Algebraic Topology

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Taught by Jacob Rasmussen
Notes taken by Leonard Tomczak

Contents

0 Homotopies

Conventions:

- *space* means topological space,
- *map* means continuous map unless otherwise stated,
- $\mathrm{Map}(X, Y) := \{f : X \to Y \mid f \text{ continuous}\}$ where X, Y are spaces.

Some spaces:

- $I = [0, 1]$,
- $I^n = I \times \cdots \times I$ closed n-cube,
- $D^n = \{v \in \mathbb{R}^n \mid \|v\| \leq 1\}$ closed n-dimensional disk,
- $S^{n-1} = \partial D^n = \{v \in \mathbb{R}^n \mid \|v\| = 1\}$.

Note that $D^n \cong I^n$, $S^{n-1} \subseteq D^n$, $D^n/S^{n-1} \cong S^n$.

Definition. *If $f_0, f_1 : X \to Y$ are continuous maps, f_0 is* homotopic *to f_1, written $f_0 \sim f_1$, if there exists a continuous map $H : X \times I \to Y$ with $H(x, 0) = f_0(x)$ and $H(x, 1) = f_1(x)$ for all $x \in X$. H is called a* homotopy*.*

Think: $f_t(x) = H(x, t)$, $f_t : X \to Y$, $t \mapsto f_t$ is a path from f_0 to f_1 in $\mathrm{Map}(X, Y)$.

Examples.

1. $\mathrm{id}_{\mathbb{R}^n} \sim 0_{\mathbb{R}^n}$.
2. $A_n : S^n \to S^n, v \mapsto -v$ antipodal map. $A_1 \sim \mathrm{id}_{S^1}$ via $f_t(z) = e^{i\pi t}z$, but $A_2 \not\sim \mathrm{id}_{S^2}$ (proven later).

Lemma 0.1. *Homotopy is an equivalence relation.*

Definition.

$$\begin{aligned}[X, Y] &:= \mathrm{Map}(X, Y)/\sim \\ &= \{\textit{homotopy classes of maps } X \to Y\} \\ \text{“} &= \{\textit{path components of } \mathrm{Map}(X, Y)\}\text{”}\end{aligned}$$

Lemma 0.2. *If $f_0, f_1 : X \to Y$, $f_0 \sim f_1$ via f_t and $g_0, g_1 : Y \to Z, g_0 \sim g_1$ via g_t, then $g_0 \circ f_0 \sim g_1 \circ f_1$ via $g_t \circ f_t$.*

Example. $f : X \to \mathbb{R}^n$, then $f = \mathrm{id}_{\mathbb{R}^n} \circ f \sim 0_{\mathbb{R}^n} \circ f = 0_X$, so $[X, \mathbb{R}^n]$ has only one element.

Definition. *A space Y is contractible if $\mathrm{id}_Y \sim c_p$ where $c_p : Y \to Y, y \mapsto p$ is the constant map with image $p \in Y$.*

Proposition 0.3. *Y is contractible iff $[X, Y]$ has one element for all (non-empty) X.*

Proof. $\Rightarrow$: as in the example with $\mathbb{R}^n$.

$\Leftarrow$: Take $X = Y$. Since $[X, Y]$ has only one element, the homotopy classes of id_Y and c_p are equal, i.e. $\mathrm{id}_Y \sim c_p$ (for any $p \in Y$). $\square$

Definition. *Spaces X and Y are* homotopy equivalent*, written $X \sim Y$, if there exist maps $f : X \to Y, g : Y \to X$ such that $f \circ g \sim \mathrm{id}_Y, g \circ f \sim \mathrm{id}_X$.*

Examples.

- $\mathbb{R}^n \sim \{0\}$.
- Y is contractible iff $Y \sim \{*\}$.
- $\mathbb{R}^n \setminus \{0\} \sim S^{n-1}$.

Basic questions of Algebraic Topology:

1. Given spaces X and Y, is $X \sim Y$?
2. What is $[X, Y]$?

Definition. *A* pair of spaces *(X, A) is a space X and a subset $A \subseteq X$. A* map of pairs *is $f : (X, A) \to (Y, B)$ is a continuous map $f : X \to Y$ such that $f(A) \subseteq B$.*

Maps of pairs $f_0, f_1 : (X, A) \to (Y, B)$ are homotopic*, written $f_0 \sim f_1$, if $f_0, f_1 : X \to Y$ are homotopic via a map of pairs $H : (X \times I, A \times I) \to (Y, B)$. Write $[(X, A), (Y, B)]$ for the set of equivalence classes of maps of pairs $(X, A) \to (Y, B)$.*

0.1 Homotopy Groups

Definition. *If X is a space and $p \in X$, the n-*th homotopy group *is*

$$\pi_n(X, p) = [(I^n, \delta I^n), (X, p)] = [(D^n, S^{n-1}), (X, p)] = [(S^n, *), (X, p)].$$

(if $n = 0$ take the last set as the definition)

Proposition 0.4.

1. *The group structure for $n \geq 1$ is given as follows: For $\varphi, \psi : (I^n, \partial I^n) \to (X, p)$ let $[\varphi] \cdot [\psi] = [\varphi \cdot \psi]$ where*

$$\varphi \cdot \psi : (I^n, \partial I^n) \to (X, p), (t_1, \ldots, t_n) \mapsto \begin{cases} \varphi(2t_1, t_2, \ldots, t_n) & 0 \leq t_1 \leq \frac{1}{2}, \\ \psi(2t_1 - 1, t_2, \ldots, t_n) & \frac{1}{2} \leq t_1 \leq 1 \end{cases}$$

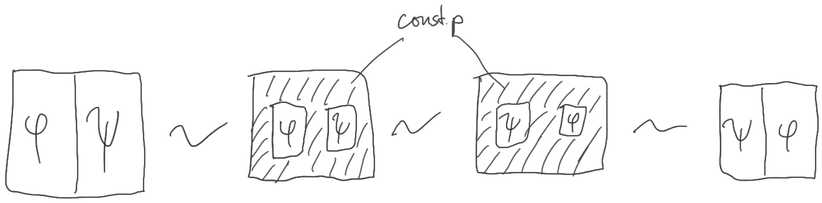
   Then:

   - $\pi_0(X, p) = \{$*path components of* $X\}$,
   - $\pi_1(X, p)$ *is a group,*
   - $\pi_n(X, p)$ *is an abelian group for* $n > 1$.

2. *Functoriality: If $f : (X, p) \to (Y, q)$ is a map of pairs, it induces $f_* : \pi_n(X, p) \to \pi_n(Y, q)$ by $f_*([\varphi]) = [f \circ \varphi]$. This satisfies $(f \circ g)_* = f_* \circ g_*$*

3. *Homotopy invariance: If $f_0, f_1 : (X, p) \to (Y, q)$ are homotopic as maps of pairs, then $f_{0*} = f_{1*}$.*

Group structure for $n = 2$



π_n is abelian for $n = 2$

Theorem 0.5. $\pi_1(S^n, *) = \begin{cases} \mathbb{Z} & n = 1, \\ 0 & \textit{otherwise.} \end{cases}$

But $\pi_n(S^k)$ is very complicated in general, e.g.:

n	1	2	3	4	5	6	7	8	9	10
$\pi_n(S^2)$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{Z}/15$

This is why we study *homology* instead of homotopy groups in this course.

1 Singular Homology

1.1 Definition of Homology

Definition. *The standard k-simplex is* $\Delta^k := \{(t_0, \dots, t_k) \in \mathbb{R}^{k+1} \mid \sum t_i = 1, t_i \geq 0\}$.

For $I \subseteq \{0, \dots, k\}$, we associate a face $f_I = \{t \in \Delta^k \mid t_i = 0$ for $i \notin I\}$. There is an obvious inclusion map $F_I : \Delta^{|I|-1} \to \Delta^k$ with image f_I.

We will write $I = i_0 \cdots i_k$ if $I = \{i_0, \dots, i_k\}$ and $i_0 < i_1 < \cdots < i_k$.

Recall that a ($\mathbb{Z}$-graded) chain complex $(C_\bullet, d)$ over a commutative ring R consists of R-modules C_k, $k \in \mathbb{Z}$ and homomorphisms $d_k : C_k \to C_{k-1}$ such that $d_k \circ d_{k+1} = 0$ for all k.

The k-th homology group of such a chain complex is the quotient $H_k(C_\bullet) = \ker d_k / \operatorname{Im} d_{k+1}$.

Elements of $\ker d$ are called *cycles*, and elements of $\operatorname{Im} d$ *boundaries*.

Definition. *The chain complex $S_\bullet(\Delta^n)$ of the n-simplex is given by $S_k(\Delta^n) = \langle f_I \mid I \subseteq \{0, \dots, n\}, |I| = k+1\rangle$. For $k > 0$ the boundary map is given by*

$$d(f_I) = \sum_{j=0}^{k} (-1)^j f_{I \setminus \{i_j\}}$$

where $I = i_0 \cdots i_k$ and we set $d(f_I) = 0$ if $I = i_0$.

It is easy to see that $d^2 = 0$, so this is indeed a chain complex.

The following is true[1]:

$$H_i(S_\bullet(\Delta^n)) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Definition. *The* reduced chain complex associated to Δ^n *is the chain complex $(\widetilde{S}_\bullet(\Delta^n), d)$ with $\widetilde{S}_k(\Delta^n) = S_k(\Delta^n)$ for $k \neq -1$ and $\widetilde{S}_{-1}(\Delta^n) = \langle f_\emptyset \rangle$. The differential is defined using the formula above above, now including $k = 0$, i.e. $df_{\{i\}} = f_\emptyset$.*

Then one has $H_*(\tilde{S}_\bullet(\Delta^n)) = 0$.

[1] Remark by L.T.: See e.g. [Rot88, Corollary 7.18]

Definition. *For a space X its* singular chain complex $(C_\bullet(X), d)$ *is defined by* $C_k(X) = \langle \sigma : \Delta^k \to X \rangle$ *for* $k \geq 0$ *and* $C_k(X) = 0$ *for* $k < 0$*. For* $\sigma : \Delta^k \to X$ *the differential* $d\sigma$ *is given by*

$$d\sigma = \sum_{j=0}^{k} (-1)^j \sigma \circ F_{\hat{j}}$$

where $F_{\hat{j}} = F_{\{0,\ldots,k\}\setminus\{j\}} : \Delta^{k-1} \to \Delta^k$ *is the inclusion onto the* j*-th face.*

Note that if $\sigma : \Delta^k \to X$, then we obtain a map $\phi_\sigma : S_\bullet(\Delta^k) \to C_\bullet(X)$ by $f_I \mapsto \sigma \circ F_I$. By definition of d this satisfies $d_C \circ \phi_\sigma = \phi_\sigma \circ d_S$. From this one easily deduces that $d_C^2 = 0$.

Definition. $H_i(X) = H_i(C_\bullet(X))$ *is the* i-th singular homology group of X*.*

Example: Let $X = \{*\}$ be a one-point space. Then for $k \geq 0$, $C_k(X) = \langle \sigma_k \rangle$ where $\sigma_k : \Delta^k \to X$ is the unique map. For $k > 0$ we have $d\sigma_k = \sum_{j=0}^{k} (-1)^j \sigma_{k-1} = \begin{cases} \sigma_{k-1} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$ For $k = 0$ we get $d\sigma_0 = 0$, thus

$$H_k(X) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Definition. *The* reduced singular chain complex *of X is defined by*

$$\widetilde{C}_k(X) = \begin{cases} C_k(X) & k \neq -1, \\ \langle \sigma_\emptyset \rangle & k = -1. \end{cases}$$

with $d\sigma = \sigma_\emptyset$ *if* $\sigma : \Delta^0 \to X$ *and* $d\sigma_\emptyset = 0$

Exercise: $\widetilde{H}_k(\{*\}) = 0$ for all k.

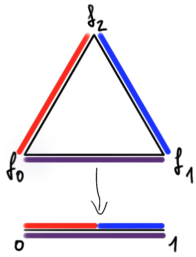
Examples.

- $\Delta^0 = \{*\}$, so elements of $\mathrm{Map}(\Delta^0, X)$ correspond to points in X.
- $\Delta^1 \cong I$, via (say) $f_0 \mapsto 0, f_1, \mapsto 1$ and then extended linearly. Then elements of $\mathrm{Map}(\Delta^1, X)$ correspond to paths $\gamma : [0,1] \to X$ with $d\gamma = \sigma_{\gamma(1)} - \sigma_{\gamma(0)}$

  Example: $X = S^1$, $\gamma : [0,1] \to S^1, t \mapsto e^{2\pi i t}$, then $d\gamma = 0$, so γ is a cycle. Define $\gamma_\pm : I \to S^1, t \mapsto e^{\pm \pi i t}$. Then $d\gamma_\pm = \sigma_{-1} - \sigma_1$, so $\gamma_+ - \gamma_-$ is a cycle in C_1.

  Claim: $[\gamma] = [\gamma_+ - \gamma_-]$. Consider $\tau : \Delta^2 \to S^1$ given by $\tau(p) = e^{2\pi i \varphi(p)}$ where $\varphi : \Delta^2 \to I$ is the affine linear map given by $f_0 \mapsto 0, f_1 \mapsto 1, f_2 \mapsto \frac{1}{2}$. Then $d\tau = \tau \circ F_{\hat{0}} - \tau \circ F_{\hat{1}} + \tau \circ F_{\hat{2}} = \gamma_- - \gamma_+ + \gamma$.

Proposition 1.1. *If X is path connected, then* $H_0(X) \cong \mathbb{Z} = \langle \sigma_p \rangle$ *for any* $p \in X$*.*



The map φ

Proof. $C_{-1}(X) = 0$, so $\ker d_0 = C_0(X)$.

$$\begin{aligned}\operatorname{Im} d_1 &= \operatorname{span}\{d\gamma \mid \gamma : I \to X\} \\ &= \operatorname{span}\{\sigma_p - \sigma_{p'} \mid p, p' \text{ joined by a path in } X\} \\ &= \operatorname{span}\{\sigma_p - \sigma_{p'} \mid p, p' \in X\}\end{aligned}$$

Then $H_0(X) = \ker d_0 / \operatorname{Im} d_1 \cong \mathbb{Z}$ via $\sum a_i \sigma_{p_i} \mapsto \sum a_i$. □

1.2 Subcomplexes, Quotient Complexes and Direct Sums

Definition. *Suppose (C, d) is a chain complex over R. A subcomplex of (C, d) consists of submodules $A_i \subseteq C_i$ for all i such that $d(A_i) \subseteq A_{i-1}$. Then $A = \bigoplus_i A_i$ is a again a chain complex with the differential being the restriction of d.*

Given a subcomplex A of C, we can form the quotient $(C/A, d)$ where $C/A = \bigoplus_i C_i/A_i$.

Example. If $A \subseteq X$ is a subspace, then $C_\bullet(A)$ is a subcomplex of $C_\bullet(X)$.

Definition. *If (X, A) is a pair of spaces, then $C_\bullet(X, A) = C_\bullet(X)/C_\bullet(A)$ is the* singular chain complex *of (X, A).*

Definition. *If $(C_\alpha, d_\alpha)_{\alpha \in A}$ are chain complexes, then their direct sum is $(\bigoplus_\alpha C_\alpha, \bigoplus_\alpha d_\alpha)$ is also a chain complex.*

Easy exercise: $H_*(\bigoplus_\alpha C_\alpha) = \bigoplus_\alpha H_*(C_\alpha)$.

Proposition 1.2. *$H_*(X) = \bigoplus_\alpha H_*(X_\alpha)$ where the X_α are the path-components of X*

Proof. Since Δ^k is (path-)connected, we have $\operatorname{Map}(\Delta^k, X) = \coprod_\alpha \operatorname{Map}(\Delta^k, X_\alpha)$, so $C_k(X) = \bigoplus_\alpha C_k(X_\alpha)$ and this decomposition respects d, so we have a direct sum of chain complexes. □

Definition. *If (C, d) and (C', d') are chain complex over R, a* chain map *$f : (C, d) \to (C', d')$ is a collection of R-linear maps $f_i : C_i \to C'$ such that $d'_i \circ f_i = f_{i-1} \circ d_i$, in other words $d'f = fd$ where $f = \bigoplus_i f_i$.*

Notation. We denote categories as follows:

$$\left\{ \begin{array}{c} \text{Objects} \\ \text{Morphisms} \end{array} \right\}$$

Note that a chain map $f : (C, d) \to (C', d')$ induces a map $f_* : H_*(C) \to H_*(C')$. So taking homology gives a functor:

$$H_* : \left\{ \begin{array}{c} \text{chain complexes over } R \\ \text{chain maps} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{(graded) } R\text{-modules} \\ \text{(graded) } R\text{-linear maps} \end{array} \right\}$$
$$(C, d) \longmapsto H_*(C)$$
$$f : C \to C' \longmapsto f_* : H_*(C) \to H_*(C')$$

Definition. *If $f : X \to Y$ is a continuous map, define $f_\# : C_\bullet(X) \to C_\bullet(Y)$ by* $\mathrm{Map}(\Delta^*, X) \ni \sigma \mapsto f_\#(\sigma) = f \circ \sigma$.

Lemma 1.3. *$f_\#$ is a chain map.*

Proof. $d(f_\#(\sigma)) = d(f \circ \sigma) = \sum_{j=0}^{k} (-1)^j f \circ \sigma \circ F_{\hat{j}} = f_\# \big(\sum_{j=0}^{k} (-1)^j \sigma \circ F_{\hat{j}} \big) = f_\# d\sigma$ $\square$

So we get a functor

$$\left\{ \begin{array}{c} \text{spaces} \\ \text{continuous maps} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{chain complexes over } \mathbb{Z} \\ \text{chain maps} \end{array} \right\}$$
$$X \longmapsto (C_\bullet(X), d)$$
$$f \longmapsto f_\#$$

Composing the functors we get the *singular homology functor*:

$$\left\{ \begin{array}{c} \text{spaces} \\ \text{continuous maps} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{graded } \mathbb{Z}\text{-modules} \\ \text{graded linear maps} \end{array} \right\}$$
$$X \longmapsto H_*(X)$$
$$f : X \to Y \longmapsto f_* : H_*(X) \to H_*(Y)$$

Suppose $f : (X, A) \to (Y, B)$. Then $f_\# : C_\bullet(X) \to C_\bullet(Y)$. If $\sigma : \Delta^k \to A$, then $f \circ \sigma : \Delta^k \to B$, so $f_\#(C_\bullet(A)) \subseteq C_\bullet(B)$. Thus $f_\#$ descends to a map $f_\# : C_\bullet(X, A) \to C_\bullet(Y, B)$. Hence we get functors:

$$\left\{ \begin{array}{c} \text{pairs of spaces} \\ \text{maps of pairs} \end{array} \right\} \xrightarrow{C_\bullet(-,-)} \left\{ \begin{array}{c} \text{chain complexes over } \mathbb{Z} \\ \text{chain maps} \end{array} \right\} \xrightarrow{H_*} \left\{ \begin{array}{c} \mathbb{Z}\text{-modules} \\ \mathbb{Z}\text{-linear maps} \end{array} \right\}$$

1.3 Homotopy Invariance

Goal: We want to prove that homotopic maps of spaces induce the same maps on homology.

Definition. *Suppose $g_0, g_1 : C \to C'$ are maps of chain complexes (over some ring R). g_0 is* chain homotopic *to g_1, written $g_0 \sim g_1$, if there are R-linear maps $h_i : C_i \to C'_{i+1}$ such that $d'h + hd = g_1 - g_0$ where $h = \oplus h_i$.*

Chain complexes C, C' are chain homotopy equivalent*, written $C \sim C'$, if there are chain maps $f : C \to C', g : C' \to C$ such that $f \circ g \sim 1_{C'}, g \circ f \sim 1_C$.*

Lemma 1.4. *Chain homotopy and chain homotopy equivalence are equivalence relations.*

Proposition 1.5. *If $g_0, g_1 : C \to C'$ are chain maps with $g_0 \sim g_1$, then*

$$g_{0*} = g_{1*} : H_*(C) \to H_*(C').$$

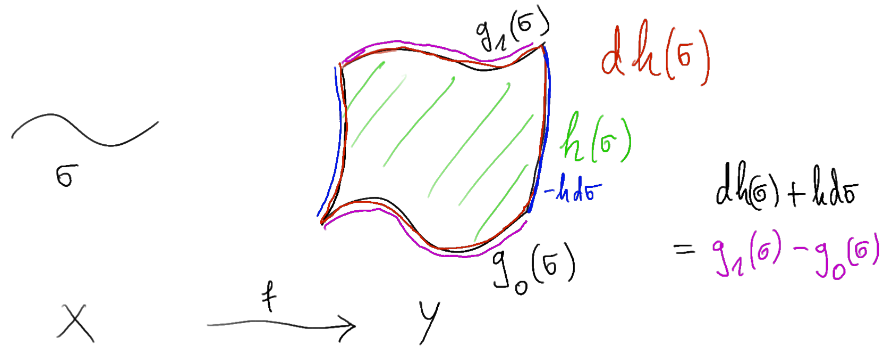
Proof. Suppose the $g_0 \sim g_1$ via h. If $[x] \in H_*(C)$, $dx = 0$, so

$$g_{1*}[x] - g_{0*}[x] = [g_1(x) - g_0(x)] = [d'h(x) + hd(x)] = [d'h(x)] = 0.$$

□

Corollary 1.6. *If $C \sim C'$, then $H_*(C) \cong H_*(C')$.*

Idea behind the definition of chain homotopy: Suppose $f_0, f_1 : X \to Y$, $f_0 \sim f_1$ via $H : X \times I \to Y$. Let $g_0(\sigma) = f_{0*}(\sigma), g_1(\sigma) = f_{1*}(\sigma)$. Want $h(\sigma) =$ "$H(\sigma \times I)$".



Idea for the chain homotopy

Recall if $\sigma : \Delta^k \to X$, there is a chain map $\varphi_\sigma : S_\bullet(\Delta^k) \to C_\bullet(X), f_I \mapsto \sigma \circ F_I$.

Define $c_0, c_1 : \Delta^n \mapsto \Delta^n \times I$ by $c_i(x) = (x, i)$, $i = 0, 1$. From this we get $\varphi_{c_0}, \varphi_{c_1} : S_\bullet(\Delta^n) \to C_\bullet(\Delta^n \times I)$.

Definition. *If $X \subseteq \mathbb{R}^N$ is convex and $v_0, \ldots, v_k \in X$, define a k-simplex in X by*

$$\begin{aligned}[v_0, \ldots, v_k] : \Delta^k &\longrightarrow X, \\ (t_i)_i &\longmapsto \sum_i t_i v_i,\end{aligned}$$

$[v_0, \ldots, v_k]$ *is the* linear simplex determined by $v_0, \ldots, v_k$.

Note that $[v_0, \ldots, v_k] \circ F_{\hat{j}} = [v_0 \ldots \widehat{v_j} \ldots v_k]$ (omit v_j), so that

$$d[v_0 \ldots v_k] = \sum_j (-1)^j [v_0 \ldots \widehat{v_j} \ldots v_k].$$

To avoid lots of indices, we use the following notation: If $f_i \in \Delta^n$, $i = 0, \ldots, n$, write $i = f_i \times 0, i' = f_i \times 1 \in \Delta^n \times I$.

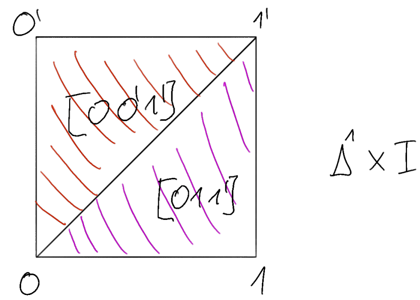
Notational warning: In the following we will use I for two different things: An index set or the interval $[0,1]$. Whenever it is used for $[0,1]$ it occurs only in the form $\Delta^n \times I$, so this will hopefully cause no confusion.

Definition. *The* universal chain homotopy $U_n : S_\bullet(\Delta^n) \to C_{\bullet+1}(\Delta^n \times I)$ *is given by*

$$U_n(f_I) = \sum_{j'=0}^{k} (-1)^{j'} [i_0 \ldots i_{j'} i'_{j'} i'_{j'+1} \ldots i'_k]$$

where $I = i_0 \ldots i_k$.

U_n "breaks up" $\Delta^n \times I$ into simplices. For example, for $n = 1$ we have $U_1(f_{01}) = [00'1'] - [011']$.



$n = 1$

Proposition 1.7. $dU_n + U_n d = \varphi_{c_1} - \varphi_{c_0}$.

Proof. Let $I = i_0 \ldots i_k$. What terms appear in $(dU_n + U_n d)(f_I)$?

$$(dU_n + U_n d)(f_I) = \sum_{j < j'} m_{jj'} [i_0 \ldots \widehat{i_j} \ldots i_{j'} i'_{j'} \ldots i'_k]$$

$$
\begin{aligned}
&+ \sum_{j'<j} n_{jj'}[i_0 \dots i_{j'} i'_{j'} \dots \widehat{i_j}' \dots i'_k] \\
&+ \sum_{j=0}^{k-1} r_j [i_0 \dots i_j i'_{j+1} \dots i'_k] \\
&+ a[i_0 \dots i_k] + b[i'_0 \dots i'_k]
\end{aligned}
$$

We have

$$
\begin{aligned}
m_{jj'} &= \underbrace{(-1)^j(-1)^{j'-1}}_{\substack{\text{delete } i_j \\ \text{split at } j'}} + \underbrace{(-1)^{j'}(-1)^j}_{\substack{\text{split at } j' \\ \text{delete } i_j}} = 0, \\
n_{jj'} &= \underbrace{(-1)^j(-1)^{j'}}_{\substack{\text{delete } i_j \\ \text{split at } j'}} + \underbrace{(-1)^{j'}(-1)^{j+1}}_{\substack{\text{split at } j' \\ \text{delete } i'_j}} = 0, \\
r_j &= \underbrace{(-1)^j(-1)^{j+1}}_{\substack{\text{split at } j \\ \text{delete } i'_j}} + \underbrace{(-1)^{j+1}(-1)^{j+1}}_{\substack{\text{split at } j+1 \\ \text{delete } i_{j+1}}} = 0, \\
a &= \underbrace{(-1)^k(-1)^{k+1}}_{\substack{\text{split at } k \\ \text{delete } i'_k}} = -1, \\
b &= \underbrace{(-1)^0(-1)^0}_{\substack{\text{split at } 0 \\ \text{delete } i_0}} = 1.
\end{aligned}
$$

So

$$(dU_n + U_n d)(f_I) = [i'_0 \dots i'_k] - [i_0 \dots i_k] = \varphi_{c_0}(f_I) - \varphi_{c_1}(f_I).$$

□

Let $i_0 \dots i_k = I \subseteq \{0, \dots, n\}$. This gives a chain map $\varphi_I : S_\bullet(\Delta^k) \to S_\bullet(\Delta^n)$ with $\varphi(f_J) = f_{i_{j_0} i_{j_1} \dots i_{j_l}}$ where $J = j_0 \dots j_l$. (i.e. the J-face of Δ^k gets mapped to the corresponding face of the I-face of Δ^n).

Let $\varphi_{\hat{j}} = \varphi_{\{0,\dots,n\}\setminus\{j\}} : S_\bullet(\Delta^{n-1}) \to S_\bullet(\Delta^n)$ and $f^n_{\text{top}} = f_{0\dots n} \in S_n(\Delta^n)$ (i.e. top face, the whole simplex). Then $df^n_{\text{top}} = \sum_j (-1)^j \varphi_{\hat{j}}(f^{n-1}_{\text{top}})$.

Lemma 1.8 (Naturality of U_n)**.** *The following square commutes:*

$$
\begin{array}{ccc}
S_\bullet(\Delta^k) & \xrightarrow{\varphi_I} & S_\bullet(\Delta^n) \\
\downarrow{\scriptstyle U_k} & & \downarrow{\scriptstyle U_n} \\
C_{\bullet+1}(\Delta^k \times I) & \xrightarrow{\overline{F_I}_\#} & C_{\bullet+1}(\Delta^n \times I)
\end{array}
$$

where $\overline{F_I} : \Delta^k \times I \to \Delta^n \times I, (x,t) \mapsto (F_I(x), t)$.

Proof. Immediate by writing out the maps. $\square$

Now suppose that $f_0, f_1 : X \to Y$ are homotopic via $H : X \times I \to Y$. Given $\sigma : \Delta^n \to X$, define $H_\sigma : \Delta^n \times I \to Y$ by $(x,t) \mapsto H(\sigma(x), t)$. Observe that $H_{\sigma \circ F_I} = H_\sigma \circ \overline{F_I}$.

Define $h : C_\bullet(X) \to C_{\bullet+1}(Y)$ by $h(\sigma) = H_{\sigma\#}(U_n(f^n_{\text{top}}))$ if $\sigma : \Delta^n \to X$.

Theorem 1.9. *$dh + hd = f_{1\#} - f_{0\#}$, so $f_{0\#} \sim f_{1\#}$.*

Proof. Let $\sigma : \Delta^n \to X$. Then

$$\begin{aligned}
hd(\sigma) &= h\Big(\sum_j (-1)^j \sigma \circ F_{\hat{j}}\Big) \\
&= \sum_j (-1)^j H_{\sigma F_{\hat{j}}\#} U_{n-1}(f^{n-1}_{\text{top}}) \\
&= \sum_j (-1)^j H_{\sigma\#} \overline{F_{\hat{j}}}_{\#} U_{n-1}(f^{n-1}_{\text{top}}) \\
&= \sum_j (-1)^j H_{\sigma\#} U_n(\varphi_{\hat{j}}(f^{n-1}_{\text{top}})) \\
&= H_{\sigma\#} U_n\Big(\sum_j (-1)^j \varphi_{\hat{j}}(f^{n-1}_{\text{top}})\Big) \\
&= H_{\sigma\#} U_n(d f^n_{\text{top}})
\end{aligned}$$

We also have $dh(\sigma) = dH_{\sigma\#}(U_n(f^n_{\text{top}})) = H_{\sigma\#}(dU_n(f^n_{\text{top}}))$. Thus

$$\begin{aligned}
(hd + dh)(\sigma) &= H_{\sigma\#}(U_n(d f^n_{\text{top}} + dU_n(f^n_{\text{top}}))) \\
&= H_{\sigma\#}(\varphi_{c_1}(f^n_{\text{top}}) - \varphi_{c_0}(f^n_{\text{top}})) \\
&= H_{\sigma\#}(c_1 \circ F_{\{0,\dots,n\}} - c_0 \circ F_{\{0,\dots,n\}}) \\
&= H_{\sigma\#}(c_1) - H_{\sigma\#}(c_0) \\
&= f_{1\#}(\sigma) - f_{0\#}(\sigma)
\end{aligned}$$

$\square$

Corollary 1.10. *If $f_0, f_1 : X \to Y$ are homotopic, then $f_{0*} = f_{1*}$.*

Corollary 1.11. *If $f : X \to Y$ is a homotopy equivalence, then $f_* : H_*(X) \to H_*(Y)$ is an isomorphism.*

Corollary 1.12. *If X is contractible, then*

$$H_i(X) \cong \begin{cases} \mathbb{Z} & i = 0, \\ 0 & i \neq 0. \end{cases}$$

1.4 Subdivision

1.4.1 Some Homological Algebra

Lemma 1.13 (Snake Lemma/Long exact sequence of Homology)**.** *Let*

$$0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$$

be a short exact sequence (SES) of chain complexes. Then there is a long exact sequence (LES)in homology:

$$\cdots \to H_{i+1}(C) \xrightarrow{\partial} H_i(A) \xrightarrow{\iota_*} H_i(B) \xrightarrow{\pi_*} H_i(C) \xrightarrow{\partial} H_{i-1}(C) \to \cdots$$

Proof. ∂ is defined as follows: Let $[c] \in H_i(C)$, so $c \in C_i$ and $dc = 0$. Then there is a $b \in B_i$ such that $\pi(b) = c$. As $\pi(db) = d(\pi b) = dc = 0$, we have $db \in \ker \pi$, so there is $a \in A_{i-1}$ with $\iota(a) = db$. Then $\iota(da) = d\iota(a) = d(db) = 0$, so $da = 0$ as ι is injective. Define $\partial[a] = [c] \in H_{i-1}(A)$. That this is well-defined and gives the exact sequence is a straightforward diagram chase... $\square$

Corollary 1.14 (LES of a pair)**.** *Let (X, A) be a pair of spaces. Then there is a long exact sequence:*

$$\cdots \to H_{i+1}(X, A) \xrightarrow{\partial} H_i(A) \xrightarrow{\iota_*} H_i(X) \xrightarrow{\pi_*} H_i(X, A) \xrightarrow{\partial} H_{i-1}(A) \to \cdots$$

Example. For $p \in X$, we have $H_i(\{p\}) = 0$ for $i \neq 0$ and $H_i(\{p\}) = \mathbb{Z}$ for $i = 0$ in which case it is generated by $[\sigma_p]$ where $\sigma_p : \Delta^0 \to X, * \mapsto p$. So the LES of the pair $(X, \{p\})$ is:

$$\cdots \to 0 = H_{i+1}(\{p\}) \to H_{i+1}(X) \to H_{i+1}(X, \{p\}) \to H_i(\{p\}) = 0 \to \cdots$$

for $i > 0$. Hence $H_{i+1}(X) \to H_{i+1}(X, \{p\})$ is an isomorphism. At $i = 0$ we have:

$$0 = H_1(\{p\}) \to H_1(X) \to H_1(X, \{p\}) \xrightarrow{\partial_1} \underbrace{H_0(\{p\})}_{\cong \mathbb{Z}} \xrightarrow{i_*} H_0(X) \to H_0(X, \{p\}) \to 0$$

Note that $i_*(n[\sigma_p]) = n[\sigma_p] \neq 0$ for $n \neq 0$, so i_* is injective and thus $\partial_1 = 0$. Hence also $H_1(X) \to H_1(X, \{p\})$ is an isomorphism. We know that $H_0(X) = \bigoplus_\alpha \mathbb{Z}$ where α runs through the set of path components of X and i_* maps onto the factor $\mathbb{Z}$ corresponding to the path component of p, hence $H_0(X) = H_0(X, \{p\}) \oplus \langle [\sigma_p] \rangle$. This discussion gives:

Corollary 1.15. *For $A = \{p\}$ a point in X we have*

$$H_i(X) \cong \begin{cases} H_i(X, p) & i > 0, \\ H_0(X, p) \oplus \mathbb{Z} & i = 0 \end{cases}.$$

Lemma 1.16. *$\widetilde{H}_i(X) \cong H_i(X, p)$ for all $i \geq 0$.*

Proof. Define $\widetilde{C}_\bullet(X,p) = \widetilde{C}_\bullet(X)/\widetilde{C}_\bullet(p) \cong C_\bullet(X)/C_\bullet(p) = C_\bullet(X,p)$, i.e. $\widetilde{H}_*(X,p) = H_*(X,p)$. We have a SES

$$0 \to \widetilde{C}_\bullet(p) \to \widetilde{C}_\bullet(X) \to \widetilde{C}_\bullet(X,p) \to 0$$

which gives a LES

$$\cdots \to \widetilde{H}_i(p) \to \widetilde{H}_i(X) \to \widetilde{H}_i(X,p) \to \widetilde{H}_{i-1}(p) \to \dots$$

We know $\widetilde{H}_*(p) = 0$, so $\widetilde{H}_i(X) \cong \widetilde{H}_i(X,p) \cong H_i(X,p)$. $\square$

1.4.2 Subdivision

Suppose $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ is an open cover of X. Define

$$C_k^{\mathcal{U}}(X) = \langle \sigma \mid \sigma : \Delta^k \to X \text{ such that } \operatorname{im}\sigma \in U_\alpha \text{ for some } \alpha \rangle.$$

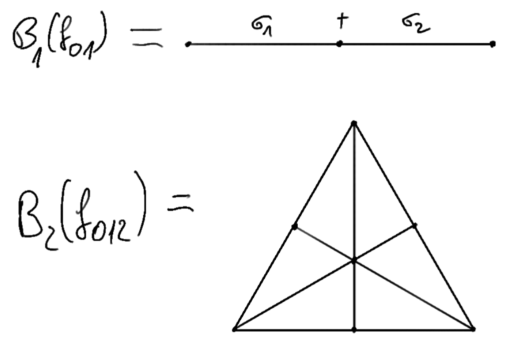
If $\operatorname{im}\sigma \in U_\alpha$, then $\operatorname{im}\sigma \circ F_{\hat{\jmath}} \subseteq U_\alpha$, so $d\sigma \in C_{k-1}^{\mathcal{U}}(X)$, i.e. $C_*^{\mathcal{U}}(X)$ is a subcomplex of $C_*(X)$. Let $i : C_*^{\mathcal{U}}(X) \to C_*(X)$ be the inclusion.

Lemma 1.17 (Subdivision lemma)**.** *If $\mathcal{U}$ is an open cover of X, then*

$$i_* : H_*^{\mathcal{U}}(X) \to H_*(X)$$

is an isomorphism.

Proof (idea only). (1) Define natural maps $B_n : S_*(\Delta^n) \to C_*(\Delta^n), H_n : C_*(\Delta^n) \to C_{*+1}(\Delta^n)$. B_n is defined inductively via barycentric subdivision. They satifsy $dH_n + H_n d = B_n - \varphi_{\mathrm{id}_{\Delta^n}}$.



Barycentric subdivision of Δ^n for $n = 1, 2$.

(2) Use B_n, H_n to define $B : C_*(X) \to C_*(X), H : S_*(X) \to C_*(X)$ with $dH + Hd = B - \mathrm{id}_{C_*(X)}$.

(3) If $c \in C_k(X)$ and $\mathcal{U}$ is an open cover of X, then there exists N such that $B^N c \in C_*^{\mathcal{U}}(X)$, so $[c] = [B^N c]$, so i_* is surjective. And similarly one shows that i_* is injective.

See handout for the details. $\square$

1.4.3 Mayer-Vietoris Sequence

Suppose $U_1, U_2 \subseteq X$ are open, $U_1 \cup U_2 = X$, so $\{U_1, U_2\} = \mathcal{U}$ is an open cover of X. We then have a commutative diagram of inclusions:

$$\begin{tikzcd} & U_1 \arrow[dr, "j_1"] & \\ U_1 \cap U_2 \arrow[ur, hook, "i_1"] \arrow[dr, "i_2"'] & & X \\ & U_2 \arrow[ur, hook, "j_1"'] & \end{tikzcd}$$

Proposition 1.18. *There is a SES*

$$0 \to C_*(U_1 \cap U_2) \xrightarrow{i} C_*(U_1) \oplus C_*(U_2) \xrightarrow{j} C_*^{\mathcal{U}}(X) \to 0$$

where $i = \begin{bmatrix} i_{1\#} \\ i_{2\#} \end{bmatrix}, j = \begin{bmatrix} j_{1\#} - j_{2\#} \end{bmatrix}$.

Proof. It is clear that $i_{1\#}, i_{2\#}$ are injective, so i is injective.

Exactness at $C_*(U_1) \oplus C_*(U_2)$: We have $j \circ i = j_{1\#}i_{1\#} - j_{2\#}i_{2\#} = 0$. Suppose $j(a, b) = 0$, $a = \sum a_i\sigma_i, a_i \neq 0, \sigma_i \neq \sigma_j$ for $i \neq j$, $\operatorname{im}\sigma_i \subseteq U_1$ and similarly $b = \sum b_j\tau_j$. But if $j(a, b) = 0$, then $\sum a_i\sigma_i = \sum b_j\tau_j$ which can only happen if (after reordering indices) if $a_i = b_i, \sigma_i = \tau_i$, so $\operatorname{im}\sigma_i \subseteq U_1 \cap U_2$, so if $c = \sum a_i\sigma_i \in C_*(U_1 \cap U_2)$, then $i(c) = (a, b)$.

Exactness at $C_*^{\mathcal{U}}(X)$: If $c \in C_k^{\mathcal{U}}(X)$, we can write $c = \sum a_i\sigma_i + \sum b_j\tau_j$ where $\operatorname{im}\sigma_i \subseteq U_1, \operatorname{im}\tau_j \subseteq U_2$, so $c = j(a, -b)$ and j is surjective. ☐

By the Subdivision Lemma we have $H_*^{\mathcal{U}}(X) = H_*(X)$, hence we obtain:

Corollary 1.19 (Mayer-Vietoris Sequence)**.** *If $U_1, U_2 \subseteq X$ are open, $U_1 \cup U_2 = X$, there is a LES*

$$\ldots \xrightarrow{\partial} H_i(U_1 \cap U_2) \xrightarrow{i} H_i(U_1) \oplus H_i(U_2) \xrightarrow{j} H_i(X) \xrightarrow{\partial} H_{i-1}(U_1 \cap U_2) \to \ldots$$

Note that

$$0 \to \widetilde{C}_*(U_1 \cap U_2) \xrightarrow{i} \widetilde{C}_*(U_1) \oplus \widetilde{C}_*(U_2) \xrightarrow{j} \widetilde{C}_*^{\mathcal{U}}(X) \to 0$$

is also exact: It only differs from the non-reduced complex in degree -1 where the sequence becomes

$$0 \to \mathbb{Z} \xrightarrow{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{bmatrix} 1 & -1 \end{bmatrix}} \mathbb{Z} \to 0$$

Hence we also get a reduced version of the Mayer-Vietoris sequence:

$$\ldots \xrightarrow{\partial} \widetilde{H}_i(U_1 \cap U_2) \xrightarrow{i} \widetilde{H}_i(U_1) \oplus \widetilde{H}_i(U_2) \xrightarrow{j} \widetilde{H}_i(X) \xrightarrow{\partial} \widetilde{H}_{i-1}(U_1 \cap U_2) \to \ldots$$

1.4.4 Homology of S^n

Proposition 1.20.

$$\widetilde{H}_i(S^n) \cong \begin{cases} \mathbb{Z} & i = n, \\ 0 & i \neq n \end{cases}$$

Proof. By induction on n. If $n = 0$, we have $S^0 = \{\pm 1\}$, so

$$H_*(S^0) = H_*(\{1\}) \oplus H_*(\{-1\}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & i = 0, \\ 0 & i \neq 0, \end{cases}$$

and therefore $\widetilde{H}_i(S^0) \cong \begin{cases} \mathbb{Z} & i = 0, \\ 0 & i \neq 0. \end{cases}$

In general, let $U_+ = S^n \setminus \{(-1, 0, \dots, 0)\}, U_- = S^n \setminus \{1, 0, \dots, 0\}$. Note that $U_\pm \cong \mathbb{R}^n \cong D^{n\circ}$ by stereographic projection, so contractible, while $U_+ \cap U_- = S^n \setminus \{(\pm 1, 0, \dots, 0)\} \cong I^\circ \times S^{n-1}$ is homotopic to S^{n-1} via

$$\begin{aligned} p : U_+ \cap U_- &\longrightarrow S^{n-1}, \\ (x_1, \dots, x_{n+1}) &\longmapsto \frac{1}{\sqrt{x_2^2 + \dots + x_{n+1}^2}} (x_2, x_3, \dots, x_{n+1}). \end{aligned}$$

The MV-sequence is

$$\dots \to \widetilde{H}_i(U_1) \oplus \widetilde{H}_i(U_2) \to \widetilde{H}_i(S^n) \xrightarrow{\partial} \widetilde{H}_{i-1}(U_+ \cap U_-) \to \widetilde{H}_{i-1}(U_+) \oplus \widetilde{H}_{i-1}(U_-) \to \dots$$

As the $U_\pm$ are contractible we get that ∂ is an isomorphism. Hence $\widetilde{H}_i(S^n) \xrightarrow{\partial} \widetilde{H}_{i-1}(U_+ \cap U_i) \xrightarrow{p_*} \widetilde{H}_{i-1}(S^{n-1})$ is an isomorphism. By induction we are done. $\square$

Define $[S^n]$, the preferred generator of $\widetilde{H}_n(S^n) \cong \mathbb{Z}$, by $[S^0] = [\sigma_1 - \sigma_{-1}]$ and then inductively by $p_*(\partial[S^n]) = [S^{n-1}]$ where $p_* \circ \partial$ is the isomorphism $\widetilde{H}_i(S^n) \xrightarrow{\partial} \widetilde{H}_{i-1}(U_+ \cap U_-) \xrightarrow{p_*} \widetilde{H}_{i-1}(S^{n-1})$.

Lemma 1.21 (Naturality of the connecting homomorphism)**.** *Suppose*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f_A"] & B \arrow[r] \arrow[d, "f_B"] & C \arrow[r] \arrow[d, "f_C"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

is a commuting diagram of chain complexes with exact rows. Then we have commuting diagram of LES

$$\begin{tikzcd} \dots \arrow[r] & H_i(B) \arrow[r] \arrow[d, "f_{B*}"] & H_i(C) \arrow[r, "\partial"] \arrow[d, "f_{C*}"] & H_{i-1}(A) \arrow[r] \arrow[d, "f_{A*}"] & \dots \\ \dots \arrow[r] & H_i(B') \arrow[r] & H_i(C') \arrow[r, "\partial'"] & H_{i-1}(A') \arrow[r] & \dots \end{tikzcd}$$

Proof. Straightforward diagram chase. $\square$

Example. Suppose $f : X \to Y$, $Y = U_1 \cup U_2$, $U_i \subseteq Y$ open. Let $V_i = f^{-1}(U_i)$, so $X = V_1 \cup V_2$, $V_i \subseteq X$ open. Then $f_\#$ induces a map of SES

$$\begin{tikzcd}
0 \arrow[r] & C_*(V_1 \cap V_2) \arrow[r] \arrow[d, "f_\#"] & C_*(V_1) \oplus C_*(V_2) \arrow[r] \arrow[d, "f_\#"] & C_*^{\mathcal{V}}(X) \arrow[r] \arrow[d, "f_\#"] & 0 \\
0 \arrow[r] & C_*(U_1 \cap U_2) \arrow[r] & C_*(U_1) \oplus C_*(U_2) \arrow[r] & C_*^{\mathcal{U}}(Y) \arrow[r] & 0
\end{tikzcd}$$

and hence we get a corresponding map of MV sequences.

Example. Define $r_n : S^n \to S^n, (x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_n, -x_{n+1})$. Let $S^n = U_+ \cup U_-$ as before. Then $r : U_+ \to U_+, U_- \to U_-$.

Proposition 1.22. *$r_{n*} : \widetilde{H}_n(S^n) \to \widetilde{H}_n(S^n)$ maps $[S^n]$ to $-[S^n]$.*

Proof. By induction on n. For $n = 0$ we have $[S^0] = [\sigma_1 - \sigma_{-1}]$, so $r_{0*}[S^0] = [r_0\sigma_1 - r_0\sigma_{-1}] = [\sigma_{-1} - \sigma_1] = -[S^0]$ since $r_0(\pm 1) = \mp 1$.

In general, r_n induces a map of MV sequences $(S^n, U_+, U_-) \to (S^n, U_+, U_-)$:

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{H}_*(S^n) \arrow[r, "\partial"] \arrow[d, "r_{n*}"] & \widetilde{H}_{*-1}(U_+ \cap U_-) \arrow[r] \arrow[d, "r_{n*}"] & 0 \\
0 \arrow[r] & \widetilde{H}_*(S^n) \arrow[r, "\partial"] & \widetilde{H}_{*-1}(U_+ \cap U_-) \arrow[r] & 0
\end{tikzcd}$$

The homotopy equivalence

$$\begin{aligned}
p : U_+ \cap U_- &\longrightarrow S^{n-1}, \\
(x_1, \dots, x_{n+1}) &\longmapsto \frac{1}{\sqrt{x_2^2 + \cdots + x_{n+1}^2}}(x_2, \dots, x_{n+1}),
\end{aligned}$$

satisfies $p \circ r_n = r_{n-1} \circ p$. So we get a commuting diagram where all maps are isomorphisms:

$$\begin{tikzcd}
\widetilde{H}_*(S^n) \arrow[r, "\partial"] \arrow[d, "r_{n*}"] & \widetilde{H}_{*-1}(U_+ \cap U_-) \arrow[r, "p_*"] \arrow[d, "r_{n*}"] & H_{*-1}(S^{n-1}) \arrow[d, "r_{n-1}"] \\
\widetilde{H}_*(S^n) \arrow[r, "\partial"] & \widetilde{H}_{*-1}(U_+ \cap U_-) \arrow[r, "p_*"] & \widetilde{H}_{*-1}(S^{n-1})
\end{tikzcd}$$

From induction hypothesis we then get $r_{n*}[S^n] = -[S^n]$. $\square$

Corollary 1.23. *If $n \geq 1$ and $v \in S^n$, let $r_v : S^n \to S^n$ be reflection across the plane perpendicular to v. Then $r_{v*}[S^n] = -[S^n]$.*

Proof. S^n is path connected, so if γ is a path from v to e_{n+1}, $r_{\gamma(v)}$ is a homotopy from r_v to $r_{e_{n+1}} = r_n$, so $r_{v*} = r_{n*}$. $\square$

1.5 Excision and Collapsing a Pair

Definition. *Suppose $A \subseteq Z$. A is a* deformation retract *of Z if there exists a map $p : (Z, A) \to (A, A)$ such that $p \circ i = 1_{(A,A)}$ and $i \circ p : (Z, A) \to (Z, A) \sim 1_{(Z,A)}$ as a map of pairs where $i : (A, A) \to (Z, A)$ is the inclusion.*

Note that if A is a deformation retract of Z, then in particular $Z \sim A$.

Example. $Y \times 0$ is a deformation retract of $Y \times D^{n\circ}$.

Definition. *A pair (X, A) is a* good pair *if there exists $U \subseteq X$ open such that $A \subseteq U$, A is a deformation retract of U and $\overline{A} \subseteq U$.*

Examples.

- $X = S^2, A = \{n, s\}$ is a good pair,
- $Y = T^2 = S^1 \times S^1, B = S^1 \times 1 \subseteq Y$ is a good pair.
- More generally, if M is a manifold, N is a submanifold, then (M, N) is a good pair.
- $(\mathbb{R}, \mathbb{Q})$ is not a good pair.

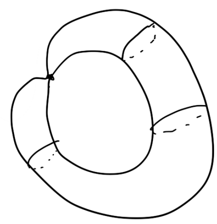
Theorem 1.24. *Suppose (X, A) is a good pair, and $\pi : (X, A) \to (X/A, A/A)$ the quotient map. Then*

$$\pi_* : H_*(X, A) \to H_*(X/A, A/A) \simeq \widetilde{H}_*(X/A)$$

is an isomorphism.

Examples.

- $X = S^2, A = \{n, s\}$, $Z = X/A$. By the Theorem $\widetilde{H}_*(Z) \simeq H_*(X, A)$. We compute



$Z = S^2/\{n, s\}$

$H_*(X, A)$ using the LES of the pair (X, A). Note that

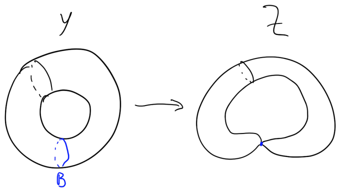
$$\widetilde{H}_*(S^2) = \begin{cases} \mathbb{Z} & * = 2, \\ 0 & * \neq 2, \end{cases} \qquad \widetilde{H}_*(A) = \begin{cases} \mathbb{Z} & * = 0, \\ 0 & * \neq 0. \end{cases}$$

So the LES is

$$0 \to \mathbb{Z} \to \widetilde{H}_2(X, A) \to 0 \to 0 \to \widetilde{H}_1(X, A) \to \mathbb{Z} \to 0 \to \widetilde{H}_0(X, A) \to 0$$

Therefore $\widetilde{H}_*(Z) = \begin{cases} \mathbb{Z} & * = 1, 2, \\ 0 & * \neq 1, 2. \end{cases}$

- $Y = S^1 \times S^1, B = S^1 \times 1$. Note that $Y/B \cong Z$. For example $Z = (S^1 \times [-1,1])/(S^1 \times$



$$Y/B \cong Z$$

S^0), and we have quotient maps $S^1 \times [-1,1] \to S^2 \to Z$ and $S^1 \times [-1,1] \to T^2 \to Z$. Since we know $H_*(B)$ and $H_*(Z) \simeq H_*(Y,B)$, we can determine $H_*(Y)$: We get the LES

$$0 \to \widetilde{H}_2(T^2) \to \mathbb{Z} \to \mathbb{Z} \xrightarrow{i_{1*}} \widetilde{H}_1(T^2) \to \mathbb{Z} \to 0 \to \widetilde{H}_0(T^2) \to 0$$

Here $i_1 : S^1 \to Y$ is the inclusion on the first factor. It has the retract $\pi_1 : T^2 \to S^1$, i.e. $\pi_1 \circ i_1 = \mathrm{id}_{S^1}$, hence $\pi_{1*} \circ i_{1*} = \mathrm{id}_{H_*(S^1)}$, so i_{1*} is injective. From this we deduce that $\widetilde{H}_2(T^2) \cong \mathbb{Z}$ and $\widetilde{H}_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$. Exercise: $H^1(T^2)$ is generated by $i_{1*}[S^1] = [S^1] \times 1, i_{2*}[S^1] = 1 \times [S^1]$

Lemma 1.25 (Five Lemma)**.** *Suppose*

$$\begin{array}{ccccccccccccc} \dots & \longrightarrow & A_{i+2} & \longrightarrow & A_{i+1} & \longrightarrow & A_i & \longrightarrow & A_{i-1} & \longrightarrow & A_{i-2} & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle f_{i+2}} & & \downarrow{\scriptstyle f_{i+1}} & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i-1}} & & \downarrow{\scriptstyle f_{i-2}} & & \\ \dots & \longrightarrow & B_{i+2} & \longrightarrow & B_{i+1} & \longrightarrow & B_i & \longrightarrow & B_{i-1} & \longrightarrow & B_{i-2} & \longrightarrow & \dots \end{array}$$

is a commuting diagram of R-modules with exact rows. If $f_{i\pm1}, f_{i\pm2}$ are isomorphisms, then also f_i is an isomorphism.

Proof. Straightforward diagram chase. □

Suppose $\mathcal{U} = \{U_j \mid j \in J\}$ is an open cover of X. If $A \subseteq X$, $\mathcal{U}_A := \{U_j \cap A \mid j \in J\}$ is an open cover of A and $C_*^{\mathcal{U}_A}(A) \subseteq C_*^{\mathcal{U}}(X)$. Define $C_*^{\mathcal{U}}(X,A) := C_*^{\mathcal{U}}(X)/C_*^{\mathcal{U}_A}(A)$. The map $i : C_*^{\mathcal{U}}(X) \to C_*(X)$ induces $i : C_*^{\mathcal{U}}(X,A) \to C_*(X,A)$.

Lemma 1.26. *$i_* : H_*^{\mathcal{U}}(X,A) \to H_*(X,A)$ is an isomorphism.*

Proof. There is a commutative diagram of SES's:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_*^{\mathcal{U}_A}(A) & \longrightarrow & C_*^{\mathcal{U}}(X) & \longrightarrow & C_*^{\mathcal{U}}(X,A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \\ 0 & \longrightarrow & C_*(A) & \longrightarrow & C_*(X) & \longrightarrow & C_*(X,A) & \longrightarrow & 0 \end{array}$$

So we get a commutative diagram of LES's:

$$\begin{tikzcd}
H_*^{\mathcal{U}_A}(A) \arrow[r] \arrow[d, "i_*"] & H_*^{\mathcal{U}}(X) \arrow[r] \arrow[d, "i_*"] & H_*^{\mathcal{U}}(X,A) \arrow[r] \arrow[d, "i_*"] & H_{*-1}^{\mathcal{U}_A}(A) \arrow[r] \arrow[d, "i_*"] & H_{*-1}^{\mathcal{U}}(X) \arrow[d, "i_*"] \\
H_*(A) \arrow[r] & H_*(X) \arrow[r] & H_*(X,A) \arrow[r] & H_{*-1}(A) \arrow[r] & H_{*-1}(X)
\end{tikzcd}$$

The four red arrows are isomorphisms by the subdivision lemma, so the blue one also is. $\square$

Theorem 1.27 (Excision)**.** *Suppose $B \subseteq A \subseteq X$, $\overline{B} \subseteq \operatorname{Int} A$, and let $j : (X \setminus B, A \setminus B) \to (X, A)$ be the inclusion. Then*

$$j_* : H_*(X \setminus B, A \setminus B) \to H_*(X, A)$$

is an isomorphism

Proof. $\overline{B} \subseteq \operatorname{Int} A$, so $\mathcal{U} = \{\operatorname{Int} A, X \setminus \overline{B}\}$ is an open cover of X. Notation: If $\sigma : \Delta^k \to X$, write $\sigma \lhd \mathcal{U}$ if $\operatorname{im} \sigma \subseteq U$ for some $U \in \mathcal{U}$.

Then

$$\begin{aligned}
C_*^{\mathcal{U}}(X) &= \langle \sigma \mid \sigma \lhd \mathcal{U} \rangle \\
&= \langle \sigma \mid \sigma \lhd \mathcal{U}, \operatorname{im} \sigma \cap B = \emptyset \rangle \oplus \langle \sigma \mid \sigma \lhd \mathcal{U} \text{ and } \operatorname{im} \sigma \cap B \neq \emptyset \rangle \\
&= C_*^{\mathcal{U}}(X \setminus B) \oplus M_B
\end{aligned}$$

where $M_B = \langle \sigma \mid \operatorname{im} \sigma \subseteq A \text{ and } \operatorname{im} \sigma \cap B \neq \emptyset \rangle$. Similarly $C_*^{\mathcal{U}_A}(A) = C_*^{\mathcal{U}_{A \setminus B}}(A \setminus B) \oplus M_B$. Now if $C' \subseteq C$, then the inclusion $C/C' \to (C \oplus M)/(C' \oplus M)$ is an isomorphism. So taking $C = C_*^{\mathcal{U}}(X \setminus B), C' = C_*^{\mathcal{U}_{A \setminus B}}(A \setminus B)$ we get that $j_\# : C_*^{\mathcal{U}}(X \setminus B)/C_*^{\mathcal{U}_{A \setminus B}}(A \setminus B) \to C_*^{\mathcal{U}}(X)/C_*^{\mathcal{U}_A}(A)$ is an isomorphism, i.e. $j_\# : C_*^{\mathcal{U}}(X \setminus B, A \setminus B) \cong C_*^{\mathcal{U}}(X, A)$, so $j_* : H_*^{\mathcal{U}}(X \setminus B, A \setminus B) \cong H_*^{\mathcal{U}}(X, A)$.

There is a commuting square

$$\begin{tikzcd}
H_*^{\mathcal{U}}(X \setminus B, A \setminus B) \arrow[r, "j_*"] \arrow[d, "i_*"] & H_*^{\mathcal{U}}(X, A) \arrow[d, "i_*"] \\
H_*(X \setminus B, A \setminus B) \arrow[r, "j_*"] & H_*(X, A)
\end{tikzcd}$$

The vertical maps and top map are isomorphisms, thus so is the bottom map. $\square$

Proposition 1.28 (LES of a triple)**.** *Suppose $Z \subseteq Y \subseteq X$. Then there is a LES:*

$$\ldots \xrightarrow{\partial} H_*(Y, Z) \xrightarrow{j_{1*}} H_*(X, Z) \xrightarrow{j_{2*}} H_*(X, Y) \xrightarrow{\partial} H_{*-1}(Y, Z) \to \ldots$$

where $j_1 : (Y, Z) \to (X, Z), j_2 : (X, Z) \to (X, Y)$ are inclusions.

Proof. There is a short exact sequence

$$0 \to C_*(Y,Z) \to C_*(X,Z) \to C_*(X,Y) \to 0$$

and the sequence in the claim is the associated long exact sequence. $\square$

Lemma 1.29. *If A is a deformation retract of U, $U \subseteq X$ and $j : (X,A) \to (X,U)$ the inclusion, then $j_* : H_*(X,A) \to H_*(X,U)$ is an isomorphism.*

Proof. Let $i : A \to U$ be the inclusion. By definition it is a homotopy equivalence, hence $i_* : H_*(A) \to H_*(U)$ is an isomorphism and so the LES of the pair (U,A) shows that $H_*(U,A) = 0$. Then the LES of the triple (X,U,A) gives

$$0 = H_*(U,A) \to H_*(X,A) \xrightarrow{j_*} H_*(X,U) \to H_{*-1}(U,A) = 0,$$

so j_* is an isomorphism. $\square$

Proof of Theorem 1.24. There is a commutative diagram

$$\begin{tikzcd}
H_*(X-A,U-A) \arrow[r, "j_*"] \arrow[d, "\pi_{1*}"] & H_*(X,U) \arrow[d, "\pi_{2*}"] & H_*(X,A) \arrow[l, "i_*"'] \arrow[d, "\pi_{3*}"] \\
H_*(X/A-A/A,U/A-A/A) \arrow[r, "j_*"] & H_*(X/A,U/A) & H_*(X/A,A/A) \arrow[l, "i_*"']
\end{tikzcd}$$

The maps j_* are isomorphisms by excision, the i_* are isomorphisms by the lemma (exercise: A/A is deformation retract of U/A). π_{1*} is induced by a homeomorphism $(X-A,U-A) \to (X/A-A/A,U/A-A/A)$, hence an isomorphism. Then π_{2*} is an isomorphism and finally also π_{3*} is an isomorphism. $\square$

Definition. *A space X is an n-*manifold *if it is metrizable (in particular Hausdorff and first-countable) and every $x \in X$ has an open neighborhood U_x homeomorphic to $\mathbb{R}^n$.*

Proposition 1.30. *If X is an n-manifold and $x \in X$, then*

$$H_*(X,X\setminus x) \cong \begin{cases} \mathbb{Z} & * = n, \\ 0 & * \neq n. \end{cases}$$

Proof. Choose $U_x \subseteq X$ as above with $U_x \cong \mathbb{R}^n, x \mapsto 0$. Then by excision and Lemma 1.29:

$$H_*(X,X\setminus p) \cong H_*(D^n,D^n\setminus 0) \cong H_*(D^n,S^{n-1}).$$

The LES of (D^n,S^{n-1}) yields $\widetilde{H}_*(D^n,S^{n-1}) = \widetilde{H}_{*-1}(S^{n-1})$ and we are done. $\square$

Corollary 1.31. *If M and N are m and n-manifolds resp. and $M \cong N$, then $n = m$.*

2 Cellular Homology

2.1 Degrees of Maps $f : S^n \to S^n$

Recall that $H_n(S^n) \cong \mathbb{Z}$ $(n > 0)$. It is generated by $[S^n]$. So if $f : S^n \to S^n$, then $f_*[S^n] = k[S^n]$ for some (unique) $k \in \mathbb{Z}$.

Definition. *If $f : S^n \to S^n$ with $f_*[S^n] = k[S^n]$, $k =: \deg f$ is the* degree *of f.*

Properties:

(1) $(1_{S^n})_* = 1_{H_*(S^n)}$, so $\deg 1_{S^n} = 1$

(2) If $f_0, f_1 : S^n \to S^n$ are homotopic, then $f_{0*} = f_{1*}$, so $\deg f_0 = \deg f_1$.

(3) If $f, g : S^n \to S^n$, then $\deg f \deg g = \deg(f \circ g)$.

(4) If $f : S^n \to S^n$ is a homeomorphism, then $\deg f = \pm 1$. We say f *is orientation preserving* if $\deg f = 1$, otherwise *orientation reversing*.

(5) If $r_v : S^n \to S^n$ is the reflection in $v^\perp$, then $\deg r_v = -1$ (Corollary 1.23)

(6) If $A : S^n \to S^n, x \mapsto -x$ is the antipodal map, then $A = r_{e_1} \circ r_{e_2} \circ \cdots \circ r_{e_{n+1}}$, so $\deg A = (-1)^{n+1}$. In particular $A \not\simeq 1_{S^n}$ if n is even.

2.1.1 Local Degree

Let $p \in S^n$. Then $S^n - p \cong D^{n\circ}$ is contractible, so $\pi_* : \widetilde{H}_n(S^n) \to H_n(S^n, S^n - p)$ is an isomorphism.Define $[S^n, S^n - p] \in H_n(S^n, S^n - p)$ as the image of $[S^n]$ under π_*.

If $U \subseteq S^n$ is open, $p \in U$, let $B = S^n \setminus U$. B is closed and $\overline{B} \subseteq \operatorname{Int}(S^n - p)$. Then $(S^n - B, S^n - p - B) = (U, U - p)$, so by excision

$$j_* : H_n(U, U - p) \to H_n(S^n, S^n - p)$$

is an isomorphism. Define $[U, U - p]$ to be the preimage of $[S^n, S^n - p]$ under j_*.

Observe: If $p \in U' \subseteq U$, we have a commutative diagram:

$$\begin{tikzcd} H_n(U, U_p) \arrow[r, "\simeq"] & H_n(S^n, S^n - p) \\ H_n(U', U' - p) \arrow[u, "\iota_*"] \arrow[ur, "\simeq"'] & \end{tikzcd}$$

So $[U', U'-p]$ gets mapped to $[U, U-p]$ under ι_*.

Suppose $f: S^n \to S^n$ and $f^{-1}(p) = \{q_1, \ldots, q_r\}$ is finite. As S^n is Hausdorff, we can find $U_i \subseteq S^n$ open such that $q_i \in U_i$ and $U_i \cap U_j = \emptyset$ for $i \neq j$. Then $f : (U_i, U_i - q_i) \to (S^n, S^n - p)$. Then $f_*[U_i, U_i - q_i] = k[S^n, S^n - p]$ for some $k \in \mathbb{Z}$.

Definition. *Under the above hypotheses we define* $\deg_{q_i} f := k$ *to be the* local degree of f at q_i.

Lemma 2.1. *The definition of the local degree does not depend on the choice of U_i.*

Proof. Suppose $q_i \in U_i' \subseteq U_i$ and $q_i \in U_i'$. Then

$$\begin{tikzcd} H_n(U_i, U_i - q_i) \arrow[r, "f_*"] & H_n(S^n, S^n - p) \\ H_n(U_i', U_i' - q_i) \arrow[u, "i_*"] \arrow[ur, "f_*'"] & \end{tikzcd}$$

commutes. We have $i_*[U_i', U_i' - q_i] = [U_i, U_i - q_i]$, so $\deg f_* = \deg f_*'$. In general, given open sets U_i, U_i' containing q_i, consider $U_i \cap U_i' \subseteq U_i, U_i'$ and use above to see that the degrees defined using $U_i, U_i', U_i \cap U_i'$ are all the same. □

Let $V = \coprod_i U_i \subseteq S^n$. By excision we have an isomorphism $j_* : H_n(V, V - f^{-1}(p) \xrightarrow{\simeq} H_n(S^n, S^n - f^{-1}(p))$. We also know that $H_n(V, V - f^{-1}(p)) = \bigoplus_{i=1}^r H_n(U_i, U_i - q) \simeq \mathbb{Z}^r$ and the $[U_i, U_i - q_i]$ form a basis of this group.

Lemma 2.2. *The map*

$$\widetilde{H}_n(S^n) \to H_n(S^n, S^n - f^{-1}(p)) \cong \bigoplus_{i=1}^r H_n(U_i, U_i - q_i)$$

is given by $[S^n] \mapsto \sum_{i=1}^r [U_i, U_i - q_i]$.

Proof. There is a commutative diagram:

$$\begin{tikzcd} H_n(S^n, S^n - f^{-1}(p)) \arrow[r] & H_n(S^n, S^n - q_j) & \\ H_n(V, V - f^{-1}(p)) \arrow[u, "\simeq"] \arrow[r] & H_n(V, V - q_j) \arrow[u, "\simeq"] \arrow[r, "\simeq"] & H_n(U_j, U_j - q_j) \end{tikzcd}$$

The vertical maps are isomorphisms, so the diagram still commutes if we reverse those arrows. Now consider the following diagram:

$$\begin{tikzcd}
\widetilde{H}_n(S^n) \arrow[d] \arrow[dr] \arrow[ddrr, bend left, "\alpha"] \arrow[ddd, bend right, "\beta"'] & & \\
H_n(S^n, S^n - f^{-1}(p)) \arrow[r] \arrow[d, "\simeq"] & H_n(S^n, S^n - q_j) \arrow[d, "\simeq"] & \\
H_n(V, V - f^{-1}(p)) \arrow[r] \arrow[d, "\simeq"] & H_n(V, V - q_j) \arrow[r, "\simeq"] & H_n(U_j, U_j - q_j) \\
\bigoplus_i H_n(U_i, U_i - q_i) \arrow[urr, "\pi_j"'] & &
\end{tikzcd}$$

Here π_j is the projection onto the j-th component. The diagram is still commutative (exercise: check the bottom triangle). Then $\alpha([S^n]) = j_*^{-1}[S^n, S^n - p] = [U_j, U_j - q_j]$, so $\pi_j \beta[S^n] = \alpha[S^n] = [U_j, U_j - q_j]$, hence $\beta[S^n] = \sum_j [U_j, U_j - q_j]$. $\square$

Theorem 2.3. *Suppose $f : S^n \to S^n$, $f^{-1}(p) = \{q_1, \ldots, q_r\}$ as above. Then $\deg f = \sum_{i=1}^r \deg_{q_i} f$.*

Proof. We have a commutative diagram:

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "f_*"] \arrow[d] & H_n(S^n) \arrow[d] \\
H_n(S^n, S^n - f^{-1}(p)) \arrow[r, "f_*"] \arrow[d, "j_*^{-1}"] & H_n(S^n, S^n - p) \\
\bigoplus H_n(U_i, U_i - q_i) \arrow[ur, "\oplus f_{i*}"'] &
\end{tikzcd}$$

Following the different paths, we see that the image of $[S^n]$ in $H_n(S^n, S^n - p)$ is both $\deg f[S^n, S^n - p]$ and $\sum f_{i*}[U_i, U_i - q_i] = (\sum \deg_{q_i} f)[S^n, S^n - p]$, so the result follows. $\square$

Example. Let $f : S^1 \to S^1, z \mapsto z^n$. Then $f^{-1}(1) = \{1, \omega, \ldots, \omega^{n-1}\}$ where $\omega = e^{2\pi i/n}$. Consider the homeomorphism $\varphi_k : S^1 \to S^1, z \mapsto \omega^k z$. Note that $\varphi_k \sim 1_{S^1}$. Let $U_k = \phi_k(U_0)$ where U_0 is a small neighborhood of 1. Then $\varphi_{k*}[U_0, U_0 - 1] = [U_k, U_k - \omega^k]$ and $f \circ \varphi_k = f$, so $f_*[U_k, U_k - \omega_k] = f_*(\varphi_{k*}[U_0, U_0 - 1]) = f_*[U_0, U_0 - 1]$. So $\deg_{\omega^k} f = \deg_1 f = 1$ (the last equality is an exercise). Therefore $\deg f = \sum_{i=0}^{n-1} 1 = n$.

2.1.2 Some Intuition

If $f : S^n \to X$, then $f_*[S^n] \in H_n(X)$ and if $f_0 \sim f_1$, then $f_{0*}[S^n] = f_{1*}[S^n]$. This can be used to define the "Hurewicz homomorphism":

$$\Phi : \pi_n(X, *) \longrightarrow H_n(X),$$

$$f \longmapsto f_*[S^n]$$

In general, this map is quite far from being an isomorphism. Example: $H_2(T^2) \simeq \mathbb{Z}$. But if $f : S^2 \to T^2$, we can factor it through the universal covering $\pi : \mathbb{R}^2 \to T^2$, i.e. $f = \widehat{f} \circ \pi$ for some $\widehat{f} : S^2 \to \mathbb{R}^2$. Then $f_*[S^2] = \pi_* \widehat{f}_*[S^2] = \pi_*(0) = 0$, since $H_2(\mathbb{R}^2) = 0$.

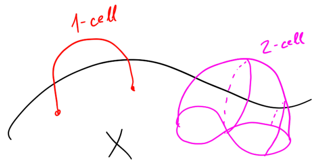
Better model: If M is a closed (i.e. without boundary and compact) connected n-manifold, we will show $H_n(M) \simeq \mathbb{Z} = \langle [M] \rangle$ such that the image of $[M]$ under $H_n(M) \to H_n(M, M - *) \simeq \mathbb{Z}$ is a generator. So if $f : M \to X$, we can consider $f_*[M] \in H_n(X)$. If W^{n+1} is a compact $n+1$-manifold, $\partial W = \coprod_{i=1}^k M_i$. Then $i : \partial W \to W$ induces $i_* : H_n(\partial W) \to H_n(W)$ with $[\partial W] = \sum_{i=1}^k [M_i] \mapsto 0$. So if $f : W \to X$, then $f_*(\sum_i [M_i]) = 0$.

This is still not an accurate model for H_n, but much better.

2.2 The Cellular Chain Complex

Definition. *Suppose $B \subseteq Y$, $f : B \to X$. Then $X \cup_f Y := (X \amalg Y)/\sim$, where $\sim$ is the smallest equivalence relation containing $b \sim f(b)$ for all $b \in B$, is the space obtained by* attaching (or gluing) *Y to X along f.*

If $(Y, B) = (D^k, S^{k-1})$, say $X \cup_f D^k$ is obtained by attaching a k-cell to X.



Attaching a 1- and a 2-cell

Definition. *A* finite cell complex (fcc) *of dimension n is a space X equipped with closed subsets $\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n$, such that for each k, X_k is obtained by attaching finitely many k-cells to X_{k-1}, i.e. $X_k \cong X_{k-1} \cup_F \coprod_{\alpha \in A_k} D^k$ where $F : \coprod_{\alpha \in A_k} S^{k-1} \to X_{k-1}$, $F = \coprod_{\alpha \in A_k} f_\alpha$, $f_\alpha : S^{k-1} \to X_{k-1}$.*

X_k is the k-skeleton *of X.*

If we drop the finiteness conditions, $X = \bigcup_{k=0}^{\infty} X_k$ and $U \subseteq X$ is open iff $U \cap X_k$ is open for all k, then this is called a CW-complex.

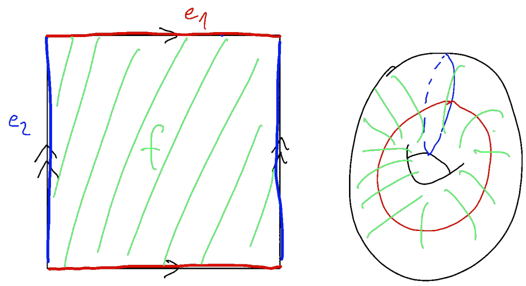
Examples.

(1) If X is a graph with v vertices and e edges, then X is a fcc with v 0-cells and e 1-cells.

(2) If X is a fcc with one 0-cell and one k-cell, then $X \cong D^k/S^{k-1} \cong S^k$.

(3) If X is a simplicial chain complex, $|X|$ is a fcc with one k-cell for each k-dimensional face of X.

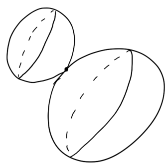
(4) T^2 is a fcc with one 0-cell P, two 1-cells e_1, e_2 and one 2-cell f.



Cell structure of T^2

Definition. *If* $(X_i, x_i), i \in I$ *are pointed spaces, their* wedge product *is*

$$\bigvee_{i\in I}(X_i, x_i) := \coprod_{i\in I} X_i/(\coprod_i x_i).$$



$S^2 \vee S^2$

If X is a fcc with one 0-cell and r k-cells, then $X \simeq \bigvee_{i=1}^r S^k$.

2.2.1 Projectives Spaces

Definition. *The* n-dimensional complex projective space *is* $\mathbb{CP}^n = (\mathbb{C}^{n+1} - 0)/\mathbb{C}^*$.

The n-dimensional real projective space *is* $\mathbb{RP}^n = (\mathbb{R}^{n+1} - 0)/\mathbb{R}^*$.

Note that $\mathbb{C}^* = \mathbb{R}_{>0} \times S^1$ and $(\mathbb{C}^{n+1} - 0)/\mathbb{R}_{>0} \simeq S^{2n+1}$, so $\mathbb{CP}^n \cong S^{2n+1}/S^1$ where $\lambda \in S^1$ acts on $z \in S^{2n+1}$ by $\lambda \cdot z = \lambda z$ (inside $\mathbb{C}^{n+1}$).

Similarly, $\mathbb{RP}^n = S^n/(\mathbb{Z}/2)$.

Definition. *The* Hopf map $p_n : S^{2n+1} \to \mathbb{CP}^n$ *is the projection.*

Proposition 2.4. $\mathbb{CP}^n \simeq \mathbb{CP}^{n-1} \cup_{p_{n-1}} D^{2n}$ *where* $p_{n-1} : S^{2n-1} \to \mathbb{CP}^{n-1}$ *is the Hopf map.*

Proof. We have maps

$$\begin{aligned} i_1 : \mathbb{CP}^{n-1} &\longrightarrow \mathbb{CP}^n \\ [z] &\longmapsto [z:0] \\ i_2 : D^{2n} = \{z \in \mathbb{C}^n : \|z\| \le 1\} &\longrightarrow \mathbb{CP}^n \\ z &\longmapsto [z : \sqrt{1-\|z\|^2}] \end{aligned}$$

Then $i_2|_{S^{2n-1}} = i_1 \circ p_{n-1}$. So i_1, i_2 glue to give $i : \mathbb{CP}^{n-1} \cup_{p_{n-1}} D^{2n} \to \mathbb{CP}^n$. i is a bijection. Indeed, the inverse is given by

$$[z_0 : \dots : z_n] \mapsto \begin{cases} (z_0, \dots, z_{n-1}) \in D^{2n} & \text{if } z_n \in \mathbb{R}_{>0},\ \|z\| = 1, \\ [z_0 : \dots : z_{n-1}] \in \mathbb{CP}^{n-1} & \text{if } z_n = 0. \end{cases}$$

Since the spaces are compact Hausdorff, it follows that i is a homeomorphism. ☐

Consequence: By induction $\mathbb{CP}^n$ is a fcc with one cell of dimension $2i$ for $0 \le i \le n$ and no other cells.[1] For example, $\mathbb{CP}^1 \simeq S^2$.

The same argument shows $\mathbb{RP}^n \cong \mathbb{RP}^{n-1} \cup_{p_{n-1}} D^n$. So $\mathbb{RP}^n$ is a fcc with 1 cell of dimension i for $0 \le i \le n$.

Proposition 2.5.

$$H_*(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & * = 0, 2, \dots, 2n, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The quotient $\mathbb{CP}^n/\mathbb{CP}^{n-1}$ is a cell complex with one 0-cell (image of $\mathbb{CP}^{n-1}$) and one $2n$-cell (image of D^{2n}), so $\mathbb{CP}^n/\mathbb{CP}^{n-1} \cong S^{2n}$. Hence

$$H_*(\mathbb{CP}^n, \mathbb{CP}^{n-1}) \simeq \widetilde{H}_*(S^{2n}) = \begin{cases} \mathbb{Z} & * = 2n, \\ 0 & \text{otherwise.} \end{cases}$$

By induction we have $H_*(\mathbb{CP}^{n-1}) = 0$ for odd $*$, hence the LES of $(\mathbb{CP}^n, \mathbb{CP}^{n-1})$ gives us SES

$$0 \to H_i(\mathbb{CP}^{n-1}) \to H_i(\mathbb{CP}^n) \to \widetilde{H}_i(S^{2n}) \to 0.$$

Hence

$$H_*(\mathbb{CP}^n) \cong H_*(\mathbb{CP}^{n-1}) \oplus \widetilde{H}_*(S^{2n})$$

and the claim then follows by induction. ☐

For $H_*(\mathbb{RP}^n)$ we need to work a little bit harder, we will compute it in the next section.

[1] This gives rise to the funny-looking formula

$$\text{“}\frac{\mathbb{C}^{n+1} - 0}{\mathbb{C} - 0} = \mathbb{C}^0 + \mathbb{C}^1 + \dots + \mathbb{C}^n\text{”}.$$

Sadly this doesn't work for $\mathbb{RP}^n$.

2.2.2 Homology of Cell Complexes

Observation: In the LES of (D^k, S^{k-1}), the map $H_k(D^k, S^{k-1}) \to \widetilde{H}_{k-1}(S^{k-1})$ is an isomorphism as $\widetilde{H}_*(D^k) = 0$. Define $[D^k, S^{k-1}]$ as the preimage of $[S^{k-1}]$.

Suppose X is a fcc. Let A_k be the set of k-cells of X, $X_k = X_{k-1} \cup_{\coprod f_\alpha} \coprod_{\alpha \in A_k} D^k$ with $f_\alpha : S^{k-1} \to X_{k-1}$. Let $U_{k-1} = X_{k-1} \cup_{\coprod f_\alpha} (\coprod_{\alpha \in A_k} D^k - 0)$. Since S^{k-1} is a deformation retract of $D^k - 0$, X_{k-1} is also a deformation retract of U_{k-1}. Hence (X_k, X_{k-1}) is a good pair. Furthermore, $X_k / X_{k-1} \simeq \coprod_{\alpha \in A_k} D^k / \coprod_{\alpha \in A_k} S^{k-1} \cong \bigvee_{\alpha \in A_k} S^k$.[2]

So

$$H_k(X_k, X_{k-1}) \simeq H_k\Big(\coprod_{\alpha \in A_k} D^k, \coprod_{\alpha \in A_k} S^{k-1}\Big) \simeq \bigoplus_{\alpha \in A_k} H_k(D^k, S^{k-1}).$$

Then $H_k(X_k, X_{k-1}) = \bigoplus_{\alpha \in A_k} e_\alpha \mathbb{Z}$ where $e_\alpha = i_{\alpha *}[D^k, S^{k-1}]$ where $i_\alpha : (D^k, S^{k-1}) \to (X_k, X_{k-1})$.

Let $p_\beta : \bigvee_{\alpha \in A_k} S^k \to \bigvee_{\alpha \in A_k} S^k / \bigvee_{\alpha \neq \beta} S^k \simeq S^k$. Then $p_{\beta *}$ is the projection onto the factor corresponding to $\langle e_\beta \rangle$.

Let $d_k : H_k(X_k, X_{k-1}) \to H_{k-1}(X_{k-1}, X_{k-2})$ be the boundary map in the long exact sequence of the triple (X_k, X_{k-1}, X_{k-2}).

Lemma 2.6. *$d_k = (\pi_{k-1})_* \circ \partial_k$ where $\partial_k : H_k(X_k, X_{k-1}) \to H_{k-1}(X_{k-1})$ is the boundary in the LES of the pair (X_k, X_{k-1}) and $\pi_{k-1} : (X_{k-1}, \emptyset) \to (X_{k-1}, X_{k-2})$.*

Proof. Look at the construction of d_k, ∂_k in the Snake Lemma. □

Corollary 2.7. *$d_k \circ d_{k+1} = 0$.*

Proof. $d_k \circ d_{k+1} = (\pi_{k-1})_* \circ \partial_k \circ \pi_{k*} \circ \partial_{k+1}$ and $\partial_k \circ \pi_{k*} = 0$ as they are two consecutive maps in the LES of (X_k, X_{k-1}). □

Definition 2.8. *If X is a fcc, $(C_*^{\text{cell}}(X), d^{\text{cell}}) = (\bigoplus_k H_k(X_k, X_{k-1}), \bigoplus_k d_k)$ is the* cellular chain complex *of X.*

Theorem 2.9. *$H_*^{\text{cell}}(X) := H_*(C_*^{\text{cell}}(X)) \simeq H_*(X)$.*

How to compute $H_*^{\text{cell}}(X)$: We have $C_k^{\text{cell}}(X) = H_k(X_k, X_{k-1}) = \langle e_\alpha \mid \alpha \in A_k \rangle$ and:

Proposition 2.10. *$d_k^{\text{cell}} : C_k^{\text{cell}}(X) \to C_{k-1}^{\text{cell}}(X)$ is given by*

$$d_k^{\text{cell}}(e_\alpha) = \sum_{\beta \in A_{k-1}} n_{\alpha\beta} e_\beta,$$

[2]Remark by L.T.: Here and in the following, the homeomorphism $D^k/S^{k-1} \cong S^k$ should probably be chosen such that $[D^k, S^{k-1}]$ corresponds to $[S^k]$ under $H_k(D^k, S^{k-1}) \cong H_k(D^k/S^{k-1}) \cong H_k(S^k)$.

where $n_{\alpha\beta} = \deg p_\beta \circ f_\alpha$ *where*

$$p_\beta \circ f_\alpha : S^{k-1} \to X_{k-1} \to X_{k-1}/X_{k-2} \simeq \bigvee_{\beta \in A_{k-1}} S^{k-1} \xrightarrow{p_\beta} S^{k-1}.$$

Proof. $d_k(e_\alpha) = (\pi_{k-1})_* \circ \partial_k(i_{\alpha *}[D^k, S^{k-1}])$. By naturality of the connecting homomorphism this is $(\pi_{k-1})_* \circ i_{\alpha *}(\partial_k[D^k, S^{k-1}]) = (\pi_{k-1})_* i_{\alpha *}[S^{k-1}]) = f_{\alpha *}[S^{k-1}]$. The coefficient of e_β in $f_{\alpha *}[S^{k-1}]$ is the coefficient of $[S^{k-1}]$ in $(p_\beta \circ f_\alpha)_*[S^{k-1}]$ this is $\deg(p_\beta \circ f_\alpha)$. $\square$

Examples.

- $\mathbb{CP}^n$ has one cell of dimension $2i$ for $0 \le i \le n$, so

  $$C_*^{\text{cell}}(\mathbb{CP}^n) = (C_{2n}^{\text{cell}}(\mathbb{CP}^n) = \mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \cdots \to 0 \to \mathbb{Z} = C_0^{\text{cell}}(\mathbb{CP}^n))$$

  The boundary maps are 0. So

  $$H_*(\mathbb{CP}^n) \simeq H_*^{\text{cell}}(\mathbb{CP}^n) = C_*^{\text{cell}}(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & * = 0, 2, \ldots, 2n, \\ 0 & \text{otherwise} \end{cases}$$

  as we already knew.

- $\mathbb{RP}^n$ has one cell of dimension k for all $0 \le k \le n$, so $C_k^{\text{cell}}(\mathbb{RP}^n) = \langle e_k \rangle$. Then

  $$C_*^{\text{cell}} = \mathbb{Z} \xrightarrow{d_n} \mathbb{Z} \xrightarrow{d_{n-1}} \mathbb{Z} \to \cdots \to \mathbb{Z} \xrightarrow{d_1} \mathbb{Z}$$

  where $d_k e_k = n_k e_{k-1}$ with $n_k = \deg g_k$,

  $$g_k : S^{k-1} \xrightarrow{f_k} \mathbb{RP}^{k-1} \xrightarrow{\pi} \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} \simeq S^{k-1}.$$

  Given $p \in S^{k-1}$, not coming from $\mathbb{RP}^{k-2}$, it has two preimages in S^{k-1}, q and Aq where $A : S^{k-1} \to S^{k-1}$ is the antipodal map. Note that $g_k = g_k \circ A$, so $\deg_{Aq} g_k = \deg_q g_k \deg A = (-1)^k \deg_q g_k =: (-1)^k \alpha$. $g_k|_U$ is a homeomorphism (where U is a small neighborhood of q), so $\deg_q g_k = \pm 1 = \alpha$. So $\deg g_k = \deg_q g_k + \deg_{Aq} g_k = \alpha + (-1)^k \alpha = \begin{cases} \pm 2 & k \text{ even}, \\ 0 & k \text{ odd}. \end{cases}$

  Summary:

  - Suppose n is even. Then:

    $$C_*^{\text{cell}}(\mathbb{RP}^n) = \mathbb{Z} \xrightarrow{\pm 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\pm 2} \mathbb{Z} \to \cdots \to \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

    So $H_*(\mathbb{RP}^n) = H_*^{\text{cell}}(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}/2 & * = 1, 3, 5, \ldots, n-1 \\ \mathbb{Z} & * = 0, \\ 0 & \text{otherwise.} \end{cases}$

– Suppose n is odd. Then:

$$C_*^{\text{cell}}(\mathbb{RP}^n) = \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\pm 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to \cdots \to \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

$$H_*(\mathbb{RP}^n) = H_*^{\text{cell}}(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}/2 & * = 1, 3, 5, \ldots, n-2 \\ \mathbb{Z} & * = 0, n, \\ 0 & \text{otherwise} \end{cases}$$

We now turn to the proof of the theorem.

Lemma 2.11. *Suppose X is a fcc with one 0-cell, and all other cells have dimension $\leq M$ and $\geq m$. Then $\widetilde{H}_*(X) = 0$ if $* < m$ or $* > M$.*

Proof. By induction on $M - m$. If $M - m = 0$, then X has one cell dimension 0 and all other cells of dimension $m = M$, so $X \simeq \bigvee_{\alpha \in A} S^m$, and therefore $\widetilde{H}_*(X) = 0$ for $* \neq m$.

Now suppose the claim is true for $M - m < k$. If X has cells of dimension $\leq m + k$ and $\geq m$, then X_{m+k-1} has cells of dimension between m and $m + k - 1$, so the induction hypothesis applies to X_{m+k-1}. (X, X_{m+k-1}) is a good pair with $X/X_{m+k-1} = \bigvee_{\alpha \in A} S^{m+k}$, so $H_*(X, X_{m+k-1}) = 0$ unless $* = m + k$ and $\widetilde{H}_*(X_{m+k-1}) = 0$ unless $m \leq * \leq m + k - 1$. Then consider the LES of the pair:

$$\widetilde{H}_*(X_{m+k-1}) \to \widetilde{H}_*(X) \to H_*(X, X_{m+k-1})$$

The two outer groups are 0 unless $m \leq * \leq m + k$. □

Lemma 2.12. *If X is a fcc, then (X, X_k) is a good pair.*

Proof. "Annoying but not terribly hard exercise" □

Corollary 2.13. *If X is a fcc, then $H_k(X_{k+1}) \simeq H_k(X)$.*

Proof. From the LES of (X, X_{k+1}) we get

$$H_{k+1}(X, X_{k+1}) \to H_k(X_{k+1}) \to H_k(X) \to H_k(X, X_{k+1}).$$

We have $H_k(X, X_{k+1}) \simeq \widetilde{H}_k(X/X_{k+1})$. X/X_{k+1} has one 0-cell (image of X_{k+1}), and all other cells have dimension $\geq k + 2$, so by the lemma $\widetilde{H}_k(X/X_{k+1}) = \widetilde{H}_{k+1}(X/X_{k+1}) = 0$, and our result follows. □

Proof of Theorem 2.9. Consider the following commutative diagram:

$$
\begin{tikzcd}
 & & & H_k(X_{k+1}, X_k) = 0 & \\
H_k(X_{k-1}) = 0 \arrow[dr] & & H_k(X_{k+1}) \arrow[ur] & & \\
 & H_k(X_k) \arrow[ur, two heads, "i"] \arrow[dr, hook, "\pi_k"] & & & \\
H_{k+1}(X_{k+1}, X_k) \arrow[ur, "\partial_{k+1}"] \arrow[rr, "d_{k+1}"] & & H_k(X_k, X_{k-1}) \arrow[rr, "d_k"] \arrow[dr, "\partial_k"] & & H_{k-1}(X_{k-1}, X_{k-2}) \\
 & & & H_{k-1}(X_{k-1}) \arrow[ur, hook, "\pi_{k-1}"] & \\
 & & H_{k-1}(X_{k-2}) = 0 \arrow[ur] & &
\end{tikzcd}
$$

The horizontal sequence in the middle is the cellular chain complex. The diagonal sequences are parts of long exact sequences of pairs.

So π_k, π_{k-1} are injections, i is a surjection. So now we have $\ker d_k = \ker \partial_k = \operatorname{im} \pi_k \cong H_k(X_k)$. Under this isomorphism, $\operatorname{im} d_{k+1} \leftrightarrow \operatorname{im} \partial_{k+1}$, so $H_k^{\text{cell}}(X) = (\ker d_k)/(\operatorname{im} d_{k+1}) \simeq H_k(X_k)/\operatorname{im} \partial_{k+1} \simeq H_k(X_{k+1}) \cong H_k(X)$ by the corollary. $\square$

2.3 Homology with Coefficients

Definition. *If G is a $\mathbb{Z}$-module, then $C_*(X;G) := C_*(X) \otimes G$ is the* singular chain complex with coefficients in G. *$H_*(X;G)$ is its homology.*

Note: If $f, g : C \to C'$ are chain homotopic via h, then $f \otimes 1, g \otimes 1 : C \otimes M \to C' \otimes M$ are chain homotopic via $h \otimes 1$.

Example. Let $C = C_*^{\text{cell}}(\mathbb{RP}^3) = (\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z})$, so $H_*(C) = (\mathbb{Z}, 0, \mathbb{Z}/2, \mathbb{Z})$.

Then $C_* \otimes \mathbb{Q} = (\mathbb{Q} \xrightarrow{0} \mathbb{Q} \xrightarrow{2} \mathbb{Q} \xrightarrow{0} \mathbb{Q})$, so

$$H_*(C_* \otimes \mathbb{Q}) = (\mathbb{Q}, 0, 0, \mathbb{Q}) = H_*(C) \otimes \mathbb{Q}.$$

And $C_* \otimes \mathbb{Z}/2 = (\mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2)$, so

$$H_*(C_* \otimes \mathbb{Z}/2) = (\mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/2) \neq H_*(C) \otimes \mathbb{Z}/2.$$

2.3.1 Euler Characteristic

Suppose C is a finite dimensional chain complex over a field. Let $c_k = \dim C_k$.

Definition. *The* Euler characteristic *of C is $\chi(C) := \sum_k (-1)^k c_k$.*

Let $h_k = \dim H_k(C)$.

Theorem 2.14. $\chi(C) = \chi(H_*(C)) = \sum_k (-1)^k h_k$.

Proof. Let $z_k = \dim \ker d_k, b_k = \dim \operatorname{im} d_k$, so $c_k = z_k + b_k$ and $h_k = z_k - b_{k+1}$. Then $\chi(C) = \sum_k (-1)^k (z_k + b_k) = \sum_k (-1)^k (z_k - b_{k+1}) = \chi(H(C))$. □

2.3.2 Eilenberg-Steenrod Axioms

Definition. *An* ordinary homology theory *with coefficients in G (abelian group) is a functor*

$$\left\{ \begin{array}{c} \textit{pairs of spaces} \\ \textit{maps of pairs} \end{array} \right\} \to \left\{ \begin{array}{c} \textit{graded } \mathbb{Z}\textit{-modules} \\ \textit{graded homomorphisms} \end{array} \right\}$$

satisfying:

(i) Homotopy invariance: If $f_0, f_1 : (X,A) \to (Y,B)$, $f_0 \sim f_1$ as maps of pairs, then $f_{0*} = f_{1*}$

(ii) LES of a pair: There is a LES

$$\ldots \to H_k(A) \to H_k(X) \to H_k(X,A) \to H_{k-1}(A) \to \ldots$$

where $H_k(X) = H_k(X, \emptyset)$. A map $(X,A) \to (Y,B)$ induces a map of LES (naturality).

(iii) Excision: If $\overline{B} \subseteq \operatorname{Int} A$, then $i_* : H_*(X \setminus B, A \setminus B) \to H_*(X,A)$ is an isomorphism.

(iv) Dimension axiom: $H_*(\{\bullet\}) = \begin{cases} G & * = 0, \\ 0 & * \neq 0. \end{cases}$

Theorem 2.15. *If X is a fcc and H_* is any functor satisfying these axioms, then $H_*(X) \simeq H_*(C_*^{\text{cell}}(X) \otimes G)$. In particular, if $H_*(X;G)$ satisfies these axioms, then $H_*(X;G) \cong H_*(C_*^{\text{cell}}(X) \otimes G)$.*

Proof idea. Go through the proof of Theorem 2.9 and the construction of $H_*^{\text{cell}}(X)$ to see that we only ever used these axioms (for the computation of $H_*(S^n)$ we used the MV-sequence which can be deduced from the axioms). □

2.3.3 More Algebra - The Universal Coefficient Theorem

Definition. *If M is an R-module, a* free resolution *of M is a free chain complex A over R such that*

(1) $A_k = 0$ for $k < 0$,

(2) $H_*(A) = \begin{cases} M & * = 0, \\ 0 & * \neq 0. \end{cases}$

Examples.

- If $a \in R$ is not a zero divisor, then $0 \to R \xrightarrow{a} R \to 0$ is a free resolution of $R/(a)$.
- $R = \mathbb{C}[x, y]$. Then $R \xrightarrow{\begin{bmatrix} y \\ -x \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} x & y \end{bmatrix}} R$ is a free resolution of $R/(x, y)$.

Definition. *If M, N are R-modules, then* $\mathrm{Tor}_i(M, N) := H_i(A \otimes N)$ *where A is a free resolution of M.*

Tor_i measures the failure of $H_*(A \otimes M) \stackrel{?}{=} H_*(A) \otimes M$.

This is well-defined due to the following fact: Any two free resolutions of M are chain homotopic.

Exercise: $\mathrm{Tor}_0(M, N) \simeq M \otimes N$.

Examples. $R = \mathbb{Z}$. Then $\mathbb{Z} \xrightarrow{a} \mathbb{Z}$ is a free resolution of $\mathbb{Z}/(a)$, so

$$\mathrm{Tor}_*(\mathbb{Z}/a, \mathbb{Z}) = \begin{cases} \mathbb{Z}/a & * = 0, \\ 0 & * \neq 0. \end{cases}$$

And

$$\mathrm{Tor}_*(\mathbb{Z}/a, \mathbb{Z}/b) = H_*(\mathbb{Z}/b \xrightarrow{a} \mathbb{Z}/b) = \begin{cases} \mathbb{Z}/(a, b) & * = 0, 1, \\ 0 & * \neq 0, 1. \end{cases}$$

E.g. $\mathrm{Tor}_1(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2$ accounts for the extra $\mathbb{Z}/2$ in $H_*(C_*^{\mathrm{cell}}(\mathbb{RP}^3))$.

Definition. *A chain complex is* short injective *if for some $k \in \mathbb{Z}$,*

(1) $C_* = 0$ *for* $* \neq k, k+1$ *and* C_k, C_{k+1} *are free.*

(2) $d : C_{k+1} \to C_k$ *is injective.*

So C is a shifted free resolution of $H_*(C) = H_k(C)$.

Theorem 2.16. *A free chain complex over a PID is a direct sum of short injective complexes.*

Proof. Fact: If R is a PID, a submodule of a free module over R is free.

For each $k \in \mathbb{Z}$ we have a SES

$$0 \to \ker d_k \to C_k \xrightarrow{d_k} \operatorname{im} d_k \to 0.$$

By the fact, $\operatorname{im} d_k$ is free. Thus, the sequence splits and we get $C_k = \ker d_k \oplus B_k$ where $d_k : B_k \xrightarrow{\simeq} \operatorname{im} d_k$. Since $d^2 = 0$, $\operatorname{im} d_k \subseteq \ker d_{k-1} =: Z_{k-1}$, so $C_* = \bigoplus_k (B_k \xrightarrow{d_k} Z_{k-1})$. $\square$

Corollary 2.17. *If two free chain complexes over a PID have isomorphic homology, they are chain homotopy equivalent.*

Proof. By the theorem, a chain complex over a PID is a direct sum of free resolutions of its homologies, so the claim follows the fact that any two free resolutions of the same module are chain homotopy equivalent. □

Corollary 2.18. *If C is a chain complex over a field $\mathbb{F}$, then $C \sim (H_*(C), 0)$.*

Proof. $H_*(C)$ is free over $\mathbb{F}$ since every module over $\mathbb{F}$ is free, and the previous corollary applies. □

Corollary 2.19 (Universal Coefficient Theorem)**.** *If C is a free chain complex over a PID, then*

$$H_k(C \otimes N) = (H_k(C) \otimes N) \oplus \mathrm{Tor}_1(H_{k-1}(C), N) = \mathrm{Tor}_0(H_k(C), N) \oplus \mathrm{Tor}_1(H_{k-1}(C), N).$$

Proof. C is a direct sum of short injective complexes (and both sides commute with direct sums in C), so it suffices to check the claim for a short injective complex, where it is the definition of Tor. □

So $H_*(X; G)$ is determined by $H_*(X)$.

3 Cohomology and Products

3.1 Cohomology

Definition. *If M, N are R-modules, then* $\operatorname{Hom}(M,N)$ *is the R module of R-linear maps $M \to N$. If $f : M_1 \to M_2$ is R-linear, we get an R-linear map $f^* : \operatorname{Hom}(M_2, N) \to \operatorname{Hom}(M_1, N)$, $\alpha \mapsto \alpha \circ f$.*

So we have a contravariant functor

$$\left\{ \begin{array}{c} R\text{-modules} \\ R\text{-linear maps} \end{array} \right\} \xrightarrow{\operatorname{Hom}(-,N)} \left\{ \begin{array}{c} R\text{-modules} \\ R\text{-linear maps} \end{array} \right\}$$

If (C,d) is a chain complex over R, then define $(\operatorname{Hom}(C,N), d^*)$ by $\operatorname{Hom}(C,N) = \bigoplus_k \operatorname{Hom}(C_k, N)$, $d_k^* : \operatorname{Hom}(C_{k-1}, N) \to \operatorname{Hom}(C_k, N)$. We say $(\operatorname{Hom}(C,N), d^*)$ is a *cochain complex* and d^* *raises homological degree* by 1.

So there is a contravariant functor

$$\left\{ \begin{array}{c} \text{chain complexes over } R \\ \text{chain maps} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{cochain complexes over } R \\ \text{cochain maps} \end{array} \right\}$$
$$(C,d) \longmapsto (\operatorname{Hom}(C,N), d^*)$$
$$f : C \to C' \longmapsto f^* : \operatorname{Hom}(C',N) \to \operatorname{Hom}(C,N)$$

If (C, d^*) is a cochain complex, its k-th cohomology is $H^k(C) = \ker d_k^* / \operatorname{im} d_{k-1}^*$

Definition. *If X is a space, its* singular cochain complex with coefficients in G *is* $(C^*(X;G), d^*)$ *where $C^*(X;G) = \operatorname{Hom}(C_*(X), G)$ and its* k-th singular cohomology *is $H^k(X;G) = H^k(C^*(X;G))$. Similarly we define the cochain complex and cohomology of a pair of spaces.*

If $f : X \to Y$, then we get the cochain map $f^\# : C^k(Y;G) \to C^k(X;G)$ given by $f^\#(\alpha)(\sigma) = \alpha(f_\#(\sigma)) = \alpha(f \circ \sigma)$ for $\sigma : \Delta^k \to X$. This induces a map $f^* : H^k(Y;G) \to H^k(X;G)$.

Hence we get a contravariant functor

$$H^*(-,-;G) : \left\{ \begin{array}{c} \text{pairs of spaces} \\ \text{maps of pairs} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{graded } \mathbb{Z}\text{-modules} \\ \mathbb{Z}\text{-linear maps} \end{array} \right\}$$

It is the composition of the following functors:

$$\begin{array}{ccc} \left\{\begin{array}{c}\text{pairs of spaces}\\ \text{maps of pairs}\end{array}\right\} \xrightarrow{C_*} \left\{\begin{array}{c}\text{chain complexes}\\ \text{chain maps}\end{array}\right\} & & \\ \Big\downarrow{\scriptstyle \mathrm{Hom}(-,G)} & & \\ \left\{\begin{array}{c}\text{cochain complexes}\\ \text{cochain maps}\end{array}\right\} \xrightarrow{H^*} \left\{\begin{array}{c}\mathbb{Z}\text{-modules}\\ \mathbb{Z}\text{-linear maps}\end{array}\right\} & & \end{array}$$

Dual to chain homotopies we have:

Definition. *If C, C' are cochain complexes (over R), $f, g : C \to C'$ cochain maps, we say f and g are* cochain homotopic *if $f - g = d^*h + hd^*$ where $h : C^k \to C'^{k-1}$ is R-linear. h is a* cochain homotopy*.*

Lemma 3.1. *Cochain homotopic maps induce the same map on cohomology.*

Lemma 3.2. *If $f, g : C \to C'$ are maps of chain complexes and $f \sim g$ via h, then $f^*, g^* : \mathrm{Hom}(C'; N) \to \mathrm{Hom}(C; N)$ are cochain homotopic via h^*.*

3.1.1 Eilenberg-Steenrod Axioms for Cohomology

Note that $C^k(X, Y; G) = \{f : C_k(X) \to G \mid f \text{ is } \mathbb{Z}\text{-linear, } f(\sigma) = 0 \text{ if } \operatorname{im}\sigma \subseteq A\}$.

For convenience we will drop the G in $H^*(X, G), H^*(X, A; G)$ in the following.

H^* satisfies the dual version of the Eilenberg-Steenrod axioms:

(i) Homtopy invariance: If $f_0, f_1 : (X, A) \to (Y, B)$ with $f_0 \sim f_1$ as maps of pairs, then $f_0^* = f_1^* : H^*(Y, B) \to H^*(X, A)$.

Proof. $f_{0\#}, f_{1\#}$ are chain homotopic, hence $f_0^\#, f_1^\#$ are cochain homotopic, hence $f_0^* = f_0^1$. □

(ii) LES of pair: We have a SES of cochain complexes

$$0 \to C^*(X, A) \to C^*(X) \to C^*(A) \to 0.$$

The associated LES is

$$\ldots \to H^k(X, A) \to H^k(X) \to H^k(A) \xrightarrow{\delta} H^{k+1}(X, A) \to \ldots$$

A map of pairs induces a map of LES's in cohomology.

(iii) Excision: If $B \subseteq A \subseteq X$, $\overline{B} \subseteq A^\circ$, then

$$i^* : H^*(X, A) \to H^*(X - B, A - B)$$

is an isomorphism.

Proof. $i_\# : C_*(X - B, A - B) \to C_*(X, A)$ induces an isomorphism on homology (ordinary excision). Since $C_*(X, A), C_*(X - B, A - B)$ are free, $i_\#$ is a chain homotopy equivalence (Sheet 1, Exercise 11). So $i^\#$ is cochain homotopy equivalence and thus i^* an isomorphism. □

(iv) Dimension: $H^*(\{\bullet\}; G) = \begin{cases} G & * = 0, \\ 0 & * \neq 0 \end{cases}$

Theorem 3.3. *Any functor satisfying these axioms is given by*

$$H^*_{\text{cell}}(X; G) = H^*(\text{Hom}(C^{\text{cell}}_*(X); G))$$

when X is a finite cell complex.

Short proof that $H^*(X; G) \cong H^*_{\text{cell}}(X; G)$ if X is a fcc:

$C_*(X), C^{\text{cell}}_*(X)$ are free chain complexes with the same homology over the PID $\mathbb{Z}$, so they are homotopy equivalent by Corollary 2.17, so $C^*(X; G), C^*_{\text{cell}}(X; G)$ are homotopy equivalent.

Example. We compute $H^*_{\text{cell}}(\mathbb{RP}^3)$. Recall that

$$C^{\text{cell}}_*(\mathbb{RP}^3) = (\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}).$$

Therefore

$$C^*_{\text{cell}}(\mathbb{RP}^3) = (\mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z}),$$

so

$$H^*_{\text{cell}}(\mathbb{RP}^3) = \begin{cases} \mathbb{Z} & * = 0, 3, \\ \mathbb{Z}/2 & * = 2, \\ 0 & \text{otherwise.} \end{cases}$$

3.1.2 Ext and the Universal Coefficient Theorems

Definition. *If M, N are R-modules, then* $\text{Ext}^i(M, N) = H^i(\text{Hom}(A, N))$ *where A is a free resolution of M.*

Again this is well-defined since any two free resolutions of the same module are chain homotopy equivalent.

Example. We compute $\text{Ext}^*(\mathbb{Z}/n, \mathbb{Z})$ for $n \neq 0$. $A = (\mathbb{Z} \xrightarrow{n} \mathbb{Z})$ is a free resolution of $\mathbb{Z}/n$, and $\text{Hom}(A, \mathbb{Z}) = \mathbb{Z} \xleftarrow{n} \mathbb{Z}$, so

$$\text{Ext}^i(\mathbb{Z}/n, \mathbb{Z}) = \begin{cases} \mathbb{Z}/n & * = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly,

$$\text{Ext}^i(\mathbb{Z}/n, \mathbb{Z}/n) = \begin{cases} \mathbb{Z}/n & * = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.4 (Universal Coefficient Theorem)**.** *Suppose X is a finite cell complex. Then*

$$H^k(X;G) \cong \mathrm{Hom}(H_k(X);G) \oplus \mathrm{Ext}^1(H_{k-1}(X);G).$$

Proof. Split $C_*^{\mathrm{cell}}(X)$ into a direct sum of short injective complexes and use definition of Ext. □

Example. If X is a fcc, $H_k(X) = \mathbb{Z}^{b_k} \oplus T_k$ by structure theorem for finitely generated abelian groups where T_k is finite. b_k is called the *k-th Betti number* of X. We get $H^k(X,\mathbb{Z}) \cong \mathbb{Z}^{b_k} \oplus T_{k-1}$.

3.1.3 Pairing

Suppose C is a chain complex over R. There is a bilinear pairing $\langle , \rangle : \mathrm{Hom}(C_k;N) \times C_k \to N$, $\langle \alpha, c\rangle = \alpha(c)$.

Lemma 3.5. *This descends to a pairing*

$$\begin{aligned} H^k(\mathrm{Hom}(C,N)) \times H_k(C) &\longrightarrow N \\ ([\alpha],[c]) &\longmapsto \langle [\alpha],[c]\rangle := \alpha(c) \end{aligned}$$

Proof. We need to check that this is well-defined. For $\beta \in \mathrm{Hom}(C,N), b \in C$ we have:

$$\begin{aligned} \langle \alpha + d^*\beta, c + db\rangle &= \alpha(c) + \alpha(db) + d^*\beta(c+db) \\ &= \alpha(c) + d^*\alpha(b) + \beta(d(c+db)) \\ &= \alpha(c) = \langle \alpha, c\rangle \end{aligned}$$

□

3.2 Cup Product

Let R be a commutative ring.

Definition. *If $\alpha \in C^k(X;R), \beta \in C^l(X;R)$, then $\alpha \smile \beta \in C^{k+l}(X;R)$ is given by*

$$\alpha \smile \beta(\sigma) = \alpha(\sigma \circ F_{0\ldots k})\beta(\sigma \circ F_{k\ldots k+l}),$$

for $\sigma : \Delta^{k+l} \to X$.

Lemma 3.6. *$\smile$ makes $C^*(X;R)$ into a (noncommutative) ring with identity $1 \in C^0(X;R)$ where $1(\sigma_p) = 1 \in R$.*

Proof. We must check

(1) $(\alpha \smile \beta) \smile \gamma = \alpha \smile (\beta \smile \gamma)$,

(2) $(\alpha_1 + \alpha_2) \smile \beta = \alpha_1 \smile \beta + \alpha_2 \smile \beta$,

(3) $\alpha \smile (\beta_1 + \beta_2) = \alpha \smile \beta_1 + \alpha \smile \beta_2$,

(4) $\alpha \smile 1 = \alpha = 1 \smile \alpha$.

These are all easy. □

Lemma 3.7. *If $\alpha \in C^k(X;R), \beta \in C^l(X;R)$, then*

$$d^*(\alpha \smile \beta) = (d^*\alpha) \smile \beta + (-1)^k \alpha \smile (d^*\beta)$$

Proof. Let $\sigma : \Delta^{k+l+1} \to X$. Then:

$$\begin{aligned} d^*(\alpha \cup \beta)(\sigma) = \alpha \smile \beta(d\sigma) = \alpha \smile \beta\Big(\sum_{j=0}^{k+l+1} (-1)^j \sigma \circ F_{\hat{j}}\Big) \\ &= \sum_{j=0}^{k+l+1} (-1)^j \alpha(\sigma \circ F_{\hat{j}} \circ F_{0\ldots k})\beta(\sigma \circ F_{\hat{j}} \circ F_{k\ldots k+l}) \\ &= \sum_{j=0}^{k+1}(-1)^j \alpha(\sigma \circ F_{0\ldots\hat{j}\ldots k+1})\beta(\sigma \circ F_{k+1\ldots k+l+1}) \\ &\quad + \sum_{j=k}^{k+l+1} (-1)^j \alpha(\sigma \circ F_{0\ldots k})\beta(\sigma \circ F_{k\ldots\hat{j}\ldots k+l+1}) \\ &= (d^*\alpha) \smile \beta(\sigma) + (-1)^k (\alpha \smile d^*\beta)(\sigma) \end{aligned}$$

(Note that here different F_I map between different simplices) □

Corollary 3.8. $\smile$ *descends to give a map*

$$\begin{aligned} \smile: H^k(X;R) \times H^l(X;R) &\longrightarrow H^{k+l}(X;R) \\ [\alpha] \times [\beta] &\longmapsto [\alpha \smile \beta] \end{aligned}$$

This makes $H^*(X;R)$ into a ring with unit $[1] \in H^0(X;R)$.

Proof. We check that this is well-defined. First note that if $[\alpha] \in H^k(X;R), [\beta] \in H^l(X;R)$, then $d^*\alpha = 0 = d^*\beta$, so $d^*(\alpha \smile \beta) = d^*\alpha \smile \beta + (-1)^k \alpha \smile d^*\beta = 0$, so $[\alpha \cup \beta] \in H^{k+l}(X;R)$. Now let $\alpha' = \alpha + d^*a, \beta' = \beta + d^*b$ with $a \in C^k(X;R), b \in C^l(X;R)$. Then

$$\begin{aligned} \alpha' \smile \beta' &= \alpha \smile \beta + (d^*a) \smile \beta + (\alpha + d^*a) \smile d^*b \\ &= \alpha \smile \beta + (d^*a) \smile \beta \pm (\alpha + d^*a) \smile d^*((\alpha + d^*a) \cup b) \end{aligned}$$

Hence $[\alpha' \smile \beta'] = [\alpha \smile \beta]$. Hence $\smile$ is well-defined on cohomology.

Note that for $\tau : \Delta^1 \to X$, we have $d^*1(\tau) = 1(d\tau) = 1(\sigma_{\tau(1)} - \sigma_{\tau(0)}) = 1 - 1 = 0$, so $d^*1 = 0$, so 1 defines a class in $H^0(X;R)$. □

Proposition 3.9. *If $f : X \to Y$, then $f^* : H^*(Y;R) \to H^*(X;R)$ is a ring homomorphism, i.e. $f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta)$ and $f^*(1) = 1$.*

Proof. Consider $f^\# : C^*(Y;R) \to C^*(X;R)$. Then

$$\begin{aligned} f^\#(\alpha \smile \beta)(\sigma) &= \alpha \smile \beta(f \circ \sigma) \\ &= \alpha(f \circ \sigma \circ F_{0\dots k})\beta(f \circ \sigma \circ F_{k\dots k+l}) \\ &= f^\#(\alpha)(\sigma \circ F_{0\dots k}) f^\#(\beta)(\sigma \circ F_{k\dots k+l}) \\ &= f^\#(\alpha) \smile f^\#(\beta)(\sigma) \end{aligned}$$

□

Notation: If $a \in H^k(X;R)$, we write $|a| := k$.

Proposition 3.10. *$H^*(X;R)$ is graded commutative, i.e. $a \smile b = (-1)^{|a||b|} b \smile a$ (But this is totally false for chains).*

We use a chain map $r : C_*(X) \to C_*(X)$ defined as follows. Let $\rho_n : \Delta^n \to \Delta^n$ be the linear map defined by $e_i \mapsto e_{n-i}$. Let $\varepsilon(j) = \frac{j(j+1)}{2} = \sum_{i=0}^{j} i$, so that $\det \rho_j = (-1)^{\varepsilon(j)}$. Define $r_j : C_j(X) \to C_j(X)$ by $r_j(\sigma) = (-1)^{\varepsilon(j)} \sigma \circ \rho_j$. Note that r also induces a map $r : C_*(X, A) \to C_*(X, A)$ for $A \subseteq X$.

Theorem 3.11. *(1) $r : C_*(X) \to C_*(X)$ is a chain map.*

(2) $r \sim 1_{C_*(X)}$.

Proof of the proposition using the theorem. Dualizing r gives $r^* : C^*(X;R) \to C^*(X;R)$ and $r^* \sim 1_{C^*(X)}$. So $[r^*(\alpha)] = [\alpha]$. By definition of r, we have

$$(-1)^{\varepsilon(|\alpha|+|\beta|)} r^*(\alpha \smile \beta) = (-1)^{\varepsilon(|\alpha|)}(-1)^{\varepsilon(|\beta|)} r^*(\beta) \smile r^*(\alpha),$$

hence

$$\begin{aligned} [\alpha \smile \beta] = [r^*(\alpha \smile \beta)] &= (-1)^{\varepsilon(|\alpha|+|\beta|)}(-1)^{\varepsilon(|\alpha|)}(-1)^{\varepsilon(|\beta|)}[r^*(\beta) \smile r^*(\alpha)] \\ &= (-1)^{|\alpha||\beta|}[\beta] \smile [\alpha]. \end{aligned}$$

□

Proof of the theorem. (1) Let $\sigma : \Delta^n \to X$. We have $\rho_n \circ F_{\hat{j}} = F_{\widehat{n-j}} \circ \rho_{n-1}$, so

$$\begin{aligned} d(r(\sigma)) &= (-1)^{\varepsilon(n)} \sum\nolimits_j (-1)^j \sigma \circ \rho_n \circ F_{\hat{j}} \\ &= (-1)^{\varepsilon(n)} \sum\nolimits_j (-1)^j \sigma \circ F_{\widehat{n-j}} \circ \rho_{n-1} \\ &= (-1)^n (-1)^{\varepsilon(n)} \sum\nolimits_j (-1)^{n-j} \sigma \circ F_{\widehat{n-j}} \circ \rho_{n-1} \end{aligned}$$

$$= (-1)^{\varepsilon(n-1)} \sum_j (-1)^j \sigma \circ F_{\widehat{\jmath}} \circ \rho_{n-1}$$
$$= r_{n-1}(d\sigma)$$

(2) One can write down an explicit chain homotopy, this is done e.g. in [Hat02]. We do it in a different way:

$C_*(X)$ is free, so it suffices to show that $r_* : H_*(X) \to H_*(X)$ is the identity on $H_*(X)$. Observations:

(i) If $f : X \to Y$, $f_\# \circ r(\sigma) = (-1)^{\varepsilon(|\sigma|)} f \circ \sigma \circ p_{|\sigma|} = r \circ f_\#(\sigma)$, so $f_* r_* = r_* f_*$.

(ii) There is a commutative diagram of SES:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_*(A) & \longrightarrow & C_*(X) & \longrightarrow & C_*(X,A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} & & \\
0 & \longrightarrow & C_*(A) & \longrightarrow & C_*(X) & \longrightarrow & C_*(X,A) & \longrightarrow & 0
\end{array}$$

This induces a map between the LES of (X, A), hence we see that $r_*\partial = \partial r_*$ where ∂ is the boundary map in the LES.

Notation: $R_n(X, A)$ is the statement $(r_*)_n = 1_{H_n(X,A)}$.

(iii) If $f_* : H_n(X, A) \to H_n(Y, B)$ is injective, then $R_n(Y, B) \implies R_n(X, A)$.

If $g_* : H_n(X, A) \to H_n(Y, B)$ is surjective, then $R_n(X, A) \implies R_n(Y, B)$.

Both statements follow from (i).

We now prove the claim in several steps:

(A) $R_0(X)$ holds. Indeed, if $[\sigma] \in H_0(X)$, then $r(\sigma) = \sigma$.

(B) By Observation (2), the following square commutes:

$$\begin{array}{ccc}
H_n(D^n, S^{n-1}) & \xrightarrow[\simeq]{\partial} & \widetilde{H}_{n-1}(S^{n-1}) \\
{\scriptstyle r_*}\downarrow & & \downarrow{\scriptstyle r_*} \\
H_n(D^n, S^{n-1}) & \xrightarrow[\simeq]{\partial} & \widetilde{H}_{n-1}(S^{n-1})
\end{array}$$

So $R_{n-1}(S^{n-1}) \implies R_n(D^n, S^{n-1})$. From the isomorphisms

$$H_n(D^n, S^{n-1}) \xrightarrow{p_*} H_n(D^n/S^{n-1}, S^{n-1}/S^{n-1}) \xleftarrow{\simeq} H_n(S^n)$$

we also get $R_n(D^n, S^{n-1}) \implies R_n(S^n)$.

Hence by induction on n, we get that $R_n(D^n, S^{n-1})$ and $R_n(S^n)$ are true for all n.

(C) $H_n(\coprod_{k=1}^r D^n, \coprod_{k=1}^r S^{n-1}) = \bigoplus_{k=1}^r H_n(D^n, S^{n-1})$. Hence $R_n(\coprod D^n, \coprod S^{n-1})$.

(D) If X is an fcc, then $R_*(X)$. Proof: We show $R_*(X_k)$ holds for all k by induction. Base case $R_*(X_0) = R_0(x_0)$ holds by (A). Suppose $R_*(X_{k-1})$ and then consider the LES of (X_k, X_{k-1}):

$$0 \to H_k(X_k) \to H_k(X_k, X_{k-1}) \to H_{k-1}(X_{k-1}) \to H_{k-1}(X_k) \to 0$$

and

$$0 \to H_i(X_{k-1}) \to H_i(X_k) \to 0$$

for $i < k-1$.

The map $F_* : H_*(\coprod D^k, \coprod S^{k-1}) \to H_*(X_k, X_{k-1})$ is an isomorphism where F is the attaching map. By (B), $R_*(X_k, X_{k-1})$ holds, hence $R_k(X_k)$ by (3) (a). By induction, $R_*(X_{k-1})$ holds, hence $R_i(X_k)$ holds for $i < k$ by (3) (b). Hence $R_*(X_k)$.

(E) For any X and $x \in H_*(X)$, there exists an fcc Y and $f : Y \to X$ with $f_*(y) = x$ (Sheet 2, Exercise 6). Then $r_*(x) = r_*(f_*(y)) = f_*(r_*(y)) = f_*(y) = x$.

$\square$

Pairs: Recall $C^*(X, A) \subseteq C^*(X)$. Let $\alpha \in C^k(X, A), \beta \in C^l(X)$. If $\operatorname{im}\sigma \subseteq A$, then $\operatorname{im}\sigma \circ F_{0\ldots k} \subseteq A$, so

$$(\alpha \smile \beta)(\sigma) = \alpha(\sigma \circ F_{0\ldots k})\beta(\sigma \circ F_{k\ldots k+l}) = 0\beta(\ldots) = 0,$$

i.e. $\alpha \smile \beta \in C^*(X, A)$.

So $\smile$ defines a map $H^*(X, A) \times H^*(X) \to H^*(X, A)$. More generally, $\smile$ defines $H^*(X, A) \times H^*(X, B) \to H^*(X, A \cup B)$ (using subdivision lemma, see example sheet).

Examples.

(1) If X is path connected, $H_0(X) = \mathbb{Z}$, so $H^0(X) \simeq \operatorname{Hom}(H_0(X), \mathbb{Z}) = \mathbb{Z}$ (as $H_{-1}(X) = 0$ using UCT) and $H^0(X) = \langle 1 \rangle$.

(2) We compute the cohomology ring of S^n for $n > 0$. Recall that

$$H_*(S^n) = \begin{cases} \mathbb{Z} & * = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

$H_*(S^n)$ is free over $\mathbb{Z}$, so by the UCT

$$H^*(S^n) = \operatorname{Hom}(H_*(S^n), \mathbb{Z}) = \begin{cases} \mathbb{Z} & * = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

We know $H^0(S^n) = \langle 1 \rangle$. Let a be a generator of $H^n(S^n)$. Then

$$1 \cup 1 = 1, a \cup 1 = a = 1 \cup a.$$

And $a \cup a \in H^{2n}(S^n) = 0$, so $H^*(S^n) = \mathbb{Z}[a]/a^2$ with $|a| = n$.

(3) If X is path connected, $p \in X$, then $\iota_* : H_0(p) \xrightarrow{\simeq} H_0(X)$, so $\mathbb{Z} \simeq H^0(X) \to H^0(p) \simeq \mathbb{Z}$ is an isomorphism, so

$$H^*(X,p) = \ker\big(H^*(X) \to H^*(p)\big) = \bigoplus_{i>0} H^i(X)$$

is an ideal in $H^*(X)$.

(4) $H^*(X \amalg Y) = H^*(X) \oplus H^*(Y)$ (direct product of rings). Proof: $C_*(X \amalg Y) = C_*(X) \oplus C_*(Y)$, so $C^*(X \amalg Y) = C^*(X) \oplus C^*(Y)$. It is easy to see that this decomposition respects both d^* and $\smile$, hence the claim.

(5) Suppose $(X, p_X), (Y, p_Y)$ are good pairs and X, Y are path-connected. By collapsing a pair, $\pi^* : H^*(X \vee Y, p) \to H^*(X \amalg Y, p_X \amalg p_Y)$ is an isomorphism. We have

$$H^*(X \amalg Y, p_X \amalg p_Y) = H^*(X, p_X) \oplus H^*(Y, p_Y) \subseteq H^*(X) \oplus H^*(Y).$$

So

$$H^i(X \vee Y) = \begin{cases} H^i(X) \oplus H^i(Y) & i > 0, \\ \langle 1 \rangle \simeq \mathbb{Z} & i = 0. \end{cases}$$

The multiplication is given by $(a_1, a_2) \smile (b_1, b_2) = (a_1 \smile b_1, a_2 \smile b_2)$ if $|a_i|, |b_i| > 0$.

Example: Let a_n denote a generator of $H^n(S^n)$. Then $H^*(S^2 \vee S^2 \vee S^4) = \langle 1, a, a', b \rangle$ where

$$\begin{aligned} a = (a_2, 0, 0), a' = (0, a_2, 0) &\in H^2(S^2) \oplus H^2(S^2) \oplus H^2(S^4) \cong \mathbb{Z}^2, \\ b = (0, 0, a_4) &\in H^4(S^2) \oplus H^4(S^2) \oplus H^4(S^4) \cong \mathbb{Z}. \end{aligned}$$

We have $a \smile a' = (a_2, 0, 0)(0, a_2, 0) = (0, 0, 0) = 0$. So there are no interesting cup products.

3.3 Exterior Products

Setup: Let (X, A) be a pair of spaces, Y a space. Let

$$\begin{aligned} \pi_1 &: (X \times Y, A \times Y) \to (X, A), \\ \pi_2 &: X \times Y \to Y \end{aligned}$$

be the projections.

Definition. *If $a \in H^k(X, A), b \in H^l(Y)$, their* exterior product *is*

$$a \times b = \pi_1^*(a) \smile \pi_2^*(b) \in H^{k+l}(X \times Y, A \times Y).$$

Remark: If C, C' are graded groups/rings, their product (resp. tensor product) is given by $(C \times C')_n = \bigoplus_{k+l=n} C_k \times C'_l$ (resp. $(C \otimes C')_n = \bigoplus_{k+l=n} C_k \otimes C'_l$).

Observations:

(1) $H^*(X,A) \times H^*(Y) \to H^*(X \times Y, A \times Y), (a,b) \mapsto a \times b$ is bilinear, so it extends to $\Phi : H^*(X,A) \otimes H^*(Y) \to H^*(X \times Y, A \times Y)$.

(2) We have $(a_1 \times b_1) \smile (a_2 \times b_2) = (-1)^{|b_1||a_2|}(a_1 \smile a_2) \times (b_1 \smile b_2)$. Proof:

$$\begin{aligned}(a_1 \times b_1) \smile (a_2 \times b_2) &= \pi_1^*(a_1) \smile \pi_2^*(b_1) \smile \pi_1^*(a_2) \smile \pi_2^*(b_2) \\ &= (-1)^{|b_1||a_2|}\pi_1^*(a_1) \smile \pi_1^*(a_2) \smile \pi_2^*(b_1) \smile \pi_2^*(b_2) \\ &= (-1)^{|b_1||a_2|}\pi_1^*(a_1 \smile a_2) \smile \pi_2^*(b_1 \smile b_2) \\ &= (-1)^{|b_1||a_2|}(a_1 \smile a_2) \times (b_1 \smile b_2)\end{aligned}$$

Theorem 3.12. *If $H^*(Y;R)$ is free over R, then*

$$\Phi : H^*(X,A;R) \otimes H^*(Y;R) \to H^*(X \times Y, A \times Y; R)$$

is an isomorphism.

Note that the hypothesis of the theorem is always satisfied if e.g. R is a field.

Consequences:

(1) This lets us compute $H^*(X \times Y;R)$ from $H^*(X;R), H^*(Y;R)$ if $H^*(Y;R)$ is free.

(2) It also tells us the ring structure on $H^*(X \times Y;R)$ (by Observation (2) above).

Examples.

- Consider $T^2 = S^1 \times S^1$. We have $H^*(S^1) = \langle 1, a_1\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$. Then

$$(H^*(S^1) \otimes H^*(S^1))_n \cong \begin{cases} \mathbb{Z} & n = 2, \\ \mathbb{Z}^2 & n = 1, \\ \mathbb{Z} & n = 1. \end{cases}$$

  Since $H^*(S^1)$ is free, we get $H^*(T^2) = H^*(S^1) \otimes H^*(S^1)$ and we obtain generators:

$$H^*(S^1 \times S^1) = \begin{cases} \mathbb{Z} = \langle a_1 \times a_1\rangle = \langle c\rangle & * = 2, \\ \mathbb{Z}^2 = \langle a_1 \times 1, 1 \times a_1\rangle = \langle a, b\rangle & * = 1, \\ \mathbb{Z} = \langle 1 \times 1\rangle = \langle 1\rangle & * = 0. \end{cases}$$

  Then $a^2 = (1_1 \times 1) \smile (a_1 \times 1) = -(a_1^2 \times 1) = 0$ as $H^2(S^1) = 0$. Similarly, $b^2 = 0$. We have $a \smile b = (a_1 \times 1) \smile (1 \times a_1) = (a_1 \times a_1) = c$ and $b \smile a = -a \smile b = -c$.

  In other words, we get $H^*(T^2) = \bigwedge^*\langle \alpha_1, \alpha_2\rangle$ with $\alpha_1 = a_1, \alpha_2 = b$ and $\alpha_i\alpha_j = -\alpha_j\alpha_i$. More generally $H^*(T^n) = H^*(S^1) \otimes \cdots \otimes H^*(S^1)$ (n times) $\simeq \bigwedge^*\langle \alpha_1, \ldots, \alpha_n\rangle$ with $\alpha_i = 1 \times 1 \times \cdots \times a_1 \times \cdots \times 1$.

- Similarly, we calculate the cohomology ring of $S^2 \times S^2$. $H^*(S^2)$ is free, so $H^*(S^2 \times S^2) = H^*(S^2) \otimes H^*(S^2)$. Let $a = a_2 \times 1, b = 1 \times a_2, c = a_2 \times a_2$. Then

$$H^*(S^2 \times S^2) = \begin{cases} \langle c \rangle = \mathbb{Z} & * = 4, \\ \langle a, b \rangle = \mathbb{Z}^2 & * = 2, \\ \langle 1 \rangle = \mathbb{Z} & * = 0. \end{cases}$$

  Again we have $a^2 = 0 = b^2, a \smile b = c$, but now $b \smile a = a \smile b = c$.

  Corollary 3.13. *$S^2 \times S^2 \not\simeq S^2 \vee S^2 \vee S^4$, even though $H_*(S^2 \times S^2) \simeq H_*(S^2 \vee S^2 \vee S^4)$.*

  Proof. We have $H^*(S^2 \times S^2) \not\simeq H^*(S^2 \vee S^2 \vee S^4)$ as rings. For example, if $a, b \in H^2(S^2 \vee S^2 \vee S^4)$, then $a \smile b = 0$, but this is not true in $H^*(S^2 \times S^2)$. □

Proof of Theorem 3.12. We drop the R in $H^*(-; R)$.

We have two contravariant functors

$$\overline{h}, \underline{h} : \left\{ \begin{array}{c} \text{pairs of spaces} \\ \text{maps of pairs} \end{array} \right\} \to \left\{ \begin{array}{c} \text{graded } \mathbb{Z}\text{-modules} \\ \text{graded } \mathbb{Z}\text{-linear amps} \end{array} \right\}$$

defined by

$$\begin{aligned} \overline{h}(X, A) &= H^*(X \times Y, A \times Y), \\ f : (X, A) \to (X', A') &\mapsto \overline{f}^* = (f \times \mathrm{id}_Y)^* : H^*(X' \times Y, A' \times Y) \to H^*(X \times Y, A \times Y) \end{aligned}$$

and

$$\begin{aligned} \underline{h}(X, A) &= H^*(X, A) \otimes H^*(Y), \\ f : (X, A) \to (X', A') &\mapsto \underline{f}^* = f^* \otimes \mathrm{id}_{H^*(Y)} : \underline{h}(X', A') \to \underline{h}(X, A). \end{aligned}$$

$\overline{h}, \underline{h}$ satisfy all Eilenberg-Steenrod axioms for cohomology except the dimension axiom (so they are *generalized cohomology theories*). They are:

(1) Homotopy invariance: Let $f_0 \sim f_1 : (X, A) \to (X', A')$. Then $f_0^* = f_1^*$, hence $\underline{f}_0^* = \underline{f}_1^*$. Also $f_0 \times 1_Y \sim f_1 \times 1_Y$, so $\overline{f}_0^* = (f_0 \times 1_Y)^* = (f_1 \times 1_Y)^* = \overline{f}_1^*$.

(2) LES of a pair: For $\overline{h}$ this is just the LES of $(X \times Y, A \times Y)$. For $\underline{h}$: $H^*(Y)$ is free by hypothesis, the LES of (X, A) stays exact after tensoring with $H^*(Y)$.

(3) Excision: If $\overline{B} \subseteq \operatorname{Int} A \subseteq A \subseteq X$, then $\overline{i}^* : \overline{h}(X, A) \to \overline{h}(X - B, A - B)$ is an isomorphism (excision for $B \times Y \subseteq A \times Y \subseteq X \times Y$). And $\underline{i}^* : \underline{h}(X, A) \to \underline{h}(X - B, A - B)$ is an isomorphism (excision for $B \subseteq A \subseteq X$).

Properties (1),(2),(3) imply that $\overline{h}, \underline{h}$ satisfy (4) “Collapsing a pair”, i.e. if (X, A) is a good pair, then $\underline{h}(\pi), \overline{h}(\pi)$ are isomorphisms where $\pi : (X, A) \to (X/A, A/A)$ is the quotient map.

Lemma 3.14. Φ *commutes with the induced maps and boundary map in the LES of a pair.*

Proof. Suppose $f : X_1 \to X_2$. Let $F = f \times 1_Y : X_1 \times Y \to X_2 \times Y$. Then

$$\begin{aligned}
\overline{f}^*(\Phi(a \otimes b)) &= F^*(a \times b) \\
&= F^*(\pi_1^*(a) \smile \pi_2^*(b)) \\
&= F^*\pi_1^*(a) \smile F^*\pi_2^*(b) \\
&= (\pi_1 \circ F)^*(a) \smile (\pi_2 \circ F)^*(b) \\
&= (f \circ \pi_1)^*(a) \smile (\pi_2)^*(b) \\
&= \pi_1^* f^*(a) \smile \pi_2^* b \\
&= f^*(a) \times b \\
&= \Phi(\underline{f}^*(a \otimes b))
\end{aligned}$$

For boundary see Sheet 3, Exercise 2. $\square$

We now prove the theorem in the case where X is a fcc. We proceed in several steps.

Let $P(X, A)$ be the statement that $\Phi : \underline{h}(X, A) \to \overline{h}(X, A)$ is an isomorphism.

(A) $P(\{\bullet\}), P(S^0)$ hold.

Proof. The map

$$H^*(\{\bullet\}) \otimes H^*(Y) = \underline{h}(\{\bullet\}) \to \overline{h}(\{\bullet\}) = H^*(\{\bullet\} \times Y)$$

is given by

$$\begin{aligned}
\mathbb{Z} \otimes H^*(Y) &\longrightarrow H^*(Y), \\
1 \otimes b &\longmapsto 1 \times b = \pi_1^*(1) \smile b = 1 \smile b = b
\end{aligned}$$

so it is an isomorphism. For S^0, use $H^*(X \amalg Y) = H^*(X) \oplus H^*(Y)$ (Exercise). $\square$

(B) If $X_1 \sim X_2$, then $P(X_1) \Leftrightarrow P(X_2)$.

Proof. If $f : X_1 \to X_2$ is a homotopy equivalence, then by the lemma there is a commuting square:

$$\begin{array}{ccc}
\underline{h}(X_2) & \xrightarrow{\underline{f}^*} & \underline{h}(X_1) \\
\big\downarrow{\scriptstyle \Phi_2} & & \big\downarrow{\scriptstyle \Phi_1} \\
\overline{h}(X_2) & \xrightarrow{\overline{f}^*} & \overline{h}(X_1)
\end{array}$$

Then $\underline{f}^*, \overline{f}^*$ are isomorphisms, so Φ_1 is an isomorphism iff Φ_2 is. $\square$

(C) If two of $P(X), P(A), P(X, A)$ hold, so does the third.

Proof. By Lemma, we have a commuting map of LESs:

$$\begin{tikzcd}
\dots \arrow[r] & \underline{h}^*(X,A) \arrow[r] \arrow[d] & \underline{h}^*(X) \arrow[r] \arrow[d] & \underline{h}^*(A) \arrow[r] \arrow[d] & \underline{h}^{*+1}(X,A) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & \overline{h}^*(X,A) \arrow[r] & \overline{h}^*(X) \arrow[r] & \overline{h}^*(A) \arrow[r] & \overline{h}^{*+1}(X,A) \arrow[r] & \dots
\end{tikzcd}$$

So the claim follows from the Five Lemma. □

(D) If (X, A) is a good pair, then $P(X, A) \Leftrightarrow P(X/A)$.

Proof. As in (B) we deduce that $P(X, A) \Leftrightarrow P(X/A, A/A)$ using (4) Collapsing a pair. $P(A/A)$ holds by (A), so $P(X/A, A/A) \Leftrightarrow P(X/A)$ by (C)

□

(E) $P(S^n)$ and $P(D^n, S^{n-1})$ hold.

Proof. We induct on n. The case $n = 0$ is (A). $D^n \sim \{\bullet\}$, so $P(D^n)$ holds by (B). So if $P(S^{n-1})$ is true, then so is $P(D^n, S^{n-1})$ by (C), hence $P(S^n)$ holds by (D).

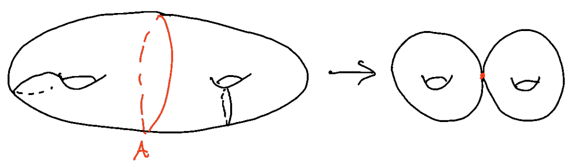
□

(F) $P(X) \implies P(X \cup_f D^n)$.

Proof. $(X \cup_f D^n)/X \simeq S^n$, so $P(X \cup_f D^n, X)$ holds by (D) and (E). So by (C) we get $P(X) \implies P(X \cup_f D^n)$. □

Using (F) and induction, $P(X)$ holds for any fcc X. □

Example. Let Σ_2 be the surface of genus 2. Let A be a closed curve in Σ_2 as in the figure



$$\Sigma_2 \to \Sigma_2/A \cong T^2 \vee T^2$$

such that $\Sigma_2/A \cong T^2 \wedge T^2$. Let $\pi : \Sigma_2 \to \Sigma_2/A$ be the quotient map. Recall from Sheet 1, Exercise 5 that

$$H_*(\Sigma_2) = \begin{cases} \mathbb{Z} & * = 0, 2, \\ \mathbb{Z}^4 & * = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore we know $H_2(T^2 \vee T^2) = \mathbb{Z} \oplus \mathbb{Z}$. On H_2 the map $\pi_* : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is given by $1 \mapsto (1, 1)$. And on H_1, $\pi_* : \mathbb{Z}^4 = H_1(\Sigma_2) \to H_1(T^2 \vee T^2)$ is an isomorphism. $H_*(\Sigma_2)$ and

$H_*(T^2 \vee T^2)$ are free over $\mathbb{Z}$, so by the UCT we have $H^*(\Sigma_2) = \mathrm{Hom}(H_*(\Sigma_2), \mathbb{Z})$, same for $T^2 \vee T^2$ and π^* is dual to π_*. So on H^2, π^* is given by $\begin{bmatrix} 1 & 1 \end{bmatrix} : H^2(T^2 \vee T^2) = \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} = H^2(\Sigma_2)$.

Let $\langle a_1', b_1' \rangle \oplus \langle a_2', b_2' \rangle = H^1(T^2) \oplus H^1(T^2)$. Let $a_i = \pi^*(a_i'), b_i = \pi^*(b_i')$, so that $H^1(\Sigma_2) = \langle a_1, b_1, a_2, b_2 \rangle$. Let $c_i = a_i' \smile b_1'$, $i = 1, 2$, be generators of the two factors $H^2(T^2)$ in $H^2(T^2 \vee T^2)$ and let $c = \pi^*(c_1) = \pi^*(c_2)$, so that $H^2(\Sigma_2) = \langle c \rangle$.

Then we have the following cup products:

$$\begin{aligned} a_i \smile b_j &= \pi^*(a_i') \smile \pi^*(b_j') \\ &= \pi^*(a_i' \smile b_j') \\ &= \pi^*(\delta_{ij} c_i) = \delta_{ij} c \end{aligned}$$

and similarly $a_i \smile a_j = 0, b_i \smile b_j = 0$.

More generally, the same argument shows that $H^1(\Sigma_g) = \langle a_i, b_i \rangle_{i=1}^g$, with

$$a_i \smile b_j = \delta_{ij} c, \quad a_i \smile a_j = b_i \smile b_j = 0$$

where $\langle c \rangle = H^2(\Sigma_g) = \mathbb{Z}$.

4 Vector Bundles

4.1 Definitions and Examples

Definition. *An n-*dimensional real vector bundle *(E, B, π) consists of two spaces E* (total space), *B* (base) *and a map $\pi : E \to B$ such that:*

(1) $\pi^{-1}(b)$ carries the structure of a real n-dimensional real vector space for each $b \in B$.

(2) There is an open cover $\{U_\alpha \mid \alpha \in A\}$ of B and homeomorphisms $f_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ such that

(a)
$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[r, "f_\alpha"] \arrow[d, "\pi"] & U_\alpha \times \mathbb{R}^n \arrow[d, "\pi_1"] \\ U_\alpha \arrow[r, "\mathrm{id}_{U_\alpha}"] & U_\alpha \end{tikzcd}$$
commutes,

(b) $\pi_2 \circ f_\alpha : \pi^{-1}(b) \to \mathbb{R}^n$ is an isomorphism of vector spaces for all $b \in U_\alpha$.

The f_α are local trivializations.

Similar one defines complex n-dimensional vector bundles.

Definition. *A morphism $f : (E, B, \pi) \to (E', B', \pi')$ is a commuting square*

$$\begin{tikzcd} E \arrow[r, "f_E"] \arrow[d, "\pi"] & E' \arrow[d, "\pi'"] \\ B \arrow[r, "f_B"] & B' \end{tikzcd}$$

such that $f_E|_{\pi^{-1}(b)} : \pi^{-1}(b) \to (\pi')^{-1}(f(b))$ is a linear map $\mathbb{R}^n \to \mathbb{R}^m$.

E is a subbundle *of E' if there is an injective morphism*

$$\begin{tikzcd} E \arrow[r, "f_E"] \arrow[d, "\pi"] & E' \arrow[d, "\pi'"] \\ B \arrow[r, "1_B"] & B' \end{tikzcd}$$

so that $\pi^{-1}(b)$ is a linear subspace of $(\pi')^{-1}(b)$.

Definition. *A* section *s of E is a continuous map $s : B \to E$ with $\pi \circ s = 1_B$. s is* non-vanishing *if $s(b) \neq 0$ for all b.*

The map $s_0 : B \to E, b \mapsto 0$ is the 0-section. To check that s_0 is continuous it is enough to check that $f_\alpha \circ s_0$ is continuous for all $\alpha \in A$ which is clearly the case.

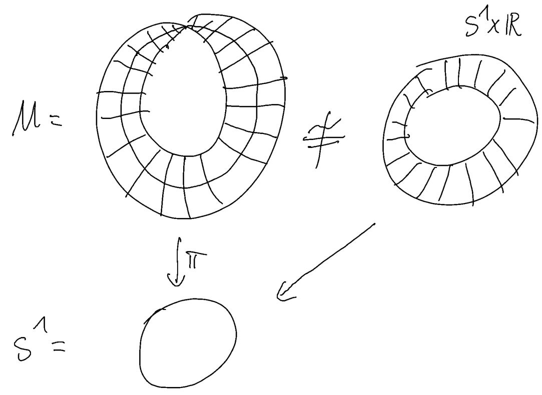
Examples.

(1) $E = B \times \mathbb{R}^n$ is an n-dimensional real vector bundle over B, $f = 1_{B\times\mathbb{R}^n} : E \to B \times \mathbb{R}^n$ is a local (here global) trivialization. $B \times \mathbb{R}^n$ is the n-dimensional *trivial bundle* on B.

In general, $\pi : E \to B$ is *trivial* if there is a bundle isomorphism $f : E \to B \times \mathbb{R}^n$.

Proposition 4.1. *E is trivial iff there exist sections $s_1, \dots, s_n : B \to E$ such that $\{s_1(b), \dots, s_n(b)\}$ is a basis for $\pi^{-1}(b)$ for all $b \in B$.*

(2) $M = [0,1] \times \mathbb{R}/\sim$ where $\sim$ is the smallest equivalence relation with $(0,x) \sim (1,-x)$. There is a natural projection $\pi : M \to S^1 = [0,1]/\sim$ where $0 \sim 1$. This is a 1-dimensional vector bundle over S^1, called the *Möbius bundle.*



Möbius bundle

A section $s : S^1 \to M$ is given by a continuous map $f : [0,1] \to \mathbb{R}$ with $f(0) = -f(1)$. Then $f(t) = 0$ for some $t \in [0,1]$, so $s(t) = 0$, so s is not a non-vanishing section. So M is non-trivial.

(3) The *tautological bundle* $\tau_{\mathbb{RP}^n} = \{([z], v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \mid v \in \langle z \rangle\}$. Then there is a projection $\pi : \tau_{\mathbb{RP}^n} \to \mathbb{RP}^n$ and $\pi^{-1}([z]) = \langle z \rangle \subseteq \mathbb{R}^{n+1}$.

The open subsets $U_i = \{[z] \in \mathbb{RP}^n \mid z_i \neq 0\}$, $i = 0, \dots, n$ cover $\tau_{\mathbb{RP}^n}$. The maps $f_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}$, $([z], v) \mapsto ([z], v_i)$ are local trivializations.

We have $\mathbb{RP}^1 \simeq S^1$ and $\tau_{\mathbb{RP}^1} \simeq M$ is non-trivial.

Similarly one can define the complex tautological bundle $\tau_{\mathbb{CP}^n}$.

(4) $TS^n = \{(x,v) \in \mathbb{S}^n \times \mathbb{R}^{n+1} \mid \langle v, x \rangle = 0\}$ is the tangent bundle of S^n. Let $\pi : TS^n \to S^n$ be the natural map. Then $\pi^{-1}(x) = x^\perp \simeq \mathbb{R}^n$. Let $U_i = \{x \in S^n \mid x_i \neq 0\}$.

Local trivializations are given by $f_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^n, (x, v) \mapsto (x, \pi_{\hat{\imath}} v)$ where $\pi_{\hat{\imath}}$ is the projection dropping the i-th coordinate.

TS^1 has a non-vanishing section $s(x, y) = ((x, y), (-y, x))$, so TS^1 is trivial. But TS^{2n} has no non-vanishing section (Sheet 1, Exercise 8), so TS^{2n} is not trivial.

More generally, any smooth manifold has a tangent bundle.

4.1.1 Pullbacks

If $\pi : E \to B$ is an n-dimensional real vector bundle and $g : B' \to B$ is continuous, let

$$g^*(E) = \{(b', b, v) \in B' \times B \times E \mid g(b') = \pi(v) = b\}.$$

We equip $g^*(E)$ with the subspace topology in $B' \times B \times E$. Let $\pi_g : g^*(E) \to B'$, $(b', b, v) \mapsto b'$. Then

$$\pi_g^{-1}(b') = \{(b', g(b), v) \mid \pi(v) = g(b)\} = \pi^{-1}(g(b))$$

has a vector space vector space structure. If $f_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ is a local trivialization for E, let $V_\alpha = g^{-1}(U_\alpha)$ and $f'_\alpha : \pi_g^{-1}(V_\alpha) \to V_\alpha \times \mathbb{R}^n$, $(b', b, v) \mapsto (b', \pi_2(f_\alpha(v)))$. This gives a local trivialization for g^*E.

Definition. *The vector bundle g^*E is the* pullback of E by g.

Lemma 4.2. $(g \circ f)^*E = f^*(g^*E)$

Definition. *If $A \subseteq B$, $i : A \hookrightarrow B$ is the inclusion, then $E|_A := i^*(E)$ is the restriction of E to A.*

If $s : B \to E$ is a section, then $g^*s : B' \to g^*E, b' \mapsto (b', g(b'), s(g(b')))$ is a section of $g^*(E)$.

Example: $\tau_{\mathbb{RP}^n}|_{\mathbb{RP}^1} \simeq \tau_{\mathbb{RP}^1}$ has no non-vanishing section, so $\tau_{\mathbb{RP}^n}$ has no non-vanishing section, so $\tau_{\mathbb{RP}^n}$ is non-trivial.

4.1.2 Products

If $\pi : E \to B, \pi' : E' \to B'$ are vector bundles of dimension n, n', their product is $\pi \times \pi' : E \times E' \to B \times B'$. The fibre $(\pi \times \pi')^{-1}(b, b') = \pi^{-1}(b) \times \pi^{-1}(b')$ is a vector space of dimension $n + n'$. If $f_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n, f'_\beta : (\pi'^{-1})(U_\beta) \to U_\beta \times \mathbb{R}^n$ are local trivializations, then

$$f_\alpha \times f'_\beta : (\pi \times \pi')^{-1}(U_\alpha \times U_\beta) \to U_\alpha \times \mathbb{R}^n \times U_\beta \times \mathbb{R}^{n'} \simeq U_\alpha \times U_\beta \times \mathbb{R}^{n+n'}$$

is a local trivialization for $E \times E'$ over $U_\alpha \times U_\beta$.

Definition. *If $B = B'$, $E \oplus E' = \Delta^*(E \times E')$, where $\Delta : B \to B \times B$, $b \mapsto (b, b)$ is the diagonal, is the* Whitney sum *of E and E'*

4.1.3 Partitions of Unity

Notation: If $\varphi : B \to \mathbb{R}$, set $\operatorname{supp} \varphi = \overline{\{b \in B \mid \varphi(b) \neq 0\}}$.

Definition. *If $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ is an open cover of B, a* partition of unity (PoU) subordinate to *$\mathcal{U}$ is a family of functions $\varphi_i : B \to \mathbb{R}$, $i \in \mathbb{N}_0$ such that*

(1) $0 \leq \varphi_i(b) \leq 1$ *for all* $b \in B$*,*

(2) $\{i \mid \varphi_i(b) \neq 0\}$ *is finite for all* b*,*

(3) $\operatorname{supp} \varphi_i \subseteq U_{\alpha_i}$ *for some* $\alpha_i \in A$*,*

(4) $\sum_{i \geq 0} \varphi_i(b) = 1$ *for all* b*.*

Definition. B admits PoU *if for every open cover $\mathcal{U}$ there is a partition of unity subordinate to $\mathcal{U}$.*

Remark: If B is compact or metrizable, then B admits PoU. More generally B admits PoU if it is paracompact and Hausdorff.

Theorem 4.3. *Suppose B admits PoU and $\pi : E \to B \times I$ is a vector bundle. Then $E|_{B \times 0} \simeq E|_{B \times 1}$.*

Lemma 4.4. *If $E|_{B \times [0, \frac{1}{2}]}$ and $E|_{B \times [\frac{1}{2}, 1]}$ are trivial, then E is trivial.*

Proof. Exercise. □

Lemma 4.5. *For each $b \in B$, b has an open neighborhood U_b such that $E|_{U_b \times I}$ is trivial.*

Proof. E is locally trivial, so for each $t \in I$ we can find open neighborhoods U_t of b in B and I_t of t in I such that $E|_{U_t \times I_t}$ is trivial. $\{I_t \mid t \in I\}$ is an open cover of the compact set I, so let $\{I_{t_0}, \ldots, I_{t_n}\}$ be a finite subcover. Then there exist $0 = s_0 < s_1 < \cdots < s_n = 1$ such that $[s_i, s_{i+1}] \subseteq I_{t_k}$ for some k. So $E|_{U_{t_k} \times [s_i, s_{i+1}]}$ is trivial. Let $U_b = \bigcap_{k=0}^n U_{t_k}$. It is an open neighborhood of b and $U|_{U_b \times [s_i, s_{i+1}]}$ is trivial for all i. By the previous lemma and induction $E|_{U_b \times [0, s_i]}$ is trivial for all $i = 0, \ldots, n$. □

Proof of Theorem 4.3. For each $b \in B$, let U_b be an open neighborhood of b as in the Lemma and pick a PoU $\{\varphi_i\}_{i \in \mathbb{N}}$ subordinate to $\{U_b \mid b \in B\}$. For $i \in \mathbb{N}$ let $b_i \in B$ such that $\operatorname{supp} \varphi_i \subseteq U_{b_i}$.

For $k \in \mathbb{N}_0$ define $\psi_k : B \to I$ by $\psi_k(b) = \sum_{i=1}^k \varphi_i(b)$. Then let

$$\begin{aligned} g_k : B &\longrightarrow B \times I, \\ b &\longmapsto (b, \psi_k(b)) \end{aligned}$$

and define

$$E_k = g_k^*(E) = \{(b, g_k(b), v) \in B \times (B \times I) \times E \mid \pi(v) = (b, \psi_k(b))\}.$$

Let $f_b : \pi^{-1}(U_b \times I) \to U_b \times I \times \mathbb{R}^n$ be a trivialization. Define $\beta_k : E_{k-1} \to E_k$ by

$$\beta_k((b, g_k(b), v)) = \begin{cases} (b, g_k(b), v) & b \notin U_{b_k}, \\ (b, f_{b_k}^{-1}(b, g_k(b), v')) & b \in U_{b_k} \end{cases}$$

where $f_{b_k}(v) = (b, g_{k-1}(b), v')$. β_k is an isomorphism.

Then $\cdots \circ \beta_3 \circ \beta_2 \circ \beta_1$ is the desired isomorphism $E|_{B\times 0} \to E|_{B\times 1}$. □

Corollary 4.6. *Suppose $\pi : E \to B$ is a vector bundle, $g_0, g_1 : B' \to B$, $g_0 \sim g_1$ via $h : B' \times I \to B$ and that B' admits PoU. Then*

$$g_0^*(E) = h^*(E)|_{B'\times 0} \simeq h^*(E)|_{B'\times 1} = g_1^*(E).$$

Corollary 4.7. *If B is contractible and admits PoU, then every vector bundle $\pi : E \to B$ is trivial.*

Proof. $1_B \sim c_{B,p}$, so $E = (1_B)^*(E) \simeq (c_{B,p})^*(E) = B \times \pi^{-1}(p)$ is trivial. □

4.1.4 Riemannian metrics

Definition. *Suppose $\pi : E \to B$ is a real (resp. complex) vector bundle. A* Riemannian *(resp.* Hermitian*) metric on E is a continuous map $g : E \oplus E \to \mathbb{R}$ (resp. $E \oplus E \to \mathbb{C}$) such that $g|_{\pi^{-1}_{E\oplus E}(b)}$ is an inner product (resp. Hermitian inner product) on $\pi_E^{-1}(b)$ for all $b \in B$.*

Example. $\tau_{\mathbb{RP}^n} = \{([z], v)) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \mid v \in \langle z \rangle\}$ has a natural Riemannian metric given by $g([z, v_1], [z, v_2]) = \langle v_1, v_2 \rangle_{\mathbb{R}^{n+1}}$. Similarly, $\tau_{\mathbb{CP}^n}$ has a natural Hermitian metric.

Definition. *Suppose E is a vector bundle with Riemannian metric g. The* unit disk *and the* unit sphere bundles *of E are given by:*

$$D_g(E) = \{v \in E \mid \langle v, v \rangle \leq 1\},$$
$$S_g(E) = \{v \in E \mid \langle v, v \rangle = 1\}.$$

Note: $D_g(E), S_g(E)$ are not vector bundles, they are *fibre bundles.*

Exercise: If g, g' are two Riemannian metrics on E, then by radial projection on fibres we get commutative diagrams:

$$\begin{tikzcd} S_g(E) \arrow[rr, "\simeq"] \arrow[dr, "\pi"'] & & S_{g'}(E) \arrow[dl, "\pi"] \\ & B & \end{tikzcd} \qquad \begin{tikzcd} D_g(E) \arrow[rr, "\simeq"] \arrow[dr, "\pi"'] & & D_{g'}(E) \arrow[dl, "\pi"] \\ & B & \end{tikzcd}$$

So we drop g from the notation and write $S(E), D(E)$.

Examples.

- $S(\tau_{\mathbb{RP}^n}) = \{([z], v) \mid v \in \langle z \rangle, \|v\| = 1\}$. We can identify this with S^n, via
$$S^n \ni v \mapsto ([v], v) \in S(\tau_{\mathbb{RP}^n}).$$
Under this identification, the projection $\pi : S^(\tau^n) \to \mathbb{RP}^n$ is just the natural projection $S^n \to \mathbb{RP}^n$.

  Similarly, $S(\tau_{\mathbb{CP}^n}) = S^{2n-1}$.

- If $\pi : E \to B$ is trivial with trivialization $f : E \to B \times \mathbb{R}^n$, then E has a Riemannian metric given by $g(v_1, v_2) = \langle \pi_2(f(v_1)), \pi_2(f(v_2)) \rangle$. So $S(B \times \mathbb{R}^n) = B \times S^{n-1}$.

  Therefore $\tau_{\mathbb{RP}^n}, \tau_{\mathbb{CP}^n}$ are non-trivial, since $\mathbb{RP}^n \times S^0 \not\cong S^n, \mathbb{CP}^n \times S^1 \not\cong S^{2n-1}$.

Proposition 4.8. *If B admits PoU and $\pi : E \to B$ is a real vector bundle, then E has a Riemannian metric.*

Proof. By the second example above, B has admits a Riemannian metric over any trivialized open subset of E, then patch them together using a PoU. □

4.2 The Thom Isomorphism

Let $\pi : E \to B$ be an n-dimensional vector bundle. If $b \in B$, let $E_b = \pi^{-1}(b)$ be the fibre of E over b. There is an inclusion $i_b : E_b \hookrightarrow E$. Let $s_0 : B \to E$ be the 0-section.

Define $E^\sharp = E \setminus \operatorname{im} s_0$, $E_b^\sharp = E_b \setminus 0$. Then
$$H_*(E_b, E_b^\sharp) \simeq H_*(\mathbb{R}^n, \mathbb{R}^n - 0) = \begin{cases} \mathbb{Z} & * = n, \\ 0 & \text{otherwise} \end{cases}$$
is free. Fix a ring R. By the UCT, we have
$$H^*(E_b, E_b^\sharp, R) = \begin{cases} R & * = n, \\ 0 & \text{otherwise.} \end{cases}$$

Definition. *$U \in H^n(E, E^\sharp; R)$ is an R-*Thom class *(or R-*orientation*) for E if $i_b^*(U)$ generates $H^n(E_b, E_b^\sharp; R)$ for all $b \in B$.*

From now on, we assume R-coefficients.

Example. Let $E = B \times \mathbb{R}^n$ be the trivial bundle. Then
$$H^*(E, E^\sharp) = H^*(B \times \mathbb{R}^n, B \times (\mathbb{R}^n - 0)) \simeq H^*(B) \otimes H^*(\mathbb{R}^n, \mathbb{R}^n - 0),^1$$

[1]Remark by L.T.: In the lecture this was justified by saying that $H^*(\mathbb{R}^n, \mathbb{R}^n - 0)$ is free, but this is not the hypothesis in our Künneth formula. There we required that the factor with the non-relative cohomology $H^*(B)$ was free. However, it should still be fine, see e.g. [Hat02, Theorem 3.18] for the case of CW-complexes.

i.e. we have an isomorphism

$$H^{k-n}(B) \xrightarrow{\simeq} H^k(E, E^\sharp), a \mapsto a \times U = \pi_1^*(a) \smile \pi_2^*(U),$$

where U is a generator of $H^n(\mathbb{R}^n, \mathbb{R}^n - 0) \simeq R$. For $k = 0$, we get $H^0(B) \simeq H^n(E, E^\sharp)$ via $r \mapsto r \times U$. Let $(B_i)_{i \in I}$ be the path components of B. Then $H^0(B) = \prod_{i \in I} H^0(B_i)$. Let $r = (r_i)_{i \in I} \in H^0(B)$.

If $b \in B_i$, $i_b^*(r \times u) = r_i U \in H^*(\mathbb{R}^n, \mathbb{R}^n - 0)$. So $r \times U$ is a Thom class iff r_i generates $H^0(B_i) \simeq R$ for all i. In particular, if $R = \mathbb{Z}/2$, there is a unique Thom class. If $R = \mathbb{Z}$, there are $2^{\#\pi_0(B)}$ Thom classes (choose $r_i = \pm 1$).

If $f : B' \to B$, there is a morphism $F : f^*(E) \to E$ over $f : B' \to B$, given by $(b', b, v) \mapsto v$. Note that $F(\operatorname{im} s_0') = \operatorname{im} s_0$, so we get a map of pairs $F : (f^*(E), f^*(E)^\sharp) \to (E, E^\sharp)$.

Lemma 4.9. *If U is an R-Thom class for E, then $F^*(U)$ is an R-Thom class for f^*E.*

Proof. Let $b' \in B'$, $b = f(b)$ and $j = F|_{f^*(E)_{b'}}$. There is a commutative square:

$$\begin{tikzcd} f^*(E) \arrow[r, "F"] & E \\ f^*(E)_{b'} \arrow[u, hook, "i_{b'}"] \arrow[r, "j"] & E_b \arrow[u, hook, "i_b"'] \end{tikzcd}$$

The bottom map is an isomorphism and $i_{b'}^*(F^*(U)) = j^*(i_b^*(U))$. Since $i_b^*(U)$ generates $H^n(E_b, E_b^\sharp)$, $i_{b'}^*(F^*(U))$ generates $H^n(f^*(E)_{b'}, f^*(E)_{b'}^\sharp)$, so $F^*(u)$ is a TC. □

Lemma 4.10. *Suppose $B = B_1 \cup B_2$, $U \in H^n(E, E^\sharp)$. For $k = 1, 2$, let $i_k : B_k \to B$ be the inclusion. Then if $i_1^*(U), i_2^*(U)$ are TC's for $E|_{B_1}, E|_{B_2}$, then U is a TC for E.*

Proof. Obvious. □

Theorem 4.11 (Thom isomorphism)**.** *If $\pi : E \to B$ is an n-dimensional real vector bundle, then:*

(a) E has a unique $\mathbb{Z}/2$ Thom class.

(b) If E has an R-Thom class U, the map

$$\begin{aligned} \Phi : H^*(B; R) &\longrightarrow H^{*+n}(E, E^\sharp; R), \\ a &\longmapsto \pi^*(a) \smile U \end{aligned}$$

is an isomorphism, called the Thom isomorphism*.*

Proof. We assume that B is compact.

Step 1 The theorem holds if $E = B \times \mathbb{R}^n$ is trivial. This is the example we did before.

Step 2 Suppose $V_1, V_2 \subseteq B$ are open. Let $E_i = E|_{V_i}, E_\cap = E|_{V_1 \cap V_2}, E_\cup = E|_{V_1 \cup V_2}$. If the theorem holds for E_1, E_2 and $E_\cap$, then it holds for $E_\cup$.

Proof. For (a) consider $\mathbb{Z}/2$ coefficients.

The MV sequence is

$$H^{n-1}(E_\cap, E_\cap^\sharp) \to H^n(E_\cup, E_\cup^\sharp) \xrightarrow{i} H^n(E_1, E_1^\sharp) \oplus H^n(E_2, E_2^\sharp) \xrightarrow{j} H^n(E_\cap, E_\cap^\sharp),$$

where

$$i = \begin{bmatrix} i_1^* \\ i_2^* \end{bmatrix}, j = \begin{bmatrix} j_1^* - j_2^* \end{bmatrix}.$$

Let $U_i \in H^n(E_i, E_i^\sharp)$ be the unique $\mathbb{Z}/2$ Thom class for E_i. By the first lemma, $j_i^*(U_i)$ is a TC for $E_\cap$. By uniqueness,

$$j_1^*(U_1) = j_2^*(U_2) = U_\cap$$

is the unique $\mathbb{Z}/2$-TC for $E_\cap$, so $(U_1, U_2) \in \ker j = \operatorname{im} i$, hence $(U_1, U_2) = i(U_\cup)$ for some $U_\cup \in H^n(E_\cup, E_\cup^\sharp)$. Then $i_i^*(U_\cup) = U_i$, so by Lemma 4.10, $U_\cup$ is a TC for $E_\cup$. It is unique, since if $U'_\cup \in H^n(E_\cup, E_\cup^\sharp)$ is a TC, then $i(U'_\cup) = (U_1, U_2)$ by the first lemma and uniqueness for E_i. Since $\ker i \subseteq H^{n-1}(E_n, E_n^\sharp) \simeq H^{-1}(V_1 \cap V_2) = 0$ (by (b)), we get $U_\cup = U'_\cup$.

For (b), we have a commuting diagram of MV sequences

$$\begin{array}{ccccc} H^*(V_1 \cup V_2) & \longrightarrow & H^*(V_1) \oplus H^*(V_2) & \longrightarrow & H^*(V_1 \cap V_2) \\ \downarrow{\scriptstyle \Phi_\cup} & & \downarrow{\scriptstyle \Phi_1 \oplus \Phi_2} & & \downarrow{\scriptstyle \Phi_\cap} \\ H^{*+n}(E_\cup, E_\cup^\sharp) & \longrightarrow & H^{*+n}(E_1, E_1^\sharp) \oplus H^{*+n}(E_2, E_2^\sharp) & \longrightarrow & H^{*+n}(E_\cap, E_\cap^\sharp) \end{array}$$

By hypothesis, $\Phi_1, \Phi_2, \Phi_\cap$ are all isomorphisms, so $\Phi_\cup$ is an isomorphism by the Five Lemma. □

Step 3 B is compact, so it has a finite open cover $\{V_1, \ldots, V_r\}$ such that $E|_{V_i}$ is trivial. Let $W_k = \bigcup_{i=1}^k V_i$. By Step 1, the theorem holds for W_1. If the theorem holds for W_k, it holds for W_{k+1} by Step 2, hence it holds for $B = W_r$ by induction.

□

4.2.1 The Gysin Sequence

Suppose $\pi : E \to B$ has an R-Thom class U. Note that $E^\sharp = E \setminus \operatorname{im} s_0 \sim S(E)$. Also $\pi : E \to B$ is a homotopy equivalence with homotopy inverse $s_0 : B \to E$. The LES of

$(E, E^\sharp)$ is

$$\begin{tikzcd}
H^*(E,E^\sharp) \arrow[r, "j^*"] & H^*(E) \arrow[r] \arrow[d, shift right, "\simeq s_0^*"'] & H^*(E^\sharp) \arrow[r] & H^{*+1}(E,E^\sharp) \\
H^{*-n}(B) \arrow[r, "\alpha"] \arrow[u, "\Phi", "\simeq"'] & H^*(B) \arrow[r] \arrow[u, shift right, "\pi^*\simeq"'] & H^*(S(E)) \arrow[r] \arrow[u, "\simeq"] & H^{*-n+1}(B) \arrow[u, "\Phi", "\simeq"']
\end{tikzcd}$$

α is defined in such a way that the diagram commutes, so for $a \in H^{*-n}(B)$, we have:

$$\begin{aligned}
\alpha(a) = s_0^*(j^*(\Phi(a))) &= s_0^* j^*(\pi^* a \smile U) \\
&= s_0^*(\pi^* a \smile j^* U) \\
&= (s_0^* \pi^* a) \smile s_0^* j^*(U) \\
&= a \smile s_0^* j^*(U).
\end{aligned}$$

Definition. *If $\pi : E \to B$ is an R-oriented n-dimensional real vector bundle with TC U, its* Euler class *is $e(E) = s_0^* j^*(U) \in H^n(B)$.*

Theorem 4.12 (Gysin sequence)**.** *There is a LES*

$$\cdots \to H^{*-n}(B) \xrightarrow{\alpha} H^*(B) \xrightarrow{\pi^*} H^*(S(E)) \to H^{*-n+1}(B) \to \cdots$$

where $\alpha(a) = a \smile e(E)$.

Proposition 4.13. *Properties of the Euler class:*

(1) If $f : B' \to B$, then $f^*(E)$ is oriented and $e(f^*(E)) = f^*(e(E))$.

(2) If E is trivial and $n > 0$, then $e(E) = 0$.

(3) $e(E_1 \oplus E_2) = e(E_1) \smile e(E_2)$.

(4) If E has a non-vanishing section, then $e(E) = 0$.

Proof.

(1) There is a commuting diagram:

$$\begin{tikzcd}
(B,\emptyset) \arrow[r, "s_0"] & (E,\emptyset) \arrow[r, "j"] & (E,E^\sharp) \\
(B',\emptyset) \arrow[r, "s_0'"] \arrow[u, "f"] & (f^*E,\emptyset) \arrow[r, "j'"] \arrow[u, "F"] & (f^*E,(f^*E)^\sharp) \arrow[u, "F"]
\end{tikzcd}$$

By Lemma 4.9, $F^*(U)$ is an orientation on $f^*(E)$, so

$$e(f^*(E)) = s_0' j'^* F * (U) = f^* s_0^* j^*(U) = f^*(e(E))$$

(2) This is true if $B = \{\bullet\}$, since $H^n(\{\cdot\}) = 0$. In general E is trivial, iff $E = f^*(E_\bullet)$ where $f : B \to \{\bullet\}$ and $E_\bullet = \mathbb{R}^n$, so $e(E) = f^*(e(E_\bullet)) = f^*(0) = 0$.

(3) Is on Example sheet 4.

(4) If s is a non-vanishing section, $\langle s\rangle$ is a trivial bundle and $E = \langle s\rangle \oplus \langle s\rangle^\perp$, so

$$e(E) = e(\langle s\rangle) \smile e(\langle s\rangle^\perp) = 0 \smile e(\langle s\rangle^\perp) = 0.$$

□

Theorem 4.14.
$$H^*(\mathbb{RP}^n; \mathbb{Z}/2) \simeq \mathbb{Z}/2[X]/(X^{n+1})$$
where $x = e(\tau_{\mathbb{RP}^n}) \in H^1(\mathbb{RP}^n; \mathbb{Z}/2)$.

By Theorem 4.11, every vector bundle is $\mathbb{Z}/2$-orientable, so $e(\tau_{\mathbb{RP}^n})$ exists.

Proof. $\mathbb{Z}/2$-coefficients everyhwere.

We have $S(\tau_{\mathbb{RP}^n}) = S^n$, so the Gysin sequence is

$$\cdots \to H^{k-1}(\mathbb{RP}^n) \xrightarrow{\alpha} H^k(\mathbb{RP}^n) \to H^k(S^n) \to H^k(\mathbb{RP}^n) \to \cdots$$

Claim: $\alpha = \cdot \smile x$ is an isomorphism for $1 \le k \le n$. Proof:

- $k = 1$ and $n > 1$. The Gysin sequence is:

  $$0 \to H^0(\mathbb{RP}^n) \to H^0(S^n) \to H^0(\mathbb{RP}^n) \xrightarrow{\alpha} H^1(\mathbb{RP}^n) \to H^1(S^n) = 0$$

  Clearly, $\pi^* : H^0(\mathbb{RP}^n) \to H^0(S^n)$ is an isomorphism, so the map $H^0(S^n) \to H^0(\mathbb{RP}^n)$ is the zero map. It follows that α is an isomorphism.

- $1 < k < n$. We get:

  $$0 = H^{k-1}(S^n) \to H^{k-1}(\mathbb{RP}^n) \xrightarrow{\alpha} H^k(\mathbb{RP}^n) \to H^k(S^n) = 0$$

  So again α is an isomorphism.

- $k = n$. Then

  $$0 = H^{n-1}(S^n) \to H^{n-1}(\mathbb{RP}^n) \xrightarrow{\alpha} H^n(\mathbb{RP}^n) \to H^n(S^n) \to H^n(\mathbb{RP}^n) \to 0$$

  Since $H^n(S^n) \to H^n(\mathbb{RP}^n)$ is surjective and both groups are $\mathbb{Z}/2$, it must be an isomorphism. Then $H^n(\mathbb{RP}^n \to H^n(S^n)$ must be the zero map, hence α is an isomorphism.

So by induction, the claim implies that x^k generates $H^k(\mathbb{RP}^n; \mathbb{Z}/2) \simeq \mathbb{Z}/2$ for $0 \le k \le n$ and $x^{n+1} \in H^{n+1}(\mathbb{RP}^n) = 0$. □

Similarly, $\tau_{\mathbb{CP}^n}$ is a complex vector bundle, so its underlying real vector bundle is $\mathbb{Z}$-orientable (Sheet 3, Exercise 10). The same arguments show that

$$H^*(\mathbb{CP}^n;\mathbb{Z}) \simeq \mathbb{Z}[X]/(X^{n+1})$$

where $x = e(\tau_{\mathbb{CP}^n}) \in H^2(\mathbb{CP}^n;\mathbb{Z})$.

Corollary 4.15. $\pi_3(S^2) \neq 0$.

Proof. Let $h : S^3 \to S^2 \cong \mathbb{CP}^1$ be the Hopf map. Then $\mathbb{CP}^2 = S^2 \cup_h D^4$, if the class of h were 0 in $\pi_3(S^2)$, we would get $\mathbb{CP}^2 \sim S^2 \vee S^4$. But $H^*(S^2 \vee S^4) \ncong H^*(\mathbb{CP}^2)$ as graded rings, for example if $x \in H^2(S^2 \vee S^4)$, then $x \smile x = 0$.

Hence the Hopf map is a non-trivial element in $\pi_3(S^2)$. □

4.2.2 Comments on Orientability

(1) Every E is $\mathbb{Z}/2$ orientable.

(2) For $p \neq 2$, E is $\mathbb{Z}/p$-orientable iff E is $\mathbb{Z}$-orientable (If so, we just say E is orientable).

(3) $\tau_{\mathbb{RP}^1} = M$ is not $\mathbb{Z}$-orientable. Indeed, we have

$$H^*(M, M^\sharp) = H^*(D(M), S(M)) \simeq H^*(\overline{M}, \partial\overline{M})$$

where $\overline{M}$ is the closed Möbius band. Then $H^2(\overline{M}, \partial\overline{M}) = \mathbb{Z}/2 \ncong \mathbb{Z} = H^1(S^1)$, so the Thom isomorphism with $\mathbb{Z}$ coefficients is false.

(4) There is a homomorphism $\varphi : \pi_1(B) \to \mathbb{Z}/2$ such that: For $\gamma : S^1 \to B$, $\varphi([\gamma]) = 0$ iff $\gamma^*(E)$ is orientable. So if $\pi_1(B) = 1$, then any $\pi : E \to B$ is orientable. See Example Sheet 4.

5 Manifolds

5.1 Definitions and Fundamental Class

Definition. *A n-*manifold *is a second countable Hausdorff space M with an open cover $\{U_\alpha \mid \alpha \in A\}$ and homeomorphisms $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$. The transition functions $\psi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ are homeomorphisms. M is* smooth *if the φ_α can be chosen so that $\psi_{\alpha\beta}$ are diffeomorphisms.*

We call a manifold M *closed* if it is compact and has no boundary. Since our definition of a manifold doesn't allow for a boundary, closed just means compact.

A smooth manifold has a tangent bundle $\pi : TM \to M$.

Notation: If $A \subseteq M$ is compact, write $(M \mid A) = (M, M - A)$. If $B \subseteq A$, we get an inclusion of pairs

$$i : (M \mid A) = (M, M - A) \to (M, M - B) = (M \mid B).$$

If $w \in H_*(M \mid A)$, then we set $w|_B := i_*(w)$.

If $x \in M$, $x \in U_\alpha \simeq \mathbb{R}^n$ for some α. By excision, we have:

$$H_*(M \mid x) \simeq H_*(U_\alpha \mid x) \overset{\varphi_{\alpha *}}{\simeq} H_*(\mathbb{R}^n \mid \varphi_\alpha(x)) = H_*(\mathbb{R}^n, \mathbb{R}^n - \varphi_\alpha(x)) = \begin{cases} \mathbb{Z} & * = n, \\ 0 & \text{otherwise.} \end{cases}$$

Now fix any ring R. Then $H_*(M \mid x; R) \simeq \begin{cases} R & * = n, \\ 0 & \text{otherwise.} \end{cases}$

Definition. *An R-*fundamental class *for $(M \mid A)$ is a class $w \in H_n(M \mid A; R)$ such that $w|_x$ generates $H_n(M \mid x)$ for all $x \in A$.*

This is an analogue of the Thom class.

Theorem 5.1. *If $A \subseteq M$ is compact, $(M \mid A)$ has a unique $\mathbb{Z}/2$-fundamental class.*

Proof. The proof is very similar to the proof of the Thom isomorphism theorem. See the handout on the Moodle page. □

A fundamental class for $(M \mid M) = (M, \emptyset)$ will be written as $[M] \in H_n(M)$.

We say M is *orientable* if it has a $\mathbb{Z}$-fundamental class.

Proposition 5.2. *A smooth manifold M is orientable iff TM is orientable.*

Definition. *A subset $N \subseteq M$ is a k-*dimensional (smooth) submanifold *of an n-manifold M, if for every $x \in N$, there is a (smooth) chart $\varphi_x : U_x \to \mathbb{R}^n$ such that $\varphi_x(U_x \cap N) = \mathbb{R}^k \times 0 \subseteq \mathbb{R}^n$.*

Note that if $N \subseteq M$ is a smooth submanifold, then TN is a subbundle of $TM|_N$.

Definition. *Let $N \subseteq M$ be a smooth submanifold. Then $\nu_{M/N} = TN^\perp \subseteq TM|_N$ is the* normal bundle *of N in M (for some fixed choice of Riemannian metric).*

So we have $TM|_N = \nu_{M/N} \oplus TN$.

Theorem 5.3 (Tubular Neighborhood Theorem)**.** *If $N \subseteq M$ is a closed smooth submanifold, there is an open neighborhood $V \subseteq M$ of N with $(\nu, N) \simeq (\nu_{M/N}, s_0(N))$.*

Lemma 5.4. *Suppose $E = E_1 \oplus E_2$ is orientable. Then E_1 is orientable iff E_2 is.*

Proof. Exercise. □

Proof of Proposition 5.2 (Idea only). If $\gamma : S^1 \to M$ is an embedding, let $V(\gamma)$ be a tubular neighborhood. Then

$$\begin{aligned} M \text{ is orientable} &\iff V(\gamma) \text{ is orientable for all } \gamma \\ &\iff \nu_{M/\gamma} \text{ is orientable for all } \gamma \\ &\iff TM|_\gamma \text{ is orientable for all } \gamma \\ &\iff TM \text{ is orientable.} \end{aligned}$$

□

Corollary 5.5. *If M is orientable, then a closed smooth submanifold $N \subseteq M$ is orientable iff $\nu_{M/N}$ is.*

5.2 Poincare Duality

From now on, we work with coefficients in a field $\mathbb{F}$, i.e. $H^k(X) = H^k(X; \mathbb{F})$. By the UCT we get $H^k(X) \simeq \mathrm{Hom}(H_k(X), \mathbb{F})$,[1] hence by dualizing we get an isomorphism[2]

$$\mathrm{Hom}(H^k(X), \mathbb{F}) \underset{\varphi}{\xrightarrow{\simeq}} H_k(X)$$

where $\langle a, \varphi(\alpha)\rangle = \alpha(a)$. Here $\langle -, - \rangle : H^k(X) \times H_k(X) \to \mathbb{F}$ is the pairing induced by $H^k(X) \simeq \mathrm{Hom}(H_k(X), \mathbb{F})$.

If $a \in H^k(X)$, we have a map $a \smile - : H^l(X) \to H^{k+l}(X)$ given by the cup product.

[1] Remark by L.T.: Our UCT only gives this in the case where X is a fcc. But it is still true, see e.g. [Hat02, Theorem 3.2]

[2] Remark by L.T.: Only if H^k, H_k are finite-dimensional...

Definition. *The* cap product $- \frown a : H_{k+l}(X) \to H_l(X)$ *is the dual of* $a \smile -$, *i.e. for* $x \in H_{k+l}(X), b \in H^l(X)$ *we have:*

$$\langle b, x \frown a \rangle = \langle a \smile b, x \rangle.$$

5.2.1 Intersection Pairing

Suppose M is an $\mathbb{F}$-oriented n-manifold with fundamental class $[M] \in H_n(M)$.

Definition. *The* intersection pairing $(-,-) : H^k(M) \times H^{n-k}(M) \to \mathbb{F}$ *is the bilinear pairing given by*

$$(a, b) = \langle a \smile b, [M] \rangle$$

It satisfies $(b, a) = (-1)^{|b||a|}(a, b) = (-1)^{k(n-k)}(a, b)$.

If $a \in H^k(M)$, then $(a, -) \in \operatorname{Hom}(H^{n-k}(M), \mathbb{F})$.

Definition. *The* (algebraic) Poincare Dual *of* $a \in H^k(M)$ *is*

$$\operatorname{PD}(a) = \varphi((a, -)) = [M] \frown a \in H_{n-k}(M).$$

So $\langle b, \operatorname{PD}(a) \rangle = (a, b) = \langle a \smile b, [M] \rangle$.

5.2.2 Geometric Poincare Dual

Theorem 5.6. *If M is a connected n-manifold and $x \in M$, the map*

$$H_n(M) \to H_n(M \mid x) = H_n(M, M - x) \simeq \mathbb{F}$$

is injective. So if M is $\mathbb{F}$-oriented, then $H_n(M) = \langle [M] \rangle \simeq \mathbb{F}$ and $H^n(M) = \langle [M]^* \rangle \simeq \mathbb{F}$ where $[M]^* \in H^n(M)$ is defined so that $\langle [M]^*, [M] \rangle = 1 \in \mathbb{F}$.

Proof. See Moodle handout. $\square$

Assume $i : N \hookrightarrow M$ is a k-dimensional smooth closed connected $\mathbb{F}$-oriented submanifold and $x \in N$. Let V be a tubular neighborhood of N. Let $\nu = \nu_{M/N}$ be the normal bundle. There is a commutative diagram:

$$\begin{tikzcd}
(M, \emptyset) \arrow[r, "j"] \arrow[dr, "j_x"'] & (M \mid N) \arrow[d] & (V \mid N) \arrow[l] \arrow[r, dash, "\simeq"] & (\nu, \nu^\sharp) \arrow[ll, bend right, "i"'] \\
& (M \mid x) & &
\end{tikzcd}$$

Since N is connected, $H^k(N) \simeq \mathbb{F} = \langle [N]^* \rangle$. Hence $H^n(\nu, \nu^\sharp) = \langle U \smile \pi^*[N]^* \rangle \simeq \mathbb{F}$ where $U \in H^{n-k}(\nu, \nu^\sharp)$ is an orientation for $\nu_{M/N}$. Then $H_n(\nu, \nu^\sharp) \simeq \mathbb{F}$

Now $i_* : H_n(\nu, \nu^{\#}) \to H_n(M \mid N) \simeq \mathbb{F}$ is an isomorphism by Excision. Also $j_{x*} : H_n(M) \to H_n(M \mid x) \simeq \mathbb{F}$ is an isomorphism, so $j_* : H_n(M) \to H_n(M \mid N)$ is an isomorphism.

So $i_*^{-1} j_*[M]$ generates $H_n(\nu, \nu^\sharp) \simeq \mathbb{F}$. So

$$\langle U \smile \pi^*[N]^*, i_*^{-1} j_*[M] \rangle =: c \in \mathbb{F}^*.$$

Remark by L.T.: Lots of missing inclusions etc., in the following...

Definition. $U_{M/N} := c^{-1}U$ *is the* orientation on $\nu_{M/N}$ induced by $[N]$ and $[M]$. *It satisfies*

$$\langle U_{M/N} \smile \pi^*[N]^*, i_*^{-1} j_*[M] \rangle = 1.$$

Definition. $\mathrm{pd}(N) := j^*(i^*)^{-1}(U_{M/N}) \in H^{n-k}(M)$ *is the* geometric Poincare dual *of* N.

Proposition 5.7. *If* $a \in H^k(M)$*, then*

$$\langle \mathrm{pd}(N) \smile a, [M] \rangle = \langle a, i_*[N] \rangle,$$

i.e. $\mathrm{PD}(\mathrm{pd}(N)) = i_*[N]$.

Lemma 5.8. *Let* $i : V \to M$ *be the inclusion. Then*

$$i^*(a) = \langle a, i_*[N] \rangle \pi^*[N]^*.$$

Proof. $\pi : V \to N$ is a homotopy equivalence, so $H^k(V)$ is generated by $\pi^*[N]^*$. So it is enough to check that $\langle i^*(a), [N] \rangle = \langle \langle a, i_*[N] \rangle \pi^*[N]^*, [N] \rangle$ (exercise). □

Proof of Proposition 5.7. If $b \in H^l(M \mid N)$, then $j^*(b \smile a) = j^*(b) \smile a$. So

$$\begin{aligned}
\langle \mathrm{pd}(N) \smile a, [M] \rangle &= \langle (i^*)^{-1}(U_{M/N}) \smile a, j_*[M] \rangle \\
&= \langle U_{M/N} \smile i^*(a), i_*^{-1}(j_*[M]) \rangle \\
&= \langle U_{M/N} \smile \langle a, i_*[N] \rangle \pi^*[N]^*, i_*^{-1} j_*[M] \rangle \\
&= \langle a, i_*[N] \rangle \cdot \langle U_{M/N} \smile \pi^*[N]^*, i_*^{-1} j_*[M] \rangle \\
&= \langle a, i_*[N] \rangle
\end{aligned}$$

□

Next we will show that PD is an isomorphism by considering the diagonal $\Delta : M \to M \times M$.

5.2.3 Homology of Products and Proof of Poincare Duality

Note that $\mathrm{Hom}(A \otimes B, \mathbb{F}) \simeq \mathrm{Hom}(A, \mathbb{F}) \otimes \mathrm{Hom}(B, \mathbb{F})$, hence

$$\begin{aligned} H_*(X \times Y) &\simeq \mathrm{Hom}(H^*(X \times Y), \mathbb{F}) \\ &\simeq \mathrm{Hom}(H^*(X) \otimes H^*(Y), \mathbb{F}) \\ &\simeq H_*(X) \otimes H_*(Y) \end{aligned}$$

Under this isomorphism $\alpha \otimes \beta \in H_*(X) \otimes H_*(Y)$ corresponds to $\alpha \times \beta \in H_*(X \times Y)$ where $\alpha \times \beta$ is defined by

$$\langle a \times b, \alpha \times \beta \rangle = \langle a, \alpha \rangle \langle b, \beta \rangle.$$

Lemma 5.9.

$$(z_1 \times z_2) \frown (a_1 \times a_2) = (-1)^{|a_2|(|z_1|-|a_1|)}(z_1 \frown a_1) \times (z_2 \frown a_2)$$

Proof. We have to check that $\langle b_1 \times b_2, \mathrm{LHS} \rangle = \langle b_1 \times b_2, \mathrm{RHS} \rangle$ (exercise). $\square$

Lemma 5.10. *If X is path-connected, $p \in X$, so $H_0(X) = \langle [p] \rangle$, and $a \in H^k(X), \alpha \in H_k(X)$, then*

$$\alpha \frown a = \langle a, \alpha \rangle [p].$$

Proof. $\langle 1, \alpha \frown a \rangle = \langle a \smile 1, \alpha \rangle = \langle a, \alpha \rangle$ and $\langle 1, [p] \rangle = 1$. $\square$

Lemma 5.11. *Let $\Delta : X \to X \times X$ be the diagonal. Then $\Delta^*(a \times b) = a \smile b$ for $a, b \in H^*(X)$.*

Proof. Let $\pi_1, \pi_2 : X \times X \to X$ be the projections. Then

$$\Delta^*(a \times b) = \Delta^*(\pi_1 * a \smile \pi_2^* b) = \Delta^* \pi_1^* a \smile \Delta^* \pi_2^* b = a \smile b.$$

$\square$

Now let M again be a closed, connected, oriented n-manifold. We orient $M \times M$ by $[M \times M] = [M] \times [M]$. Let $\widetilde{U} = \mathrm{pd}(\Delta) \in H^n(M \times M)$.

Proposition 5.12. $\langle \widetilde{U}, [M] \times [p] \rangle = (-1)^n$.

Proof.

$$\begin{aligned} \langle \widetilde{U} \smile (1 \times [M]^*), [M] \times [M] \rangle &= (-1)^n \langle (1 \times [M]^*) \smile \widetilde{U}, [M] \times [M] \rangle \\ &= (-1)^n \langle \widetilde{U}, ([M] \times [M]) \frown (1 \times [M]^*) \rangle \\ &= (-1)^n \langle \widetilde{U}, ([M] \frown 1) \times ([M] \frown [M]^*) \rangle \\ &= (-1)^n \langle \widetilde{U}, [M] \times [p] \rangle \end{aligned}$$

On the other hand, since $\widetilde{U} = \mathrm{pd}(\Delta)$, we have by Proposition 5.7:

$$\begin{aligned}
\langle \widetilde{U} \smile (1 \times [M]^*), [M] \times [M] \rangle &= \langle 1 \times [M]^*, [\Delta] \rangle \\
&= \langle \pi_2^*[M]^*, \Delta_*[M] \rangle \\
&= \langle [M]^*, \pi_{2*}\Delta_*[M] \rangle \\
&= \langle [M]^*, [M] \rangle \\
&= 1
\end{aligned}$$

The claim follows. $\square$

Proposition 5.13.

$$\widetilde{U} \smile (a \times b) = (-1)^{|a||b|} \widetilde{U} \smile (b \times a)$$

Proof. Let V be a tubular neighborhood of Δ in $M \times M$. We have a commutative diagram:

$$\begin{tikzcd}
& M \arrow[dl, "j_\Delta"'] \arrow[d, "\Delta"] \\
V \arrow[r, "i'"] \arrow[d, "j'"] & M \times M \arrow[d, "j"] \\
(V \mid \Delta) \arrow[r, "i"] & (M \times M \mid \Delta)
\end{tikzcd}$$

Let $\pi : V \to \Delta$ be the projection in the normal bundle, so π and j_Δ are homotopy inverses. Hence

$$\begin{aligned}
U \smile i'^*(a \times b) &= U \smile \pi^* j_\Delta^* i'^*(a \times b) \\
&= U \smile \pi^* \Delta^*(a \times b) \\
&= U \smile \pi^*(a \smile b) \\
&= (-1)^{|a||b|} U \smile \pi^*(b \smile b) \\
&= (-1)^{|a||b|} U \smile i'^*(b \times a)
\end{aligned}$$

Now apply $j^*(i^*)^{-1}$ to both sides. $\square$

Proposition 5.14. *For $a \in H^k(M), y \in H_k(M)$ we have*

$$\langle \widetilde{U}, \mathrm{PD}(a) \times y \rangle = (-1)^{n(n-|a|)} \langle a, y \rangle.$$

Proof.

$$\begin{aligned}
\langle \widetilde{U}, \mathrm{PD}(a) \times y \rangle &= \langle \widetilde{U}, ([M] \frown a) \times (y \frown 1) \rangle \\
&= (-1)^0 \langle \widetilde{U}, ([M] \times y) \frown (a \times 1) \rangle = \langle (a \times 1) \smile \widetilde{U}, [M] \times y \rangle \\
&= \langle (1 \times a) \smile \widetilde{U}, [M] \times y \rangle = \langle \widetilde{U}, ([M] \times y) \frown (1 \times a) \rangle
\end{aligned}$$

$$= (-1)^{n|a|}\langle \widetilde{U}, ([M] \frown 1) \times (y \frown a)\rangle = (-1)^{n|a|}\langle \widetilde{U}, [M] \times [p]\rangle\langle a, y\rangle$$
$$= (-1)^n(-1)^{n|a|}\langle a, y\rangle$$
$$= (-1)^{n(n-|a|)}\langle a, y\rangle.$$

□

Theorem 5.15 (Poincare duality)**.** PD : $H^k(M) \to H_{n-k}(M)$ *is an isomorphism.*

Proof. If $0 \neq a \in H^k(M)$, choose $y \in H_k(M)$ with $\langle a, y\rangle \neq 0$. Then $\mathrm{PD}(a) \times y \neq 0$, so $\mathrm{PD}(a) \neq 0$. Hence PD is injective. Applying this twice we get

$$\dim H^k(M) \leq \dim H_{n-k}(M) = \dim H^{n-k}(M) \leq H_k(M),$$

hence $H^k(M)$ and $H_{n-k}(M)$ have the same (finite) dimension, so PD is an isomorphism. □

Corollary 5.16. $(-,-)$ *is nondegenerate, i.e. if* $0 \neq a \in H^k(M)$*, there exists* $b \in H^{n-k}(M)$ *such that* $(a, b) \neq 0$.

If $\{a_i\}$ is a basis for $H^*(M)$, let $\{b_i\}$ be the dual basis w.r.t. $(-,-)$, i.e. $(a_i, b_j) = \delta_{ij}$.

Then $\langle b_j, \mathrm{PD}(a_i)\rangle = (a_i, b_j) = \delta_{ij}$, so $\mathrm{PD}(a_i) = b_i^*$ and $\langle a_i, \mathrm{PD}(b_j)\rangle = (b_j, a_i) = (-1)^{|a_i||b_j|}\delta_{ij}$, hence $\mathrm{PD}(b_j) = (-1)^{|a_i||b_i|}a_i^*$.

Corollary 5.17. $\widetilde{U} = \sum_i (-1)^{|a_i|} a_i \times b_i$.

Proof.

$$\langle \widetilde{U}, a_i^* \times b_j^*\rangle = (-1)^{|a_i|(n-|a_i|)}\langle \widetilde{U}, \mathrm{PD}(b_i) \times \mathrm{PD}(a_j)\rangle$$
$$= (-1)^s\langle b_i, \mathrm{PD}(a_j)\rangle = (-1)^s(a_j, b_i) = (-1)^s\delta_{ij}$$

where $s = |a_i|(n - |a_i|) + n|a_i| \equiv |a_i| \bmod 2$. □

5.2.4 Intersection Pairing on Homology

Definition. *If* $N_1, N_2 \hookrightarrow M$ *are smooth submanifolds, then* N_1 *is transverse to* N_2*, written* $N_1 \pitchfork N_2$*, if* $TN_1|_x + TN_2|_x = TM|_x$ *for all* $x \in N_1 \cap N_2$.

If If $N_1, N_2 \hookrightarrow M$ are smooth transverse submanifolds, then:

(1) $N_1 \cap N_2$ is a smooth submanifold of dimension $\dim N_1 + \dim N_2 - \dim M$,

(2) $T(N_1 \cap N_2)|_x = TN_1|_x \cap TN_2|_x$,

(3) $\nu_{M/N_1 \cap N_2} = \nu_{M/N_1} \oplus \nu_{M/N_2}$,

(4) $\mathrm{pd}(N_1 \cap N_2) = \mathrm{pd}(N_1) \smile \mathrm{pd}(N_2)$.

Definition.

$$[N_1] \cdot [N_2] := (\mathrm{pd}(N_1), \mathrm{pd}(N_2)) = \langle \mathrm{pd}(N_1) \smile \mathrm{pd}(N_2), [M] \rangle = \langle \mathrm{pd}(N_1 \cap N_2), [M] \rangle$$

is the number of points in $N_1 \cap N_2$*, counted with intersection sign.*

Let $j : N_1 \hookrightarrow M$ be the inclusion.

Proposition 5.18.

$$j^*(\mathrm{pd}(N_2)) = \mathrm{pd}_{N_1}(N_1 \cap N_2)$$

Proof. $\nu_{N_1/N_1 \cap N_2} \simeq \nu_{M/N}$, so $U_{N_1/N_1 \cap N_2} = j^* U_{M/N}$. □

Proposition 5.19. *Suppose* $\pi : E \to M$ *is an oriented vector bundle,* $s : M \to E$ *a section,* $s \pitchfork s_0$*. Then*

$$e(E) = \mathrm{pd}_M(s \cap s_0) = \mathrm{pd}_M(s^{-1}(0)).$$

Proof. $(i^*)^{-1}(U_E) = \mathrm{pd}_E(s_0) = \mathrm{pd}_E(s)$ since $s \sim s_0$, so $e(E) = s_0^*(i^*)^{-1}(U_E)) = s_0^*(\mathrm{pd}_E(s)) = \mathrm{pd}_M(s_0 \cap s)$. □

Corollary 5.20. $\langle e(TM), [M] \rangle = \chi(M)$*.*

Proof. In $M \times M$, we have $\nu_{M \times M/\Delta} \simeq TM$, so $\langle e(TM), [M] \rangle = \Delta \cdot \Delta = (\widetilde{U}, \widetilde{U}) = \chi(M)$. For the last equality, recall that $\widetilde{U} = \sum_i (-1)^{|a_i|} a_i \times b_i = \sum_i (-1)^{|b_i|} b_i \times a_i$. □

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