{n+1})$ which is non-zero (hence surjective) everywhere except at 0. So for all $\lambda \in \mathbb{R}_{>0}$ is a codimension 1 submanifold of $\mathbb{R}^{n+1}$, in particular $S^n = F^{-1}(1)$ is a submanifold.

Theorem 1.12 (Sard's theorem)**.** *The set of critical values of a smooth map $F : X \to Y$ has measure 0 in Y (i.e. for any chart $\psi : S \to T$ on Y, $\psi(\{$critical values$\} \cap S) \subseteq T \subseteq \mathbb{R}^{\dim Y}$ has Lebesgue-measure 0).*

Corollary 1.13. *The set of regular values is dense in Y.*

Warning. This only concerns regular *values*, e.g. if $\dim X < \dim Y$ there are no regular *points*.

Definition. *Submanifolds $Y, Z \subseteq X$ are* transverse *if for all $p \in Y \cap Z$ we have*

$$T_pY + T_pZ = T_pX.$$

Theorem 1.14. *If Y, Z are submanifolds of X of codimensions k, l, intersecting transversely, then $Y \cap Z$ then is a submanifold of codimension $k + l$.*

Proof. Pick $p \in Y \cap Z$. Since Y, Z are submanifolds, there exist coordinates $y_1, \dots, y_n$, $z_1, \dots, z_n$ about p such that $Y = \{y_1 = \cdots = y_k\}, Z = \{z_1 = \cdots = z_l = 0\}$. Let U be an open neighborhood of p on which y and z are defined. Consider $F : U \to \mathbb{R}^{k+l}$ given by $(y_1, \dots, y_k, z_1, \dots, z_l)$. Y and Z being transverse at p is equivalent to D_pF being surjective. So there exist local coordinates $x_1, \dots, x_n$ on U about p such that $x_1 = y_1, \dots, x_k = y_k, x_{k+1} = z_1, \dots, x_{k+l} = z_l$. Then in these coordinates, $Y \cap Z = \{x_1 = \cdots = x_{k+l} = 0\}$. $\square$

2 Vector bundles and tensors

2.1 Vector bundles

Definition. *A* vector bundle *of rank k over a manifold B consists of the following information:*

- *A manifold E,*
- *A smooth map $\pi : E \to B$,*
- *An open cover $(U_\alpha)_{\alpha \in A}$ of B*
- *For each $\alpha \in A$ a diffeomorphism $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ such that*
  - $\mathrm{pr}_1 \circ \Phi_\alpha = \pi$ *on* $\pi^{-1}(U_\alpha)$
  - *For all $\alpha, \beta \in A$ the map $\Phi_\beta \circ \Phi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$ has the form $(b, v) \mapsto (b, g_{\beta\alpha} v)$ for some (necessarily smooth) map $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}_k(\mathbb{R})$*

  E is called the total space*, π the* projection*, B the* base space*, Φ_α the* local trivializations *and $g_{\alpha\beta}$ the transition functions.The* fibres *$\pi^{-1}(b)$ are denoted by E_b.*

Via each Φ_α the fibres E_b carry the structure of a k-dimensional vector space, independent of the local trivialization chosen.

Examples.

(i) The *trivial bundle* (of rank k over B) has $E = B \times \mathbb{R}^k$ as the total space with the obvious projection $E \to B$ and global trivialization $\Phi : E \to B \times \mathbb{R}^k$.

(ii) The *tautological bundle* over $\mathbb{RP}^n$ is the line bundle (i.e. rank 1 vector bundle) over $\mathbb{RP}^n$ given by:

$$E = \{(p, v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \mid v \text{ lies on the line described by } p\}$$

It is a submanifold of $\mathbb{RP}^n \times \mathbb{R}^{n+1}$. Define π by $\pi(p, v) = p$.

Open cover: $U_i = \{[x_0 : \cdots : x_n] \mid i \neq 0\}$. $\Phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}$ is given by $([x_0 : \cdots : x_n], (y_0, \ldots, y_n)) \mapsto ([x_0 : \cdots : x_n], y_i)$. On $U_i \cap U_j$ we have $\Phi_j \circ \Phi_i^{-1}([x_0 : \cdots : x_n], t) = \Phi_j([x], t(x_0/x_i, \ldots, x_n/x_i)) = ([x], tx_j/x_i)$, so $g_{ji} = x_j/x_i \in \mathbb{R}^* = \mathrm{GL}_1(\mathbb{R})$

(iii) The tautological complex line bundle over $\mathbb{CP}^n$.

(iv) The *tangent bundle* of an n-manifold X is given by

- Total space: $TX = \bigsqcup_{p\in X} T_pX$. This is a manifold via a pseudo-atlas: Given a coordinate patch U on X with coordinates x_i we get a pseudo-chart $\varphi : \bigsqcup_{p\in U} T_pX \to U \times \mathbb{R}^n$ given by $(p, \sum a_i \partial_{x_i}) \mapsto (p, (a_1, \dots, a_n))$. These make TX into a manifold.
- Projection $\pi : (p, v) \mapsto p$.
- The pseudo-charts give the local trivializations.

Definition. *A* section *of a vector bundle $\pi : E \to B$ is a smooth map $s : B \to E$ such that $\pi \circ s = \mathrm{id}_B$.*

Examples.

(i) Every vector bundle has a zero section given by $s(b) = 0$ for all $b \in B$.

(ii) A vector field on X is a section of the tangent bundle TX

Definition. *Given vector bundles $\pi_i : E_i \to B_i$ ($i = 1, 2$) and a smooth map $F : B_1 \to B_2$, a* morphism of vector bundles $E_1 \to E_2$ covering F *is a smooth map $G : E_1 \to E_2$ such that $\pi_2 \circ G = F \circ \pi_1$ and for all $p \in B_1$ the restricted map $G_b : (E_1)_p \to (E_2)_{F(p)}$ is linear.*

If $B_1 = B_2 = B$, an isomorphism of vector bundles over B *is a morphism covering* id_B *with a two-sided inverse. Equivalently, a diffeomorphism of the total space that induces linear isomorphisms $(E_1)_p \to (E_2)_p$.*

Example. Consider $S^1 = \{e^{i\theta} \mid \theta \in \mathbb{R}\} \subseteq \mathbb{C}$. The vector field ∂_θ is defined and non-zero in each fibre. So we get an isomorphism

$$\begin{aligned} &S^1 \times \mathbb{R} \to TS^1 \\ &(e^{i\theta}, a) \mapsto (e^{i\theta}, a\partial_\theta) \end{aligned}$$

So TS^1 is trivial (i.e. isomorphic to the trivial bundle).

For the trivial bundle of rank k over some (fixed) base we also write $\underline{\mathbb{R}}^k$.

Remark. A morphism $G : B \times \mathbb{R} \to E$ (covering the identity) is the same thing as a section s of E. More generally, a morphism $G \times \mathbb{R}^k \to E$ is the same as a k-tuple of sections. The morphism is an isomorphism iff the sections form a basis in each fibre.

Definition. *Given a vector bundle $\pi : E \to B$ of rank k, a* subbundle *of rank l is a subset $F \subseteq E$ such that B can be covered by local trivializations $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ under which $F \cap \pi^{-1}(U_\alpha) = U_\alpha \times (\mathbb{R}^l \times \{0\})$. This is naturally a vector bundle of rank l. Similarly we can define quotient bundle E/F of rank $k - l$ and we have morphisms $F \to E \to E/F$.*

Example. $\mathcal{O}_{\mathbb{RP}^n}(-1)$ (the tautological bundle) is (by construction) a subbundle of $\underline{\mathbb{R}}^{n+1}$ over $\mathbb{RP}^n$. We get the *Euler sequence* $\mathcal{O}_{\mathbb{RP}^n}(-1) \to \underline{\mathbb{R}}^{n+1} \to \underline{\mathbb{R}}^{n+1}/\mathcal{O}_{\mathbb{RP}^n}(-1) \cong T\mathbb{RP}^n(-1)$.

2.2 Vector bundles by gluing

To define a vector bundle over B (of rank k) it suffices to give an open cover $\{U_\alpha\}$ of B and for all α, β a smooth map $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}_k(\mathbb{R})$ satisfying

(i) $g_{\alpha\alpha}$ = constant map with value $\mathrm{id}_{\mathbb{R}^k}$.

(ii) $g_{\gamma\alpha} = g_{\gamma\beta} g_{\beta\alpha}$ for all α, β, γ (cocycle condition)

Note that from (i) and (ii) it follows that $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$.

Given this data, define

$$E = \coprod_\alpha U_\alpha \times \mathbb{R}^k / (\underbrace{b}_{\in U_\alpha}, v) \sim (\underbrace{b}_{\in U_\alpha}, g_{\beta\alpha}(b)v)$$

$\pi : E \to B$ is the obvious map. There are identifications $\pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{R}^k$. These define pseudo-charts and trivializations.

Example. For $r \in \mathbb{Z}$ can define a line bundle $\mathcal{O}_{\mathbb{RP}^n}(r)$ on $\mathbb{RP}^n$ to be trivialized over the $U_i = \{[x] \mid x_i \neq 0\}$ with transition functions $g_{ji} = \left(\frac{x_j}{x_i}\right)^{-r}$. Note that $\mathcal{O}_{\mathbb{RP}^n}(-1)$ is the tautological bundle.

Proposition 2.1. *If $\pi : E \to B$ is a vector bundle of rank k, trivialized over $\{U_\alpha\}$ with transition functions $g_{\beta\alpha}$, then*

(a) The $g_{\beta\alpha}$ satisfy (i) and (ii) above.

(b) E is isomorphic to the bundle constructed above.

Proof. (a) The $g_{\beta\alpha}$ are defined via $\Phi_\beta \Phi_\alpha^{-1}$, so (i) and (ii) are immediate.

(b) The trivializations of E and their inverses define diffeomorphisms $E \xrightarrow{\sim} \coprod_\alpha U_\alpha \times \mathbb{R}^k / \sim$. This is linear on fibres, hence a bundle isomorphism. $\square$

Corollary 2.2. *Two bundles are isomorphic iff they can be trivialized over a common open cover with the same transition functions.*

Proof. If both can be trivialized over $\{U_\alpha\}$ with transition functions $g_{\beta\alpha}$, then they are both isomorphic to the above construction. Converse is clear. $\square$

Example. Define the *Möbius line bundle* $M \to \mathbb{RP}^1$ to be trivialized over U_0, U_1 with $g_{10} = \mathrm{sign}\left(\frac{x_1}{x_0}\right)$.

Claim: This is isomorphic to $\mathcal{O}_{\mathbb{RP}^1}(-1)$. Suffices to show we can modify the trivializations of M to make the transition function become $\frac{x_1}{x_0}$ instead of sign $\frac{x_1}{x_0}$. Let's rescale the trivialization Φ_i of M by a smooth map $\psi_i : U_i \to \mathbb{R}^* = \mathrm{GL}_1(\mathbb{R})$. Excplicitly, consider the trivialization

$$(p, v) \mapsto (p, \psi_i(p)\, \mathrm{pr}_2(\Phi_i(p, v)))$$

This changes g_{10} to $\frac{\psi_1}{\psi_0} g_{10}$. Left to choose ψ_0, ψ_1 such that $\psi_1/\psi_0 = |x_1|/|x_0|$. One choice that works is $\psi_1 = \sqrt{\frac{x_1^2}{x_0^2+x_1^2}}, \psi_0 = \sqrt{\frac{x_0^2}{x_0^2+x_1^2}}$.

Definition. *Given a vector bundle $\pi : E \to B$ and a smooth map $F : B' \to B$ the* pullback bundle *F^*E defined as follows: Suppose E is trivialized over $\{U_\alpha\}$ with transition functions $g_{\beta\alpha}$, then F^*E is trivialized over $\{F^{-1}(U_\alpha)\}$ with transition functions $F^*g_{\beta\alpha} = g_{\beta\alpha} \circ F$. The fibre $(F^*E)_p$ is $E_{f(p)}$.*

Example. Consider the Hopf map $H : S^{2n+1} \to \mathbb{CP}^n$. Claim: $H^*\mathcal{O}_{\mathbb{CP}^n}(-1)$ is trivial. Proof: It is trivialized by the section $S^{2n+1} \ni p \mapsto p \in$ line through p.

Definition. *Given a vector bundle $\pi : E \to B$, the* dual bundle *$E^\vee \to B$ has total space $\coprod_{p\in B}(E_p)^\vee$ trivialized over $\{U_\alpha\}$ with transition functions $(g_{\beta\alpha}^\vee)^{-1}$*

If E is trivialized over $U \subseteq B$ by a fibrewise basis of sections $\sigma_1, \ldots, \sigma_k$. Then the fibrewise dual basis $\sigma_1^\vee, \ldots, \sigma_k^\vee$ give smooth sections of $E^\vee$ which trivialize it over U.

2.3 Cotangent bundle

Fix a n-manifold X.

Definition. *The* cotangent bundle *of X, denoted T^*X is the dual of the tangent bundle. The fibre of p is T_p^*X, the* cotangent space *at p.*

Dual to the picture of T_pX via curves $\mathbb{R} \to X$ we can describe T_p^*X via functions $X \to \mathbb{R}$: Say that functions f_1, f_2 about p *agree to first order* if $D_pf_1 = D_pf_2$

Proposition 2.3. *There is a natural isomorphism*

$$\{\textit{functions about } p\}/{}_{\substack{\textit{agreement to}\\ \textit{first order}}} \xrightarrow{\sim} T_p^*X$$

Proof. Define the map $e : \{\text{functions about } p\} \to T_p^*X$, $f \mapsto ([\gamma] \mapsto (f \circ \gamma)'(0)) = D_pf$. In local coordinates: $f \mapsto (\sum a_i \partial_{x_i} \mapsto \sum a_i \frac{\partial f}{\partial x_i})$. We see that $e(x_j)$ are the dual basis to ∂_{x_j}, so e is surjective and by definition $e(f_1) = e(f_2)$ iff f_1, f_2 agree to first order about p. $\square$

So for any smooth function $f : U \to \mathbb{R}$ we get an element of T_p^*X at each $p \in U$. This is denoted d_pf.

Lemma 2.4. *The d_pf define a smooth section of T^*X, denoted df, the differential of f.*

Proof. We have $df = \sum \frac{\partial f}{\partial x_i} dx_i$. We saw above that the dx_i are fibrewise dual to the ∂_{x_i}, so the dx_i are smooth. So df is a smooth linear combination of smooth sections, hence smooth itself. $\square$

Note, by construction $df(v)$ = directional derivative of f in direction v.

Definition. *A section of T^*X is a* 1*-form.*

*Un*warning! dx_i only depends on x_i, not the other x_j (unlike ∂_{x_i}).

Definition. *Given a smooth map $F : X \to Y$. The map $(D_pF)^\vee : T^*_{F(p)}Y \to T^*_pX$ is called* pullback by F*, denoted F^*.*

Lemma 2.5. *Given $F : X \to Y$ and a smooth function g on Y, we have $F * dg = d(F * g) := d(g \circ F)$.*

Proof. Given $[\gamma] \in T_pX$, we have $(F^*dg)[\gamma] = dg(D_pF[\gamma]) = dg([F \circ \gamma]) = (g \circ F \circ \gamma)'(0) = d(g \circ F)[\gamma]$. $\square$

2.4 Multilinear algebra

See handout.

2.5 Tensors and forms

We can apply any functorial operations to transition functions of existing bundles to build new ones.

Example. Dual bundle above.

Example. Given vector bundles $E, F \to B$, trivialized over $\{U_\alpha\}$ with transition functions $g_{\beta\alpha}, h_{\beta\alpha}$, can define $E \oplus F \to B$ with fibres $E_p \oplus F_p$, trivialized over $\{U_\alpha\}$ with transition functions $g_{\beta\alpha} \oplus h_{\beta\alpha}$.

Similarly can define $E \otimes F$.

Given a smooth map $F : X \to Y$, DF defines a section of $T^*X \otimes F^*TY$. For each $p \in X$, $D_pF \in \mathrm{Hom}_{\mathbb{R}}(T_pX, T_{F(p)}Y) = T^*_pX \otimes T_{F(p)}Y$.

Similarly can take tensor or exterior powers of a given vector bundle.

Definition. *A* tensor (field) *of type (p, q) on X is a section of $(TX)^{\otimes p} \otimes (T^*X)^{\otimes q}$. An* r-form *is a section of $\bigwedge^r T^*X$. The space of r-forms on an open set $U \subseteq X$ is denoted $\Omega^r(U)$.*

Examples. A tensor of type

- $(0, 0)$ is a section of $\underline{\mathbb{R}}$, i.e. a smooth function (or scalar field).
- $(1, 0)$ is a vector field.
- $(0, q)$ is something which "eats q vectors multilinearly and spits out a number".

2.6 Index notation

From now on, indices on local coordinates will be superscripts: $x^1, \dots, x^n$.

A section T of $TX \otimes T^*X \otimes TX$ (a specific kind of tensor of type $(2,1)$) can be written in local coordinates x^i uniquely as

$$T = \sum_{i,j,k} T^i{}_j{}^k \partial_i \otimes dx^j \otimes \partial_k$$

for locally defined smooth functions $T^i{}_j{}^k$.

Horizontal position of indices refer to the ordering of tensor factors. Vertical position denotes TX vs T^*X. We will often use summation convention where repeated indices once up and once down are summed over, e.g.

$$T = T^i{}_j{}^k \partial_i \otimes dx^j \otimes \partial_k$$

Often we just write $T^i{}_j{}^k$ for T.

Tensor product corresponds to juxtaposition, e.g.

$$(T^i{}_j{}^k \partial_i \otimes dx^j \otimes \partial_k) \otimes (S_{lm} dx^l \otimes dx^m) = T^i{}_j{}^k S_{lm} \partial_i \otimes dx^j \otimes \partial_k \otimes dx^l \otimes dx^m$$

or $(T \otimes S)^i{}_j{}^k{}_{sm} = T^i{}_j{}^k S_{lm}$.

Contraction corresponds to summation. E.g. contraction of third factor of T with second factor of S is

$$T^i{}_j{}^k S_{lm} \underbrace{dx^m(\partial_k)}_{\partial^m{}_k} \partial_i \otimes dx^j \otimes dx^l = T^i{}_j{}^k S_{lk} \partial_i \otimes dx^j \otimes dx^l,$$

i.e. the result is $T^i{}_j{}^k S_{lk} = \sum_k T^i{}_j{}^k S_{lk}$.

Similarly, in local coordinates x^i an r-form α can be written uniquely as

$$\sum \alpha_I dx^I = \sum_I \alpha_I dx^{i_1} \wedge \cdots \wedge dx^{i_r}$$

where the sum is over multi-indices $I = (i_1 < i_2 < \cdots < i_r)$.

Given r vectors $v_{(1)}, \dots, v_{(r)}$, we can feed them to α to give the number

$$\sum_{\substack{I \\ \sigma \in S_r}} \varepsilon(\sigma) \alpha_I v_{(1)}^{i_{\sigma(1)}} \dots v_{(r)}^{i_{\sigma(r)}}$$

This is equivalent to viewing α as the tensor

$$\sum_{\substack{I \\ \sigma \in S_r}} \varepsilon(\sigma) \alpha_I dx^{i_{\sigma(1)}} \otimes \cdots \otimes dx^{i_{\sigma(r)}}$$

of type $(0, r)$ and contracting with $v_{(1)}, \dots, v_{(r)}$.

Warning. Some people include the factor $\frac{1}{r!}$.

We can refer to the components of this tensor as $\alpha_{i_1 \dots i_r}$ (this is the coefficient of $dx^{i_1} \otimes \dots \otimes dx^{i_r}$). When the i_j form a multi-index I, i.e. $i_1 < i_2 \dots < i_r$, this agrees with α_I.

Example. On $\mathbb{R}^2$ we view $dx^1 \wedge dx^2$ as $dx^1 \otimes dx^2 - dx^2 \otimes dx^1$. A general 2-form looks like $\alpha_{12} dx^1 \wedge dx^2 = \alpha_{ij} dx^i \otimes dx^j$ where $\alpha_{21} = -\alpha_{12}$ and $\alpha_{11} = \alpha_{22} = 0$.

In summation convention, it is correct to say $\alpha = \alpha_{i_1 \dots i_r} dx^{i_1} \otimes \dots \otimes dx^{i_r}$ but **NOT** $\alpha = \alpha_{i_1 \dots i_r} dx^{i_1} \wedge \dots \wedge dx^{i_r}$ (the last sum would be $r!\alpha$).

If $\alpha = \alpha_1 \wedge \dots \wedge \alpha_r$, then $\alpha(v) = \det(\alpha_i(v_{(j)}))$.

2.7 Pushforward and pullback

Fix manifolds X, Y and a smooth map $F : X \to Y$.

- Given $p \in X$ and a tensor of type $(r, 0)$ at p (i.e. an element of $(T_p X)^{\otimes r}$). We can push this forward to $(T_{F(p)} Y)^{\otimes r}$ by applying $D_p F$ on each tensor factor. Denoted F_*.
- Given $p \in X$ and a tensor of type $(0, r)$ at $F(p)$, we can pullback to $(T_p^* X)^{\otimes r}$ using $((D_p F)^\vee)^{\otimes r}$. We can do the same for r-forms at $F(p)$ using $\wedge^r (D_p F)^\vee$. Denoted F^*
- Given a tensor T of type $(0, r)$ on Y can pull back to a tensor $F^* T$ on X by $(F^* T)_p = F^*(T_{F(p)})$. Similarly for r-forms.

Summary: Can pushforward "up" tensors at a point and can pullback "down" tensors or forms at a point or across the whole manifold.

If F is a diffeomorphism, then can pushforward or pullback any tensor over the whole manifold, e.g. let T be of type $(1, 1)$ on X. Then

$$(F_* T)_q = F_*(T_{F^{-1}(q)})$$

where we apply $D_{F^{-1}(q)} F$ on the TX factor and $(D_q(F^{-1}))^\vee$ on the $T^* X$ factor.

In this setting $F_* = (F^{-1})^*$ and vice versa.

3 Differential forms

3.1 Exterior derivative

Take a 1-form $\alpha = \alpha_i dx^i$ on X. Let us try to differentiate naively. We get:

$$\frac{\partial \alpha_i}{\partial x^j} dx^j \otimes dx^i.$$

Suppose we change to different local coords y^i. Then $\alpha = \alpha'_i dy^i = \alpha'_i \frac{\partial y^i}{\partial x^j} dx^j$, so $\alpha_j = \alpha'_i \frac{\partial y^i}{\partial x^j}$. Hence

$$\begin{aligned}\frac{\partial \alpha_i}{\partial x^j} dx^j \otimes dx^i = \frac{\partial}{\partial x^j}\left(\alpha'_i \frac{\partial y^k}{\partial x^i}\right) dx^j \otimes dx^i &= \frac{\partial \alpha'_k}{\partial x^j} \frac{y^k}{\partial x^i} dx^j \otimes dx^i + \alpha'_k \frac{\partial^2 y^k}{\partial x^j \partial x^i} dx^j \otimes dx^i \\ &= \frac{\partial \alpha'_k}{\partial y^j} dy^j \otimes dy^k + \alpha'_k \frac{\partial^2 y^k}{\partial x^j \partial x^i} dx^j \otimes dx^i\end{aligned}$$

Definition. *The* exterior derivative $d\alpha$ *is* $\frac{\partial \alpha_i}{\partial x^j} dx^j \wedge dx^i = d\alpha_i \wedge dx^i$*. By the above calculation this is well-defined (independent of local coordinates).*

More general, given a p-form $\alpha = \alpha_I dx^I$ *we define* $d\alpha := d\alpha_I \wedge dx^I = \frac{\partial \alpha_I}{\partial x^j} dx^j \wedge dx^I$*. Again, this is well-defined.*

Proposition 3.1. *d satisfies the following:*

(i) It is $\mathbb{R}$-linear.

(ii) It agrees with the differential on 0-forms.

(iii) $d^2 = 0$.

(iv) It commutes with pullback, i.e. $F^*(d\alpha) = d(F^*\alpha)$.

(v) Graded Leibniz rule: Given a p-form and a q-form β*:*

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge (d\beta)$$

Proof. (i) and (ii) are immediate from the definition.

(iii) Let $\alpha = \alpha_I dx^I$. We have

$$d^2\alpha = d\left(\frac{\partial \alpha_I}{\partial x^j} dx^j \wedge dx^I\right) = \frac{\partial^2 \alpha_I}{\partial x^k \partial x^j} dx^k \wedge dx^j \wedge dx^I = 0$$

since $\frac{\partial^2 \alpha_I}{\partial x^k \partial x^j}$ is symmetric in j, k, but $dx^k \wedge dx^j$ is antisymmetric.

(iv) Write α locally as $\alpha_I dy^I$. Then

$$\begin{aligned}
F^*(d\alpha) &= F^*(d\alpha_I \wedge dy^{i_1} \wedge \cdots \wedge dy^{i_p}) \\
&= F * (d\alpha_I) \wedge F^*(dy^{i_1}) \wedge \cdots \wedge F^*(dy^{i_p}) \\
&= d((F^*\alpha_I) d(F^* y^{i_1}) \wedge \cdots \wedge d(F^* y^{i_p}) \\
&= d(F^*\alpha)
\end{aligned}$$

using (iii) and (v) (we sum over multi-indices $i_1 < \cdots < i_p$).

(v) Let $\alpha = \alpha_I dx^I, \beta = \beta_J dx^J$. Then

$$\begin{aligned}
d(\alpha \wedge \beta) &= d(\alpha_I \beta_J dx^I \wedge dx^J) \\
&= d(\alpha_I \beta_J) \wedge dx^I \wedge dx^J \\
&= (d\alpha_I)\beta_J \wedge dx^I \wedge dx^J + \alpha_I d\beta_J \wedge dx^I \wedge dx^J \\
&= (d\alpha_I)\beta_J \wedge dx^I \wedge dx^J + (-1)^p (\alpha_I \wedge dx^I)(d\beta_J \wedge dx^J) \\
&= (d\alpha) \wedge \beta + (-1)^p \alpha \wedge d\beta
\end{aligned}$$

□

Definition. *A form α is* closed *if $d\alpha = 0$,* exact *if there exists β such that $\alpha = d\beta$. We write $Z^r(X), B^r(X) \subseteq \Omega^r(X)$ for the spaces of closed resp. exact r-forms.*

Aside: The exteriors derivative is the *unique* map $\Omega^*(X) \to \Omega^{*+1}(X)$ satisfying the properties in the proposition.

3.2 De Rham cohomology

Since $d^2 = 0$, we have $B^r(X) \subseteq Z^r(X)$.

Definition. *The* r-th de Rham cohomology group *of X, denoted $H^r_{\mathrm{dR}}(X)$, is $Z^r(X)/B^r(X)$.*

Note that $H^r_{\mathrm{dR}}(X) = 0$ for $r > \dim X$ or $r < 0$.

Example. (trivial cases)

(i) $H^0_{\mathrm{dR}}(X) = Z^0(X)/B^0(X) = \{\text{functions } f : df = 0\}/0 = \{\text{locally constant functions}\}$, so $H^0_{\mathrm{dR}}(X) = \mathbb{R}^{\{\text{components of } X\}}$.

(ii) $H^0_{\mathrm{dR}}(\text{point}) = \begin{cases} \mathbb{R} & r = 0, \\ 0 & r \neq 0. \end{cases}$

Example. Let $X = S^1$. We know that

$$H^r_{\mathrm{dR}}(S^1) = \begin{cases} \mathbb{R} & r = 0, \\ ? & r = 1, \\ 0 & r \neq 0, 1. \end{cases}$$

A 1-form on S^1 can be written uniquely as $f(\theta)d\theta$. All 1-forms are closed. Define a map

$$\begin{aligned} I : \Omega^1(S^1) &\longrightarrow \mathbb{R}, \\ f(\theta)d\theta &\longmapsto \int_0^{2\pi} f(\theta)d\theta \end{aligned}$$

This is linear and non-zero, hence surjective. Claim: $\ker I = B^1(S^1)$. Proof: If $fd\theta = dg$, then $f = \frac{\partial g}{\partial \theta}$, so $I(fd\theta) = g(2\pi) - g(0) = 0$. Conversely, if $I(fd\theta) = 0$, define $g(\theta) = \int_0^\theta f(t)dt$. Then we have $fd\theta = dg$. Thus we get an isomorphism $I : H^1_{\mathrm{dR}}(S^1) \simeq \mathbb{R}$.

Proposition 3.2. *If $F : X \to Y$ is smooth, then F^* induces a linear map $F^* : H^*_{\mathrm{dR}}(Y) \to H^*_{\mathrm{dR}}(X)$.*

Proof. Immediate from the fact that d commutes with pullback. □

E.g. consider $F : S^1 \to S^1$ given by $e^{i\theta} \mapsto e^{in\theta}$. The map $F^* : H^1_{\mathrm{dR}}(S^1) \to H^1_{\mathrm{dR}}(S^1)$ is multiplication by n.

Proposition 3.3. *Wedge product of forms descends to $H^*_{\mathrm{dR}}(X)$. This makes $H^*_{\mathrm{dR}}(X)$ into a unital, graded-commutative associative algebra.*

Proof. Given $[\alpha], [\beta] \in H^*_{\mathrm{dR}}(X)$, we need to show that $d(\alpha \wedge \beta) = 0$ and $[\alpha \wedge \beta]$ depends only on $[\alpha]$ and $[\beta]$. $d(\alpha \wedge \beta) = 0$ follows from the Leibniz rule. If $\alpha' = \alpha + d\gamma, \beta' = \beta + d\delta$, then $\alpha' \wedge \beta' = \alpha \wedge \beta + \alpha \wedge d\delta + (d\gamma) \wedge \beta + (d\gamma) \wedge (d\delta) = \alpha \wedge \beta + d((-1)^{|\alpha|}\alpha \wedge \delta + \gamma \wedge \beta + \gamma \wedge (d\delta))$, so $[\alpha \wedge \beta]$ only depends on the classes. □

Since F^* commutes with $\wedge$ and $F^*1 = 1$, the map $F^* : H^*_{\mathrm{dR}}(Y) \to H^*_{\mathrm{dR}}(X)$ is a unital algebra homomorphism.

Proposition 3.4 (Homotopy invariance)**.** *If $F_0, F_1 : X \to Y$ are smoothly homotopic., then the maps $F_0^*, F_1^* : H^*_{\mathrm{dR}}(Y) \to H^*_{\mathrm{dR}}(X)$ are equal.*

Proof. See Section 5.3. □

Corollary 3.5. *If $F : X \to Y$ is a homotopy equivalence, then $F^* : H^*_{\mathrm{dR}}(Y) \to H^*_{\mathrm{dR}}(X)$ is an isomorphism.*

Example. $H^*_{\mathrm{dR}}(\mathbb{R}^n) = H^*_{\mathrm{dR}}(\text{point})$.

3.3 Orientations

Definition. *An* orientation *of a n-dimensional vector space is a non-zero element of $\bigwedge^n V$ modulo positive rescalings.*

An orientation *of a vector bundle $E \to X$ of rank k is a nowhere-zero section of $\bigwedge^k E$, modulo rescaling by positive smooth functions. E is* orientable *if there exists an orientation for it, and* oriented *if it is equipped with a choice of orientation.*

Note: E is orientable iff $\bigwedge^k E$ is trivial. E.g. any trivial bundle is orientable. The tautological bundle over $\mathbb{RP}^n$ is not orientable.

Definition. *A manifold X is* orientable/oriented *if $TX \to X$ is.*

E.g. S^n is orientable for all n. $\mathbb{RP}^n$ is orientable iff n is odd.

Definition. *A volume form on n-manifold X is a nowhere-zero n-form.*

A volume form ω defines an orientation (basis $e_1, \dots, e_n$ for T_pX is positively oriented iff $\omega(e_1, \dots, e_n) > 0$) and conversely an orientation defines a volume form modulo rescaling by positive smooth functions.

3.4 Partitions of unity

Definition. *Given an open over $\{U_\alpha\}$ of X, a* partition of unity subordinate to the cover *is a collection of smooth functions $\{\rho_\alpha : X \to \mathbb{R}_\geq\}$ such*

- supp $\rho_\alpha \subseteq U_\alpha$.
- *locally finite: For all $p \in X$ there exists an open neighborhood V of p such that on V all but finitely many ρ_α are $\equiv 0$.*
- $\sum_\alpha \rho_\alpha = 1$.

Lemma 3.6. *For any open cover $\{U_\alpha\}$ of X, there exists a partition of unity subordinate to it.*

3.5 Integration

Fix an oriented n-manifold X and a compactly supported n-form ω on X.

Definition. *The* integral of ω over X*, denoted $\int_X \omega$, is defined as follows:*

- *Cover X by coordinate patches U_α with coordinates x^i_α. WLOG these are positively oriented, i.e. $\partial_{x^1_\alpha} \wedge \cdots \wedge \partial_{x^n_\alpha}$ represents the orientation.*
- *Pick a subordinate partition of unity $\{\rho_\alpha\}$. Write $\rho_\alpha \omega = f_/alphadx^1_\alpha \wedge \cdots \wedge dx^n_\alpha$. Define $\int_X \omega = \sum_\alpha \int_{\mathbb{R}^n} f_\alpha dx^1_\alpha \dots dx^n_\alpha$.*

Lemma 3.7. *The integral $\int_X \omega$ is well-defined.*

Proof. Suppose we cover X by patches V_β with coords $y^1_\beta, \dots, y^n_\beta$. Take a partition of unity σ_β subordinate to this cover. Locally write $\sigma_\beta \omega = g_\beta dy^1_\beta \wedge \dots \wedge dy^n_\beta$. We want to show $\sum_\alpha \int_{\mathbb{R}^n} f_\alpha dx^1_\alpha \dots dx^n_\alpha = \sum_\beta \int_{\mathbb{R}^n} g_\beta dy^1_\beta \dots dy^n_\beta$. On overlaps $U_\alpha \cap V_\beta$ we have

$$\sigma_\beta f_\alpha dx^1_\alpha \wedge \dots \wedge dx^n_\alpha = \sigma_\beta \rho_\alpha \omega = \rho_\alpha g_\beta dy^1_\beta \wedge \dots \wedge dy^n_\beta.$$

So $\sigma_\beta f_\alpha = \rho_\alpha g_\beta \det(\frac{\partial y^j_\beta}{\partial x^i_\alpha})$. Since both y and x are oriented in the same way, we have $\det(\frac{\partial y^j_\beta}{\partial x^i_\alpha}) = |\det(\frac{\partial y^j_\beta}{\partial x^i_\alpha})|$. So

$$\begin{aligned}
\sum_\alpha \int_{\mathbb{R}^n} f_\alpha dx^1_\alpha \dots dx^n_\alpha &= \sum_{\alpha,\beta} \int_{\mathbb{R}^n} \sigma_\beta f_\alpha dx^1_\alpha \dots dx^n_\alpha \\
&= \sum_{\alpha,\beta} \int_{\mathbb{R}^n} \rho_\alpha g_\beta \det(\frac{\partial y^j_\beta}{\partial x^i_\alpha}) dx^1_\alpha \dots dx^n_\alpha \\
&= \sum_{\alpha,\beta} \int_{\mathbb{R}^n} \rho_\alpha g_\beta dy^1_\beta \dots dy^n_\beta \\
&= \sum_{\beta} \int_{\mathbb{R}^n} \rho_\alpha g_\beta dy^1_\beta \dots dy^n_\beta
\end{aligned}$$

□

Note: Since ω is compactly supported and partitions of unity are locally finite, all sums appearing are actually finite.

3.6 Stokes's Theorem

Definition. *A (smooth) n-*manifold-with-boundary *X is defined in the same way as an ordinary n-manifold, except the codomain of each chart $\varphi : U \to V$ may be an open set in $\mathbb{R}^n$ or in $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$. Given $p \in X$ and a chart $\varphi : U \to V$ at p, say p is in the* boundary, *∂X, if $V \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$ and $\varphi(p) \in \{0\} \times \mathbb{R}^{n-1}$. Otherwise p is in the* interior *X°.*

The notion of boundary/interior is independent of the chart.

Examples.

(i) An ordinary n-manifold X is a manifold-with-boundary with $\partial X = \emptyset$.

(ii) The closed ball $X = \{p \in \mathbb{R}^n \mid \|p\| \leq 1\}$ is a manifold-with-boundary with $\partial X = S^{n-1}$ and $X^\circ = \{p \mid \|p\| \leq 1\}$.

(iii) If X is a m-w-b and Y is an ordinary manifold, then $X \times Y$ is a manifold with boundary. Then $X \times Y$ is a manifold-with-boundary. $\partial(X \times Y) = (\partial X) \times Y$ and $(X \times Y)^\circ = X^\circ \times Y$.

If both X and Y are manifolds-with-boundary, then in general $X \times Y$ is a manifold-with-corners.

Definition. *If X is an oriented n-manifold with boundary, then ∂X is oriented as follows. Given $p \in \partial X$, pick $o_X \in \bigwedge^n T_p X$ representing the orientation of X. Pick a vector $\mathbf{n} \in T_p X$ transverse to ∂X and pointing outwards. Orient ∂X at p by the unique $o_{\partial X} \in \bigwedge^{n-1} T_p \partial X \leq \bigwedge^{n-1} T_p X$ satisfying $o_X = \mathbf{n} \wedge o_{\partial X}$.*

Theorem 3.8 (Stokes's Theorem)**.** *Given an oriented n-manifold-with-boundary X, and a compactly supported $(n-1)$-form ω on X, we have*

$$\int_X d\omega = \int_{\partial X} \omega := \int_{\partial X} i^*\omega.$$

Proof. Step 1: Cover X by coordinate patches and pick a subordinate partition of unity ρ_α. Then

$$\int_{\partial X} \omega = \int_{\partial X} \sum_\alpha \rho_\alpha \omega = \sum_\alpha \int_{\partial X} \rho_\alpha \omega = \sum_\alpha \int_{\partial U_\alpha} \rho_\alpha \omega$$

and

$$\int_X d\omega = \int_X d(\sum_\alpha \rho_\alpha \omega) = \sum_\alpha \int_{U_\alpha} d(\rho_\alpha \omega).$$

So it suffices to prove the result when X is a coordinate patch. WLOG $X = \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$.

Step 2: Take $\omega = \sum_i \omega_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$. Want to show that

$$\int_{\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}} \sum_i (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} dx^1 \dots dx^n = \int_{\partial(\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1})} \omega$$

The left side is

$$\int_{\mathbb{R}^{n-1}} \left(\int_0^\infty \frac{\partial \omega_1}{\partial x^1} dx^1 \right) dx^2 \dots dx^n + \sum_{i \geq 2} \int_{\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-2}} \left(\int_{-\infty}^\infty \frac{\partial \omega_i}{\partial x^i} dx^i \right) dx^1 \dots \widehat{dx^i} \dots dx^n$$

$$= \int_{\mathbb{R}^{n-1}} -\omega_1 dx^2 \dots dx^n$$

Orientation of $\partial(\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1})$ is $-\partial_{x^2} \wedge \cdots \wedge \partial_{x^n}$. So the right side above is exactly this. □

Historical fact: Stokes put this theorem as a problem in the Cambridge exam (basically Part III).

Example. $X = \{x \in \mathbb{R}^2 : \|x\| \leq a\}$. Area of $X =$

$$\int_X dx \wedge dy = \int_X \frac{1}{2} d(xdy - ydx)$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\partial X} xdy - ydx \\
&= \frac{1}{2}\int_{\partial X} r^2 d\theta \\
&= \frac{a^2}{2}\int_{\partial X} d\theta \\
&= \pi a^2
\end{aligned}
$$

3.7 Applications of Stokes

Proposition 3.9 (Integration by parts)**.** *Given an oriented n-manifold-with-boundary X, a $(p-1)$-form α on X and an $(n-p)$-form β such that at least one is compactly supported, we have*

$$\int_X (d\alpha)\wedge\beta = \int_{\partial X} \alpha\wedge\beta + (-1)^p \int_X \alpha\wedge(d\beta)$$

Proof. By the Leibniz rule $d(\alpha\wedge\beta) = (d\alpha)\wedge\beta + (-1)^{p-1}\alpha\wedge d\beta$. Integrating and applying Stokes gives

$$\int_{\partial X} \alpha\wedge\beta = \int_X (d\alpha)\wedge\beta + (-1)^{p-1}\int_X \alpha\wedge d\beta$$

□

Proposition 3.10. *If X is a compact oriented n-manifold (without boundary), then integration over X defines a linear map*

$$\int_X : H^n_{\mathrm{dR}}(X) \to \mathbb{R}.$$

Corollary 3.11. *If X is a compact, orientable n-manifold, then $H^n_{\mathrm{dR}}(X) \neq 0$.*

Proof. Fix an orientation on X and choose a volume form ω representing this orientation. Then ω integrates to a positive number in every chart, hence $\int_X \omega \neq 0$ and thus $0 \neq [\omega] \in H^n_{\mathrm{dR}}(X)$. □

4 Connections on vector bundles

Notation and terminology:

- Given a vector bundle $E \to B$, an *E-valued r-form* is a section of $E \otimes \bigwedge^r T^*B$.
- Given a vector space V, a *V-valued r-form* is $\underline{V}$-valued r-form.
- $\Omega^r(E)$ is the space of E-valued r-forms. We write $\Gamma(E)$ for the space of sections of E (i.e. $\Omega^0(E)$)
- Write $\mathfrak{gl}(k, \mathbb{R})$ for the space of $k \times k$-real matrices.

4.1 Connections

Fix a rank k vector bundle $\pi : E \to B$. Given a section s, we can view it locally under each trivialization Φ_α as an $\mathbb{R}^k$-valued function which we will denote by v_α $(= \mathrm{pr}_2 \circ \Phi_\alpha \circ s|_{U_\alpha})$. The naive derivative is dv_α, which we can view as a local E-valued 1-form via Φ_α^{-1}. Under a different trivialization Φ_β, s becomes $v_\beta = g_{\beta\alpha} v_\alpha$. Taking the naive derivative and transferring the answer to the α-trivialization gives

$$g_{\beta\alpha}^{-1} d(g_{\beta\alpha} v_\alpha) = g_{\beta\alpha}^{-1}(dg_{\beta\alpha}) v_\alpha + dv_\alpha$$

So the answer is trivialization-dependent via the action of the $\mathfrak{gl}(k, \mathbb{R})$-valued 1-form $g_{\beta\alpha}^{-1} dg_{\beta\alpha}$ on v_α.

Definition. *A* connection $\mathcal{A}$ *on E comprises a $\mathfrak{gl}(k, \mathbb{R})$-valued 1-form A_α on U_α for each trivialization $\Phi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{R}^k$ such that on overlaps we have*

$$A_\alpha = g_{\beta\alpha}^{-1} A_\beta g_{\beta\alpha} + g_{\beta\alpha}^{-1} dg_{\beta\alpha} \quad (*)$$

Given a connection $\mathcal{A}$ on E, the covariant derivative *of a section s is the E-valued 1-form $d^{\mathcal{A}} s$ given under Φ_α by $dv_\alpha + A_\alpha v_\alpha$.*

By the calculations with the naive derivative, this is well-defined (i.e. consistent on overlaps). The section s is *horizontal* or *covariantly constant* if $d^{\mathcal{A}} s = 0$

The A_α are the *local connection 1-forms.* Note that the zero section is always horizontal. But non-zero horizontal sections may not exist, even locally.

Example (Trivial connection)**.** Suppose $E \to B$ admits a global trivialization Φ_α. We can define a connection $\mathcal{A}$ by $A_\alpha = 0$, then defining A_η for all other trivializations by $(*)$. A section is horizontal iff it is locally constant under Φ_α.

Lemma 4.1. *Given a connection $\mathcal{A}$ on $E \to B$, the covariant derivative*

$$d^{\mathcal{A}} : \Gamma(E) \to \Omega^1(E)$$

is $\mathbb{R}$-linear and satisfies the Leibniz-rule $d^{\mathcal{A}}(fs) = fd^{\mathcal{A}}s + s \otimes df$.

Conversely, any $\mathbb{R}$-linear map $\Gamma(E) \to \Omega^1(E)$ satisfying this, arises from a unique connection in this way.

Proof. $\mathbb{R}$-linearity os obvious. We can check Leibniz under trivializations:

$$\text{LHS} = d(fv_\alpha) + a_\alpha f v_\alpha = v_\alpha \otimes df + f dv_\alpha + f A_\alpha v_\alpha = f(dv_\alpha + A_\alpha v_\alpha) + v_\alpha \otimes df = \text{RHS}$$

The converse is on sheet 3. □

Example. Given a submanifold $i : X \hookrightarrow \mathbb{R}^N$, $\iota^* T\mathbb{R}^N$ has a standard trivialization and hence a trivial connection $\mathcal{A}_0$. Now consider

$$\Gamma(TX) \hookrightarrow \Gamma(\iota^* T\mathbb{R}^N) \xrightarrow{d^{\mathcal{A}_0}} \Omega^1(\iota^* T\mathbb{R}^N) \xrightarrow{\text{orthogonal projection}} \Omega^1(TX)$$

It is clearly $\mathbb{R}$-linear and inherits the Leibniz rule from $d^{\mathcal{A}_0}$. So it corresponds to a unique connection on TX.

Lemma 4.2. *Any vector bundle admits a connection.*

Proof. Trivialize E over U_α with transition functions $g_{\beta\alpha}$ as usual. Pick a partition of unity ρ_α subordinate to this cover. Now define

$$A_\alpha = \sum_\gamma \rho_\gamma g_{\gamma\alpha}^{-1} dg_{\gamma\alpha}$$

It remains to prove that this satisfies the transformation law $(*)$. We have

$$\begin{aligned} g_{\beta\alpha}^{-1} A_\beta g_{\beta\alpha} &= \sum_\gamma \rho_\gamma g_{\beta\alpha}^{-1} (g_{\gamma\beta} dg_{\gamma\beta}) g_{\beta\alpha} \\ &= \sum_\gamma \rho_\gamma g_{\gamma\alpha}^{-1} (d(g_{\gamma\beta} g_{\beta\alpha}) - g_{\gamma\beta} dg_{\beta\alpha}) \\ &= \sum_\gamma \rho_\gamma g_{\gamma\alpha}^{-1} dg_{\gamma\alpha} - \sum_\gamma \rho_\gamma g_{\beta\alpha}^{-1} dg_{\beta\alpha} \\ &= A_\alpha - g_{\beta\alpha}^{-1} dg_{\beta\alpha} \end{aligned}$$

□

4.2 Connections vs $\operatorname{End}(E)$

Fix rank k vector bundle $E \to B$. Let $\rho : \mathrm{GL}(k, \mathbb{R}) \to \mathrm{GL}(\mathfrak{gl}(k, \mathbb{R}))$ be the representation $\rho(A)(M) = AMA^{-1}$ for $A \in \mathrm{GL}(k, \mathbb{R})$ and $M \in \mathfrak{gl}(k, \mathbb{R})$.

Definition. $\operatorname{End}(E)$ *is the vector bundle over* B *of rank* k^2 *with total space*

$$\coprod_{b \in B} \operatorname{End}(E_b)$$

If E *is trivialized over* U_α *with transition functions* $g_{\beta\alpha}$*, then* $\operatorname{End}(E)$ *is trivialized over the same sets with transition functions* $\rho(g_{\beta\alpha})$*.*

A section M of $\operatorname{End}(E)$ is locally a $\mathfrak{gl}(k, \mathbb{R})$-valued function M_α such that $M_\beta = g_{\beta\alpha} M_\alpha g_{\beta\alpha}^{-1}$. Equivalently $\operatorname{End}(E) = E \otimes E^\vee$.

Lemma 4.3. *Given a connection* $\mathcal{A}$ *on* E*, and a section* Δ *of* $\Omega^1(\operatorname{End}(E))$*, there exists a connection* $\mathcal{A} + \Delta$*, defined locally by* $A_\alpha + \Delta_\alpha$*. Conversely, every connection* $\mathcal{A}'$ *on* E *can be written uniquely as* $\mathcal{A} + \Delta$ *for some* Δ*. Hence the set of connections on* E *is an affine space for* $\Omega^1(\operatorname{End}(E))$*.*

Proof. Just prove that everything is compatible with the transition functions.

$$A_\alpha + \Delta_\alpha = g_{\beta\alpha}^{-1} A_\beta g_{\beta\alpha} + g_{\beta\alpha}^{-1} dg_{\beta\alpha} + g_{\alpha\beta} \Delta_\beta g_{\alpha\beta}^{-1} = g_{\beta\alpha}^{-1}(A_\beta + \Delta_\beta) g_{\beta\alpha} + g_{\beta\alpha}^{-1} dg_{\beta\alpha}$$

For the other direction, verify that $\mathcal{A}' - \mathcal{A}$ transforms correctly, i.e. like a section of $\Omega^1(\operatorname{End}(E))$. □

4.3 Curvature algebraically

Definition. *The* exterior covariant derivative *is the unique* $\mathbb{R}$*-linear map* $d^{\mathcal{A}} : \Omega^\bullet(E) \to \Omega^{\bullet+1}(E)$ *satisfying the Leibniz rule*

$$d^{\mathcal{A}}(s \otimes \omega) = (d^{\mathcal{A}} s) \wedge \omega + s \otimes d\omega$$

for sections s *of* E *and forms* ω*. Locally in trivializations, an* E*-valued* p*-form* σ *becomes an* $\mathbb{R}^k$*-valued* p*-form* σ_α*, then* $d^{\mathcal{A}}\sigma$ *is given by* $d\sigma_\alpha + A_\alpha \wedge \sigma_\alpha$*.*

Warning. $(d^{\mathcal{A}})^2 \neq 0$ in general.

Proposition 4.4. *There exists a unique* $\operatorname{End}(E)$*-valued* 2*-form* F *on* B *such that for all* E*-valued forms* σ*:*

$$(d^{\mathcal{A}})^2 \sigma = F \wedge \sigma.$$

Proof. Locally in a trivialization $(d^{\mathcal{A}})^2\sigma$ is given by

$$d(d\sigma_\alpha + A_\alpha \wedge \sigma_\alpha) + A_\alpha \wedge (d\sigma_\alpha + A_\alpha \wedge \sigma_\alpha) = (dA_\alpha) \wedge \sigma_\alpha - A_\alpha \wedge d\sigma_\alpha + A_\alpha \wedge d\sigma_\alpha + A_\alpha \wedge A_\alpha \wedge \sigma_\alpha$$

$$= F_\alpha \wedge \sigma_\alpha$$

where $F_\alpha = dA_\alpha + A_\alpha \wedge A_\alpha$. Then one can check that this transforms like a $\mathrm{End}(E)$-valued 2-form. $\square$

Definition. *F is the* curvature *of $\mathcal{A}$. $\mathcal{A}$ is* flat *if $F = 0$.*

Examples.

(i) Trivial connections are flat. Conversely, if $\mathcal{A}$ is flat, then it is locally trivial.

(ii) Consider $\underline{\mathbb{R}}^2 \to \mathbb{R} \times S^1$ with a connection $\mathcal{A}$ given by $A_\alpha = f \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} dx + g \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} d\theta$ under the standard trivialization. Then

$$\begin{aligned} F_\alpha = dA_\alpha + A_\alpha \wedge A_\alpha &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} df \wedge dx + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} dg \wedge d\theta + 2fg \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} dx \wedge d\theta \\ &= \left(-\frac{\partial f}{\partial \theta} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\partial g}{\partial x} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - 2fg \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) dx \wedge d\theta \end{aligned}$$

4.4 Parallel transport

Fix $E \to [0,1]$ with connection $\mathcal{A}$.

Lemma 4.5. *Given $v_0 \in E_0$, there exists a unique horizontal section s with $s(0) = v_0$. This s depends linearly on v_0.*

Proof. Locally in trivializations the condition that s is horizontal says $dv_\alpha + A_\alpha v_\alpha = 0$ $(*)$. We have $A_\alpha = M_\alpha dt$ for some $\mathfrak{gl}(k, \mathbb{R})$-valued function M_α where t is the coordinate on $[0,1]$. Then $(*) \Leftrightarrow \frac{v_\alpha}{dt} + M_\alpha v_\alpha = 0$. This is a linear ODE. By standard ODE theory solutions exist locally and are unique (locally, hence globally). The solution depends linearly on the initial condition. Left to prove global existence: Local existence says that for all $p \in [0,1]$ there exists a fibrewise basis of horizontal sections locally about p. By compactness of $[0,1]$ there exist $0 = a_0 < a_1 < \cdots < a_N = 1$ such that on $[a_i, a_{i+1}]$ we have such a local fibrewise basis $s_i^1, \ldots, s_i^k$. Write $v_0 = \sum_{j=1}^k \lambda_{0j} s_0^j(0)$. Then define s on $[a_0, a_1]$ by $\sum_j \lambda_{0j} s_0^j$. Now write $s(a_1) = \sum_{j=1}^k \lambda_{1j} s_0^j(0)$ and extend s to $[a_1, a_2]$ as $s(a_1) = \sum_{j=1}^k \lambda_{1j} s_0^j$. Then keep going. $\square$

Definition. *The linear map $E_0 \to E_1$, $v_0 \mapsto s(1)$ is the* parallel transport *of v_0 along $[0,1]$ (w.r.t. $\mathcal{A}$).*

Now go back to general vector bundles $E \to B$ with connection $\mathcal{A}$. Suppose $\gamma : [0,1] \to B$ is a path. Then $\gamma^* A_\alpha$ defines a connection on γ^E, denoted $\gamma^* \mathcal{A}$.

Definition. *Given a vector $v_0 \in E_{\gamma(0)}$, the unique horizontal section s of γ^*E starting at v_0 is the* horizontal lift *of γ to E (starting at v_0). The vector $s(1) \in E_{\gamma(1)}$ is the* parallel transport *of v_0 along γ. Doing this for all v_0 gives a linear map $P_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$. If γ is a loop, i.e. $\gamma(0) = \gamma(1)$, then $P_\gamma : E_{\gamma(0)} \to E_{\gamma(0)}$ is the* monodromy *or* holonomy *of $\mathcal{A}$ around γ.*

Examples.

(i) Consider TS^2 with the "orthogonal projection" connection. Given path γ on S^2 and $v_0 \in T_{\gamma(0)}S^2$, the horizontal lift is the map $v : [0,1] \to TS^2$ such that

- $v(t) \in T_{\gamma(t)}S^2$ for all t.
- $\dot{v}(t)$ in $\mathbb{R}^3$ is orthogonal to $T_{\gamma(t)}S^2$, so that the orthogonal projection to $T_{\gamma(t)}S^2$ is 0

(ii) Returning to $\underline{\mathbb{R}}^2 \to \mathbb{R} \times S^1$ with connection $A_\alpha = f \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} dx + g \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} d\theta$. Consider $\gamma(t) = (t, 0)$. Horizontal lift v of γ starting at v_0 satisfies $\dot{v} + f \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v = 0$. So $v(t) = \begin{pmatrix} e^{-\lambda} & 0 \\ 0 & e^{\lambda} \end{pmatrix} v_0$ where $\lambda = \int_0^t f(x,0)dx$. Similarly, the monodromy around $\gamma(t) = (0, 2\pi t)$ is $\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$ where $\varphi = \int_0^{2\pi} g(0,\theta)d\theta$.

4.5 Curvature geometrically

Fix $E \to B$, with connection $\mathcal{A}$. Fix also a point $p \in B$, a trivialization Φ_α around p, and local coordinates x^i about p.

Let $A_\alpha = A_i dx^i$. where the A_i are $\mathfrak{gl}(k,\mathbb{R})$-valued functions. Similarly let $F_\alpha = F_{ij} dx^i \otimes dx^j$. WLOG $p = (0, \dots, 0)$. For $a, b \in \mathbb{R}$ small, let $\gamma_1(t) = ate_i, \gamma_2(t) = ae_i + bte_j$ in x coordinates. Then let γ_3, γ_4 be the other two sides of the rectangle.

Proposition 4.6. *Letting $P_{a,b} = P_{\gamma_4} P_{\gamma_3} P_{\gamma_2} P_{\gamma_1} \in \mathrm{End}(E_p)$ we have*

$$\frac{\partial^2 P_{a,b}}{\partial a \partial b}|_{a=b=0} = -F_{ij}(p)$$

Proof. For formal proof see Sheet 3. We will give an intuitive sketch proof ignoring analysis details, but these details can be filled in (e.g. see Nicolaescu Proposition 3.3.14).

Parallel transport in the x^i direction satisfies $\dot{v} = -A_i v$, so $P_{\gamma_1} = I - aA_i(p) + \dots$ where $\dots$ means higher order terms which will wash out. Similarly $P_{\gamma_2} = I - bA_j(\gamma_1(1)) + \dots = I - b(A_j(p) + a\frac{\partial A_j}{\partial x^i}(p)) + \dots$. So $P_{\gamma_2} \circ P_{\gamma_1} = I - aA_i(p) - bA_j(p) + abA_j(p)A_i(p) - ab\frac{\partial A_i}{\partial x^i}(p) + \dots$.

Similarly $P_{\gamma_4} \circ P_{\gamma_3} = I + aA_i(p) + bA_j(p) + ab\frac{\partial b_i}{\partial x_j}(p) + abA_jA_i(p) + \dots$. So

$$P_{a,b} = I + ab\left(\frac{\partial A_i}{\partial x^j}(p) - \frac{\partial A_j}{\partial x^i}(p) + A_j(p)A_i(p) - A_i(p)A_j(p)\right) + a(\dots) + b(\dots) + \dots$$

So
$$\frac{\partial^2 P_{a,b}}{\partial a \partial b}|_{a=b=0} = \frac{\partial A_i}{\partial x^j}(p) - \frac{\partial A_j}{\partial x^i}(p) + A_j(p)A_i(p) - A_i(p)A_j(p) = -F_{ij}(p)$$

□

Corollary 4.7. *If $v \in E_p$ is such that $F(p)v \neq 0$ in $E_p \otimes \bigwedge^2 T_p^* B$, then there does no exist a local horizontal section s about p with $s(p) = v$.*

Proof. If such an s exists, then $P_{\gamma_1}(v) = s(\gamma_1(1))$, similarly for the other paths, so $P_{a,b}(v) = s(p) = v$ for all a, b. So by the Proposition for all i, j we have $-F_{ij}(p)v = 0$, hence $F(p)v = 0$. □

Example. Consider $\underline{\mathbb{R}} \to \mathbb{R}^2$ with $A_\alpha = Cx^1dx^2$. Let $\gamma_1, \dots, \gamma_4$ be as before. Then $P_{\gamma_1} = \mathrm{id} = P_{\gamma_3} = P_{\gamma_4}$ and $P_{\gamma_2} = e^{-Cab}$. Hence $P_{a,b} = e^{-Cab}$. Then $\frac{\partial^2 P_{a,b}}{\partial a \partial b}|a = b = 0 = -C$. So $F_{12} = C$ which is of course also clear from $F = Cdx^1 \wedge dx^2 = dA_\alpha + A_\alpha \wedge A_\alpha$.

Explicitly, if s were a local horizontal section about p, given by v_α in our trivialization, then we would have $dv_\alpha + A_\alpha v_\alpha = 0$, i.e. $dv_\alpha + Cx^1 v_\alpha dx^2 = 0$, i.e. $\frac{\partial v_\alpha}{\partial x^1} = 0$, $\frac{\partial v_\alpha}{\partial x^2} = -Cx^1 v_\alpha$. Hence $0 = \frac{\partial^2 v_\alpha}{\partial x^2 \partial x^1} = -Cv_\alpha$. If $C \neq 0$, then the only horizontal local section is 0.

Example. Consider $\underline{\mathbb{R}} \to S^1$ with $A_\alpha = Cd\theta$. Local horizontal sections exist and have the form $v_\alpha = Ke^{-c\theta}$. This extends to a global section if $C = 0$. If $C \neq 0$, then this does not extend, due to the presence of non-trivial monodromy $e^{-2\pi C}$.

Summary. Curvature is the *local* obstruction from the existence of horizontal sections, monodromy is the *global* obstruction.

5 Flows and Lie derivatives

5.1 Flows

Fix a manifold X and a vector field v on X. Given a point $p \in X$, can try to flow along v from p, i.e. solve the ODE $\dot{\gamma}(t) = v(\gamma(t))$ and $\gamma(0) = p$. By standard ODE theory, solutions exist locally and are unique. Solutions are called *integral curves* of v.

Definition (Non-standard)**.** *A* flow domain *is an open neighborhood U of $X \times 0$ in $X \times \mathbb{R}$ such that for all $p \in X$ the set $U \cap (p \times \mathbb{R})$ is connected.*

Definition. *A* local flow *of v comprises a flow domain U and a smooth map $\Phi : U \to X$ such that*

- $\Phi(-, 0) = \mathrm{id}_X$*.*
- $\frac{d}{dt}\Phi(p, t) = v(\Phi(p, t))$ *for all* $(p, t) \in U$*.*

We write $\Phi^t(p)$ for $\Phi(p, t)$.

Previous ODE discussion plus smooth dependence on initial conditions, tells us that ocal flows exist and are unique in the sense that if (U_1, Φ_1) and (U_2, Φ_2) are local flows, then $\Phi_1 = \Phi_2$ on $U_1 \cap U_2$.

A vector field is *complete* if it has a *global* flow, i.e. one with $U = X \times \mathbb{R}$. Not all vector fields are complete, e.g. $x^2\partial_x$ on $\mathbb{R}$. But if v is compactly supported, then v is complete. (Idea: for each $p \in X$, there exists U_p neighborhood of p and $\varepsilon_p > 0$ such that a flow exists on $U_p \times (-\varepsilon_p, \varepsilon_p)$. By compactness, get local flow on $U = X \times (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. Can then define a global flow by $\Phi^t = (\Phi^{t/N})^N$ for $n \gg 0$.)

Lemma 5.1. *If Φ is a local flow of v, then $\Phi^{s+t} = \Phi^s \circ \Phi^t$ whenever this makes sense. So in particular $(\Phi^{t/N})^N = \Phi^t$ when this makes sense, and $\Phi^{-t} = (\Phi^t)^{-1}$.*

Proof. Fix $p \in X$, fix t. Let $q \in \Phi^t(p)$. Then $\gamma_1(s) := \Phi^{s+t}(p), \gamma_2(s) = \Phi^s \circ \Phi^t(p)$. These two curves both satisfy $\dot{\gamma}_i = v \circ \gamma_i$ and $\gamma_i(0) = q$. So by uniqueness of solutions to ODEs get $\gamma_1 = \gamma_2$. □

5.2 Lie Derivatives

Fix X and v, and let Φ be a local flow of v.

Definition. *For a tensor or form T on X, its* Lie derivative *is*

$$\mathcal{L}_v T = \frac{d}{dt}|_{t=0} (\Phi^t)^* T$$

Lemma 5.2. *We have*

$$\frac{d}{dt} (\Phi^t)^* T = (\Phi^t)^* \mathcal{L}_v T$$

Proof. We have

$$\frac{d}{dt}(\Phi^t)^* T = \frac{d}{dh}|_{h=0}(\Phi^{t+h})^* T = \frac{d}{dh}|_{h=0}(\Phi^t)^*(\Phi^h)^* T = (\Phi^t)^* \mathcal{L}_v T$$

□

Lemma 5.3. *For a function f we have*

$$\mathcal{L}_v f = df(v).$$

For a 1-form α we have

$$\mathcal{L}_v \alpha = \left(v^i \frac{\partial \alpha_j}{\partial x^i} + \alpha_i \frac{\partial v^i}{\partial x^j} \right) dx^j.$$

Proof. At each point p we have $\mathcal{L}_v f = \frac{d}{dt}|_{t=0} f(\Phi^t(p)) = df(\frac{d}{dt}|_{t=0}\Phi^t(p)) = df(v)$.

We have

$$\begin{aligned}
\mathcal{L}_v \alpha &= \frac{d}{dt}|_{t=0} (\Phi^t)^* \alpha \\
&= \frac{d}{dt}|_{t=0} (\alpha_i \circ \Phi^t) d(x^i \circ \Phi^t) \\
&= (\mathcal{L}_v \alpha_i) dx^i + \alpha_i d(\mathcal{L}_v x^i) \\
&= v^j \frac{\partial \alpha_i}{\partial x^j} dx^i + \alpha_i dv^i \\
&= (v^j \frac{\partial \alpha_i}{\partial x^j} + \alpha_j \frac{\partial v^j}{\partial x^i}) dx^i
\end{aligned}$$

□

Lemma 5.4. *For a 1-form α and a vector field w we have*

$$d(\alpha_i w^i)(v) = \mathcal{L}_v(\alpha_i w^i) = (\mathcal{L}_v \alpha)_i w^i + \alpha_i (\mathcal{L}_v w)^i$$

For any tensors S, T we have

$$\mathcal{L}_v (S \otimes T) = (\mathcal{L}_v S) \otimes T + S \otimes (\mathcal{L}_v T)$$

Proof. Pullback by Φ^t commutes with contraction and with $\otimes$. Then proceed as in the proof of the ordinary Leibniz rule. $\square$

Corollary 5.5. *For a vector field w we have*

$$\mathcal{L}_v w = \left(v^j \frac{\partial w^i}{\partial x^j} - w^j \frac{\partial v^i}{\partial x^j} \right) \partial_{x^i}$$

Proof. By first part of the previous lemma, for any 1-form α we have $\mathcal{L}_v(\alpha_i w^i) = (\mathcal{L}_v \alpha)_i w^i + \alpha_i (\mathcal{L}_v w)^i$, so we get by the lemma before

$$v^j \frac{\partial(\alpha_i w^i)}{\partial x^j} = \left(v^j \frac{\partial \alpha_i}{\partial x^j} + \alpha^j \frac{\partial v^j}{\partial x^i} \right) w^i + \alpha_i (\mathcal{L}_v w)^i$$

Hence

$$v^j w^i \frac{\partial \alpha_i}{\partial x^j} + v^j \alpha_i \frac{\partial w^i}{\partial x^j} = v^j w^i \frac{\partial \alpha_i}{\partial x^j} + \alpha_j w^i \frac{\partial v^j}{\partial x^i} + \alpha_i (\mathcal{L}_v w)^i$$

and thus

$$\alpha_i (\mathcal{L}_v w)^i = v^j \alpha_i \frac{\partial w^i}{\partial x^j} - \alpha_j w^i \frac{\partial v^j}{\partial x^i}.$$

This holds for all α, so

$$(\mathcal{L}_v w)^i = v^j \frac{\partial w^i}{\partial x^j} - w^j \frac{\partial v^i}{\partial x^j}.$$

$\square$

Definition. *The* Lie bracket *of v and w is*

$$[v, w] := \mathcal{L}_v w = -\mathcal{L}_w v.$$

This operation makes the $\Gamma(TX)$ into a Lie algebra, i.e. a vector space equipped with a bilinear operation $[\cdot, \cdot]$ satisfying

- $[v, v] = 0$ for all v (alternating)
- $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$ for all u, v, w (Jacobi identity)

Lemma 5.6. *If $F : X \to Y$ is a diffeomorphism, then for any vector field v on Y, and any tensor T on Y, we have*

$$F^*(\mathcal{L}_v T) = \mathcal{L}_{F^* v}(F^* T)$$

Proof. We have

$$\begin{aligned} F^*(\mathcal{L}_v T) &= F^* \frac{d}{dt}|_{t=0} (\Phi^t)^* T \\ &= \frac{d}{dt}|_{t=0} F^* (\Phi^t)^* T \end{aligned}$$

$$\begin{aligned}
&= \frac{d}{dt}|_{t=0} F^*(\Phi^t)^*(F^*)^{-1}F^*T \\
&= \frac{d}{dt}|_{t=0} (F^{-1}\circ\Phi^t\circ F)^*F^*T
\end{aligned}$$

But $F^{-1}\circ\Phi^t\circ F$ is a flow of F^*v. □

5.3 Homotopy invariance of de Rham cohomology

Definition. *Given an r-form α and a vector field v, the $(r-1)$ form $\iota_v\alpha$ or $v\lrcorner\alpha$ is defined to by*

$$(\iota_v\alpha)_{i_1\cdots i_{r-1}} = v^j\alpha_{ji_1\cdots i_{r-1}}$$

The Lie derivative and exterior derivative are related as follows:

Proposition 5.7 (Cartan's magic formula)**.** *For a vector field v, an r-form α, we have*

$$\mathcal{L}_v\alpha = (d\iota_v\alpha) + \iota_v d\alpha$$

Proof. Example Sheet 3. □

Proof of Proposition 3.4 (Homotopy Invariance of de Rham cohomology). Let $F:[0,1]\times X\to Y$ be a homotopy between F_0,F_1. Write F_t for $F(t,-)$. Let $i_t : X\to[0,1]\times X$ be the inclusion $x\mapsto(t,x)$. Note that $i_t=\Phi^t\circ i_0$ where Φ^t is the flow of ∂_t. Note that $F_t = F\circ i_t$. For any form α on Y we have

$$\begin{aligned}
F_1^*\alpha - F_0^*\alpha &= \int_0^1 \frac{d}{dt}F_t^*\alpha dt \\
&= \int_0^1 \frac{d}{dt}(F\circ\Phi^t\circ i_0)^*\alpha dt \\
&= \int_0^1 i_0^*\frac{d}{dt}(\Phi^t)^*F^*\alpha dt \\
&= i_0^*\int_0^1 (\Phi^t)^*\mathcal{L}_{\partial_t}(F^*\alpha)dt
\end{aligned}$$

Now suppose α is closed. By Cartan's magic formula we have

$$\mathcal{L}_v(F^*\alpha) = d(\iota_{\partial_t}F^*\alpha) + 0$$

So

$$\begin{aligned}
F_1^*\alpha - F_0^*\alpha &= i_0^*\int_0^1 (\Phi^t)^*d(\iota_{\partial_t}F^*\alpha)dt \\
&= \int_0^1 i_t^*d(\iota_{\partial_t}F^*\alpha)dt
\end{aligned}$$

$$= d\int_0^1 i_t^* \iota_{\partial_t} F^* \alpha dt$$

So $F_1^*\alpha - F_0^*\alpha$ is exact and thus F_0, F_1 induce the same map on de Rham cohomology. $\square$

6 Foliation and Frobenius integrability

6.1 Foliations

If $F : X \to Y$ is a submersion, then X decomposes into slices $F^{-1}(q)$ which are submanifolds of dimension $\dim X - \dim Y$.

A k-foliation on X is a local decomposition of X into k-dimensional slices, but the slices need not globally form submanifolds.

Example. Consider $X = T^2 = \mathbb{R}^2/\mathbb{Z}^2$. For any $\alpha \in \mathbb{R}$, we can locally slice X into lines of slope α. If α is irrational, then the slices do not globally form submanifolds.

Definition. *An atlas on X is* k-foliated *if the transition functions $\varphi_\beta \circ \varphi_\alpha^{-1}$ are locally of the form $\mathbb{R}^k \times \mathbb{R}^{n-k} \ni (x, y) \mapsto (\zeta(x, y), \eta(y)) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$. Two k-foliated atlases are* equivalent *if their union is k-foliated, and a k-foliation is an equivalence class of k-foliated atlases. We will usually write associated local coordinates as $x^1, \dots, x^k, y^1, \dots, y^{n-k}$. Slices are given locally by $y =$ const.*

Example. If $F : X \to Y$ is a submersion, then foliated charts correspond to local coordinates on X in which F corresponds to projection onto the last $n - k$ components.

6.2 Distributions

Fix an n-manifold X.

Definition. *A k-*plane distribution *on X is a rank k subbundle D of TX.*

Example. $\langle \partial_x, \partial_y \rangle$ and $\langle \partial_x + y\partial_z, \partial_y \rangle$ each define a 2-plane distribution on $\mathbb{R}^3$. Note that $\langle \partial_x, \partial_y \rangle = \ker dz$ and $\langle \partial_x + y\partial_z, \partial_y \rangle = \ker(dz - ydx)$.

In general a k-plane distribution can be written locally as the kernel as the kernel of $n - k$ 1-forms.

Example. If X is equipped with a k-foliation, with foliated coordinates x, y as usual, then $\langle \partial_{x^1}, \dots, \partial_{x^k} \rangle = \bigcap_{i=1}^{n-k} \ker dy^i$ is a k-plane distribution. It describes the tangent spaces to the slices.

Definition. *A k-plane distribution D is* integrable *if it arises from a k-foliation in this way*

If $k = 1$, then every distribution is integrable: Locally $D = \langle v \rangle$ for some vector field v, then X is foliated by integral curves of v.

6.3 Frobenius integrability

Theorem 6.1 (Frobenius integrability). *A distribution D on X is integrable iff D is closed under $[\cdot,\cdot]$, i.e. for all vector fields v, w tangent to D, $[v,w]$ is also tangent to D.*

Proof. Both conditions are local, so it suffices to work near a point p. If D is integrable with local foliated coordinates x, y, then $D = \langle \partial_{x^1}, \ldots, \partial_{x^k} \rangle$. Can easily check by hand that for any smooth coefficients f^i, g^i, $[f^i \partial_{x^i}, g^j \partial_{x^j}] \in D$.

Conversely, suppose D is closed under $[\cdot,\cdot]$. We want to show that there exist local coordinates x, y such that $D = \langle \partial_{x^1}, \ldots, \partial_{x^k} \rangle$. First choose local coordinates $s^1, \ldots, s^k, t^1, \ldots, t^{n-k}$ about p such that $D = \langle \partial_{s^1}, \ldots, \partial_{s^k} \rangle$ at p. WLOG p corresponds to $s = 0, t = 0$. Locally there exist uniquely determined smooth functions a_{ij} such that $v_i := \partial_{s^i} + \sum a_{ij} \partial_{t^j}$ lies in D. Let Φ_i^t be the flow of v_i. Now define F : open neighborhood of 0 in $\mathbb{R}^n \to X$ by $F(x,y) = \Phi_1^{x^1} \circ \cdots \circ \Phi_k^{x^k}(s = 0, t = y)$. This has $F(0) = p$ and $D_0 F(\partial_{x^i}) = v_i = \partial_{s^i}$ and $D_0 F(\partial_{y^i}) = \partial_{t^j}$. So $D_0 F$ is an isomorphism, hence F defines a parametrization near p. It suffices to show that $\partial_{x^i} = v_i$. Suppose the Φ_i all commute, then for each i we would have

$$\begin{aligned}
\partial_{x^i} &= \frac{d}{dh}|_{h=0} \Phi_1^{x^1} \cdots \Phi_i^{x^i+h} \cdots \Phi_k^{x^k}(0, y) \\
&= \frac{d}{dh}|_{h=0} \Phi_i^{x^i+h} \Phi_1^{x^1} \cdots \widehat{\Phi_i} \ldots \Phi_k^{x^k}(0, y) \\
&= v_i(\Phi_i^{x^i} \Phi_1^{x^1} \cdots \widehat{\Phi_i} \ldots \Phi_k^{x^k}(0, y)) \\
&= v_i
\end{aligned}$$

so we would be done.

Left to show: $\Phi_i^{x^i} \circ \Phi_j^{x^j} = \Phi_j^{x^j} \circ \Phi_i^{x^i}$ for all i, j, i.e. that $[v_i, v_j] = 0$ for all i, j. We know that D is closed under $[\cdot,\cdot]$, so there exist b_{ijl} such that $[v_i, v_j] = \sum_l b_{ijl} v_l$. Equate coefficients of ∂_{s^l} to get all $b_{ijl} = 0$. $\square$

Example. Consider $D = \langle \partial_x + y\partial_z, \partial_y \rangle$ on $\mathbb{R}^3$. This is not closed under $[,]$ since $[\partial_x + y\partial_z, \partial_y] = -\partial_z \notin D$. So D is not integrable.

By hand: If D were tangent to a surface $f = \text{const}$, then we would have $\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial z} = \frac{\partial f}{\partial y} = 0$. So $0 = \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial z} + y\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial f}{\partial z}$. Then also $\frac{\partial f}{\partial x} = -y\frac{\partial f}{\partial z} = 0$ and so $df = 0$. So $\{f = \text{const}\}$ is not a surface!

7 Connections on vector bundles with extra structure

7.1 Connections on TX

Suppose $\mathcal{A}$ is a connection on $E = TX \to X$. Given local coordinates $x^1, \ldots, x^n$ on X, we get a trivialization of E by $\partial_{x^1}, \ldots, \partial_{x^n}$. Call this a *coordinate trivialization.* We typically write the induced local connection 1-form as $\Gamma^i{}_{jk} dx^k$ where i, j are the matrix indices on $\mathfrak{gl}(n, \mathbb{R})$. So for a vector field v we have $(d^{\mathcal{A}} v)^i = dv^i + \Gamma^i{}_{jk} v^j dx^k$.

Warning. The $\Gamma^i{}_{jk}$ do not transform like a tensor of type $(1,2)$. But the space of connections on E is an affine space for $\Omega^1(\operatorname{End} E) = \Gamma(E \otimes E^\vee \otimes T^*X) = \Gamma(TX \otimes T^*X \otimes T^*X)$, i.e. the space of tensors of type $(1,2)$.

Definition. *The* solder form θ *is the* E*-valued* 1*-form that corresponds to the fibrewise identity map under* $E \otimes T^*X = TX \otimes T^*X = \operatorname{End}(TX)$*.*

The torsion T *of* $\mathcal{A}$ *is the* E*-valued* 2*-form* $d^{\mathcal{A}}\theta$*.* $\mathcal{A}$ *is* torsion-free *if* $T = 0$*.*

In a coordinate trivialization $\theta = e_i \otimes dx^i$, so $T = d(e_i \otimes dx^i) + A_\alpha \wedge (e_i \otimes dx^i) = \Gamma^j{}_{ik} e_j \otimes dx^k \wedge dx^i$.

So $\mathcal{A}$ is torsion-free iff $\Gamma^i{}_{jk} = \Gamma^i{}_{kj}$.

Proposition 7.1 ((First) Bianchi identity)**.** *$d^{\mathcal{A}} T = F \wedge \theta$.*

Proof. We have $d^{\mathcal{A}} T = (d^{\mathcal{A}})^2 \theta = F \wedge \theta$. □

Definition. *A curve* γ *in* X *is a* geodesic *(w.r.t.* $\mathcal{A}$*) if* $\dot\gamma$ *is covariantly constant as a section of* $\gamma^* TX$*. This is equivalent to the* geodesic equation

$$\ddot\gamma^i + \Gamma^i{}_{jk} \dot\gamma^j \dot\gamma^k = 0.$$

Note that

- A connection on TX induces connections on T^*X and all bundles of tensors and forms. If we had taken the covariant derivative of θ as a tensor of type $(1,1)$, we would have got 0 automatically.

- The curvature of $\mathcal{A}$ is an $\mathrm{End}(E)$-valued 2-form, which we can view as a tensor of type $(1,3)$ F^i_{jkl} that is antisymmetric in k, l.
- Often $d^{\mathcal{A}}$ or $\mathcal{A}$ itself is called ∇ and the contraction of $d^{\mathcal{A}}$ with a vector or vector field v is written ∇_v.

7.2 Orthogonal vector bundles

Fix a vector bundle $E \to B$.

Definition. *An* inner product *on E is a section of $(E^\vee)^{\otimes 2}$ which is fibrewise symmetric and positive definite.*

Lemma 7.2. *E admits an inner product.*

Proof. Define locally and glue using a partition of unity. □

Definition. *An* orthogonal vector bundle *is a vector bundle equipped with an inner product g. A trivialization Φ_α is* orthogonal *if under Φ_α, g becomes the standard inner product on $\mathbb{R}^k$.*

Note: Transition functions between orthogonal trivializations take values in $O(k)$.

Fix an orthogonal vector bundle $(E, g) \to B$.

Lemma 7.3. *E can be covered by orthogonal trivializations.*

Proof. We can locally trivialize E by sections $s_1, \dots, s_k$. Apply Gram-Schmidt fibrewise to make the s_i orthonormal. The corresponding trivialization is then orthogonal. □

Definition. *A connection $\mathcal{A}$ on E is* orthogonal *if g is covariantly constant w.r.t. to the induced connection on $(E^\vee)^{\otimes 2}$.*

Lemma 7.4. *E admits an orthogonal connection, and the space of orthogonal connections on E is an affine space for $\Omega^1(\mathfrak{o}(E)) \subseteq \Omega^1(\mathrm{End}(E))$ where $\mathfrak{o}(E) \leq \mathrm{End}(E)$ is the bundle of skew-adjoint endomorphisms of E*

Lemma 7.5. *If $\mathcal{A}$ is an orthogonal connection on (E, g), then its curvature is an $\mathfrak{o}(E)$-valued 2-form.*

8 Riemannian geometry

8.1 Riemannian metrics

Fix an n-manifold X.

Definition. *A* (Riemannian) metric *on X is an inner product on TX. A* Riemannian manifold *is a pair (X, g) where X is a manifold and g is a Riemannian metric on X.*

Since every vector bundle admits an inner product, every manifold admits a Riemannian metric.

Given a Riemannian metric g_{ij}, we write g^{ij} for the dual metric on T^*X. This satisfies (and is defined by) $g^{ij} = g^{ji}$ and $g^{ij} g_{jk} = \delta^i{}_k$. We denote contraction with g_{ij} or g^{ij} by raising or lowering indices, e.g. $g_{il} T^{ij}{}_k = T_l{}^j{}_k$ or $g^{ik} S_{ij} = S^k{}_j$.

A section T^i_j of $\mathrm{End}(TX)$ lies in $\mathfrak{o}(TX)$ iff $T^i{}_j g_{ik} = -T^i{}_k g_{ji}$, i.e. $T_{kj} = -T_{jk}$. When writing coordinate expressions, we use $dx^i dx^j$ to mean $\frac{dx^i \otimes dx^j + dx^j \otimes dx^i}{2}$, e.g. the standard Riemannian metric on $\mathbb{R}^n$ is $g_{\mathrm{Eucl}} = \sum_i (dx^i)^2$

8.2 The Levi-Civita connection

Fix a Riemannian manifold (X, g).

Theorem 8.1 (Fundamental theorem of Riemannian geometry)**.** *There exists a unique torsion-free orthogonal connection on TX.*

Proof. We will prove the more generally statement that the map $\{\text{orthogonal connections}\} \to \Omega^2(TX)$ sending a connection to its torsion, is a bijection.

Fix an arbitrary orthogonal connection $\mathcal{A}_0$. Any other orthogonal connection $\mathcal{A}$ can be written uniquely as $\mathcal{A}_0 + \Delta$ for an $\mathfrak{o}(E)$-valued 1-form Δ. We will show that the map $\Omega^1(\mathfrak{o}(E)) \to \Omega^2(TX), \Delta \mapsto T_{\mathcal{A}_0+\Delta} - T_{\mathcal{A}_0}$ is a bijection. This map sends Δ to $\Delta \wedge \theta$, i.e $\Delta^i{}_{kj} - \Delta^i{}_{jk}$. (If $\mathcal{A}_0$ is locally $\Gamma^i{}_{jk}$, then $\mathcal{A}_0 + \Delta$ is $\Gamma^i{}_{jk} + \Delta^i{}_{jk}$, so $(T_{\mathcal{A}_0+\Delta} - T_{\mathcal{A}_0})^i{}_{jk} = (\Gamma + \Delta)^i{}_{kj} - (\Gamma + \Delta)^i{}_{jk} - (\Gamma^i{}_{kj} - \Gamma^i{}_{jk})$).

It is induced by the bundle morphism $F : \mathfrak{o}(TX) \otimes T^*X \to TX \otimes \wedge^2 T^*X$ given by wedging with θ. So it suffices to show that F is an isomorphism which we can do fibrewise. Note both bundles have rank $n\binom{n}{2}$ since $\mathfrak{o}(TX) \otimes T^*X = \{\Delta^i_{jk} \mid \Delta_{ijk} = -\Delta_{jik}\}$ and

$TX \otimes \wedge^2 T^*X = \{T^i{}_{jk} : T^k_{jk} = -T^i{}_{kj}\}$. So it is enough to show that $\Delta \mapsto \Delta \wedge \theta$ is injective, i.e. that if $\Delta^i{}_{jk}$ satisfies $\Delta_{ijk} = -\Delta_{jik}$ and $\Delta^i{}_{jk} = \Delta^i{}_{kj}$ ($\Delta \in \ker$), then $\Delta = 0$. But if Δ satisfies these two conditions, then $\Delta_{ijk} = -\Delta_{jik} = -\Delta_{jki} = \Delta_{kji} = \Delta_{kij} = -\Delta_{ikj} = -\Delta_{ijk}$. ☐

Definition. *This is the* Levi-Civita connection *on* (X, g)*. Its components* $\Gamma^i{}_{jk}$ *are called* Christoffel symbols*.*

The explicit coordinate expressions are

$$\Gamma_{ijk} = \frac{1}{2}\left(\partial_j g_{ik} + \partial_k g_{ji} - g_i g_{jk}\right).$$

Proposition 8.2. *If* $\iota : X \hookrightarrow \mathbb{R}^N$ *is an embedding, then*

- *X inherits a metric $\iota^* g_{\mathrm{Eucl}}$, hence has an induced Levi-Civita connection.*
- *TX carries the "orthogonally project from $\iota^* T\mathbb{R}^N$" connection.*

The connections coincide.

Proof. Example Sheet 4. ☐

8.3 The Riemann tensor

Fix (X, g).

Definition. *The curvature of the Levi-Civita connection* ∇ *is the* Riemann tensor $R^i{}_{jkl}$*. This is an* $\mathfrak{o}(TX)$*-valued* 2*-form, viewed as a tensor of type* $(1, 3)$*.*

The Riemann tensor has the following properties:

- $R^i{}_{jkl} = -R^i{}_{jkl}$ since it is a 2-form.
- $R_{ijkl} = -R_{jikl}$ since it takes values in $\mathfrak{o}(TX)$.
- First Bianchi identity $R \wedge \theta = d^\nabla T = 0$, i.e. $R^i{}_{jkl} + R^i{}_{klj} + R^i{}_{ljk} = 0$.
- Second Bianchi $d^{\mathrm{End}\,\nabla} R = 0$.

8.4 Hodge theory

Let (X, g) be an oriented Riemannian manifold. The dual metric g^{ij} gives an inner product on T^*X and induces inner products on $\wedge^p T^*X$ for all p. Explicitly, if $\alpha^1, \ldots, \alpha^n$ is a local fibrewise orthonormal basis of 1-forms, then the $\alpha^I = \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_p}$ are a fibrewise orthonormal basis of p-forms.

In particular, there is a distinguished unit volume form ω.

Given a p-form β there exists a unique $(n-p)$-form $*\beta$ such that for all p-forms α

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \omega$$

E.g. $*\alpha^I = \pm\alpha^J$ where $J = \{1, \dots, n\} \setminus I$.

Definition. *The map*

$$* : \Omega^p(X) \to \Omega^{n-p}(X)$$

is the Hodge star operator*.*

By considering its action on the α^I, can see that it is a fibrewise isometry and $*^2 = (-1)^{p(n-p)} \operatorname{id}_{\Omega^p(X)}$.

Example. Take $\mathbb{R}^3$ with the standard metric and orientation. Then $\omega = dx^1 \wedge dx^2 \wedge dx^3$, so $*dx^1 = dx^2 \wedge dx^3$, $*(dx^2 \wedge dx^3) = dx^1$ and cyclically.

Now assume X is compact. Define an inner product on $\Omega^p(X)$ by $\langle \alpha, \beta \rangle_X = \int_X \langle \alpha, \beta \rangle \omega = \int_X \alpha \wedge *\beta$. For $(p-1)$-form α, p-form β we have

$$\begin{aligned}
\langle d\alpha, \beta \rangle_X &= \int_X (d\alpha) \wedge *\beta \\
&= \int_X d(\alpha \wedge *\beta) - (-1)^{p-1} \alpha \wedge d * \beta \\
&= (-1)^p \int_X \alpha \wedge d * \beta \\
&= \langle \alpha, (-1)^p *^{-1} d * \beta \rangle_X
\end{aligned}$$

So the operator $\delta : \Omega^p \to \Omega^{p-1}(X)$ given by $(-1)^p *^{-1} d*$ is adjoint to d.

Definition 8.3. *This δ is the* codifferential*. A form α is* coclosed *if $\delta\alpha = 0$,* coexact *if $\exists \beta$ such that $\alpha = \delta\beta$.*

NB: $\delta = (-1)^{np+n+1} * d*$ and the definition of δ also makes sense for non-compact X.

Notice $\delta^2 = - *^{-1} d * *^{-1} d* = - * d^2 * = 0$.

Definition. *The* Laplace-Beltrami operator $\Delta : \Omega^p(X) \to \Omega^p(X)$ *is defined by $d\delta + \delta d = (d+\delta)^2$.*

A form α is harmonic *if $\Delta\alpha = 0$. This is equivalent to α being closed and coclosed (Sheet 4). We denote the space of harmonic forms by $\mathcal{H}^p(X)$.*

Theorem 8.4. *The map*

$$\begin{aligned}
\mathcal{H}^p(X) &\longrightarrow H^p_{\mathrm{dR}}(X) \\
\alpha &\longmapsto [\alpha]
\end{aligned}$$

is an isomorphism, i.e. every cohomology class has a unique harmonic representative.

Idea: $\mathcal{H}^p(X) = \ker \Delta = \ker d \cap \ker \delta = \ker d \cap (\operatorname{im} d)^\perp \cong \ker d / \operatorname{im} d = H^p_{\mathrm{dR}}(X)$

Theorem 8.5 (Hodge decomposition)**.** *The space $\mathcal{H}^p(X)$ is finite-dimensional and we have orthogonal decompositions*

$$\begin{aligned}\Omega^p(X) &= \mathcal{H}^p(X) \oplus d\delta\Omega^p(X) \oplus \delta d\Omega^p(X) \\ &= \mathcal{H}^p(X) \oplus \delta\Omega^{p-1}(X) \oplus \delta\Omega^{p+1}(X)\end{aligned}$$

Proof. See Section 10.4.3 in Nicolaescu. □

Proof of Theorem 8.4. It suffices to show that

$$\ker d = \mathcal{H}^p(X) \oplus d\Omega^{p-1}(X)$$

LHS⊇RHS: harmonic and exact forms are both closed.

LHS⊆RHS: by Hodge decomposition RHS $= (\operatorname{im} \delta)^\perp$, so it suffices to prove $\langle \ker d, \operatorname{im} \delta \rangle = 0$. Given $\alpha \in \ker d$, we have for all β, $\langle \alpha, \delta\beta \rangle = \langle d\alpha, \beta \rangle = 0$. □

9 Lie groups and principal bundles

9.1 Lie groups and Lie algebras

Definition. *A* Lie group *is a manifold G equipped with a group structure such that multiplication and inversion $m : G \times G \to G$, $i : G \to G$ are smooth.*

An embedded Lie subgroup of *G is a submanifold H that is also a subgroup. The restrictions of the operations from G to h make H into a Lie group.*

Examples. $\mathrm{GL}(n, \mathbb{R})$ is a Lie group. $\mathrm{SL}(n, \mathbb{R}), O(n), SO(n)$ are embedded Lie subgroups. Similarly $\mathrm{SL}(n, \mathbb{C}), U(n), SU(n)$ are embedded Lie subgroups of $\mathrm{GL}(n, \mathbb{C})$.

Definition. *For each $g \in G$ we get diffeomorphisms $L_g, R_g, C_g : G \to G$ defined by $L_g(h) = gh, R_g(h) = hg, C_g(h) = ghg^{-1}$ for all h.*

A tensor T is left/right/conjugation invariant *iff $(L_g)_*T = T$ for all g etc. It is* bi-invariant *if it is both left and right invariant.*

Lemma 9.1. *For any $h \in G$, the map*

$$\begin{aligned} \{\textit{left-invariant tensor field of type } (p,q)\} &\longrightarrow \{\textit{tensors of type } (p,q) \textit{ at } h\} \\ T &\longmapsto T_h \end{aligned}$$

is an isomorphism. Similarly for right-invariant.

Proof. The inverse map is define by $T_g = (L_{gh^{-1}})_* T_h$. □

Definition. *The* Lie algebra $\mathfrak{g}$ *of a Lie group G is T_eG.*

Examples.

- $\mathfrak{gl}(n, \mathbb{R}) = \{n \times n \text{ matrices}\}$.
- $\mathfrak{sl}(n, \mathbb{R}) = \{A \in \mathfrak{gl}(n, \mathbb{R}) \mid \operatorname{tr} A = D_I \det A = 0\}$
- $\mathfrak{o}(n) = \{A \in \mathfrak{gl}(n, \mathbb{R}) \mid A^T + A = 0\}$.

For $\xi \in \mathfrak{g}$ let ℓ_ξ denote the corresponding left-invariant vector field, i.e. $\ell_\xi(g) = (L_g)_*\xi$.

Lemma 9.2. *The Lie bracket of left-invariant vector fields is left-invariant.*

Proof. Given left-invariant vector fields v, w, we have for all $g \in G$ that

$$(L_g)_*[v, w] = [(L_g)_* v, (L_g)_* w] = [v, w]$$

where we used the diffeomorphism-invariance of the Lie derivative. $\square$

Definition. *The* Lie bracket *on* $\mathfrak{g}$ *is defined by* $[\xi, \eta] = \zeta$ *where* ζ *is the unique element of* $\mathfrak{g}$ *such that* $[\ell_\xi, \ell_\eta] = \ell_\zeta$. *It inherits alternating, bilinear, Jacobi from the Lie bracket of vector fields.*

9.2 Lie group actions

Definition. *An action of* G *on a manifold* X *is* smooth *if the action map* $\sigma : G \times X \to X$ *is smooth. Similarly for right actions.*

E.g. $\mathrm{GL}(n, R)$ acting on $\mathbb{R}^n$, G acting on itself by conjugation, $O(n)$ acting on $S^{n-1} \subseteq \mathbb{R}^n$.

Example. The *adjoint action/representation* of G on $\mathfrak{g}$ is

$$\mathrm{Ad}_g(\xi) := (C_g)_* \xi.$$

Definition. *Given a smooth left action of* G *on* X*, the* infinitesimal action *of* $\xi \in \mathfrak{g}$ *on* $x \in X$ *is*

$$\xi \cdot X := D_{(e,g)}\sigma(\xi, 0) = [\gamma(t)x]$$

where γ *is any curve representing* ξ. *Similarly for right actions but with* $[x\gamma(t)]$.

9.3 Principal bundles

Fix a Lie group G.

Definition. *A* (principal) G-bundle P *over* B *is defined the same way as a vector bundle except trivializations are* $\Phi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\simeq} U_\alpha \times G$ *and on overlaps* $\Phi_\beta \Phi_\alpha^{-1}(b, g) = (b, g_{\beta\alpha}(b)g)$ *for (necessarily smooth) maps* $g_{\beta\alpha} : U_\alpha \cap U_\beta \to G$.

Example. Given a rank k vector bundle $E \to B$, its *frame bundle* $F(E) \to B$ is the principal $\mathrm{GL}(k, \mathbb{R})$-bundle with $F(E)_b := \{\text{ordered bases in } E_b\}$. Similarly, if E has an inner product, can consider the *orthonormal frame* bundle $F_0(E)$, which is a principal $O(k)$-bundle.

Note that

- Many definitions transfer from vector bundles, e.g. sections, constructions by gluing etc.
- P admits a right G-action, defined in trivializations, i.e. $\Phi_\alpha^{-1}(b, x)g := \Phi_\alpha^{-1}(b, xg)$.

- Sections s over $U \subseteq B$ correspond to trivializations Φ over U:
  - Given Φ, define s by $s(b) = \Phi^{-1}(b, e)$
  - Given s, define Φ by $\Phi(s(b)g) = (b, g)$.

9.4 Connections

Fix a principal G-bundle $P \to B$. Write $R_g : P \to P$ for the right action of g.

Definition. *A* connection *on P is a $\mathfrak{g}$-valued 1-form $\mathcal{A}$ on P, satisfying:*

- *$\mathcal{A}(p \cdot \xi) = \xi$ for $p \in P, \xi \in \mathfrak{g}$.*
- *$R_g^* \mathcal{A} = \mathrm{Ad}_{g^{-1}} \mathcal{A}$ ($\mathcal{A}$ is equivariant).*

Given a local section s_α (or equivalently a trivialization Φ_α), the *local connection 1-form* A_α is $s_\alpha^* \mathcal{A}$.

Lemma 9.3. *On overlaps we have $A_\alpha = \mathrm{Ad}_{g_{\beta\alpha}^{-1}} A_\beta + (L_{g_{\beta\alpha}^{-1}})_* dg_{\beta\alpha}$.*

Conversely, given A_α transforming this way, they arise from a unique connection $\mathcal{A}$ on P.

Proof. Sheet 4. $\square$

N.B. If $P = F(E)$, then a connection on P is equivalent to a connection on E.

Definition. *The curvature of $\mathcal{A}$ is the $\mathfrak{g}$-valued 2-form $\mathcal{F}$ on P given by $\mathcal{F} = d\mathcal{A} + \frac{1}{2}[\mathcal{A} \wedge \mathcal{A}]$ where $[(\sum_i \xi_i \otimes \alpha_i) \wedge (\sum_j \eta_j \otimes \beta_j)] = \sum_{i,j} [\xi_i, \eta_j] \otimes (\alpha_i \wedge \beta_j)$.*

$\mathcal{A}$ is flat *if $\mathcal{F} = 0$.*