Model Theory

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Contents

1 Preliminaries and Review

Definition. *A (first order)* language L *consists of*

(i) a set $\mathcal{F}$ of function symbols *and for each $f \in \mathcal{F}$ a positive integer n_f, the* arity *of f,*

(ii) a set $\mathcal{R}$ of relation symbols *and for each $R \in \mathcal{R}$ a positive integer n_R, the* arity *of R,*

(iii) a set $\mathcal{C}$ of constant symbols.

Remark. Constant symbols could bee seen as function symbols of arity 0. So some authors only include (i) and (ii) in the definition and allow $n_f = 0$ in (i).

Examples.

(a) L_{gp} is the language of groups, it has two function symbols $\cdot$ and $^{-1}$ of arity 2 resp. 1, a constant symbol 1 and no relation symbols.

(b) L_{lo} is the language of linear orders. It has only one binary relation symbol $<$.

Definition. *Given a language $L = (\mathcal{F}, \mathcal{R}, \mathcal{C})$, an L-*structure *consists of*

(i) a non-empty set M, called the domain*,*

(ii) for each function symbol $f \in \mathcal{F}$, a function $f^M : M^{n_f} \to M$,

(iii) for each relation symbol $R \in \mathcal{R}$, a relation $R^M \subseteq M^{n_R}$,

(iv) for each constant symbol $c \in \mathcal{C}$, an element $c^M \in M$.

f^M, R^M, c^M *are called the* interpretations *of the symbols f, R, c resp. in M.*

Remarks.

1. We sometimes ignore the distinction between an L-structure and its domain, and between symbols in L and their interpretations in the structure when it is clear from the context.
2. We write $\mathcal{M} = (M, \{f_i\}_{i\in I}, \{R_j\}_{j\in J}, \{c_k\}_{k\in K})$ for a structure in $L = (\{f_i\}_{i\in I}, \{R_j\}_{j\in J}, \{c_k\}_{k\in K})$.

Examples.

(a) $(\mathbb{R}^+, \{\cdot, ^{-1}\}, \{1\})$ is an L_{gp}-structure.

(b) $(\mathbb{Z}, \{+, -\}, 0)$ is another L_{gp}-structure.

(c) $(\mathbb{Q}, \{<\})$ is an L_{lo}-structure.

Using

- the symbols of L,

- connectives $\wedge$, $\neg$ (and consequently also $\vee, \rightarrow, \leftrightarrow$),
- quantifiers $\exists$ (and consequently also $\forall$),
- variables $x_0, x_1, x_2, \ldots, y, z$ etc. (arbitrarily many),
- punctuation (,),
- $\perp$,
- equality

define recursively L-terms and L-formulas.

Notation. The letters u, v, x, y, z usually stand for variables while a, b, c stand for constants. If φ is a formula, $\varphi(x_0, \ldots, x_n)$ indicates that the x_i are free variables in φ, same for terms. We write $\overline{x} = x_0, \ldots, x_n$ for an $(n+1)$-tuple of variables and same for constants.

2 Embeddings

Definition. *Let $L = (\mathcal{F}, \mathcal{R}, \mathcal{C})$ be a language and M, N be L-structures. An* embedding *of M into N is an injective map $\alpha : M \to N$ such that:*

(i) for all $f \in \mathcal{F}$, and $a = a_1, \ldots, a_{n_f} \in M$,

$$\alpha(f^M(a_1, \ldots, a_{n_f})) = f^N(\alpha(a_1), \ldots, \alpha(n_f)),$$

(ii) for all $R \in \mathcal{R}$, and $a_1, \ldots, a_{n_R} \in M$,

$$(a_1, \ldots, a_{n_R}) \in R^M \iff (\alpha(a_1), \ldots, \alpha(a_{n_R})) \in R^N,$$

(iii) for each $c \in \mathcal{C}$,

$$\alpha(c^M) = c^N.$$

A bijective embedding $\alpha : M \to N$ is called an isomorphism*. If there exists an isomorphism between M and N, we write $M \simeq N$.*

Examples.

(i) Let G_1, G_2 be groups, viewed as L_{gp}-structures, then $\alpha : G_1 \to G_2$ is an embedding iff it is an injective group homomorphism.

(ii) If A, B are linear orders, viewed as L_{op}-structures, then $\alpha : A \to B$ is an embedding iff α is injective and such that for $a, b \in A$, $a < b$ iff $\alpha(a) < \alpha(b)$.

Proposition 2.1. *Let M, N be L-structures, $\alpha : M \to N$ an embedding. Let $\overline{a} \in M^k$, and $t(\overline{x})$ a term with $|\overline{x}| = k$. Then*

$$\alpha(t^M(\overline{a})) = t^N(\alpha(\overline{a})),$$

where $\alpha(\overline{a}) = (\alpha(a_1), \ldots, \alpha(a_k))$.

Proof. This is a standard proof by induction on the complexity of the term $t(\overline{x})$.

- Case 1: t is a variable x_i. Then $\alpha(t^M(\overline{a})) = \alpha(a_i)$ and $t^N(\alpha(\overline{a})) = \alpha(a_i)$.
- Case 2: t is a constant c. Then it follows from (iii) in the definition of embeddings.
- Case 3: Let $t(\overline{x}) = f(t_1(\overline{x}), \ldots, t_{n_f}(\overline{x}))$. Then $\alpha(t_i^M(\overline{a})) = t_i^N(\alpha(\overline{a}))$ by induction and then $\alpha(t^M(\overline{a})) = t^N(\alpha(\overline{a}))$ by (i) in the definition of embeddings.

$\square$

Notation. Recall that if $\phi(\overline{x})$ is an L-formula, M is an L-structure and $\overline{a} \in M^{|\overline{x}|}$, then $M \models \phi(\overline{a})$ means that ϕ holds in M under the assignment $x_i \mapsto a_i$ (defined recursively). Also recall that *atomic* L-formulas are those of one of the following two forms:

(i) $t_1 = t_2$ where t_1, t_2 are L-terms,

(ii) $R(t_1, \dots, t_{m_R})$ where R is a relation symbol and $t_1, \dots, t_{m_R}$ are terms.

Proposition 2.2. *Let M, N be L-structures, $\alpha : M \to N$ an embedding. Let $\varphi(\overline{x})$ be an atomic formula and $\overline{a} \in M^{|x|}$. Then*

$$M \models \varphi(\overline{a}) \iff N \models \varphi(\alpha(\overline{a})).$$

Proof. Immediate from the definitions and Proposition 2.1. □

Exercise: Show that the same holds more generally for quantifier-free formulas instead of just atomic ones.

Warning. Embeddings do not necessarily preserve all formulas. Consider e.g. $(\mathbb{Z}, <)$ and $(\mathbb{Q}, <)$ as L_{lo}-structures. Then the map $\alpha : \mathbb{Z} \to \mathbb{Q}$, $n \mapsto n$ is an embedding. Let $\varphi(x_1, x_2)$ be the formula $\exists z(x_1 < z \wedge z < x_2)$. Then $\mathbb{Z} \not\models \varphi(1, 2)$, but $\mathbb{Q} \models \varphi(1, 2) = \varphi(\alpha(1), \alpha(2))$.

Exercise: Let M, N be L-structures, $\alpha : M \to N$ an isomorphism. Let $\varphi(\overline{x})$ be any formula and $\overline{a} \in M^{|x|}$. Then

$$M \models \varphi(\overline{a}) \iff N \models \varphi(\alpha(\overline{a})).$$

Remark. The converse of Proposition 2.2 also holds, i.e. a map $\alpha : M \to N$ that preserves atomic formulas is an embedding (exercise).

3 Theories and Elementarity

Let L be a fixed language. Recall that a *sentence* is a formula with no free variables.

Definition. *An L-*theory *T is a set of L-sentences. An L-structure M is a* model *of T if all sentences in T hold in M, i.e. $M \models \sigma$ for all $\sigma \in T$. We write* $\mathrm{Mod}(T)$ *for the class of all models of T.*

If M is a L-structure, then the theory *of M is*

$$\mathrm{Th}(M) = \{\sigma \mid \sigma \text{ is an } L\text{-sentence and } M \models \sigma\}.$$

Example. Consider $L = L_{\mathrm{gp}}$. Let T_{gp} be the theory consisting of

(i) $\forall x, y, z\,((x \cdot y) \cdot z = x \cdot (y \cdot z))$,

(ii) $\forall x\,(x \cdot 1 = 1 \cdot x = x)$,

(iii) $\forall x\,(x \cdot x^{-1} = x^{-1} \cdot x = 1)$.

If G is a group, clearly $G \models T_{\mathrm{gp}}$, but $\mathrm{Th}(G) \supsetneq T_{\mathrm{gp}}$.

Definition. *L-structures M, N are* elementary equivalent *if*

$$\mathrm{Th}(M) = \mathrm{Th}(N).$$

In this case we write $M \equiv N$.

Remark. If $M \simeq N$, then $M \equiv N$, but the converse does not hold in general. E.g. we will later, see Corollary 4.7, show that

$$(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$$

as L_{lo}-structures, but they are clearly not isomorphic.

Definition. *Let M, N be L-structures. Then:*

(i) An embedding $\beta : M \to N$ is elementary *if for all L-formulas $\varphi(\overline{x})$ and $\overline{a} \in M^{|a|}$,*

$$M \models \varphi(\overline{a}) \iff N \models \varphi(\beta(\overline{a})).$$

(ii) When M is a subset of N and the inclusion map $M \hookrightarrow N$ is an embedding, then M is a substructure *of N, written $M \subseteq N$.*

(iii) When M is a subset of N and the inclusion map $M \hookrightarrow N$ is an elementary embedding, then M is an elementary substructure *of N, written $M \preceq N$.*

Example. Let $\mathcal{M} = ([0, 1], <)$ and $\mathcal{N} = ([0, 2], <)$ be L_{lo}-structures. Then $\mathcal{M} \subseteq \mathcal{N}$. Also $\mathcal{M} \simeq \mathcal{N}$ (e.g. via $x \mapsto 2x$), hence $\mathcal{M} \equiv \mathcal{N}$. But $\mathcal{M} \not\preceq \mathcal{N}$! Indeed, consider the formula $\varphi(x) = \forall y\,(y < x \vee y = x)$. Then $\mathcal{M} \models \varphi(1)$, but $\mathcal{N} \not\models \varphi(1)$.

Definition. *Let M be an L-structure, $A \subseteq M$ a subset. Then we define the language*

$$L(A) := L \cup \{\textit{constant symbols } c_a \mid a \in A\}.$$

We interpret M as an $L(A)$-structure by $c_a^M := a$. In this context, the elements of A are called parameters*.*

Notation. Let M, N be L-structures and $A \subseteq M \cap N$ a subset. Then we write $M \equiv_A N$ and say that M is elementary equivalent to N *over* A, if M, N satisfy exactly the same $L(A)$-sentences.

Remark. If $M \preceq N$, then $M \equiv_M N$.

Lemma 3.1 (Tarski-Vaught Test)**.** *Let N be an L-structure, $A \subseteq N$ a subset. TFAE:*

(i) A is the domain of an elementary substructure of N.

(ii) For all $L(A)$-formulas $\varphi(x)$ with one free variable x,

$$N \models \exists x\, \varphi(x) \Longrightarrow N \models \varphi(b) \quad \textit{for some } b \in A. \tag{$*$}$$

Proof. "$(i) \Rightarrow (ii)$" is easy: By elementarity,

$$\begin{aligned} N \models \exists x\, \varphi(x) &\Longrightarrow A \models \exists x\, \varphi(x) \\ &\Longrightarrow A \models \varphi(b) \quad \text{for some } b \in A \\ &\Longrightarrow N \models \varphi(b) \quad \text{for some } b \in A. \end{aligned}$$

"$(ii) \Rightarrow (i)$" First show that A is the domain of a substructure. It suffices to show (exercise)

(a) for all $c \in \mathcal{C}$, $c^N \in A$. [Use $(*)$ with $\exists x\,(x = c)$. Then $N \models \exists x\,(x = c)$, so $N \models b = c$ for some $b \in A$, so $c^N = b \in A$.]

(b) for $f \in \mathcal{F}, \overline{a} \in A^{n_f}$, we have $f(\overline{a}) \in A$. [Similar to (a) with $\exists x\, f(\overline{a}) = x$.]

So $A \subseteq N$ is a substructure. Next let $\chi(\overline{x})$ be an L-formula and $\overline{a} \in A^{|\overline{x}|}$. We have to show $A \models \chi(\overline{a}) \Longleftrightarrow N \models \chi(\overline{a})$. We argue by induction on the complexity of $\chi(\overline{x})$.

- If $\chi(\overline{x})$ is atomic, the claim follows from $A \subseteq N$ and Proposition 2.2.
- If $\chi(\overline{x}) = \neg\psi(\overline{x})$. Then

$$\begin{aligned} A \models \chi(\overline{a}) &\Longleftrightarrow A \not\models \psi(\overline{a}) \\ &\Longleftrightarrow N \not\models \psi(A) \\ &\Longleftrightarrow N \models \chi(\overline{a}). \end{aligned}$$

- If $\chi(\overline{x}) = \psi(\overline{x}) \wedge \xi(\overline{x})$. Similar as before.

- If $\chi(\overline{x}) = \exists y\,\psi(\overline{x}, y)$. Then for $\overline{a} \in A^{|x|}$, $\psi(\overline{a}, y)$ is an $L(A)$-formula with one free variable. Then

$$\begin{aligned} A \models \chi(\overline{a}) &\Longleftrightarrow A \models \exists y\,\psi(\overline{a}, y) \\ &\Longrightarrow A \models \psi(\overline{a}, b) \ \text{ for some } b \in A \\ &\Longrightarrow N \models \psi(\overline{a}, b) \ \text{ for some } b \in A \\ &\Longrightarrow N \models \exists y\,\psi(\overline{a}, y) \\ &\Longrightarrow N \models \chi(\overline{a}). \end{aligned}$$

  For the converse we need to use (∗), so suppose $N \models \exists y\,\psi(\overline{a}, y)$. Then $N \models \psi(\overline{a}, b)$ for some $b \in A$. By induction hypothesis $A \models \psi(\overline{a}, b)$, so $A \models \exists y\,\psi(\overline{a}, y)$.

□

Definition. *We define the* cardinality *of the language L to be*

$$|L| := |\{\varphi(\overline{x}) \mid \varphi(\overline{x}) \textit{ is an } L\textit{-formula}\}|.$$

Note that always $|L| \geq \omega$ (we use ω both for the ordinal and the cardinality). Also $|L(A)| = |L| + |A| (= \max\{|L|, |A|\})$ for parameter sets A.

Definition. *Let λ be an ordinal. Then a* chain of sets *of length λ is a sequence $(A_i)_{i<\lambda}$ where the A_i are sets such that $A_i \subseteq A_j$ whenever $i \leq j < \lambda$.*

Similarly, a chain of L-structures *of length λ is a sequence $(M_i)_{i<\lambda}$ such that $M_i \subseteq M_j$ is a substructure whenever $i \leq j < \lambda$. The* union *of the chain $(M_i)_{i<\lambda}$ is defined as follows:*

- *the domain is $M = \bigcup_{i<\lambda} M_i$.*
- *if $c \in \mathcal{C}$, $c^M := c^{M_i}$ for any $i < \lambda$.*
- *if $f \in \mathcal{F}$, $\overline{a} \in M^{n_f}$, then $f^M(\overline{a}) = f^{M_i}(\overline{a})$ where i is large enough such that $\overline{a} \in M_i^{n_f}$.*
- *if $R \in \mathcal{R}$, then $\mathcal{R}^M = \bigcup_{i<\lambda} R^{M_i}$.*

Note that these interpretations are well-defined because $M_i \subseteq M_j$ is a substructure for $i \leq j$.

Theorem 3.2 (Downward Löwenheim-Skolem)**.** *Let N be an L-structure with $|N| \geq |L|$ and $A \subseteq N$ a subset. Then for any cardinal λ such that $|L| + |A| \leq \lambda \leq |N|$ there is an elementary substructure $M \preceq N$ such that*

(i) $|M| = \lambda$,

(ii) $A \subseteq M$.

Proof. We build inductively a chain $(A_i)_{i<\omega}$ of subsets of N containing A such that $\bigcup A_i$ is the required substructure M. Let $A_0 \supseteq A$ be any subset of N with $|A_0| = \lambda$. Suppose we already constructed A_i (with $|A_i| = \lambda$). Let $(\varphi_k(x))_{k<\lambda}$ be an enumeration of $L(A_i)$-formulas with one free variable and such that $N \models \exists x\, \varphi_k(x)$. Then let

$$A_{i+1} := A_i \cup \{a_k \in N \mid N \models \varphi_k(a_k),\, k < \lambda\}.^1$$

Now let $M = \bigcup_{i<\omega} A_i$. Claim: $M \preceq N$. We use TVT (Lemma 3.1). Let $\varphi(\overline{x}, y)$ be an L-formula. Claim: If $N \models \exists y\, \varphi(\overline{a}, y)$ for $\overline{a} \in M^{|x|}$, then $N \models \varphi(\overline{a}, b)$ for some $b \in M$. Let $i < \lambda$ be such that $\overline{a} \in A_i$. Then $\varphi(\overline{a}, y)$ is among the formulas considered at stage $i + 1$ in the construction of M, hence there is a witness to $\exists y\, \varphi(\overline{a}, y)$ in $A_{i+1} \subseteq M$. □

Remark. We have the following special case: If L is a countable language, T an L-theory with an infinite model, then T has a countable model.

[1]Remark by L.T.: This should probably mean that that we choose one a_k for each $k < \lambda$ such that $N \models \varphi_k(a_k)$, instead of taking all of them. Otherwise it would not be clear why the cardinality is bounded by λ.

4 Two Relational Structures

Definition. *An L_{lo}-structure is a* linear order *if it satisfies*

1. $\forall x \neg(x < x)$,
2. $\forall x, y, z\,((x < y \wedge y < z) \to x < z)$,
3. $\forall x, y\,(x = y \vee x < y \vee y < x)$.

A linear order is dense *if it satisfies*

4. $\exists x, y\,(x < y)$,
5. $\forall x, y, (x < y \to \exists z\,(x < z \wedge z < y))$.

A linear order has no endpoints *if*

6. $\forall x\,(\exists y\,(x < y) \wedge \exists z\,(z < x))$.

We let T_{lo} be the theory consisting of 1,2,3 and T_{dlo} be the theory consisting of 1-6.

Remark. If $M \models T_{\mathrm{dlo}}$, then $|N| \geq \omega$.

Let L be any language.

Definition. *A* partial embedding *between L-structures M, N is an injective map $p : \mathrm{dom}(p) \subseteq M \to N$, where $\mathrm{dom}(p)$ is a subset of M, such that p preserves functions, relations and constants as in the definition of embeddings.*

M and N are said to be partially isomorphic *if there is a non-empty collection I of partial embeddings from M to N such that*

(1) if $p \in I$, $a \in M$, then there is $\widehat{p} \in I$ such that $p \subseteq \widehat{p}$ and $a \in \mathrm{dom}\,\widehat{p}$.

(2) if $p \in I$, $b \in N$, then there is $\widehat{p} \in I$ such that $p \subseteq \widehat{p}$ and $b \in \mathrm{ran}\,\widehat{p}$.

We sometimes write "$p : M \to N$ is partial map" for a partial map instead of $p : \mathrm{dom}\,p \subseteq M \to N$.

Lemma 4.1 ("Back and Forth")**.** *If $|M| = |N| = \omega$ and M, N are partially isomorphic via I, then $M \simeq N$.*

Proof. Enumerate M and N, say $M = \{a_i \mid i < \omega\}$, $N = \{b_i \mid i < \omega\}$. We define inductively a chain $(p_i)_{i<\omega}$ of elements of I such that $a_{i-1} \in \mathrm{dom}(p_i)$ and $b_{i-1} \in \mathrm{ran}(p_i)$. Let p_0 be any element in I. Suppose p_i is given. Use (1) in the definition to get $\widehat{p} \in I$ such that $\widehat{p} \supseteq p_i$ and $a_i \in \mathrm{dom}\,\widehat{p}$. Then use (2) to find $p_{i+1} \in I$ such that $p_{i+1} \supseteq \widehat{p}$ and $b_i \in \mathrm{ran}\,p_{i+1}$. Then $\pi = \bigcup_{i<\omega} p_i$ is the required isomorphism. □

Lemma 4.2 (Extension)**.** *Let $M \models T_{\text{lo}}$ and $N \models T_{\text{dlo}}$. Let $p : \text{dom}(p) \subseteq M \to N$ be a* finite *partial embedding, i.e.* $\text{dom}\, p$ *is finite. Let $c \in M$. Then there is a finite partial embedding $\widehat{p}$ such that $\widehat{p} \supseteq p$ and $c \in \text{dom}(\widehat{p})$.*

Proof. Let $\text{dom}\, p = \{a_0, \ldots, a_n\}$ with $a_i < a_j$ if $i < j$.

- Case 1: $c < a_0$. Since N has no endpoints, we find $d \in N$ such that $d < p(a_0)$.
- Case 2: $a_i < c < a_{i+1}$ for some i. We find $d \in \mathbb{N}$ such that $p(a_i) < d < p(a_{i+1})$ by density of N.
- Case 3: $a_n < c$. Similar to 1.

Now define $\widehat{p}$ by $\widehat{p}(c) = d$ on $\text{dom}\, \widehat{p} = \text{dom}\, p \cup \{c\}$. □

Theorem 4.3. *Let $M, N \models T_{\text{dlo}}$ be such that $|M| = |N| = \omega$. Then $M \simeq N$.*

Proof. Let $I = \{q : M \to N \mid q \text{ is finite partial embedding}\}$. Then I is non-empty as it contains the empty map. By Lemma 4.2, I satisfies properties (1) and (2) in the definition of partial isomorphism. Hence Lemma 4.1 applies, i.e. $M \simeq N$. □

Definition. *An L-theory T is* consistent *if there is an L-structure M that models T. If σ is an L-sentence, write $T \vdash \sigma$ if for all L-structures M we have*

$$M \models T \implies M \models \sigma.$$

The theory T is complete *if for all L-stentences σ, either $T \vdash \sigma$ or $T \vdash \neg\sigma$.*

Remark. $\text{Th}(M)$ is complete for all L-structures M. We often seek $S \subseteq \text{Th}(M)$ such that S is complete. Then S is an *axiomatisation* of $\text{Th}(M)$.

Definition. *If $|L| = \omega$, an L-theory T is* ω-categorical *if whenever $M, N \models T$ and $|M| = |N| = \omega$, then $M \simeq N$.*

So by Theorem 4.3, T_{dlo} is ω-categorical.

Theorem 4.4. *If T is an ω-categorical theory with no finite models, then T is complete.*

Proof. Let $M, N \models T$ and φ be an L-sentence such that $M \models \varphi$. We have to show that $N \models \varphi$. By the Downward Löwenheim-Skolem theorem there are elementary substructures $M' \preceq M, N' \preceq N$ with $|M'| = |N'| = \omega$. By ω-categoricity, $M' \simeq N'$. Then $M' \models \varphi$, so $N' \models \varphi$ and then $N \models \varphi$. □

Corollary 4.5. T_{dlo} *is complete.*

Definition. *Let $f : \text{dom}(f) \subseteq M \to N$ be a partial map. f is* elementary *if for all L-formulas $\varphi(\overline{x})$ and $\overline{a} \in (\text{dom}\, f)^{|x|}$, we have*

$$M \models \varphi(\overline{a}) \iff N \models \varphi(f(\overline{a})).$$

Remark. A map f is elementary iff every finite restriction of f is elementary.

Proposition 4.6. *Let $M, N \models T_{\mathrm{dlo}}$ and let $p : M \to N$ be a partial embedding. Then p is an elementary map.*

Proof. By the above remark we may assume that p is a finite partial embedding. By Donward Löwenheim-Skolem, there are $M' \preceq M, N' \preceq N$ with $|M'| = |N'| = \omega$ and $\operatorname{dom} p \subseteq M'$, $\operatorname{ran} p \subseteq N'$. By an argument identical to the proof of Lemma 4.1 with $p_0 = p$ and I the collection of finite partial embeddings between M' and N', we can extend p to an isomorphism $\pi : M' \simeq N'$. In particular, π is an elementary map, therefore so is its restriction p. □

Corollary 4.7. $(\mathbb{Q}, <) \preceq (\mathbb{R}, <)$.

Proof. The inclusion map is an embedding, therefore it is elementary by the proposition. □

Definition. *Let $L_{\mathrm{gph}} = \{R\}$ where R is a binary relation symbol. A* graph *is an L_{gph}-structure M which satisfies*

1. $\forall x\, (\neg R(x, x))$,
2. $\forall x, y\, (R(x, y) \to R(y, x))$.

Elements of M are called vertices*, elements of R^M* edges*.*

Let T_{gph} be the theory consisting of the two axioms above.

We want to formalise the following properties of a graph G: However we choose finite subsets $U, V \subseteq G$, we can find $z \in G \setminus (U \cup V)$ such that z is R-related to all vertices in U and not R-related to any vertex in V.

A graph is called a *random graph* if it satisfies $\exists x, y\, (x \neq y)$ (non-triviality) and for each $n \in \mathbb{N}$, the axiom

$$\forall x_0 \dots x_n, y_0 \dots y_n \left(\bigwedge_{i,j=0}^{n} x_i \neq y_j \to \exists z \left(\bigwedge_{i=0}^{n} z \neq y_i \wedge \bigwedge_{i=0}^{n} R(x_i, z) \wedge \bigwedge_{i=0}^{n} \neg R(z, y_i) \right) \right) \quad (r_n)$$

T_{rg} is the theory that says that R is a graph relation that is non-trivial in the above sense and satisfies r_n for all $n \in \mathbb{N}$.

Proposition 4.8. *T_{rg} is consistent.*

Proof. Define R on ω as follows: For $i, j \in \omega$ with $i < j$, $R(i, j)$ holds, i.e. $\{i, j\}$ is an edge, iff the i-th digit in the binary expansion of j is 1.

Exercise: Prove (ω, R) is a model for T_{rg}. □

Lemma 4.9 (Extension)**.** *Let $M \models T_{\mathrm{gph}}, N \models T_{\mathrm{rg}}$. Let $p : \mathrm{dom}(p) \subseteq M \to N$ be a finite partial embedding and $c \in M$. Then there is a finite partial embedding $\widehat{p} : M \to N$ such that $\widehat{p} \supseteq p$ and $c \in \mathrm{dom}\,\widehat{p}$.*

Proof. We may assume $c \notin \mathrm{dom}\,p$. Let $U = \{a \in \mathrm{dom}(p) \mid R(a, c)\}$ be the set of neighbors of c in $\mathrm{dom}\,p$ and $V = \{b \in \mathrm{dom}\,p \mid \neg R(b, c)\}$. By a suitable instance of (r_n), we find $d \in N$ such that $R(d, p(a))$ for all $a \in U$ and $\neg R(d, p(b))$ for all $b \in V$. Then let $\widehat{p} = p \cup \{(c, d)\}$. □

Theorem 4.10. *Let $M, N \models T_{\mathrm{rg}}$ with $|M| = |N| = \omega$. Then $M \simeq N$.*

Proof. Same as Theorem 4.3 but with Lemma 4.9 instead of Lemma 4.2. □

Theorem 4.11. *T_{rg} is ω-categorical and complete. Every partial embedding between models of T_{rg} is elementary.*

Remark. The unique countable model of T_{rg} is called the countable random graph, or Rado's graph. Rado's graph is *universal* for finite graphs, i.e. every finite graph embeds into it, and *ultrahomogeneous*, i.e. every isomorphism between finite induced subgraphs extends to an automorphism.

5 Compactness

Definition. *Let I be a set. A* filter *on I is a subset $F \subseteq \mathcal{P}(I)$ such that*

1. *$I \in F$,*
2. *$X \cap Y \in F$ whenever $X, Y \in F$,*
3. *if $X \in F$, $X \subseteq Y \subseteq I$, then also $Y \in F$.*

F is proper *if $F \neq \mathcal{P}(I)$ or, equivalently, if $\emptyset \notin F$. An* ultrafilter *is a proper filter U such that for all $X \subseteq I$, either $X \in U$ or $I \setminus X \in U$.*

Proposition 5.1. *Let U be a proper filter on I. TFAE:*

(a) U is an ultrafilter.

(b) U is maximal among all proper filters.

(c) If $X \cup Y \in U$, then $X \in U$ or $Y \in U$.

Proof. Exercise. □

Definition. *Let $(M_i)_{i \in I}$ of L-structures. The* direct product *of the M_i is the set*

$$X = \prod_{i \in I} M_i = \{f : I \to \bigcup_{i \in I} M_i \mid f(i) \in M_i \, \forall i \in I\}.$$

We write $a = \langle a_i \mid i \in I \rangle$ for $a \in X$.

Let U be an ultrafilter on I. We define the relation $\sim_U$ on X by

$$a \sim_U b \iff \{i \in I \mid a(i) = b(i)\} \in U.$$

Proposition 5.2. *$\sim_U$ is an equivalence relation.*

Proof. Reflexivity and symmetry are immediate. For transitivity let $a, b, c \in X$ such that $a \sim_U b$, $b \sim_U c$. Let $A = \{i \in I \mid a(i) = b(i)\}$, $B = \{i \in I \mid b(i) = c(i)\}$ and $C = \{i \in I \mid a(i) = c(i)\}$. Then $A, B \in U$ and thus $A \cap B \in U$. Since $A \cap B \subseteq C$, we obtain $C \in U$, hence $a \sim_U c$. □

Write a_U for the equivalence class $[a]_{\sim_U}$ under the relation $\sim_U$.

Proposition 5.3. *Let $a^k, b^k \in X$ for $k = 1, \ldots, n$, be such that $a^k \sim_U b^k$. Then*

(a) if f is an n-ary function symbol, then

$$\langle f^{M_i}(a^1(i), \ldots, a^n(i)) \mid i \in I \rangle \sim_U \langle f^{M_i}(b^1(i), \ldots, b^n(i)) \mid i \in I \rangle$$

(b) if R is an n-ary relation symbol, then

$$\{i \in I \mid (a^1(i), \ldots, a^n(i)) \in R^{M_i}\} \in U \iff \{i \in I \mid (b^1(i), \ldots, b^n(i)) \in R^{M_i}\} \in U$$

Proof. To simplify notation assume $n = 1$ and let $a = a^1, b = b^1$.

(a) Let $A = \{i \in I \mid a(i) = b(i)\}$ and $C = \{i \in I \mid f^{M_i}(a(i)) = f^{M_i}(b(i))\}$. Clearly $A \subseteq C$ and so $C \in U$ as $A \in U$, hence $\langle f^{M_i}(a(i)) \mid i \in I\rangle \sim_U \langle f^{M_i}(b(i)) \mid i \in I\rangle$.

(b) is similar (exercise).

□

Definition. *Given a set I, $(M_i)_{i \in I}$ a family of L-structures, U an ultrafilter on I, we define an L-structure on the* ultraproduct

$$\prod_{i \in I} M_i \Big/ {\sim_U} = X/{\sim_U} =: X_U$$

as follows:

(i) if $c \in \mathcal{C}$, then $c^{X_u} := \langle c^{M_i}(i) \mid i \in I\rangle_U$.

(ii) if $f \in \mathcal{F}$ and $a_U^1, \ldots, a_U^{n_f} \in X_U^{n_f}$, we define

$$f^{X_U}(a_U^1, \ldots, a_U^{n_f}) = \langle f^{M_i}(a_U^1(i), \ldots, a_U^{n_f}(i)) \mid i \in I\rangle.$$

(iii) if $R \in \mathcal{R}$, and $a_U^1, \ldots, a_U^{n_R} \in X_U$, then

$$(a_U^1, \ldots, a_U^{n_R}) \in R^{X_U} \iff \{i \in I \mid (a^1(i), \ldots, a^{n_R}(i)) \in R^{M_i}\} \in U.$$

Proposition 5.3 shows that the L-structure on X_U is well-defined. So far we have not used that U is an *ultrafilter* and not merely a filter. However, we will finally need this in the following theorem:

Theorem 5.4 (Łoś)**.** *In the above setting the following is true:*

(i) For all terms $t(x_1, \ldots, x_n)$, $a_U^1, \ldots, a_U^n \in X_U$, we have

$$t^{X_U}(a_U^1, \ldots, a_U^n) = \langle t^{M_i}(a^1(i), \ldots, a^n(i)) \mid i \in I\rangle_U.$$

(ii) For all L-formulas $\varphi(x_1, \ldots, x_n)$ and $a_U^1, \ldots, a_U^n \in X_U$, we have

$$X_U \models \varphi(a_U^1, \ldots, a_U^n) \iff \{i \in I \mid M_i \models \varphi(a^1(i), \ldots, a^n(i))\} \in U.$$

(iii) For all L-sentences σ,

$$X_U \models \sigma \iff \{i \in I \mid M_i \models \sigma\} \in U.$$

Proof.

(i) The usual argument via induction over the complexity of the term.

(ii) By induction on $\varphi(\overline{x})$. The base case $\varphi(\overline{x})$ atomic follows from (i).

Suppose $\varphi \equiv \neg\chi$ for some L-formula $\chi(x_1, \dots, x_n)$. Let $A_\chi = \{i \in I \mid M_i \models \chi(a^1(i), \dots, a^n(i))\}$. By induction hypothesis, $X_U \models \chi(a_U^1, \dots, a_U^n) \iff A_\chi \in U$. Then

$$\chi_U \not\models \chi(a_U^1, \dots, a_U^n) \iff A_\chi \notin U \overset{U \text{ ultrafilter}}{\iff} I \setminus A_\chi \in U.$$

Hence

$$X_U \models \neg\chi(a_U^1, \dots, a_U^n) \iff \{i \in I \mid M_i \models \neg\chi(a^1(i), \dots, a^n(i))\} \in U.$$

The case $\varphi \equiv \chi \wedge \psi$ is an exercise.

Finally, consider the case $\varphi(\overline{x}) = \exists y\, \psi(\overline{x}, y)$. To simplify notation assume $|\overline{x}| = 1$. Define $A_\varphi = \{i \in I \mid M_i \models \exists y\, \varphi(a(i), y)\}$. We have to show

$$X_U \models \varphi(a_U) \iff A_\varphi \in U.$$

For "$\Rightarrow$" assume $X_U \models \exists y\, \psi(a_U, y)$, i.e. $X_U \models \psi(a_U, b_U)$ for some $b_U \in X_U$. Let $A_\psi := \{i \in I \mid M_i \models \psi(a(i), b(i))\}$. Then $A_\psi \in U$ by induction hypothesis and so $A_\varphi \in U$ as $A_\psi \subseteq A_\varphi$.

For "$\Leftarrow$" let $i \in A_\varphi$. Then $M_i \models \exists y\, \psi(a(i), y)$. Pick a witness $b(i)$. For $i \in I \setminus A_\varphi$, let $b(i)$ be arbitrary in M. Define $b_U = \langle b(i) \mid i \in I \rangle_U$. Let $A_\psi = \{i \in I \mid M_i \models \psi(a(i), b(i))\}$. Then $A_\psi \supseteq A_\varphi$ by our choice of the $b(i)$. Since $A_\varphi \in U$, also A_ψ. By the induction hypothesis, $X_U \models \psi(a_U, b_U)$ and therefore $X_U \models \exists y\, \psi(a_U, y)$.

(iii) Immediate from (ii).

□

Definition. *A subset $S \subseteq \mathcal{P}(I)$ has the* finite intersection property *(FIP) if for all $n \in \mathbb{N}$, $A_0, \dots, A_n \in S$, we have $\bigcap_{i=0}^n A_i \neq \emptyset$.*

Remark. Proper filters on I have the FIP.

Lemma 5.5.

1. *If $S \subseteq \mathcal{P}(I)$ has the FIP, then S can be extended to a proper filter.*
2. *Any proper filter can be extended to an ultrafilter.*

Proof.

1. Let $F \subseteq S$ be defined as

$$F = \{X \subseteq I \mid X \supseteq \bigcap_{i=0}^{n} A_i, \text{ for some } n \in \mathbb{N} \text{ and } A_i \in S\}.$$

Then check that this works.

2. Immediate from Zorn's lemma noting that the union of a chain of filters is again a filter.

□

Definition. *An L-theory T is* finitely consistent *if every finite subset of T is consistent, i.e. has a model.*

Theorem 5.6 (Compactness)**.** *A theory T is consistent if and only if it is finitely consistent.*

Proof. "⇒" is clear.

"⇐" Let $S \subseteq T$ be finite. Let M_S be any L-structure such that $M_S \models S$. Let I be the set of finite subsets of T. For $\varphi \in T$, let $A_\varphi = \{S \in I \mid \varphi \in S\}$. We claim that the set

$$\{A_\varphi \mid \varphi \in T\}$$

has the FIP. Indeed, let $\varphi_1, \ldots, \varphi_n$. Then $\{\varphi_1, \ldots, \varphi_n\} \in I$ and $\{\varphi_1, \ldots, \varphi_n\} \in \bigcap_{i=1}^n A_{\varphi_i}$, so the intersection is non-empty. Therefore there is an ultrafilter U on I with $A_\varphi \in U$ for all $\varphi \in T$. Then let $X_U = \prod_{S \in I} M_S \big/ \sim_U$ be the ultraproduct of the M_S w.r.t. this ultrafilter. Claim: If $\varphi \in T$, then $X_U \models \varphi$. To prove this we use Łoś' theorem: $X_U \models \varphi$ iff $\{S \in I \mid M_S \models \varphi\} \in U$. But $A_\varphi \in U$, so $A_\varphi = \{S \in I \mid \varphi \in S\} \subseteq \{S \in I \mid M_S \models \varphi\}$, so $\{S \in I \mid M_S \models \varphi\} \in U$. □

Definition. *A* type $p(\overline{x})$ *in L is a set of L-formula whose free variables are among $\overline{x} = (x_i)_{i<\lambda}$. A type $p(\overline{x})$ is*

- satisfiable in an L-structure M *if there is a tuple $\overline{a} \in M^{|\overline{x}|}$ such that $M \models \varphi(\overline{a})$ for all $\varphi(\overline{x}) \in p(\overline{x})$. In this case we write $M \models p(\overline{a})$, $M \models p(\overline{x})$ or $M, \overline{a} \models p(\overline{x})$. We say $\overline{a}$* realises *or* witnesses *the type $p(\overline{x})$ in M.*
- satisfiable *if there is an L-structure M such that $M \models p(\overline{x})$.*
- finitely satisfiable in M *if every finite subset of $p(\overline{x})$ is satisfiable in M.*
- finitely satisfiable *if every finite subset of $p(\overline{x})$ is satisfiable.*

We sometimes say (finitely) consistent instead of (finitely) satisfiable.

Remark. $p(\overline{x})$ may be finitely satisfiable in M, but not satisfiable in M. E.g. let $M = (\omega, <)$. Let $\varphi_n(x)$ say "there are at least n distinct elements less than x". Then take $p(x) = \{\varphi_n(x) \mid n \in \omega\}$. It is finitely satisfiable in M, but not satisfiable in M.

Theorem 5.7 (Compactness for types)**.** *Every finitely satisfiable type is satisfiable.*

Proof. Let $p(\overline{x})$ be an L-type with $\overline{x} = (x_i)_{i<\lambda}$. Expand L to $L' = L \cup \{c_i \mid i \in \lambda\}$ where the c_i are new constant symbols. Then $p(\overline{c})$ is a finitely consistent theory in L'. By compactness, there is an L'-structure M such that $M \models p(\overline{c})$. But M is also an L-structure by forgetting the interpretations of the c. Then $M, \overline{c}^M \models p(\overline{x})$. □

Lemma 5.8. *Let M be an L-structure and $\overline{a} = (a_i)_{i<\lambda}$ an enumeration of M. Let $q(\overline{x}) = \{\varphi(\overline{x}) \mid M \models \varphi(\overline{a})\}$ where $|\overline{x}| = \lambda$*[1] *Then $q(\overline{x})$ is satisfiable in an L-structure N iff M embeds elementarily into N.*

Proof. "⇒" Let $q(\overline{x})$ be satisfiable in N, i.e. there is $\overline{b} \in N^\lambda$ such that $N \models q(\overline{b})$, i.e. $N \models \varphi(\overline{b})$ for any $\varphi(\overline{x}) \in q(\overline{x})$. Then for any L-formula $\chi(\overline{x})$,

$$M \models \chi(\overline{a}) \iff \chi(\overline{x}) \in q(\overline{x}) \iff N \models \chi(\overline{b}).^2$$

Define $\beta : M \to N$ by $\beta : a_i \mapsto b_i$. Then β is an elementary embedding.

"⇐" is clear. □

Remark. Let $A \subseteq M$ be a subset. We can works with types in $L(A)$. In particular we can work with types in $L(M)$. A type in $L(A)$ is said to have *parameters in* A, or to be *over* A. Also, if $p(\overline{x})$ is a type in $L(M)$, there is an enumeration $\overline{a}$ of M and and L-type $q(\overline{x}, \overline{z})$ such that $p(\overline{x}) = q(\overline{x}, \overline{a})$. We obtain the following restatement of the lemma:

Lemma 5.9. *Let $\mathrm{Th}(M_M)$ be the $L(M)$-theory of M. Suppose $N \models \mathrm{Th}(M_M)$, then M embeds elementarily in N.*

Theorem 5.10. *If M is an L-structure and $p(\overline{x})$ a type in $L(M)$ that is finitely satisfiable in M, then $p(\overline{x})$ is realised (satisfiable) in some elementary extension $N \succeq M$.*

Example. Let $M = ((0,1) \cap \mathbb{Q}, <)$. Let $a_n = 1 - \frac{1}{n}$ with $n \in \omega \setminus \{0\}$. Let $\varphi_n(x) = (x > a_n)$. Let $p(\overline{x}) = \{\varphi_n(\overline{x}) \mid n \in \omega \setminus \{0\}\}$. Then $p(\overline{x})$ is a type in $L(M)$ that is finitely satisfiable, but not satisfiable. However, $(\mathbb{Q}, <) \models p(1)$, and $M \preceq (\mathbb{Q}, <)$ by Proposition 4.6.

Proof of Theorem 5.10. Let $\overline{a} = (a_i)_{i<\lambda}$ be an enumeration of M and let $q(\overline{z}) = \{\varphi(\overline{z}) \mid M \models \varphi(\overline{a})\}$ where $|\overline{z}| = \lambda$ and $\overline{z} \cap \overline{x} = \emptyset$. Write $p(\overline{x}) = p'(\overline{x}, \overline{a})$ where $p'(\overline{x}, \overline{z})$ is an L-type. Now $p'(\overline{x}, \overline{z}) \cup q(\overline{z})$ is finitely satisfiable in M. By compactness for types, there are an L-structure N and $\overline{c} \in N^{|\overline{x}|}, b \in N^\lambda$ such that $N \models p'(\overline{c}, \overline{b}) \cup q(\overline{b})$. In particular, $N \models q(\overline{b})$, so by Lemma 5.8, $a_i \mapsto b_i$ is an elementary embedding $M \to N$. We may assume $M \preceq N$. □

Theorem 5.11 (Upward Löwenheim-Skolem)**.** *Let M be an infinite L-structure and $\lambda \geq |M| + |L|$. Then there is N such that $M \preceq N$ and $|N| = \lambda$.*

[1] Here we use the convention that $\varphi(\overline{x})$ only uses finitely many variables in $\overline{x}$.

[2] Remark by L.T.:To see "⇐" note that if $M \not\models \chi(\overline{a})$, then $M \models \neg\chi(\overline{a})$, so $\neg\chi(\overline{x}) \in q(\overline{x})$ and thus $N \models \neg\chi(\overline{b})$, so $N \not\models \chi(\overline{b})$.

Proof. Let $(x_i)_{i<\lambda}$ be distinct variables. Let $p(\overline{x}) = \{x_i \neq x_j \mid i < j < \lambda\}$. Then $p(\overline{x})$ is finitely satisfiable in M, so $p(\overline{x})$ is realised in some $N \succeq M$ by Theorem 5.10. In particular, $|N| \geq \lambda$. Now by Downward Löwenheim-Skolem, we may assume that in fact $|N| = \lambda$. □

6 Saturation

Definition. *Let λ be an infinite cardinal, M an infinite L-structure. Then M is* λ-saturated *if it realises every type $p(x) \in L(A)$ such that*

(i) $p(x)$ is finitely satisfiable in M,

(ii) $A \subseteq M$ is such that $|A| < \lambda$,

(iii) x is a single variable.

M is saturated *if it is λ-saturated for $\lambda = |M|$.*

Remark. If $\lambda > |M|$, then M cannot be λ-saturated. Indeed, consider the type $p(x) = \{x \neq a \mid a \in M\}$, then $p(x)$ is finitely satisfiable in M, but not satisfiable in M.

Definition. *Let M be an L-structure, $A \subseteq M$ a subset, $\bar{b}$ a tuple in M. Then the* type *of $\bar{b}$ in M over A is*

$$\mathrm{tp}_M(\bar{b}/A) := \{\varphi(\bar{x}) \text{ type in } L(A) \mid M \models \varphi(\bar{b})\}.$$

We sometimes omit the M if it is clear from the context.

Remarks.

(i) $\mathrm{tp}_M(\bar{b}/A)$ is complete, i.e. for all $\varphi(\bar{x})$ in $L(A)$, either $\varphi(\bar{x}) \in \mathrm{tp}(\bar{b}/A)$ or $\neg\varphi(\bar{x}) \in \mathrm{tp}(\bar{b}/A)$.

(ii) If $M \preceq N$, $A \subseteq M$, $\bar{b} \in M^{|\bar{b}|}$, then $\mathrm{tp}_M(\bar{b}/A) = \mathrm{tp}_N(\bar{b}/A)$.

There is a relation between types and elementary maps:

Proposition 6.1. *If $f : A \subseteq M \to N$ is an elementary map. Then*

(a) $M \equiv N$ (and if $M \equiv N$, then the empty map $\emptyset : \emptyset \subseteq M \to N$ is elementary).

(b) If $\bar{a}$ is an enumeration of dom f*, then*

$$\mathrm{tp}_M(\bar{a}/\emptyset) = \mathrm{tp}_N(f(\bar{a})/\emptyset).$$

More generally, if $B \subseteq \mathrm{dom}(f) \cap N$ and $f|_B = \mathrm{id}_B$, then for every $\bar{b} \in \mathrm{dom}(f)^{|\bar{b}|}$,

$$\mathrm{tp}(\bar{b}/B) = \mathrm{tp}(f(\bar{b})/B).$$

(c) Let $\bar{a}$ enumerate dom(f) *and let $p(\bar{x}, \bar{a})$ be finitely satisfiable in M. Then $p(\bar{x}, f(\bar{a}))$ is finitely satisfiable in N.*

Proof. Easy from the definitions. For (c) let $\{\varphi_1(\bar{x}, \bar{a}), \ldots, \varphi_n(\bar{x}, \bar{a})\} \subseteq p(\bar{x}, \bar{a})$. Then $M \models \exists \bar{x} \bigwedge_{i=1}^n \varphi_i(\bar{x}, \bar{a})$, so by elementarity $N \models \exists \bar{x} \bigwedge_{i=1}^n \varphi(\bar{x}, f(\bar{a}))$. □

If $p(\overline{x}, \overline{a})$ is satisfiable in M, then $p(\overline{x}, f(\overline{a}))$ need not be satisfiable in N.

Theorem 6.2. *Let N, λ be such that $|L| \leq \lambda \leq |N|$. Then TFAE:*

(i) *N is λ-saturated.*

(ii) *If $f : M \to N$ is a partial elementary map such that $|f| < \lambda$, and $b \in M$, then there is $\widehat{f} \supseteq f$, elementary and such that $b \in \operatorname{dom} \widehat{f}$.*

(iii) *If $p(\overline{z})$ is a type in $L(A)$ with $A \subseteq N$, $|A| < \lambda$, $|\overline{z}| \leq \lambda$, and $p(\overline{z})$ is finitely satisfiable in N, then it is satisfiable in N.*

Proof. "(*i*) $\Rightarrow$ (*ii*)" Let M, f, b be as in (*ii*). Let $\operatorname{dom} f = \overline{a} = (a_i)_{i<\lambda}$ be an enumeration of $\operatorname{dom} f$. Let $p(x, \overline{a}) = \operatorname{tp}_M(b/\overline{a})$. Since $p(x, \overline{a})$ is satisfiable in M, $p(x, f(\overline{a}))$ is finitely satisfiable in N and hence satisfiable in N since N is λ-saturated. Let $c \in N$ be such that $N \models p(c, f(\overline{a}))$. Then $\widehat{f} = f \cup \{(b, c)\}$ is the required elementary map.

"(*ii*) $\Rightarrow$ (*iii*)" Let $p(\overline{z})$ be as in (*iii*). By Theorem 5.10, $p(\overline{z})$ is realised in some $M \succeq N$ by some $\overline{a}$, say, so $|\overline{a}| = |\overline{z}| \leq \lambda$. Since $A \subseteq N \preceq M$, the partial map $\operatorname{id}_A : A \subseteq M \to N$ is an elementary map. Idea: Extend id_A to a partial elementary map $f : M \to N$ such that $\operatorname{dom} f \supseteq \overline{a}$. Build f in stages. Let $f_0 = \operatorname{id}_A$. At stage $i+1$, use (*ii*) to define f_{i+1} on a_i. At limit stages $\mu < |a|$, let $f_\mu = \bigcup_{i<\mu} f_i$. Eventually $f = \bigcup_{i<|a|} f_i$ is the required extension of id_A.

"(*iii*) $\Rightarrow$ (*i*)" is trivial. □

Corollary 6.3. *Let M, N be saturated models of the same cardinality. If there is a partial elementary map $f : M \to N$ such that $|f| < |M|$, then $M \simeq N$. In particular, if $M \equiv N$, then $M \simeq N$.*

Proof. Given $f : M \to N$, use Theorem 6.2 (*ii*) to extend f to $\alpha : M \simeq N$ by a back-and-forth argument.

If $M \equiv N$, then $\emptyset : M \to N$ is elementary. □

Corollary 6.4. *Models of T_{dlo} and T_{rg} are ω-saturated.*

Proof. This follows from Lemma 4.2 and Lemma 4.9 using Theorem 6.2 "(*ii*) $\Rightarrow$ (*i*)". □

So $(\mathbb{Q}, <)$ is saturated, and $(\mathbb{R}, <)$ is ω-saturated. But $(\mathbb{R}, <)$ is not saturated. E.g. consider $p(x) = \{x > q \mid q \in \mathbb{Q}\}$. Then $p(x)$ is finitely satisfiable in $\mathbb{R}$ and $p(x) \in L_{\text{lo}}(\mathbb{Q})$, but is not satisfiable in $\mathbb{R}$.

Definition. *An isomorphism $\alpha : M \to M$ is called an* automorphism*. The collection of automorphisms of M is a group, denoted* $\operatorname{Aut}(M)$*. Given a subset $A \subseteq M$, we let* $\operatorname{Aut}(M/A) := \{\alpha \in \operatorname{Aut}(M) \mid \alpha|_A = \operatorname{id}_A\}$.

Definition. *The L-structure N is said to be* *(i)* λ-universal *if for every* M *such that* $|M| \leq \lambda$ *and* $M \equiv N$*, there is an elementary embedding* $\beta : M \to N$*.* N *is* universal *if it is* $|N|$*-universal.*

(ii) λ-homogeneous *if every elementary map* $f : N \to N$ *with* $|f| < \lambda$ *extends to an automorphism of* N*.* N *is* homogeneous *if it is* $|N|$*-homogeneous.*

Warning. For some authors property (i) is called λ^+*-universality* and (ii) is called *strong* λ*-homogeneity* (cf. ultrahomogeneity vs. homogeneity).

Theorem 6.5. *Let* N *be such that* $|N| \geq |L|$*. Then*

$$N \text{ is saturated} \Longleftrightarrow N \text{ is homogeneous and universal}$$

Proof. "$\Rightarrow$" Assume that N is saturated and let $M \equiv N$ with $|M| \leq |N|$. Let $\overline{a} = (a_i)_{i<|M|}$ enumerate M, and let $p(\overline{x}) = \operatorname{tp}(\overline{a}/\emptyset)$. Then $p(\overline{x})$ is finitely satisfiable in M (since it is satisfiable in M), hence $p(\overline{x})$ is finitely satisfiable in N as $M \equiv N$. By saturation, there is $\overline{b} \in N^{|\overline{x}|}$ such that $N \models p(\overline{b})$. Then $a_i \mapsto b_i$ is an elementary embedding $M \to N$. So N is universal. For homogeneity, use Corollary 6.3 with $M = N$.

"$\Leftarrow$" We show that if $M \equiv N$, $b \in M$, $f : M \to N$ elementary with $|f| < |N|$, then there is $\widehat{f} \supseteq f$ with $b \in \operatorname{dom} \widehat{f}$. By Theorem 6.2 this then shows that N is saturated. By Downward Löwenheim-Skolem, we may assume $|M| \leq |N|$. Since $M \equiv N$, there is an elementary embedding $\beta : M \to N$ by universality. Then $f \circ \beta^{-1} : \beta(\operatorname{dom}(f)) \to \operatorname{ran} f$ is an elementary map $N \to N$ and satisfies $|f \circ \beta^{-1}| < |N|$. By homogeneity of N, $f \circ \beta^{-1}$ extends to $\alpha \in \operatorname{Aut}(N)$. Then $f \cup \{(b, \alpha(\beta(b)))\}$ is the required extension $\widehat{f}$. Note that $\widehat{f}$ is elementary as it is a restriction of $\alpha \circ \beta$. $\square$

Definition. *Let* $\overline{a} \in N^{|\overline{a}|}$*,* $A \subseteq N$*. Then*

$$O_N(\overline{a}/N) := \{\alpha(\overline{a}) \mid \alpha \in \operatorname{Aut}(N/A)\}$$

is the orbit *of* $\overline{a}$ *over* A*.*

If $\varphi(\overline{x})$ *is an* $L(A)$*-formula, then*

$$\varphi(N) := \{\overline{b} \in N^{|\overline{x}|} \mid N \models \varphi(\overline{b})\}$$

is the set defined by $\varphi(\overline{x})$*. A subset of* N *is* definable over A *if it defined by some formula in* $L(A)$*.*

There are analogous notions for "type-definable" sets.

Remark. If $\overline{a}, \overline{b}$ are tuples in N, $A \subseteq N$ and $|\overline{a}| = |\overline{b}|$, then TFAE:

(i) $\operatorname{tp}(\overline{a}/A) = \operatorname{tp}(\overline{b}/A)$

(ii) $\langle a_i \mapsto b_i \mid i < |\overline{a}|\rangle \cup \operatorname{id}_A$ is an elementary map.

Proposition 6.6. *Let* N *be* λ*-homogeneous,* $A \subseteq N$ *such that* $|A| < \lambda$*, and* $\overline{a} \in N^{|\overline{a}|}$ *such that* $|\overline{a}| < \lambda$*. Then* $O_N(\overline{a}/A) = p(N)$*, where* $p(\overline{x}) = \operatorname{tp}(\overline{a}/A)$ *and* $p(N) = \{\overline{b} \mid N \models p(\overline{b})\}$*.*

Proof. "$O_N(\overline{a}/A) \subseteq p(N)$" is clear, since if $\overline{b} = \alpha(\overline{a})$ for some $\alpha \in \mathrm{Aut}(N/A)$, then $\mathrm{tp}_N(\overline{b}/A) = \mathrm{tp}_N(\overline{a}/A)$.

"$O_N(\overline{a}/A) \supseteq p(N)$". If $N \models p(\overline{b})$, then the map $\{(a_i, b_i) \mid i < |\overline{a}|\} \cup \mathrm{id}_A$ is elementary, hence by λ-homogeneity of N, the map extends to $\alpha \in \mathrm{Aut}(N)$. In particular, $\alpha \in \mathrm{Aut}(N/A)$ and $\alpha(\overline{a}) = \overline{b}$. □

7 The Monster Model

Let T be a complete theory without finite models. Idea: Work in a "large" saturated model of T that embeds elementary every model of T that you might be interested in. Such a "large", "very" saturated structure is called the *monster model* of T, and is usually denoted by U; or $\mathbb{M}$.

Terminology and Notation.

When working in $U \models T$, we say

- "$\varphi(\overline{x})$ holds", written $\models \varphi(\overline{x})$, when $U \models \forall \overline{x}\, \varphi(\overline{x})$.
- "$\varphi(\overline{x})$ is consistent" if $U \models \exists \overline{x}\, \varphi(\overline{x})$.
- A type $p(\overline{x})$ is *consistent* or *satisfiable* if $p(U) \neq \emptyset$, i.e. $\exists \overline{a} \in U^{|\overline{x}|}$ such that $U \models p(\overline{a})$.
- If $|U| = \kappa$, a cardinality is *small* if it is $< \kappa$. Sets, tuples etc. are *small* if they have small cardinality.
- A *model* is $M \preceq U$ with small cardinality.

Conventions.

- Tuples have small length
- Formulas have parameters in U.
- Definable sets have the form $\varphi(U)$ for $\varphi(\overline{x})$ in $L(U)$.
- Type-definable sets have the form $p(U)$ for some type $p(\overline{x})$ in $L(A)$ where $A \subseteq U$ is small.

Notation.

- A, B, C will denote parameter sets (small).
- $\mathrm{tp}(\overline{a}/A) = \mathrm{tp}_U(\overline{a}/A)$.
- $O(\overline{a}/A) = O_U(\overline{a}/A)$.
- If $p(\overline{x}), q(\overline{x})$ are types, then "$p(\overline{x}) \to q(\overline{x})$" means that $p(U) \subseteq q(U)$.

Informally, one can think of a type as an infinite conjunction of formulas.

Proposition 7.1. *Let $p(\overline{x}), q(\overline{x})$ be satisfiable (i.e. satisfiable in U) and in $L(A), L(B)$ resp. Suppose that $p(U) \cap q(U) = \emptyset$. Then there are $\varphi_1(\overline{x}), \ldots, \varphi_n(\overline{x}) \in p(\overline{x})$, $\psi_1(\overline{x}), \ldots, \psi_n(\overline{x}) \in q(\overline{x})$ such that*

$$\bigwedge_{i=1}^{n} \varphi_i(\overline{x}) \longrightarrow \neg \bigwedge_{i=1}^{n} \psi_i(\overline{x})$$

Proof. If $p(U) \cap q(U) = \emptyset$, then $p(\overline{x}) \cup q(\overline{x})$ is not satisfiable. Then, by saturation of U, $p(\overline{x}) \cup q(\overline{x})$ is not finitely satisfiable. □

Remark. Let $\varphi(U, \overline{b})$ be a definable set and $\alpha \in \mathrm{Aut}(U)$. Then $\alpha[\varphi(U, \overline{b})] = \varphi(U, \alpha(\overline{b}))$. For "$\subseteq$", let $\overline{c} = \alpha(\overline{a})$, with $\overline{a} \in U^{|\overline{a}|}$ and $\models \varphi(\overline{a}, \overline{b})$. Then $\models \varphi(\alpha(\overline{a}), \alpha(\overline{b})) = \varphi(\overline{c}, \alpha(\overline{b}))$. "$\supseteq$" is similar.

Similarly, if $p(\overline{x}, \overline{z})$ is a type in L and $\overline{b} \in U^{|\overline{z}|}$, then $\alpha[p(U, \overline{b})] = p(U, \alpha(\overline{b}))$.

Definition. *A set $\mathcal{D} \subseteq U^{\lambda}$ with $\lambda < |U|$ is* invariant under $A \subseteq U$ *if it satisfies one of the following equivalent properties:*

- *For all $\alpha \in \mathrm{Aut}(U/A)$, we have $\alpha[\mathcal{D}] = \mathcal{D}$.*
- *For all $\alpha \in \mathrm{Aut}(U/A)$ and for all $a \in \mathcal{D}^{|\overline{a}|}$, $O(a/A) \subseteq \mathcal{D}$.*
- *For all $\alpha \in \mathrm{Aut}(U/A)$ and for all $\overline{a} \in \mathcal{D}^{|\overline{a}|}$, $\overline{b} \models \mathrm{tp}(\overline{a}/A) \Rightarrow \overline{b} \in \mathcal{D}$.*

For the equivalence of the last two statements see Proposition 6.6.

Proposition 7.2. *Let $A \subseteq U$ be small. For $\varphi(\overline{x})$ in $L(U)$, TFAE:*

(i) There is $\psi(\overline{x})$ in $L(A)$ such that

$$\models \forall \overline{x}\, [\varphi(\overline{x}) \leftrightarrow \psi(\overline{x})].$$

(ii) $\varphi(U)$ is invariant under A.

Proof. "*(i)* $\Rightarrow$ *(ii)*" is clear since $\varphi(U) = \psi(U)$ and $\psi(U)$ is invariant over A, see e.g. the above remark.

"*(ii)* $\Rightarrow$ *(i)*" Let $\varphi = \varphi(\overline{x}, \overline{z})$ be an L-formula such that $\varphi(U, \overline{b})$ is invariant over A for some $\overline{b} \in U^{|\overline{z}|}$. Let $q(\overline{z}) = \mathrm{tp}(\overline{b}/A)$ and $\overline{c} \in q(U)$ so that $\overline{c} \models q(\overline{z})$. Then $\{(b_i, c_i) \mid i < |\overline{b}|\} \cup \mathrm{id}_A$ is an elementary map, so by homogeneity there is $\alpha \in \mathrm{Aut}(U/A)$ such that $\alpha(\overline{b}) = \overline{c}$. Then $\varphi(U, \overline{b}) = \alpha[\varphi(U, \overline{b})] = \varphi(U, \overline{c})$. Therefore $q(\overline{z}) \to \forall \overline{x}\, [\varphi(\overline{x}, \overline{z}) \leftrightarrow \varphi(\overline{x}, \overline{b})]$. By a version of Proposition 7.1 (exercise), there is $\chi(\overline{z}) \in q(\overline{z})$ such that

$$\models \chi(\overline{z}) \to [\varphi(\overline{x}, \overline{z}) \leftrightarrow \varphi(\overline{x}, \overline{b})].$$

Then $\exists \overline{z}\, [\chi(\overline{z}) \wedge \varphi(\overline{x}, \overline{z})]$ is the required formula in $L(A)$. □

Proposition 7.3. *For $\varphi(\overline{x})$, a formula in L, TFAE:*

(i) There is a quantifier-free formula $\psi(\overline{x})$ such that

$$\models \forall \overline{x}\, [\varphi(\overline{x}) \leftrightarrow \psi(\overline{x})].$$

(ii) For all partial embeddings $g : U \to U$, for all $\overline{a} \in \mathrm{dom}(g)^{|\overline{a}|}$, we have

$$\models \varphi(\overline{a}) \leftrightarrow \varphi(g(\overline{a})).$$

Proof. "*(i)* ⇒ *(ii)*" is clear since partial embeddings preserve quantifier-free formulas.

"*(ii)* ⇒ *(i)*" For $\overline{a} \in U^{|\overline{a}|}$, let

$$\mathrm{qftp}(\overline{a}) = \{\psi(\overline{x}) \in \mathrm{tp}(\overline{a}) \mid \psi(\overline{x}) \text{ is quantifier-free}\}.$$

Let $\mathcal{D} = \{q(\overline{x}) \mid q(\overline{x}) = \mathrm{qftp}(\overline{a}) \text{ for some } \overline{a} \in \varphi(U)\}$. Claim: $\varphi(U) = \bigcup_{q(\overline{x}) \in \mathcal{D}} q(U)$. The inclusion "⊆" is clear by definition. For the other containment let $q(\overline{x}) = \mathrm{qftp}(\overline{a})$ with $\overline{a} \in \varphi(U)$. Let $\overline{b} \models q(\overline{x})$. Then $a_i \mapsto b_i$ is a partial embedding and so by assumption in *(ii)*, $\varphi(\overline{b})$ holds. Hence $\overline{b} \in \varphi(U)$ and thus $q(U) \subseteq \varphi(U)$. This proves the claim.

Then in particular, $q(\overline{x}) \to \varphi(\overline{x})$. By a version of Proposition 7.1 there is $\psi_q(\overline{x}) \in q(\overline{x})$ such that $\psi_q(\overline{x}) \to \varphi(\overline{x})$. Also $\varphi(\overline{x}) \to \psi_q(\overline{x})$ for some q. Then

$$\varphi(\overline{x}) \longleftrightarrow \bigvee_{q \in \mathcal{D}} \{\psi_q(\overline{x}) \mid \psi_q(x) \to \varphi(\overline{x}) \text{ and } \psi_q(\overline{x}) \in q(\overline{x})\}$$

Again by a version of Proposition 7.1 there are $q_1, \ldots, q_n \in \mathcal{D}$ such that

$$\models \varphi(\overline{x}) \longleftrightarrow \bigvee_{i=1}^{n} \psi_{q_i}(\overline{x})$$

and so $\bigvee_{i=1}^{n} \psi_{q_i}(\overline{x})$ is the required quantifier-free formula. □

Definition. *An L-theory T has* quantifier elimination *if for every $\varphi(\overline{x})$ in L there is a quantifier-free formula $\psi(\overline{x})$ such that*

$$T \vdash \forall \overline{x}\, [\varphi(\overline{x}) \leftrightarrow \psi(\overline{x})].$$

Theorem 7.4. *Let T be a complete theory with an infinite model. TFAE:*

(i) T has quantifier elimination.

(ii) Every partial embedding $p : U \to U$ is elementary.

(iii) For every partial embedding $p : U \to U$ such that $|p| < |U|$ and $b \in U$, there is a partial embedding $\widehat{p} \supseteq p$ such that $b \in \mathrm{dom}(\widehat{p})$.

Proof. "*(i)* ⇒ *(ii)*" is clear since partial embeddings preserve quantifier-free formulas.

"*(ii)* ⇒ *(i)*" All partial embeddings are elementary, so any $\varphi(\overline{x})$ is preserved by all partial embeddings, so $\varphi(\overline{x}) \leftrightarrow \psi(\overline{x})$ for some quantifier-free $\psi(\overline{x})$ by Proposition 7.3.

"*(ii)* ⇒ *(iii)*" Let $p : U \to U$ be a partial embedding such that $|p| < |U|$. Then p is elementary, so there is $\alpha \in \mathrm{Aut}(U)$ such that $p \subseteq \alpha$. For $b \in U$, $p \cup \{(b, \alpha(b))\}$ is the required $\widehat{p}$.

"*(iii)* ⇒ *(ii)*" Let $p : U \to U$ be a partial embedding, and let $p_0 \subseteq p$ be finite (or small). Extend p_0 to $\alpha \in \mathrm{Aut}(U)$ by *(iii)* using a back-and-forth argument. Then p_0 is the restriction of an isomorphism, hence elementary. □

Remark. A fourth equivalent condition is *(iii)* with p finite (exercise).

It follows that T_{rg} and T_{dlo} have quantifier elimination.

Definition. *An element $a \in U$ is* definable over $A \subseteq U$ *if there is $\varphi(x)$ in $L(A)$ such that $\varphi(U) = \{a\}$. a is* algebraic over A *if there is $\varphi(x)$ in $L(A)$ such that $|\varphi(U)| < \omega$ and $a \in \varphi(U)$. A formula $\varphi(x)$ such that $|\varphi(U)| < \omega$ is said to be* algebraic*.*

The algebraic closure *of $A \subseteq U$ is*

$$\mathrm{acl}(A) = \{a \in U \mid a \textit{ is algebraic over } A\}.$$

If $\mathrm{acl}(A) = A$*, A is* algebraically closed*. The* definable closure of *A is*

$$\mathrm{dcl}(A) = \{a \in U \mid a \textit{ is definable over } A\}.$$

Remark. Any finite set is definable: $\{a_1, \ldots, a_n\}$ is defined by $\bigvee_{i=1}^{n}(a_i = x)$ (in $L(\{a_1, \ldots, a_n\})$?).

Proposition 7.5. *For $a \in U$, $A \subseteq U$, TFAE:*

(i) $a \in \mathrm{dcl}(A)$.

(ii) $O(\overline{a}/A) = \{a\}$.

Proof. "*(i)* ⇒ *(ii)*" Let $\varphi(x)$ in $L(A)$ define a over A. Then $\varphi(U)$ is invariant under $\mathrm{Aut}(U/A)$ and so $O(a/A) \subseteq \{a\} = \varphi(U)$.

"*(ii)* ⇒ *(i)*" $O(a/A)$ is definable (in $L(A \cup \{a\})$) and invariant over A, so by Proposition 7.2, $O(a/A)$ is defined by a formula in $L(A)$. □

Theorem 7.6. *Let $a \in U, A \subseteq U$. TFAE:*

(i) $a \in \mathrm{acl}(A)$.

(ii) $|O(a/A)| < \omega$

(iii) $a \in M$ *for any model M such that $A \subseteq M$.*

Proof. "*(i)* ⇒ *(ii)*" If $a \in \mathrm{acl}(A)$, there is $\varphi(x)$ in $L(A)$ such that $\varphi(a)$ holds and $|\varphi(U)| < \omega$. Since $\varphi(U)$ is invariant over A, $O(a/A) \subseteq \varphi(U)$.

"*(ii)* ⇒ *(i)*" If $|O(a/A)| < \omega$, then $O(a/A)$ is definable. But $O(a/A)$ is invariant under A, so by Proposition 7.2, there is $\varphi(x)$ in $L(A)$ such that $\varphi(U) = O(a/A)$, so $|\varphi(U)| < \omega$. Since $a \in \varphi(U)$, $a \in \mathrm{acl}(A)$.

"*(i)* ⇒ *(iii)*" Let $\varphi(x)$ in $L(A)$ such that $U \models \varphi(a) \wedge \exists^{=n} x\, \varphi(x)$. In particular, $U \models \exists^{=n} x\, \varphi(x)$. Now let $M \preceq U$, $A \subseteq M$. Then $M \models \exists^{=n} x\, \varphi(x)$. But then $\varphi(M) = \varphi(U)$ since both sets are finite of the same size, so $a \in \varphi(M) \subseteq M$.

"*(iii)* ⇒ *(i)*" Let $a \notin \mathrm{acl}(A)$, and $\mathrm{tp}(a/A) = p(x)$. Then for $\varphi(x) \in p(x)$, we have $|\varphi(U)| \geq \omega$. We can showthat $|p(U)| \geq \omega$ and then $|p(U)| = |U|$ (see Example Sheet 2).

Let $M \supseteq A$ be a model. Then $p(U) \setminus M \neq \emptyset$ (by cardinality). Let $b \in p(U) \setminus M$. By homogeneity there is $\alpha \in \mathrm{Aut}(U/A)$ such hat $\alpha(b) = a$. Then αM is a model that contains A, but not a. □

Proposition 7.7. *Let $a \in U$, $A \subseteq U$ small. Then*

(i) If $a \in \mathrm{acl}(A)$, then $a \in \mathrm{acl}(A_0)$ for some finite subset $A_0 \subseteq A$.

(ii) $A \subseteq \mathrm{acl}(A)$.

(iii) If $A \subseteq B$, then $\mathrm{acl}(A) \subseteq \mathrm{acl}(B)$.

(iv) $\mathrm{acl}(\mathrm{acl}(A)) = \mathrm{acl}(A)$.

(v) $\mathrm{acl}(A) = \bigcap_{M \supseteq A} M$ where M ranges over models containing A.

Proof.

(i) Clear.

(ii) In fact $A \subseteq \mathrm{dcl}(A) \subseteq \mathrm{acl}(A)$.

(iii) Clear.

(iv) By (ii) and (iii), $\mathrm{acl}(A) \subseteq \mathrm{acl}(\mathrm{acl}(A))$. For the other inclusion let $a \in \mathrm{acl}(\mathrm{acl}(A))$. By Theorem 7.6, $a \in M$ for all $M \supseteq \mathrm{acl}(A)$. But $M \supseteq \mathrm{acl}(A) \Leftrightarrow M \supseteq A$ by the same theorem, hence $a \in M$ for all models M containing A, so $a \in \mathrm{acl}(A)$.

(v) Clear by Theorem 7.6.

□

Proposition 7.8. *Let $\beta \in \mathrm{Aut}(U)$, and $A \subseteq U$. Then $\beta[\mathrm{acl}(A)] = \mathrm{acl}(\beta[A])$.*

Proof. Suppose $a \in \mathrm{acl}(A)$, so $\models \varphi(a, \bar{b})$ where $\bar{b} \in A^{|\bar{b}|}$ and $|\varphi(U, \bar{b})| < \omega$. Then $\models \varphi(\beta(a), \beta(\bar{b}))$ and $|\varphi(U, \beta(\bar{b}))| < \omega$ and so $\beta(a) \in \mathrm{acl}(\beta[A])$. The other inclusion is similar, or apply what we just proved to $\beta^{-1}, \beta[A]$ instead of β, A. □

8 Strongly Minimal Theories

Definition. *Let M be an infinite L-structure. A subset $A \subseteq M$ is called* cofinite *if $|M \setminus A| < \omega$.*

Remark. Finite and cofinite sets are always definable in any structure.

We will only be concerned with infinite M.

Definition. *Let M be an L-structure. Then M is* minimal *if all its definable subsets are finite or cofinite. M is* strongly minimal *if it is minimal, and so are all its elementary extensions. If T is a consistent theory without finite models, T is* strongly minimal *if for every L-formula $\varphi(x, \overline{z})$, there is $n \in \omega \setminus \{0\}$ such that*

$$T \vdash \forall \overline{z} \, [\exists^{\leq n} x \, \varphi(x, \overline{z}) \vee \exists^{\leq n} x \, \neg\varphi(x, \overline{z})].$$

Example. Let $L = \{E\}$ where E is a binary relation symbol. Let M be an L-structure where E is interpreted as an equivalence relation with exactly one equivalence class of size n for each $n \in \omega \setminus \{0\}$ and no infinite equivalence classes. We can prove (exercise) that $\mathrm{Th}(M)$ has quantifier elimination. Also it is not difficult to see that there is an elementary extension $M \preceq N$ that has an infinite equivalence class. So M is minimal (definable sets are boolean combinations of equivalence classes thanks to quantifier elimination), but N is not.

From now on, T is a complete, strongly minimal theory without finite models.

Definition. *If $a \in U$, $B \subseteq U$, then a* is independent from *B if $a \notin \mathrm{acl}(B)$. The set B is* independent *if for all $b \in B$, $b \notin \mathrm{acl}(B \setminus \{b\})$.*

Notation. We will often write Ab for $A \cup \{b\}$, $A \setminus b$ for $A \setminus \{b\}$, etc.

Theorem 8.1. *Let $B \subseteq U$, $a, b \in U \setminus \mathrm{acl}(B)$, then*

$$a \in \mathrm{acl}(Bb) \Longleftrightarrow b \in \mathrm{acl}(Ba).$$

Proof. Assume that $a \in \mathrm{acl}(Bb)$, but $b \notin \mathrm{acl}(Ba)$. Let $\varphi(x, y) \in L(B)$ be such that

$$\models \varphi(a, b) \wedge \exists^{\leq n} x \, \varphi(x, b)$$

for some $n \in \omega \setminus \{0\}$. Consider $\psi(a, y) = \varphi(a, y) \wedge \exists^{\leq n} x \, \varphi(x, y)$ in $L(Ba)$. Now $\models \varphi(a, b)$, so $|\psi(a, U)| \geq \omega$ as $b \notin \mathrm{acl}(Ba)$. By strong minimality, $|\neg\psi(a, U)| < \omega$. Let M be a model such that $B \subseteq M$. Then $M \cap \psi(a, U) \neq \emptyset$ (by cardinality). Let $c \in M \cap \psi(a, U)$. Then $a \in \mathrm{acl}(Bc)$, and $B \subseteq M, c \in M$, so $\mathrm{acl}(Bc) \subseteq M$ and thus $a \in M$. Then $a \in \bigcap_{M \supseteq B} = \mathrm{acl}(B)$, a contradiction. □

Main examples.

1. Let K be an infinite field. The language of K-vector spaces is $L_K = \{+, -, 0, \{\lambda\}_{\lambda \in K}\}$ where the λ's are unary function symbols. Interpretations of $+, -, 0$ are obvious and interpretation of λ is multiplication by the scalar λ, we write λx for $\lambda(x)$. The theory $T_{\mathrm{VS}K}$ includes the following axioms:

   - axioms for abelian groups for $+, -, 0$.
   - axioms for scalar product, e.g.
     - for each $\lambda \in K$,
       $$\forall x, y\, [\lambda(x + y) = \lambda x + \lambda y.]$$
     - for each $\lambda_1, \lambda_2, \mu \in K$ such that $\lambda_1 \lambda_2 = \mu$,
       $$\forall x\, [\lambda_1(\lambda_2 x) = \mu x.]$$
     - etc.
   - We also require non-triviality: $\exists x\, [x \neq 0]$.

   We can prove (with some work) that $T_{\mathrm{VS}K}$ is complete and has quantifier elimination.

   Then:

   - a term is a linear combination: $\lambda_1 x_1 + \cdots + \lambda_n x_n$.
   - atomic formulas are equalities between terms.
   - atomic formulas with one free variable and parameters are equivalent to formulas of the form $\lambda x = a$. Therefore such formulas define singletons.
   - quantifier-free formulas with one variable and parameters define finite or cofinite sets.

   By quantifier elimination, a model of $T_{\mathrm{VS}K}$ is strongly minimal. Moreover, for $A \subseteq M \models T_{VSK}$, $\mathrm{acl}(A) = \langle A \rangle$, the linear span. Also $a \notin \mathrm{acl}(A)$ iff a is linearly independent from A. A set A is independent iff it is linearly independent.

   Remark. If K is finite, one can define $T^{\infty}_{\mathrm{VS}K}$, the theory of infinite-dimensional vector space over K (more later).

2. The language of rings is $L_{\mathrm{ring}} = \{+, \cdot, -, 0, 1\}$. Then ACF is the L_{ring}-theory that includes:

   - axioms for abelian group using $+, -, 0$.
   - axioms for commutative monoids $(\cdot, 1)$.
   - field axioms
   - For each $0 < n < \omega$, the axiom
     $$\forall x_0 \ldots x_n\, \exists y\, [x_0 + x_1 y + \cdots + x_n y^n = 0].$$

For p prime, let χ_p be the sentence $1 + 1 + \cdots + 1 = 0$ where there are p 1's on the left hand side.

Then $ACF_p = ACF \cup \{\chi_p\}$ and $AFC_0 = ACF \cup \{\not\chi_p \mid p \text{ prime}\}$.

ACF_0 and ACF_p for given p are both complete and have quantifier elimination. Then

- atomic formulas are polynomial equations.
- If $A \subseteq M \models ACF_{0/p}$, an atomic formula in $L_{\text{ring}}(A)$ with one free variable is equivalent to $p(x) = 0$ where $p(x) \in F[x]$ where F is the subfield generated by A.
- Therefore, atomic formulas as above define finite sets
- Quantifier-free formulas define finite/cofinite sets.

By quantifier elimination, ACF_0, ACF_p are strongly minimal.

Definition. *Let $B \subseteq C \subseteq U$. Then B is a* basis *of C if B is independent and $C \subseteq \operatorname{acl}(B)$.*

Lemma 8.2. *If B is independent and $a \notin \operatorname{acl}(B)$, then $\{a\} \cup B$ is independent.*

Proof. Assume that $a \cup \{B\}$ is not independent. Let $b \in B$ such that $b \in \operatorname{acl}(aB \setminus b)$. Since B is independent, $b \notin \operatorname{acl}(B \setminus b)$. We assumed $a \notin \operatorname{acl}(B \setminus b)$. Then $a \in \operatorname{acl}(bB \setminus b) = \operatorname{acl} B$ by Theorem 8.1, a contradiction. □

Corollary 8.3. *If $B \subseteq C \subseteq U$, TFAE:*

(i) B is a basis of C.

(ii) B is a maximal independent subset.

Theorem 8.4. *Let $C \subseteq U$ small. Then*

(i) any independent $B \subseteq C$ extends to a basis of C.

(ii) if A, B are bases of C, then $|A| = |B|$.

Proof.

(i) Immediate from Zorn's lemma.

(ii) Assume that $|A| < |B|$.

Suppose first that $|B| \geq \omega$. Assume $|A| < |B|$. For $a \in A$, let $D_a \subseteq B$ be finite such that $a \in \operatorname{acl}(D_a)$. Let $D = \bigcup_{a \in A} D_a$. Then $A \subseteq \operatorname{acl}(D)$, and $|D| < |B|$. Then $A \subseteq \operatorname{acl}(D)$ and A is a basis, so $C \subseteq \operatorname{acl}(D)$ and $B \subseteq \operatorname{acl}(D)$ which contradicts the independence of B.

Now suppose $|B| < \omega$. Among those B, choose B such that $|B \setminus A|$ is minimal. Let $b \in B \setminus A$. Let B' be a maximal independent subset of $A \cup B \setminus b$ containing $B \setminus b$. Then B' is a basis of $\mathrm{acl}(AB \setminus b)$. Since $C \subseteq \mathrm{acl}(A)$, we have $C \subseteq \mathrm{acl}(AB \setminus b) \subseteq \mathrm{acl}(B')$. So $B' \subseteq C$, B' is independent and $\mathrm{acl}(C) \subseteq \mathrm{acl}(B)$, hence B' is a basis of C. But $|B' \setminus A| = |(B' \setminus b) \setminus A| < |B \setminus A|$, contradicting the minimality of $|B \setminus A|$.

□

Definition. *Let $C \subseteq U$, $\mathrm{acl}(C) = C$. Then the* dimension *of C, denoted* $\dim C$*, is the cardinality of a basis of C.*

Proposition 8.5. *Let $f : U \to U$ be partial elementary, $b \notin \mathrm{acl}(\mathrm{dom}\, f), c \notin \mathrm{acl}(\mathrm{ran}\, f)$. Then $f \cup \{(b, c)\}$ is elementary.*

Proof. Let $\overline{a}$ enumerate $\mathrm{dom}\, f$, let $\varphi(\overline{x}, \overline{a})$ be a formula in $L(\overline{a})$. Claim: $\models \varphi(b, \overline{a}) \leftrightarrow \varphi(c, f(\overline{a}))$.

Case 1: $|\varphi(U, \overline{a})| < \omega$. Then $|\varphi(U, f(\overline{a}))| < \omega$. Since $b \notin \mathrm{acl}(\overline{a})$ and $c \notin \mathrm{acl}(f(\overline{a}))$, we have

$$\models \neg\varphi(b, \overline{a}) \wedge \neg\varphi(c, f(\overline{a})).$$

Case 2: $|(U, \overline{a})| \geq \omega$, then $|\neg\varphi(U, \overline{a})| < \omega$. As in case 1, we conclude that

$$\models \varphi(b, \overline{a}) \wedge \varphi(c, f(\overline{a})).$$

□

Corollary 8.6. *Every bijection between independent subsets of U is elementary.*

Proof. Let $A, B \subseteq U$ with $|A| = |B|$. Let $f : A \to B$ be a bijection. Let $\overline{a}$ enumerate A, so $\overline{b} = f(\overline{a})$ enumerates B. Then $a_0, b_0 \notin \mathrm{acl}(\emptyset)$. Then by Proposition 8.5, $a_0 \mapsto b_0$ is elementary. The step $i + 1$ similar, since $a_{i+1} \notin \mathrm{acl}(a_0, \ldots, a_i)$ and $b_{i+1} \notin \mathrm{acl}(b_0, \ldots, b_i)$. The limit case is clear. □

Remark. If $M \preceq U$ is a model, then $\mathrm{acl}(M) = M$ by Proposition 7.7. So models of a strongly minimal theory have a dimension.

Theorem 8.7. *Let $M, N \preceq U$ be models such that $\dim(M) = \dim(N)$. Then $M \simeq N$.*

Proof. Let A, B be bases of M, N resp. Let $f : A \to B$ be a bijection. Then f is elementary, so there is $\alpha \in \mathrm{Aut}(U)$ such that $\alpha \supseteq f$. Then $\alpha[M] = \alpha[\mathrm{acl}(A)] = \mathrm{acl}(\alpha[A]) = \mathrm{acl}(B) = N$. □

Corollary 8.8. *Let $\lambda > |L|$ be a cardinal. Then T is λ-categorical.*

Proof. If $A \subseteq U$, then $|\operatorname{acl}(A)| \leq |L(A)|$ because there are at most $|L(A)|$ algebraic formulas and each such formula contributes only finitely many elements to $\operatorname{acl}(A)$. Therefore, if $|M| = \lambda > |L|$, then a basis of M must have cardinality λ. By the previous theorem, M is then unique up to isomorphism. □

Recall $T_{\mathrm{VS}K}$, the theory of vector spaces over an infinite field K. If $|K| = \omega$, then $T_{\mathrm{VS}K}$ is λ-categorical for every uncountable λ. However, $T_{\mathrm{VS}K}$ is not ω-categorical. Each $n \in \omega \setminus \{0\}$ determines a countable model of $T_{\mathrm{VS}K}$ of dimension n, unique up to isomorphism. There is also a model of dimension ω. These models have the same cardinality.

Now let K be a finite field and let $T^{\infty}_{\mathrm{VS}K}$ be $T_{\mathrm{VS}K}$ plus axioms that ensure that models are infinite. One can show that $T^{\infty}_{\mathrm{VS}K}$ is strongly minimal. $T^{\infty}_{\mathrm{VS}K}$ has a countable model. Every countable model has dimension ω, so $T^{\infty}_{\mathrm{VS}K}$ is ω-categorical. So $T^{\infty}_{\mathrm{VS}K}$ is *totally categorical.*

Theorem 8.9. *Let $N \models T$ (still assumed to be strongly minimal) and $|N| \geq |L|$. Then*

$$N \textit{ is saturated} \iff \dim N = |N|$$

Proof. Exercise. □