Group Cohomology

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Contents

1 Basic definitions and resolutions

1.1 Some definitions and examples

Let G be a group.

Definition. *The* integral group ring $\mathbb{Z}G$ *is the free abelian group on the elements of* G *together with multiplication defined by*

$$\Big(\sum_{h\in G} m_h h\Big)\Big(\sum_{k\in G} n_k k\Big) = \sum_g \Big(\sum_{hk=g} m_h n_k\Big) g.$$

A *module* over $\mathbb{Z}G$ will usually be a left module over $\mathbb{Z}G$. A $\mathbb{Z}G$-module M is trivial, if $gm = m$ for all $m \in M$, $g \in G$. The *trivial module* is $\mathbb{Z}$ (with G acting trivially).

The free $\mathbb{Z}G$-module on X will be denoted by $\mathbb{Z}G\{X\}$.

Definition. *A* $\mathbb{Z}G$*-map (or morphism) of* $\mathbb{Z}G$*-modules* M_1, M_2 *is a homomorphism* $\alpha : M_1 \to M_2$ *of abelian groups such that* $\alpha(rm_1) = r\alpha(m_1)$ *for all* $r \in G$.

Example. The *augmentation map* $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$, $\sum_{g\in G} n_g g \mapsto \sum_g n_g$ is a $\mathbb{Z}G$-map where we regard $\mathbb{Z}G$ as a left module and $\mathbb{Z}$ is the trivial module.

We write $\mathrm{Hom}_G(M, N)$ for the set of $\mathbb{Z}G$-maps where M, N are $\mathbb{Z}G$-modules. It is an abelian group under addition.

Example. Note that $\mathrm{Hom}_G(\mathbb{Z}G, M)$ can be given a $\mathbb{Z}G$-module structure by $(s\phi)(r) := \phi(rs)$ (essentially since $\mathbb{Z}G$ is a *bimodule* over itself). We have $\mathrm{Hom}_G(\mathbb{Z}G, M) \cong M$ as $\mathbb{Z}G$-modules where the isomorphism is given by $\phi \mapsto \phi(1)$. In particular, $\mathrm{Hom}_G(\mathbb{Z}G, \mathbb{Z}G) \cong \mathbb{Z}G$ where $\phi : \mathbb{Z}G \to \mathbb{Z}G$ corresponds to $\phi(1) \in \mathbb{Z}G$. So as $\phi(r) = r\phi(1)$, ϕ is multiplication on the right by $\phi(1)$.

Note that Hom_G is functorial:

Definition. *If* $f : M_1 \to M_2$ *is a* $\mathbb{Z}G$*-map and* N *a* $\mathbb{Z}G$*-module, then the* dual map *is*

$$\begin{aligned} f^* : \mathrm{Hom}_G(M_2, N) &\longrightarrow \mathrm{Hom}_G(M_1, N), \\ \phi &\longmapsto \phi \circ f \end{aligned}$$

Similarly if $f : N_1 \to N_2$ *is a* $\mathbb{Z}G$*- map and* M *a* $\mathbb{Z}G$*-module, then the* induced map *is*

$$\begin{aligned} f_* : \mathrm{Hom}_G(M, N_1) &\longrightarrow \mathrm{Hom}_G(M, N_2), \\ \phi &\longmapsto f \circ \phi \end{aligned}$$

Example. Let $G = \langle t \rangle$ be infinite cyclic, acting on the real line where t is translation by $+1$. We view this as follows: Let $V = \{v_i\}_{i \in \mathbb{Z}}$ be a set of vertices and let G act on V by $t(v_i) = v_{i+1}$. For each pair (v_i, v_{i+1}) consider an edge between them and let E be the set of these edges. Let e be the edge $v_0 \to v_1$. Then we can regard formal integral sums $\mathbb{Z}V$ and $\mathbb{Z}E$ as $\mathbb{Z}G$-modules. They are both free of rank one and $\mathbb{Z}V = \mathbb{Z}G\{v_0\}$, $\mathbb{Z}E = \mathbb{Z}G\{e\}$. There is a $\mathbb{Z}G$-map corresponding to the augmentation map $\mathbb{Z}V \to \mathbb{Z}$.

Definition. *A* chain complex *of $\mathbb{Z}G$-modules is a sequence*

$$M_s \xrightarrow{d_s} M_{s_1} \xrightarrow{d_{s-1}} \dots \xrightarrow{d_{t+2}} M_{t+1} \xrightarrow{d_{t+1}} M_t$$

with $s > t$ such that for every $t < n < s$, $d_n d_{n+1} = 0$, i.e. $\operatorname{im} d_{n+1} \subseteq \ker d_n$. *We write $M_\bullet = (M_n, d_n)_{t \le n \le s}$. $M_\bullet$ is* exact at M_n *if* $\operatorname{im} d_{n+1}) = \ker d_n$, *it is* exact *if it is exact at all M_n with $t < n < s$.*

The homology *of the chain complex $M_\bullet$ is $H_s(M_\bullet) = \ker d_s$, $H_n(M_\bullet) = \ker d_n / \operatorname{im} d_{n+1}$ for $t < n < s$ and $H_t(M_\bullet) = M_t / \operatorname{im} d_{t+1}$.*

Example. Let $G = \langle t \rangle$ be infinite cyclic. There is an short exact sequence

$$0 \to \mathbb{Z}G \xrightarrow{\cdot(t-1)} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

corresponding to

$$0 \to \mathbb{Z}E \to \mathbb{Z}V \to \mathbb{Z} \to 0$$

Definition. *A $\mathbb{Z}G$-module P is* projective *if for every surjective $\mathbb{Z}G$-map $\alpha : M_1 \to M_2$ and every $\mathbb{Z}G$-map $\beta : P \to M_2$, then there exists $\overline{\beta} : P \to M - 1$ such that $\alpha \circ \overline{\beta} = \beta$.*

$$\begin{array}{ccccc} & & P & & \\ & \exists\overline{\beta} \swarrow & \downarrow \beta & & \\ M_1 & \xrightarrow{\alpha} & M_2 & \longrightarrow & 0 \end{array}$$

Let

$$0 \to N \xrightarrow{f} M_1 \xrightarrow{\alpha} M_2 \to 0$$

be a short exact sequence and consider the sequence

$$0 \to \operatorname{Hom}_G(P, N) \xrightarrow{f_*} \operatorname{Hom}_G(P, M_1) \xrightarrow{\alpha_*} \operatorname{Hom}(P, M_2) \to 0 \qquad (*)$$

Then (by definition) P is projective if and only if $(*)$ is exact at $\operatorname{Hom}_G(P, M_2)$ for all short exact sequences $0 \to N \to M_1 \to M_2 \to 0$.

Note that we always have exactness elsewhere in $(*)$.

Lemma 1.1. *Free modules are projective.*

Proof. In the notation from the definition, define $\overline{\beta}$ on a basis X by setting $\overline{\beta}(x) = y$ for $x \in X$ where $y \in M_1$ is such that $\alpha(y) = \beta(x)$. ☐

Definition. *A* projective (resp. free) resolution *of the trivial module* $\mathbb{Z}$ *is an exact sequence*

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \mathbb{Z} \to 0$$

with all P_i *projective (resp. free).*

Note that the sequence can be of infinite length.

Examples.

1. Let $G = \langle t \rangle$ be again the infinite cyclic group. Then

$$0 \to \mathbb{Z}G \xrightarrow{\cdot(t-1)} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

   is a free resolution of $\mathbb{Z}$.

2. Let $G = \langle t \rangle$ be cyclic of order n. Then

$$\cdots \xrightarrow{\alpha} \mathbb{Z}G \xrightarrow{\beta} \mathbb{Z}G \xrightarrow{\alpha} \mathbb{Z}G \xrightarrow{\beta} \mathbb{Z}G \xrightarrow{\alpha} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

   is a free resolution where the maps α, β are given by

$$\begin{aligned} \alpha(x) &= x(t-1) \\ \beta(x) &= x(1 + t + \cdots + t^{n-1}) \end{aligned}$$

   Exercise: Show this is indeed exact.

3. If we take a partial free/projective resolution

$$P_s \xrightarrow{d_s} \cdots \to P_1 \to P_0 \to \mathbb{Z}$$

   (so this is exact, P_i free/projective), set $X_{s+1} = \ker d_s$ and $P_{s+1} = \mathbb{Z}G\{X_{s+1}\}$. Then define $d_{s+1} : P_{s+1} \to P_s$ by $\sum r_x x \mapsto r_x x \in P_s$. This gives us a longer partial resolution

$$P_{s+1} \xrightarrow{d_{s+1}} P_s \to \cdots \to 0$$

   This shows that free (so in particular projective) resolutions always exist. But note that P_{s+1} is free of perhaps infinite rank. We could do a littple better by taking X_{s+1} to be a $\mathbb{Z}G$-module generating set of $\ker d_s$.

From algebraic topology: Let X be a connected simplicial complex X with fundamental group G so that the universal cover $\widetilde{X}$ is contractible. Then X contains information about G and we will be trying to replicate the study of cohomology of the space X algebraically.

Definition. *G is* of type FP_n *if the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ has a projective resolution*

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \mathbb{Z} \to 0$$

such that $P_n, P_{n-1}, \ldots, P_0$ are finitely generated as $\mathbb{Z}G$-modules. G is of type FP_∞ *if there is a such a resolution with all P_n finitely generated. G is* of type FP *if there is a projective resolution of $\mathbb{Z}$ of finite length, i.e. $P_s = 0$ for all s large enough, and all P_n are finitely generated.*

Examples.

1. The infinite cyclic group is of type FP.
2. The cyclic group of order n is of type FP_∞. We will see later that it is not of type FP.

The FP_n analogous to G being a fundamental group of a simplicial complex X with $\widetilde{X}$ contractible and X has finite n-skeleton.

Definition. *Let $G^{(n)} = \{[g_1|g_2|\ldots|g_n] \mid g_1, \ldots, g_n \in G\}$ for $n \geq 1$ and $G^{(0)} = \{[]\}$. The $[g_1|g_2|\ldots|g_n]$ are called symbols and $[]$ is the empty symbol. Set $F_n = \mathbb{Z}G\{G^{(n)}\}$ and define the $\mathbb{Z}G$-map $d_n : F_n \to F_{n-1}$ on symbols by*

$$\begin{aligned} d_n([g_1 \mid \cdots \mid g_n]) =& g_1[g_2 \mid \cdots \mid g_n] - [g_1g_2 \mid g_3 \mid \cdots \mid g_n] + [g_1 \mid g_2g_3 \mid \cdots \mid g_n] \\ &+ \cdots + (-1)^{n-1}[g_1 \mid g_2 \mid \cdots \mid g_{n-1}g_n] + (-1)^n[g_1 \mid g_2 \mid \cdots \mid g_{n-1}]. \end{aligned}$$

Then

$$\cdots \to F_n \xrightarrow{d_n}\to F_{n-1} \to \cdots \to F_0 \xrightarrow{[]\mapsto 1} \mathbb{Z}$$

is the standard (or bar) resolution *of group G.*

It is easily verified that $d_{n-1} \circ d_d = 0$.

Lemma 1.2. *The standard resolution is in fact a resolution, i.e. exact.*

Proof. Note that F_n is a free abelian group on $G \times G^{(n)} = \{g_0[g_1 \mid \cdots \mid g_n] \mid g_0, \ldots, g_n \in G\}$. Let $s_n : F_n \to F_{n+1}$ be the map of abelian groups given by $s_n(g_0[g_1 \mid \cdots \mid g_n]) = [g_0 \mid g_1 \mid \cdots \mid g_n]$. Then it is straightforward to check that s_n satisfies

$$\mathrm{id}_{F_n} = d_{n+1}s_n + s_{n-1}d_n.$$

(I.e. s_n gives a chain homotopy equivalence $\mathrm{id}_F \sim 0$) Hence if $x \in \ker d_n$, then $x = \mathrm{id}\, x = d_{n+1}s_n(x) + s_{n-1}d_n(x) = d_{n+1}(s_n(x))$, so $x \in \mathrm{Im}\, d_{n+1}$. □

Corollary 1.3. *A finite group G is of type FP_∞.*

Proof. Indeed, the standard resolution is free with all terms of finite rank. □

1.2 Cohomology

Definition. *Take a projective resolution*

$$\cdots \to P_{n+1} \xrightarrow{d_{n+1}} P_n \to \dots \xrightarrow{d_1} P_0 \to \mathbb{Z}$$

of $\mathbb{Z}$ *by* $\mathbb{Z}G$*-modules. Let* M *be a* $\mathbb{Z}G$*-module. Apply* $\mathrm{Hom}_G(-, M)$ *to get a sequence*

$$\cdots \leftarrow \mathrm{Hom}_G(P_{n+1}, M) \xleftarrow{d^{n+1}} \mathrm{Hom}_G(P_n, M) \leftarrow \dots \xleftarrow{d^1} \mathrm{Hom}_G(P_0, M)$$

where $d^n = d_n^*$*. Then the* n-th cohomology group $H^n(G, M)$ *with coefficients in* M *is then the abelian group*

$$H^n(G, M) = \frac{\ker d^{n+1}}{\operatorname{im} d^n} \qquad n \geq 1$$
$$H^0(G, M) = \ker d^1$$

Remarks.

1. We have dropped the $\mathbb{Z}$ on the RHS.
2. Those are the homology groups of the chain complex $C_n = \mathrm{Hom}_G(F_{-n}, M)$ defined for $-\infty < n \leq n$.
3. Those are independent of the choice of projective resolution, see Theorem 1.5

Example. Let $G = \langle t \rangle$ be infinite cyclic. Then we had the resolution

$$0 \to \mathbb{Z}G \xrightarrow{\cdot(t-1)} \mathbb{Z}G \to \mathbb{Z} \to 0.$$

If $\phi \in \mathrm{Hom}_G(\mathbb{Z}G, M)$, $x \in \mathbb{Z}G$, then

$$d^1(\phi)(x) = \phi(d_1(x)) = \phi(x(t-1)).$$

Recall that we have an isomorphism $i : \mathrm{Hom}_G(\mathbb{Z}G, M) \to M$, $\theta \mapsto \theta(1)$. In particular, $d^1(\phi) \mapsto d^1(\phi)(1) = \phi(t-1) = (t-1)\phi(1) = (t-1)i(\phi)$. So the dual chain complex $\mathrm{Hom}_G(P_\bullet, M)$ is

$$0 \leftarrow M \xleftarrow{(t-1)\cdot} M$$

Hence,

$$H^0(G, M) = \ker((t-1)\cdot) = \{m \in M \mid tm = m\} = M^G$$
$$H^1(G, M) = \frac{M}{\{(t-1)m \mid m \in M\}} =: M_G$$
$$H^n(G, M) = 0 \qquad \text{if } n \geq 2$$

Here M^G is the group of *invariant*, the largest submodule fixed by G, and M_G is the group of *co-invariants*, the largest quotient fixed by G.

Remarks.

1. $H^0(G,M) = M^G$ is true in general. $H^1(G,M) = M_G$ is special to the the infinite cyclic group and does not hold in general.

2. If G is of type FP, then $H^n(G,M) = 0$ for all $n \geq s$ for some s.

Definition. *G is of* cohomological dimension M *(over $\mathbb{Z}$) if there is some $\mathbb{Z}G$-module M such that $H^m(G,M) \neq 0$ and for all modules M we have $H^n(G,M) = 0$ for $n > m$.*

E.g. the infinite cyclic group is of cohomological dimension 1. More generally, if G is free and non-trivial, then it is of cohomological dimension 1. The converse is also true:

- (Stallings 1968) If G is finitely generated, then G is free if it has cohomological dimension 1.
- (Swan 1969) Removed the f.g. condition.

Definition. *Let (A_n, α_n) and (B_n, β_n) be chain complexes of $\mathbb{Z}G$-modules. Then a* chain map $A_\bullet \to B_\bullet$ *is a family (f_n) where each $f_n : A_n \to B_n$ is a $\mathbb{Z}G$-map such that*

$$\begin{array}{ccc} A_n & \xrightarrow{\alpha_n} & A_{n-1} \\ \big\downarrow{\scriptstyle f_n} & & \big\downarrow{\scriptstyle f_{n-1}} \\ B_n & \xrightarrow{\beta_n} & B_{n-1} \end{array}$$

commutes for all n.

Lemma 1.4. *Given a chain map (f_n) as above, it induces a well-defined map on the homology groups*

$$f_* : H_n(A_\bullet) \to H_n(B_\bullet).$$

Proof. Clear. □

Theorem 1.5. *The definition of $H^n(G,M)$ is independent of the choice of resolution.*

Proof. Let (P_n, d_n) and (P'_n, d'_n) be two projective resolutions of $\mathbb{Z}$ by $\mathbb{Z}G$-modules. We will produce various $\mathbb{Z}G$-maps:

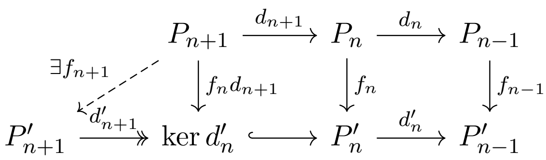
- Chain map $(f_n) : P_\bullet \to P'_\bullet$,
- Chain map $(g_n) : P'_\bullet \to P_\bullet$,
- $s_n : P_n \to P_{n+1}$ such that $d_{n+1}s_n + s_{n-1}d_n = g_nf_n - \mathrm{id}$ (i.e. $(g_nf_n) \sim \mathrm{id}$)
- $s'_n : P'_n \to P'_{n+1}$ such that $d'_{n+1}s'_n + s'_{n-1}d'_n = f_ng_n - \mathrm{id}$.

Assume we have constructed these. Then (f_n^*) gives a chain map $\mathrm{Hom}_G(P'_\bullet, M) \to \mathrm{Hom}_G(P_\bullet, M)$ and similarly (g_n^*) gives a chain map $\mathrm{Hom}_G(P_\bullet, M) \to \mathrm{Hom}_G(P'_\bullet, M)$. They induce maps between the (co)homology groups. Now observe that if $\phi \in \ker d^{n+1}$. Then

$$\begin{aligned}(f_n^* g_n^*)(\phi)(x) &= \phi(g_n f_n(x)) \\ &= \phi(x) + \phi(d_{n+1}s_n(x)) + \phi(s_{n-1}d_n(x)) \\ &= \phi(x) + s_n^* d^{n+1}\phi(x) + d^n s_{n-1}^*(\phi)(x) \\ &= \phi(x) + d^n(s_{n-1}^*(\phi))(x).\end{aligned}$$

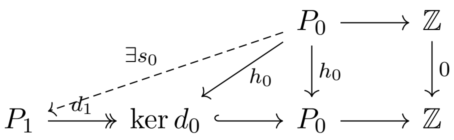
Hence $f_n^* g_n^*(\phi) = \phi + d^n(s_{n-1}^*(\phi))$, so $f_n^* g_n^*$ induces the identity on the homology group. Similarly for $g_n^* f_n^*$ and so the g_n, f_n induces isomorphisms on the homologies.

So all we have to do is to construct these maps. Consider the end of the resolutions and let $f_{-1} : \mathbb{Z} \to \mathbb{Z}$ be the identity and $f_{-2} : 0 \to 0$. Now suppose we have defined f_{n-1} and f_n. Then $f_n d_{n+1} : P_{n+1} \to P'_n$ and $d'_n(f_n d_{n+1}) = f_{n-1} d_n d_{n+1} = 0$. So the image of $f_n d_{n+1}$ lies in $\ker d'_n$. Consider the diagram:

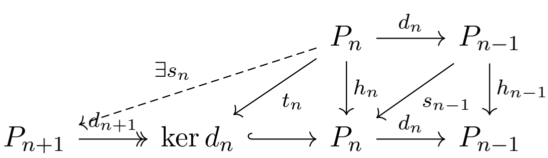


Since P_{n+1} is projective, the arrow f_{n+1} as indicated in the diagram exists. This shows the existence of the chain map (f_n) and similarly one gets the g_n.

To define (s_n), first set $h_n = g_n f_n - \mathrm{id} : P_n \to P_n$. Then (h_n) is a chain map with $h_{-1} = 0$. Set $s_{-1} : \mathbb{Z} \to P_0$ to be the zero map. Note that $d_0 h_0 = h_{-1} d_0 = 0$ and so $\mathrm{im}\, h_0 \subseteq \ker d_0$. As before $d_1 : P \to \ker d_0$ is surjective. Then consider:



Now for induction suppose s_{n-1} and s_{n-2} have been defined. Consider $t_n = h_n - s_{n-1} d_n : P_n \to P_n$. We have $d_n t_n = d_n h_n - d_n s_{n-1} d_n = h_{n-1} d_n - (h_{n-1} - s_{n-2} d_{n-1}) d_n = 0$. So $\mathrm{im}\, t_n \subseteq \ker d_n$. Now look again at the diagram:



Then we get (s_n) and similarly we get (s'_n). □

Remark. If we use free/projective resolutions of any $\mathbb{Z}G$-module N (instead of $\mathbb{Z}$), then our definitions give us

$$\operatorname{Ext}^n_{\mathbb{Z}G}(N, M).$$

Thus $\operatorname{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M) = H^n(G, M)$.

Now consider the definition of $H^n(G, M)$ as applied to the standard resolution

$$\cdots \to \mathbb{Z}G\{G^{(1)}\} \to \mathbb{Z}G\{G^{(0)}\} \to \mathbb{Z}.$$

We have

$$\operatorname{Hom}_G(\mathbb{Z}G\{G^{(n)}\}, M) \cong \{\text{functions } \phi : G^n \to M\} =: C^n(G, M)$$

and $C^0(G, M) \cong M$.

Definition. *The group of* n-cochains *of G with coefficients in M is $C^n(G, M)$ under addition. The n-th* coboundary map *is $d^n : C^{n-1}(G, M) \to C^n(G, M)$ dual to d_n in the standard resolution. Then*

$$\begin{aligned}(d^n\phi)(g_1, \dots, g_n) =& g_1\phi(g_2, \dots, g_n) - \phi(g_1g_2, g_3, \dots, g_n) + \phi(g_1, g_2g_3, g_4, \dots, g_n) \\ & - \cdots + (-1)^{n-1}\phi(g_1, g_2, \dots, g_{n-2}, g_{n-1}g_n) + (-1)^n\phi(g_1, \dots, g_{n-1}).\end{aligned}$$

The group of n-cocycles *is $Z^n(G, M) = \ker d^{n+1} \subseteq C^n(G, M)$ and the group of* n-coboundaries *is $B^n(G, M) = \operatorname{Im} d^n \subseteq C^n(G, M)$. Then $H^n(G, M) = \frac{Z^n(G,M)}{B^n(G,M)}$.*

Relationship between our standard resolution and the usual one in algebraic topology: Let G^{n+1} be the set of $n+1$-tuples and consider the free abelian group $\mathbb{Z}G^{n+1}$ on these. G acts on G^{n+1} via $g(g_0, g_1, \dots, g_n) = (gg_0, \dots, gg_n)$. Thus $\mathbb{Z}G^{n+1}$ becomes a free $\mathbb{Z}G$-modules with basis given by the $n + 1$-tuples with $g_0 = 1$. The symbol $[g_1 \mid \cdots \mid g_n]$ corresponds to $(1, g_1, g_1g_2, \dots, g_1g_2 \cdots g_n)$. Note that in the usual resolution in algebraic topology we have the boundary map where there is an alternating sum of n-tuples where we miss out one of the entries in turn. If we take $(1, g_1, g_1g_2, \dots)$ and miss out the first entry, we get $(g_1, g_1g_2, \dots) = g_1(1, g_2, g_2g_3, \dots)$ which corresponds to $g_1[g_2 \mid \cdots \mid g_n]$. If we miss out the second entry, we get $(1, g_1g_2, g_1g_2g_3, \dots)$, this corresponds to $[g_1g_2 \mid g_3 \mid \cdots \mid g_n]$.

2 Low degree cohomology and group extensions

Let G be a group and M a $\mathbb{Z}G$-module.

Corollary 2.1. $H^0(G, M) = M^G$.

Proof. Immediate from the definitions. □

2.1 H^1 - splittings of extensions

Definition. *A* derivation *(or crossed homomorphism) of G with coefficients in M is a function $\phi : G \to M$ such that*

$$\phi(gh) = g\phi(h) + \phi(g)$$

for all $g, h \in G$. An inner derivation *is one of the form $\phi(g) = gm - m$ for some fixed $m \in M$.*

Notice that $Z^1(G, M)$ is the abelian group of derivations (under addition) and $B^1(G, M)$ is the subgroup of inner derivations. Hence

$$H^1(G, M) = \frac{\{\text{derivations } G \to M\}}{\text{inner derivations } G \to M}.$$

In particular, if M is a trivial $\mathbb{Z}G$-module, then $H^1(G, M) = \operatorname{Hom}(G, M)$ (group homomorphisms $G \to M$).

We recall the definition of the semidirect product:

Definition. *Let G be a group, M be a left $\mathbb{Z}G$-module. We construct the* semidirect product $M \rtimes G$ *as follows: The underlying set is $M \times G$ and the multiplication is given by*

$$(m_1, g_1) * (m_2, g_2) = (m_1 + g_1 m_2, g_1 g_2)$$

In this case $M \cong \{(m, 1) \mid m \in M\}$ is an abelian normal subgroup and $G \cong \{(0, g) \mid g \in G\}$ is a subgroup. Conjugation of G on $M \subseteq M \rtimes G$ corresponds to our $\mathbb{Z}G$-module action. This is an example of an extension of G by M.

Note that there is a group homomorphism $s : G \to M \rtimes G$, $g \mapsto (0, g)$ such that the composite $G \xrightarrow{s} M \rtimes G \xrightarrow{\pi} G$ is the identity map. This is called a *splitting* of the extension, and the semidirect product is a *split* extension of G by M.

Let $E = M \rtimes G$. Now consider another splitting $s_1 : G \to E$ such that $G \xrightarrow{s_1} E \xrightarrow{\pi} G$ is the identity. Define $\psi_{s_1} : G \to M$ by $s_1(g) = (\psi_{s_1}(g), g)$. Then $\psi_{s_1} \in Z^1(G, M)$ (easy check). Now suppose we have two splittings s_1 and s_2. Then $\psi_{s_1} - \psi_{s_2} \in B^1(G, M)$ if and only if there exists $m \in M$ such that $(m, 1)s_1(g)(m, 1)^{-1} = s_2(g)$ for all $g \in G$. We obtain a bijection:

$$H^1(G, M) \longleftrightarrow \{M\text{-conjugacy classes of splittings}\}$$

See Example Sheet 1, Exercise 3 for details.

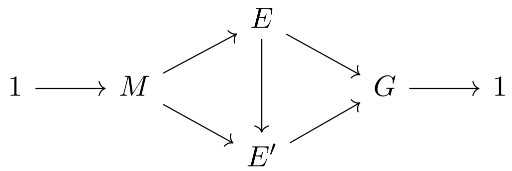
2.2 H^2 - group extensions

Now let us consider a group theoretic interpretation of $H^2(G, M)$ and for that we consider other extensions of G by an abelian group M, i.e. short exact sequences

$$1 \to M \to E \to G \to 1$$

where the maps are group homomorphisms. Thus M embeds in E as a normal subgroup and $E/M \cong G$. Then E acts on M by conjugation, with M acting trivially on itself since it is abelian. So we may regard M as a $\mathbb{Z}G$-module since $G \cong E/M$.

Definition. *Two extensions E, E' are* equivalent *if there is a commuting diagram of group homomorphisms:*



E is a central extension *if M is a trivial $\mathbb{Z}G$-module (via conjugation within E).*

Exercise: Equivalent extensions E and E' are isomorphic, but the converse is not necessarily true, see example sheet.

Proposition 2.2. *Let E be an extension of G by M. If there is a splitting $s : G \to E$ which is a group homomorphism, then E is equivalent to the semidirect product.*

Proof. Exercise. □

For other extensions there is a set-theoretic section $s : G \to E$, but it fails to be a homomorphism. Wlog, assume $s(1) = 1$.

Define $\phi(g_1, g_2) = s(g_1)s(g_2)s(g_1g_2)^{-1}$. This gives an indication of the failure of s to be a group homomorphism.

Then, writing $\pi : E \to G$ for the quotient map, we have $\pi(\phi(g_1, g_2)) = 1$ and so $\phi(g_1, g_2) \in M$ and so $\phi : G^2 \to M$ is a 2-cochain. In fact ϕ is a 2-cocycle: Consider $s(g_1)s(g_2)s(g_3)$ in two different ways. It is

$$\begin{aligned} &= \phi(g_1, g_2)s(g_1g_2)s(g_3) \\ &= \phi(g_1, g_2)\phi(g_1g_2, g_3)s(g_1g_2g_3) \end{aligned} \qquad (\dagger)$$

Also

$$\begin{aligned} &= s(g_1)\phi(g_2, g_3)s(g_2g_3) \\ &= s(g_1)\phi(g_2, g_3)s(g_1)^{-1}s(g_1)s(g_2g_3) \\ &= s(g_1)\phi(g_2, g_3)s(g_1)^{-1}\phi(g_1, g_2g_3)s(g_1g_2g_3) \end{aligned} \qquad (\dagger\dagger)$$

Equating (†) and (††) and cancelling $s(g_1g_2g_3)$ and converting to additive notation, we get

$$-d^3\phi(g_1, g_2, g_3) = \phi(g_1, g_2) + \phi(g_1g_2, g_3) - g_1\phi(g_2, g_3) - \phi(g_1, g_2g_3) = 0.$$

So $\phi \in Z^3(G, M)$. Note that ϕ is a *normalised* cocycle, meaning that $\phi(1, g) = \phi(g, 1) = 0$ for all $g \in G$.

Now take a different choice of section $s' : G \to E$ with $s'(1) = 1$. Then $\pi(s(g)s'(g)^{-1}) = 1$ for all g and so $s'(g)s(g)^{-1} =: \psi(g) \in M$. So we get a map $\psi : G \to M$. Then

$$\begin{aligned} s'(g_1)s'(g_2) &= \psi(g_1)s(g_1)\psi(g_2)s(g_2) \\ &= \psi(g_1)s(g_1)\psi(g_2)s(g_1)^{-1}s(g_1)s(g_2) \\ &= \psi(g_1)s(g_1)\psi(g_2)s(g_1)^{-1}\phi(g_1, g_2)s(g_1g_2) \\ &= \psi(g_1)s(g_1)\psi(g_2)s(g_1)^{-1}\phi(g_1, g_2)\psi(g_1g_2)^{-1}s'(g_1g_2). \end{aligned}$$

Hence (in additive notation)

$$\begin{aligned} \phi'(g_1, g_2) &= \psi(g_1) + g_1\psi(g_2) + \phi(g_1, g_2) - \psi(g_1g_2) \\ &= \phi(g_1, g_2) + (d^2\psi)(g_1, g_2). \end{aligned}$$

Thus ϕ and ϕ' differ by a coboundary. So we have shown how to construct a map

$$\text{extensions} \longrightarrow H^2(G, M).$$

We are aiming for:

Theorem 2.3. *Let G be a group, M a $\mathbb{Z}G$-module. Then there is a bijection:*

$$\left\{ \begin{array}{c} \textit{equivalence classes of} \\ \textit{extensions of } G \textit{ by } M \end{array} \right\} \longleftrightarrow H^2(G, M).$$

One has to show:

1. Equivalent extensions yield same cohomology class.
2. Construct the inverse map, i.e. given a cohomology class construct the associated extension.
3. Show these maps are inverse to each other.

To produce the inverse map, we need a lemma first.

Lemma 2.4. *Let $\phi \in Z^2(G, M)$. Then there is a cochain $\psi \in C^1(G, M)$ such that $\phi + d^2\psi$ is normalised. Hence every cohomology class can be represented by a normalised cocycle.*

Proof. Let $\psi(g) = -\phi(1, g)$. Then

$$\begin{aligned}(\phi + d^2\psi)(1, g) &= \phi(1, g) - (\phi(1, g) - \phi(1, g) + \phi(1, 1)) \\ &= \phi(1, g) - \phi(1, 1) \qquad (*) \\ (\phi + d^2\psi)(g, 1) &= \phi(g, 1) - (g\phi(1, 1) - \phi(1, g) + \phi(1, g)) \\ &= \phi(g, 1) - g\phi(1, 1) \qquad (**)\end{aligned}$$

We know $d^3\phi(1, 1, g) = 0 = d^3\phi(g, 1, 1)$ since ϕ is a cocycle. Writing this out shows that both $(*)$ and $(**)$ are 0. □

Now take a normalised cocycle $\phi \in Z^2(G, M)$ representing our given cohomology class. Define a group E_ϕ on the set $M \times G$ by

$$(m_1, g_2) *_\phi (m_2, g_2) = (m_1 + g_1 m_2 + \phi(g_1, g_2), g_1 g_2).$$

Now check that this indeed defines a group. For this we need that ϕ is normalised. Then $M \cong \{(m, 1) \mid m \in M\}$ and the quotient is $\cong G$.

Finally notice that if ϕ' is a different normalised cocycle representing the same cohomology class, then $\phi - \phi' = d^2\psi$ for some $\psi \in C^1(G, M)$. Then we define

$$\begin{aligned} E_\phi &\longrightarrow E_{\phi'}, \\ (m, g) &\longmapsto (m + \psi(g), g) \end{aligned}$$

This is a group homomorphism and gives us the equivalence the extensions.

2.2.1 Example: Central extensions of $\mathbb{Z}^2$ by $\mathbb{Z}$

Let us find all the central extensions of $\mathbb{Z}^2$ by $\mathbb{Z}$. We certainly know of two such:

- The direct product
$$0 \to \mathbb{Z} \to \mathbb{Z}^3 \to \mathbb{Z}^2 \to 0.$$

- The (integral) Heisenberg group

$$0 \to \mathbb{Z} \xrightarrow{b \mapsto X_{0,b,0}} H \xrightarrow{X_{a,b,c} \mapsto (a,c)} \mathbb{Z}^2 \to 0$$

where $H = \left\{ X_{a,b,c} := \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| a, b, c \in \mathbb{Z} \right\}$.

Write $T = \mathbb{Z}^2$, generated by a, b. What are the equivalence classes of extensions? We have a free resolution

$$0 \to \mathbb{Z}T \xrightarrow{\beta} (\mathbb{Z}T)^2 \xrightarrow{\alpha} \mathbb{Z}T \xrightarrow{\varepsilon} \mathbb{Z}$$

of the trivial $\mathbb{Z}T$-module $\mathbb{Z}$ where

$$\begin{aligned} \beta(z) &= (z(1-b), z(a-1)) \\ \alpha(x,y) &= x(a-1) + y(b-1) \end{aligned}$$

and ε is the augmentation map. Check that this indeed is an exact sequence. Then apply $\mathrm{Hom}_T(-, \mathbb{Z})$ to get the chain complex

$$0 \leftarrow \mathrm{Hom}_T(\mathbb{Z}T, \mathbb{Z}) \xleftarrow{\beta^*} \mathrm{Hom}_T((\mathbb{Z}T)^2, \mathbb{Z}) \xleftarrow{\alpha^*} \mathrm{Hom}_T(\mathbb{Z}T, \mathbb{Z}).$$

We show that both α^* and β^* are the zero maps and so

$$H^2(T, \mathbb{Z}) = \mathrm{Hom}_T(\mathbb{Z}T, \mathbb{Z}) \cong \mathbb{Z}$$

with generator represented by the augmentation map ε.

To show $\beta^* = 0$ take a $\mathbb{Z}T$-map $f : (\mathbb{Z}T)^2 \to \mathbb{Z}$ and $z \in \mathbb{Z}T$. Then

$$\begin{aligned} (\beta^* f)(z) &= f(\beta(z)) = f(z(1-b), z(a-1)) \\ &= f((z - bz, 0) + (0, za - z)) \\ &= (1-b)f(z, 0) + (a-1)f(0, z) \\ &= 0 \end{aligned}$$

since T acts trivially on $\mathbb{Z}$. Similarly for α^*.

Next we must interpret $h^2(T, \mathbb{Z})$ in terms of cocycles, in particular what cocycle corresponds to the generator. So we construct a chain map between our resolution above and the standard resolution. Consider:

$$\begin{array}{ccccccccc} \mathbb{Z}T\{T^{(2)}\} & \xrightarrow{d_2} & \mathbb{Z}T\{T^{(1)}\} & \xrightarrow{d_1} & \mathbb{Z}T\{T^{(0)}\} & \xrightarrow{\varepsilon} & \mathbb{Z} & \longrightarrow & 0 \\ \downarrow f_2 & & \downarrow f_1 & & \downarrow \mathrm{id} & & \| & & \\ \mathbb{Z}T & \xrightarrow{\beta} & (\mathbb{Z}T)^2 & \xrightarrow{\alpha} & \mathbb{Z}T & \xrightarrow{\varepsilon} & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

In degree -1 and 0 we have take the identity maps. Next we construct $f_1 : \mathbb{Z}T\{T^{(1)}\} \to \mathbb{Z}T^2$ such that $\alpha f_1 = d_1$. We just need to give the image of symbols $[a^r b^s]$ where $r, s \in \mathbb{Z}$. We let $f_1([a^r b^s]) = (x_{r,s}, y_{r,s}) \in \mathbb{Z}T^2$ so that

$$\alpha(x_{r,s}, y_{r,s}) = d_1([a^r b^s]) = a^r b^s - 1 = (a^r - 1)b^s + (b^s - 1).$$

Define

$$S(a,r) = \begin{cases} 1 + a + \cdots + a^{r-1} & r > 0 \\ -a^{-1} - \cdots - a^r & r \leq 0 \end{cases}$$

so that $S(a,r)(a-1) = a^r - 1$ in both cases. Then $\alpha(S(a,r)b^s, S(b,s)) = d_1([a^r b^s])$ as required and we let $x_{r,s} = S(a,r)b^s, y_{r,s} = S(b,s)$. Now define f_2 for each $[a^r b^s \mid a^t b^u]$. We find $z_{r,s,t,u} \in \mathbb{Z}T$ such that $f_1 d_2([a^r b^s \mid a^t b^u]) = \beta(z_{r,s,t,u})$. Note that $z_{r,s,t,u} = S(a,r)b^s S(b,u)$ works. Then define $f_2([a^r b^s \mid a^t b^u]) = S(a,r)b^s S(b,u)$.

Now we find a cochain $\phi : T^2 \to \mathbb{Z}$ representing the cohomology class $p \in \mathbb{Z} = \mathrm{Hom}_T(\mathbb{Z}T, \mathbb{Z}) = H^2(T, \mathbb{Z})$. Let ϕ be the composition $T^2 \xrightarrow{f_2} \mathbb{Z}T \xrightarrow{p\varepsilon} \mathbb{Z}$. Since $\varepsilon(S(a,r)) = r$, we find

$$\phi(a^r b^s, a^t b^u) = p\varepsilon(z_{r,s,t,u}) = pru.$$

The group structure on $\mathbb{Z} \times T$ corresponding to ϕ is:

$$(m, a^r b^s) * (n, a^t b^u) = (m + n + pru, a^{r+t} b^{s+u}).$$

Note that for $p \neq 0$ these correspond to

$$\left\{ \begin{pmatrix} 1 & pr & m \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \middle| r, s, m \in \mathbb{Z} \right\}.$$

2.3 Group presentations

Consider group extensions by using group presentations. Express G in terms of generators and relations. Let F be the free group on a set X of generators of G. So we get a surjective group homomorphism $F \to G$. Let R be its kernel.

$$1 \to R \to F \to G \to 1$$

Often it is useful just to take a generating set of R. If G is generated by a finite set X such that R is also finitely generated, then G is *of finite presentation.*

Let $R_{\mathrm{ab}} = R/R'$ be the abelianisation of R. F acts on R by conjugation and one has an inherited action of R on R_{ab}. Note that R acts trivially on R_{ab} under this and so R_{ab} may be regarded as a $\mathbb{Z}(F/R)$-module, i.e. a $\mathbb{Z}G$-module. Then

$$1 \to R_{\mathrm{ab}} \to F/R' \to G \to 1$$

is an extension of G by R_{ab}. R_{ab} is called the *relation module.*

For a central extension rather than using $R/[R,R]$ one can use $R/[R,F]$. Then

$$1 \to R/[R,F] \to F/[R,F] \to G \to 1$$

is a central extension. Is there in some sense a largest or universal central extension? No, we can always take a direct product with an arbitrary abelian group, but we do have:

Theorem 2.5 (MacLane)**.** *Given a presentation $G = \langle X \mid R\rangle$, let F be the free group on X and let M be a $\mathbb{Z}G$-module. Then there is an exact sequence*

$$H^1(F,M) \to \operatorname{Hom}_G(R_{\text{ab}}, M) \to H^2(G,M) \to 0.$$

Here we regard M as an $\mathbb{Z}F$-module via $F \to G$.

Thus any extension of M corresponding to a cohomology class arises from taking a $\mathbb{Z}G$-map $R_{\text{ab}} \to M$.

Corollary 2.6. *In the above, if M is a trivial module, we get*

$$\operatorname{Hom}(F,M) \to \operatorname{Hom}_G(R/[R,F], M) \to H^2(G,M) \to 0.$$

Proof. Recall that for trivial modules $H^1(F,M) = \operatorname{Hom}(F,M) = \operatorname{Hom}(F_{\text{ab}}, M)$ and also $\operatorname{Hom}_G(R_{\text{ab}}, M) = \operatorname{Hom}_G(R/[R,F], M)$. □

There is also a connection with group homology. Given a projective resolution of $\mathbb{Z}$, we can apply $\mathbb{Z} \otimes_{\mathbb{Z}G} -$ to it and consider the homology groups of the resulting chain complex. The homology groups are $H_n(G,\mathbb{Z})$.

Definition. *The* Schur multiplier *(or multiplicator) is the second homology group*

$$M(G) := H_2(G,\mathbb{Z}).$$

The Schur multiplier is important when considering central extensions.

Theorem 2.7 (Universal Coefficients Theorem)**.** *Let G be a group and M a trivial $\mathbb{Z}G$-module. Then there is a short exact sequence of abelian groups:*

$$0 \to \operatorname{Ext}^1(G_{\text{ab}}, M) \to H^2(G,M) \to \operatorname{Hom}(M(G), M) \to 0.$$

Corollary 2.8. *Suppose G_{ab}, i.e. G is* perfect*, then $H^2(G,M) \cong \operatorname{Hom}(M(G), M)$.*

Remark. Some authors call $H^2(G, \mathbb{C}^\times)$ the Schur multiplier, rather than $M(G)$.

There is a formula for $M(G)$:

Theorem 2.9 (Hopf)**.** *Given a presentation $G = \langle X \mid R \rangle$, then*

$$M(G) = \frac{F' \cap R}{[R, F]}.$$

Remarks.

1. We are not taking all of $R/[R, F]$.
2. This shows that $\frac{F' \cap R}{[R,F]}$ is independent of the choice of presentation.

Remark. From geometric group theory, we know that all subgroups of free groups are free. Thus the module R of relations is a free group, say with basis Y. Hence R_{ab} is a free abelian group on Y.

Proposition 2.10. *Given a presentation $G = \langle X \mid R \rangle$, there is an exact sequence*

$$\frac{\overline{I}_R}{\overline{I}_R^2} \xrightarrow{d_2} \frac{I_F}{\overline{I}_R I_F} \xrightarrow{d_1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 1$$

where $I_F = \ker(\mathbb{Z}F \xrightarrow{\varepsilon} \mathbb{Z})$ and $\overline{I}_R = \ker(\mathbb{Z}F \to \mathbb{Z}G)$. Moreover, $\frac{I_F}{\overline{I}_R I_F}$ and $\frac{\overline{I}_R}{\overline{I}_R^2}$ are free $\mathbb{Z}G$-modules with bases $\{x - 1 \mid x \in X\}$ resp. $\{y - 1 \mid y \in Y\}$. Also $\operatorname{im} d_2 \cong R_{\mathrm{ab}}$.

Lemma 2.11. *Let G be a group and M a $\mathbb{Z}G$-module. Then:*

(a) I_G under addition is the free abelian group on $\{g - 1 \mid g \in G \setminus \{1\}\}$.

(b) $I_G/I_G^2 \cong G_{\mathrm{ab}}$.

(c) $\mathrm{Der}(G, M) \cong \mathrm{Hom}_G(I_G, M)$ where $\mathrm{Der}(G, M)$ is the abelian group of derivations $G \to M$.

Proof.

(a) $\mathbb{Z}G$ is free abelian on $\{g \mid g \in G\}$ and $I_G = \ker \varepsilon = \{\sum n_g g \mid \sum n_g = 0\}$. So if $\sum n_g g \in I_G$, then $\sum n_g g = \sum n_g(g - 1)$ and clearly any element of the form $\sum n_g(g-1)$ lies in $\ker \varepsilon = I_G$. Also $\{g-1 \mid g \in G \setminus \{1\}\}$ is linearly independent as the elements $g \in G$ are. Hence $I_G = \{\sum n_g(g-1) \mid n_g \in \mathbb{Z}\}$ is free on $\{g-1 \mid g \in G \setminus \{1\}\}$.

(b) Since I_G is free abelian on $\{g-1 \mid g \in G \setminus \{1\}\}$, we can define a group homomorphism $\theta : I_G \to G_{\mathrm{ab}}$ by defining the image of $g - 1$ to be gG' for $g \in G \setminus \{1\}$. Since $(g_1 - 1)(g_2 - 1) = (g_1 g_2 - 1) - (g_1 - 1) - (g_2 - 1)$, we have $I_G^2 \subseteq \ker \theta$. So θ induces a map $\overline{\theta} : I_G/I_G^2 \to G_{\mathrm{ab}}$. Conversely, $\phi : G \to I_G/I_G^2$, $g \mapsto (g - 1) + I_G^2$ is a group homomorphism and this induces a map $\overline{\phi} : G_{\mathrm{ab}} \to I_G/I_G^2$. The two maps $\overline{\theta}$ and $\overline{\phi}$ are clearly inverse to each other.

(c) Define maps:

$$\mathrm{Der}(G, M) \longleftrightarrow \mathrm{Hom}_G(I_G, M)$$

$$\phi \longmapsto (\theta : g-1 \mapsto \phi(g))$$
$$(\phi : g \mapsto \theta(g-1)) \longleftarrow \theta$$

They are inverse to each other.

□

Lemma 2.12.

(a) *Let F be a free group on X. Then I_F is a free $\mathbb{Z}F$-module on $\widetilde{X} = \{x-1 \mid x \in X\}$.*

(b) *Let R be a normal subgroup of the free group F, so it is free on Y, say. Then $\overline{I}_R$ is a free $\mathbb{Z}F$-module on basis $\widetilde{Y} = \{y-1 \mid y \in Y\}$.*

Proof.

(a) Let $\alpha : \widetilde{X} \to M$ be a map to some $\mathbb{Z}F$-module M. To establish freeness it suffices to show that α extends uniquely to a $\mathbb{Z}F$-map $I_F \to M$. First let $\alpha' : F \to M \rtimes F$ be defined by $x \mapsto (\alpha(x-1), x)$ on $\widetilde{X}$. Thus for each $f \in F$, $f \mapsto (a, f)$ for some $a \in M$. There is a function $\overline{\alpha} : F \to M$, $f \mapsto a$ so that $\alpha'(f) = (\overline{\alpha}(f), f)$. Then

$$\begin{aligned}\alpha'(f_1 f_2) &= \alpha'(f_1) * \alpha'(f_2)\\ &= (\overline{\alpha}(f_1), f_1) * (\overline{\alpha}(f_2), f_2)\\ &= (\overline{\alpha}(f_1) + f_1\overline{\alpha}(f_2), f_1 f_2).\end{aligned}$$

Hence $\overline{\alpha}$ is a derivation $F \to M$. We take the corresponding $\mathbb{Z}F$-map $I_F \to M$ as in Lemma 2.11 (c). Check uniqueness[1].

(b) Suppose that $\sum_{y\in Y} r_y(y-1) = 0$ where $r_y \in \mathbb{Z}F$. Choose a transversal T to the cosets of R in F. We can write $r_y = \sum_{t\in T} t s_{t,y}$ where $s_{t,y} \in \mathbb{Z}R$. So $\sum_{y\in Y, t\in T} t s_{t,y}(y-1) = 0$ and so $\sum_{y\in Y} s_{t,y}(y-1) = 0$ for each t since I_F is free abelian on $\{f-1 \mid f \in F \setminus \{1\}\}$. But I_R is a free $\mathbb{Z}R$-module on $\{y-1 \mid y \in Y\}$ by (a), hence $s_{t,y} = 0$ for all $t \in T, y \in Y$.

Also check that the $y-1$ generate $\overline{I}_R$?

□

Proof of Proposition 2.10. I_F is the free $\mathbb{Z}F$-module on $\{x-1 \mid x \in X\}$ by the lemma. So $I_F/(\overline{I}_R I_F)$ is a free $\mathbb{Z}(F/R)$-module, i.e. $\mathbb{Z}G$-module, on $\{x-1 \mid x \in X\}$. Similarly it follows that $\overline{I}_R/\overline{I}_R^2$ is a free $\mathbb{Z}G$-module on $\{y-1 \mid y \in Y\}$. Consider the image of d_2. It is $\overline{I}_R/(\overline{I}_R I_F)$. Consider $\overline{I}_R$ as a right $\mathbb{Z}F$-module (note that $\overline{I}_R$ is the kernel of a ring map, hence a two-sided ideal). By the right version of the lemma, it is a free right $\mathbb{Z}F$-module on

[1] This amounts to showing that the $\mathbb{Z}F$-submodule A generated by $\widetilde{X}$ is I_F itself. To see this note first that we know that I_F is generated over $\mathbb{Z}$ by $\{f-1 \mid f \in F\}$. From $S(x,r)(x-1) = x^r - 1$, $r \in \mathbb{Z}$ we see that $x^r - 1$ whenever $x \in X$. Then from $(f-1)(g-1) = (fg-1) - (f-1) - (g-1)$ we get inductively that $f - 1 \in A$ for all $f \in F$.

$\{y-1 \mid y \in Y\}$. So $\overline{I}_R/(\overline{I}_R I_F)$ is a free abelian group on $\{y-1 \mid y \in Y\}$, hence isomorphic to R_{ab}. For the left $\mathbb{Z}G$-action note that $g(y-1) = (gyg^{-1}-1)g \equiv (gyg^{-1}-1) \bmod \overline{I}_R I_F$, so the left $\mathbb{Z}G$-action corresponds to the G action on R_{ab} inherited from the conjugation action. $\square$

This partial free resolution can be extended to give a full resolution:

Theorem 2.13 (Gruenberg resolution)**.** *Let $G = \langle X \mid R\rangle$ be a presentation of G. Then there is a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$:*

$$\to \frac{\overline{I}_R^n}{\overline{I}_R^{n+1}} \to \frac{\overline{I}_R^{n-1} I_F}{\overline{I}_R^n I_F} \to \frac{\overline{I}_R^{n-1}}{\overline{I}_R^n} \to \cdots \to \frac{\overline{I}_R}{\overline{I}_R^2} \to \frac{I_F}{\overline{I}_R I_F} \to \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 1$$

Proof. Use the two lemmas. $\square$

Lemma 2.14. *Given a projective resolution*

$$\cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{P_0} \mathbb{Z} \to 0,$$

denote $J_n = \operatorname{im} d_n \subseteq P_{n-1}$ and let $\psi : P_n \to J_n$ be d_n with its image restricted to J_n.

(a) For a $\mathbb{Z}G$-module M there is an exact sequence

$$\operatorname{Hom}_G(P_{n-1}, M) \xrightarrow{\text{res}} \operatorname{Hom}_G(J_n, M) \to H^n(G, M) \to 0.$$

(b) There is an exact sequence

$$0 \to H_n(G, \mathbb{Z}) \to \mathbb{Z} \otimes_{\mathbb{Z}G} J_n \to \mathbb{Z} \otimes_{\mathbb{Z}G} P_{n-1}.$$

Proof.

(a) We have

$$\begin{array}{ccccccc} P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \xrightarrow{\psi} & J_n & \longrightarrow & 0 \\ & & & \searrow^{d_n} & \downarrow{\scriptstyle i} & & \\ & & & & P_{n-1} & & \end{array}$$

with the row exact. Then take duals and we get

$$\begin{array}{ccccccc} \operatorname{Hom}_G(P_{n+1}, M) & \xleftarrow{d^{n+1}} & \operatorname{Hom}_G(P_n, M) & \xleftarrow{\psi^*} & \operatorname{Hom}_G(J_n, M) & \longleftarrow & 0 \\ & & & \nwarrow^{d^n} & \uparrow{\scriptstyle i^* = \text{res}} & & \\ & & & & \operatorname{Hom}_G(P_{n-1}, M) & & \end{array}$$

still with the row exact. Then $\ker d^{n+1} = \operatorname{im} \psi^* \cong \operatorname{Hom}_G(J_n, M)$. Thus $H^n(G, M) = \ker d^{n+1} / \operatorname{im} d^n \cong \operatorname{Hom}_G(J_n, M) / \operatorname{im} \text{res}$.

(b) Follows similarly.

□

Proof of Theorem 2.5. We apply the last lemma to our partial resolution in Proposition 2.10 to get:

$$\mathrm{Hom}_G(I_F/(\overline{I}_R I_F), M) \xrightarrow{\text{res}} \mathrm{Hom}_G(R_{\mathrm{ab}}, M) \to H^2(G, M) \to 0$$

But

$$\begin{aligned}\mathrm{Hom}_G(I_F/(\overline{I}_R I_F), M) &= \mathrm{Hom}_F(I_F/(\overline{I}_R I_F), M) \\ &= \mathrm{Hom}_F(I_F, M) \\ &= H^1(F, M)\end{aligned}$$

For the second equality note that any $\mathbb{Z}F$-map $I_F \to M$ will factor through $I_F/(\overline{I}_R I_F)$ as R acts trivially on M. Why does the last equality hold? □

Proof of Theorem 2.9. Again apply the lemma to our partial resolution in Proposition 2.10. We get:

$$0 \to H_2(G, \mathbb{Z}) \to \mathbb{Z} \otimes_{\mathbb{Z}G} R_{\mathrm{ab}} \to \mathbb{Z} \otimes_{\mathbb{Z}G} I_F/(\overline{I}_R I_F).$$

Note that tensoring with $\mathbb{Z} \cong \mathbb{Z}G/I_G$ is equivalent to taking coinvariants. So

$$\begin{aligned}\mathbb{Z} \otimes_{\mathbb{Z}G} R_{\mathrm{ab}} &= R/[R, F], \\ \mathbb{Z} \otimes_{\mathbb{Z}G} (I_F/(\overline{I}_R I_F)) &= I_F/I_F^2 = F/[F, F].\end{aligned}$$

Now the kernel of the right hand map $R/[R, F] \to F/[F, F]$ is exactly $\frac{F' \cap R}{[R,F]}$. □

3 General Theory

3.1 Long exact sequence

In any cohomology theory one has a long exact sequence. Given a short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

of modules, we would like some relationship between the cohomology with coefficients in M_2 and that of M_1 and M_3. Recall that if we apply $\mathrm{Hom}(P, -)$ to short exact sequences the result is always a short exact sequence only if P is projective.

Proposition 3.1 (Long exact sequence of cohomology). *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence. Then there is a long exact sequence:*

$$\cdots \to H^n(G, M_1) \to H^n(G, M_2) \to H^n(G, M_3) \to H^{n+1}(G, M_1) \to \dots$$

Lemma 3.2 (Snake lemma). *Let $0 \to A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \to 0$ be a short exact sequence of chain complexes (i.e. $f_\bullet, g_\bullet$ are chain maps and the corresponding sequences of abelian groups are exact in every degree). Then there exist maps $\delta_n : H_{n+1}(C_\bullet) \to H_n(A_\bullet)$ such that the sequence*

$$\cdots \to H_{n+1}(C) \xrightarrow{\delta_n} H_n(A_\bullet) \to H_n(B_\bullet) \to H_n(C_\bullet) \to \dots$$

is exact.

Proof. Easy diagram chase. □

Proof of Proposition 3.1. Consider a projective resolution $P_\bullet$ of $\mathbb{Z}$. Then since the modules in the resolution are projective, we have a short exact sequence of chain complexes

$$0 \to \mathrm{Hom}_G(P_\bullet, M_1) \to \mathrm{Hom}_G(P_\bullet, M_2) \to \mathrm{Hom}_G(P_\bullet, M_3) \to 0$$

Now apply the Snake lemma (relabel to convert to chain complex). □

3.2 Five term exact sequence

If we want to consider the relationship between cohomology of a group G with that of subgroups and quotients we have the following:

Theorem 3.3 (Five term exact sequence)**.** *Let H be a normal subgroup of G. Let $Q = G/H$ and M be a $\mathbb{Z}G$-module. Then there is an exact sequence*

$$0 \to H^1(Q, M^H) \to H^1(G, M) \to H^1(H, M)^Q \to H^2(Q, M^H) \to H^2(G, M).$$

Remarks.

1. There is no $\to 0$ at the end - we will see more when thinking about spectral sequences.
2. $H^1(H, M)$ may be regarded as a $\mathbb{Z}Q$-module, as we will see shortly, so that $H^1(H, M)^Q$ is defined.

Corollary 3.4. *If $G = \langle X \mid R \rangle$ is a presentation, M a $\mathbb{Z}G$-module, then there is an exact sequence*

$$0 \to H^1(G, M) \to H^1(F, M) \to \mathrm{Hom}_G(R_{\mathrm{ab}}, M) \to H^2(G, M) \to 0$$

Remark. This is a continuation of the sequence in MacLane's theorem to the left.

Proof. Set $Q = G$, $G = F$ and $H = R$ in Theorem 3.3 to get

$$0 \to H^1(G, M^R) \to H^1(F, M) \to H^1(R, M)^G \to H^2(G, M^R) \to H^2(F, M).$$

Note that we regard M as a $\mathbb{Z}F$-module via $F \to G$. Then M is a trivial $\mathbb{Z}R$-module, so $M^R = M$ and $H^1(R, M) = \mathrm{Hom}(R_{\mathrm{ab}}, M)$. Note that $H^2(F, M) = 0$ by Question 8 on Example Sheet 1 (free groups have cohomological dimension 1). Also $H^1(R, M)^G = \mathrm{Hom}(R_{\mathrm{ab}}, M)^G = \mathrm{Hom}_G(R_{\mathrm{ab}}, M)$ where G acts on $\mathrm{Hom}(R_{\mathrm{ab}}, M)$ by $(g\phi)(x) = g\phi(g^{-1}x)$. The fixed points under this action are the $\mathbb{Z}G$-maps. □

Corollary 3.5. *If $G = G'$ and M is a trivial $\mathbb{Z}G$-module, there is a short exact sequence*

$$0 \to \mathrm{Hom}(F_{\mathrm{ab}}, M) \to \mathrm{Hom}_G(R_{\mathrm{ab}}, M) \to H^2(G, M) \to 0$$

and so $H^2(G, M) \cong \frac{\mathrm{Hom}_G(R_{\mathrm{ab}}, M)}{\mathrm{Hom}(F_{\mathrm{ab}}, M)}$.

Proof. Follows from the previous corollary. □

Now back to understanding the maps and actions in Theorem 3.3

Lemma 3.6. *Let H be a normal subgroup of G, and M a $\mathbb{Z}G$-module. Let G act on the set of cochains $C^n(H, M)$ by $(g\phi)(h_1, \ldots, h_n) = g\phi(g^{-1}h_1 g, \ldots, g^{-1}h_n g)$. Then this action descends to an action of G on $H^n(H, M)$. Moreover, the action of H on $H^n(H, M)$ is trivial and so we have an induced action of $Q = G/H$ on cohomology groups, so the cohomology groups $H^n(H, M)$ are $\mathbb{Z}Q$-modules.*

Proof. To have an action induced on the cohomology groups, we need to check that the action of $g \in G$ is a chain map, i.e. $g(d^n\phi) = d^n(g\phi)$ for $\phi \in C^{n-1}(H, M)$:

$$\begin{aligned}(g(d^n\phi))(h_1, \dots, h_n) &= g(g^{-1}h_1g)\phi(g^{-1}h_2, g, \dots, g^{-1}h_ng) \\ &\quad - g\phi(g^{-1}h_1gg^{-1}h_2g, \dots, g^{-1}h_ng) + \dots \\ &= h_1g\phi(g^{-1}h_2g, \dots, g^{-1}h_ng) \\ &\quad - g\phi(g^{-1}h_1h_2g, \dots, g^{-1}h_ng) + \dots \\ &= h_1(g\phi)(h_2, \dots, h_n) - (g\phi)(h_1h_2, \dots, h_n) + \dots \\ &= d^n(g\phi)(h_1, \dots, h_n).\end{aligned}$$

To show that H acts trivially, we must take a cocycle and show that applying $h \in H$ only adds a coboundary. E.g. for 1-cocycles, let $\phi \in Z^1(H, M)$ and $h, h_1 \in H$. Then

$$\begin{aligned}(h\phi)(h_1) - \phi(h_1) &= h\phi(h^{-1}h_1h) - \phi(h_1) \\ &= h(h^{-1}\phi(h_1h) + \phi(h^{-1})) - \phi(h_1) \\ &= h_1\phi(h) + \phi(h_1) + h\phi(h^{-1}) - \phi(h_1) \\ &= h_1\phi(h) - \phi(h) \\ &= (h_1 - 1)\phi(h).\end{aligned}$$

So $h\phi - \phi$ is indeed a coboundary. Higher degrees are messier but true. □

The maps in Theorem 3.3:

- **Restriction maps:** $H^n(G, M) \to H^n(H, M)^Q$. We define these via definition on cochains which descends to cohomology. Let $f : G^n \in M$ be a cochain. Then let $\operatorname{Res} f : H^n \to M$ be the composition of f with the inclusion $H^n \hookrightarrow G^n$. This gives a map $\operatorname{Res} : C^n(G, M) \to C^n(H, M)$ which induces a map $\operatorname{Res} : H^n(G, M) \to H^n(H, M)$ whose image lies in $H^n(H, M)^G$.
- **Inflation maps:** $H^n(Q, M^H) \to H^n(G, M)$. Again we define them on cochain. Given a cochain $f : Q^n \to M^H$, we let $\operatorname{Inf} f : G^n \to M$ be the composition $G^n \to Q^n \xrightarrow{f} M^H \hookrightarrow M$. Again this map $\operatorname{Inf} : C^n(Q, M^H) \to C^n(G, M)$ descends to cohomology.
- **Transgression maps:** $\operatorname{Tg} : H^1(H, M)^Q \to H^2(Q, M^H)$. Let $s : Q \to G$ be a set-theoretic section with $s(1) = 1$. Define $\rho : G \to H$ by $\rho(g) = gs(gH)^{-1}$ where gH is the coset of g in G/H. Take a 1-cohomology class invariant under Q and $f : H \to M$ a cocycle representing it. Then define $\operatorname{Tg}(f) : G^2 \to M$ by

  $$(g_1, g_2) \mapsto f(\rho(g_1)\rho(g_2)) - f(\rho(g_1g_2)).$$

  Changing g_1 and g_2 by multiplying by elements of H doesn't change this cochain, so we can define a cochain $Q^2 \to M$.

To prove Theorem 3.3 we need to check all these maps give well-defined maps on cohomology and check exactness.

3.3 Transfer map (or corestriction)

When $K \leq G$ is a subgroup and M a $\mathbb{Z}G$-module, there is a map $H^n(K, M) \to H^n(G, M)$. Note the direction is opposite to that of the restriction map. Recall from Example Sheet 1, Question 9, the *coinduced module* is

$$\operatorname{coind}_G^K(M) = \operatorname{Hom}_K(\mathbb{Z}G, M)$$

with G-action $(gf)(x) = f(xg)$ for $f \in \operatorname{Hom}_K(\mathbb{Z}G, M)$, $x \in \mathbb{Z}G$.

Lemma 3.7 (Shapiro's Lemma)**.** *For any $K \leq G$,*

$$H^n(K, M) \cong H^n(G, \operatorname{coind}_G^K(M)).$$

Proof. Example Sheet 1, Question 9. Take a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$. It is also a free $\mathbb{Z}K$-resolution. But $\operatorname{Hom}_K(F, M) \cong \operatorname{Hom}_G(F, \operatorname{coind}_G^K(M))$ for a $\mathbb{Z}G$-module F. Now apply $\operatorname{Hom}_K(-, M)$ and $\operatorname{Hom}_G(-, \operatorname{coind}_G^K(M))$ to our resolution. □

Definition. *Given any $\mathbb{Z}K$-module V, we can define the* induced $\mathbb{Z}G$-module

$$\operatorname{ind}_K^G(V) = \mathbb{Z}G \otimes_{\mathbb{Z}K} V = \bigoplus_{t \in T} t \otimes V$$

where T is a transversal to the cosets of K in G. The G-action is given by $g(t \otimes v) = t' \otimes kv$ where $gt = t'k$ for some $t' \in T, k \in K$.

Observe that if one has a $\mathbb{Z}G$-module M, generated by a $\mathbb{Z}K$-module V (i.e. $M = \mathbb{Z}G \cdot V$), then there is a canonical map

$$\begin{aligned} \operatorname{ind}_K^G(V) &\longrightarrow M, \\ t \otimes v &\longmapsto tv \end{aligned}$$

Lemma 3.8. *When $|G : K| < \infty$ and M is a $\mathbb{Z}G$-module, then*

$$\operatorname{coind}_G^K(M) \cong \operatorname{ind}_K^G(M).$$

Proof. There is a $\mathbb{Z}K$-map

$$\begin{aligned} \phi_0 : M &\longrightarrow \operatorname{Hom}_K(\mathbb{Z}G, M) \\ m &\longmapsto \left(g \mapsto \begin{cases} gm & \text{if } g \in K, \\ 0 & \text{otherwise.} \end{cases} \right) \end{aligned}$$

This extends to a $\mathbb{Z}G$-map

$$\phi : \mathbb{Z}G \otimes_{\mathbb{Z}K} M \to \operatorname{Hom}_K(\mathbb{Z}G, M).$$

There is an inverse:

$$\begin{aligned}\psi : \operatorname{Hom}_K(\mathbb{Z}G, M) &\longrightarrow \mathbb{Z}G \otimes_{\mathbb{Z}K} M \\ f &\longmapsto \sum_{t\in T} t \otimes f(t^{-1})\end{aligned}$$

Thus we have an isomorphism. □

Definition. *If $K \leq G$ is of finite index, the* transfer *(or* corestriction*) map is the composition:*

$$\operatorname{cores}_K^G : H^n(K, M) \cong H^n(G, \operatorname{coind}_G^K(M)) \cong H^n(G, \operatorname{ind}_K^G(M)) \xrightarrow{\alpha_*} H^n(G, M)$$

where $\alpha : \operatorname{ind}_K^G(M) \to M$ is the canonical map.

Lemma 3.9. *If $z \in H^n(G, M)$, then $\operatorname{cores}_K^G \operatorname{res}_K^G(z) = |G : K|z$.*

Proof. Example Sheet 2. □

3.4 Products

Let G be a group and M, N $\mathbb{Z}G$-modules.

Definition. *Given $[u] \in H^p(G, M)$ and $[v] \in H^q(G, N)$, we define the* cup product

$$[u \smile v] \in H^{p+q}(G, M \otimes_{\mathbb{Z}} N)$$

on cochains in the standard resolution of $\mathbb{Z}$. If $u \in C^p(G, M)$ and $v \in C^q(G, N)$, then $u \smile v \in C^{p+q}(G, M \otimes N)$ is defined by

$$(u \smile v)(g_1, \ldots, g_{p+q}) = (-1)^{pq} u(g_1, \ldots, g_p) \otimes g_1 \cdots g_p v(g_{p+1}, \ldots, g_{p+q})$$

This induces the cup product on cohomology.

Here $M \otimes_{\mathbb{Z}} N$ is a $\mathbb{Z}G$-module via the diagonal action, i.e. $g(m \otimes n) = (gm) \otimes (gn)$.

Some properties:

- In degree 0 the cup produt $H^0(G, M) \times H^0(G, N) \to H^0(G, M \otimes N)$ is the map

$$M^G \otimes N^G \longrightarrow (M \otimes N)^G$$

  induced by the inclusions $M^G \to M, N^G \to N$.

- **Naturality:** The cup product is natural in the following sense: Given $\mathbb{Z}G$-maps $f : M \to M', g : N \to N'$ and elements $u \in H^*(G, M), v \in H^*(G, N)$ we have

$$(f \otimes g)_*(u \smile v) = f_* u \smile g_* v$$

- **Identity:** The element $1 \in H^0(G,\mathbb{Z}) = \mathbb{Z}$ satisfies $1 \cup u = u = u \cup 1$ for all $u \in H^*(G,M)$ using $\mathbb{Z} \otimes M = M = M \otimes \mathbb{Z}$.
- **Associativity:** Given $u_i \in H^*(G,M_i)$, $i = 1,2,3$, then

$$(u_1 \smile u_2) \smile u_3 = u_1 \smile (u_2 \smile u_3) \in H^*(G, M_1 \otimes M_2 \otimes M_3).$$

- **Commutativity:** For any $u \in H^p(G,M)$, $v \in H^q(G,N)$ we have

$$u \smile v = (-1)^{pq} \alpha_*(v \smile u)$$

  where α is the natural map $N \otimes M \to M \otimes N$.

These properties yield that $H^*(G,\mathbb{Z})$ is a graded commutative associative ring (here graded commutative means $xy = (-1)^{pq} yx$ where x, y are of degree p, q). There is a commutative subring by taking the sum of even degree terms. The whole cohomology ring is a module for this subring.

More naturality properties:

- **Change of groups:** Given a group homomorphism $\alpha : H \to G$, then we have

$$\alpha^*(u \smile v) = \alpha^* u \smile \alpha^* v.$$

  Thus $\alpha^* : H^*(G,\mathbb{Z}) \to H^*(H,\mathbb{Z})$ is a ring homomorphism.
- **Transfer:** When $H \leq G$ is a subgroup of finite index, $u \in H^*(G,M)$, $v \in H^*(H,N)$, then

$$\mathrm{cores}_H^G(\mathrm{res}_H^G(u) \smile v) = u \smile \mathrm{cores}_H^G v.$$

  Thus the transfer map $H^*(H,\mathbb{Z}) \to H^*(G,\mathbb{Z})$ is a homomorphism of $H^*(G,\mathbb{Z})$-modules.

Recall we defined $\mathrm{Ext}^n_{\mathbb{Z}G}(M,N)$ by taking a resolution for M and applying $\mathrm{Hom}_G(-,N)$ to it. The homology groups arising are the abelian groups $\mathrm{Ext}^n_{\mathbb{Z}G}(M,N)$. Now take $N = M$. We find $\mathrm{Ext}^n_{\mathbb{Z}G}(M,M)$ is a module for the cohomology ring $H^*(G,\mathbb{Z})$. There is quite a lot of work studying $\mathbb{Z}G$-modules M via this module $\mathrm{Ext}^n_{\mathbb{Z}G}(M,M)$ over the cohomology ring.

4 Brauer groups

Definition. *A* simple algebra *A is one where the only two-sided ideals are* 0 *and* A. *A* central simple algebra *A over a field k is one which is simple, finite-dimensional, and the centre is $Z(A) = k$.*

Examples.

1. The set of $n \times n$-matrices $M_n(K)$ forms a central simple k-algebra.
2. The quaternions $\mathbb{H}$ form a central simple $\mathbb{R}$-algebra. Recall that $\mathbb{H}$ has $\mathbb{R}$-basis $1, i, j, k$ where $ij = k = -ji$ and $i^2 = j^2 = k^2 = -1$. In fact, this is a division algebra, i.e. every non-zero element has a multiplicative inverse.

Basic question: Classify central simple algebras over a specified field k.

Theorem 4.1 (Artin-Wedderburn)**.** *A finite dimensional simple k-algebra A is isomorphic to a matrix ring over a division algebra D.*

Note that if D is a division-algebra over k, then $Z(M_n(D)) = \{\lambda I \mid \lambda \in Z(D)\}$. So the classification problem boils down to classifying central division k-algebras.

We define an equivalence relation on central simple k-algebras: Two such algebras A, B are equivalent, written $A \sim B$, if $A \otimes_k M_n(k) \cong B \otimes_k M_m(k)$ for some m, n. We write $[A]$ for the equivalence class. So by the Artin-Wedderburn, $[A] = [D]$ for some division algebra D.

Definition. *The* Brauer group $\mathrm{Br}(k)$ *of k is the set* $\{[A] \mid A \text{ central simple } k\text{-algebra}\}$ *together with the group operation* $[A][B] = [A \otimes_k B]$.

We will soon prove that this is well-defined, i.e. $A \otimes_k B$ is again central simple. Assuming this we show that this satisfies the abelian groups axioms:

Abelian: Clear from $A \otimes_k B \cong B \otimes_k A$.

Associativity: Also clear.

Identity: Take $[k]$.

Inverses: $[A]^{-1} = [A^{\mathrm{op}}]$ where A^{op} is the *opposite algebra.* It has the same underlying set as A, but the multiplication is defined by $a \cdot_{A^{\mathrm{op}}} b = b \cdot_A a$. Note that a right A-module may be regarded as a left A^{op}-module. That $[A^{\mathrm{op}}]$ indeed gives the inverse follows from the following lemma:

Lemma 4.2. *$A \otimes_k A^{\mathrm{op}} \cong M_n(k)$ where $n = \dim_k A$.*

Examples.

1. If k is algebraically closed, then $\mathrm{Br}(k)$ is trivial, since any division k-algebra, finite-dimensional over k, has all elements algebraic over k, hence in k (using that every non-zero element is invertible).
2. $\mathrm{Br}(\mathbb{R}) = \{[\mathbb{R}], [\mathbb{H}]\}$. We will prove this later as a consequence of knowing some 2-cohomology groups.

Definition. *If L/k is a field extension, the subgroup $\mathrm{Br}(L/k)$ is the group of classes represented by central simple k-algebras A such that $A \otimes_k L \cong M_n(L)$ for some n. In this case we say A is* split by *L.*

We will see that given A there are such field extensions L/k, in fact:

Proposition 4.3.

$$\mathrm{Br}(k) = \bigcup_{\substack{L/K \ \text{Galois} \\ [L:k]<\infty}} \mathrm{Br}(L/k)$$

Theorem 4.4. *Let L/k be finite Galois, then*

$$\mathrm{Br}(L/k) \cong H^2(\mathrm{Gal}(L/k), L^\times).$$

Example. Let $k = \mathbb{R}, L = \mathbb{C}$. Then $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is cyclic of order 2 generated by complex conjugation σ. Take $A = \mathbb{H}$. Then $\mathbb{R} \oplus \mathbb{R}i = \mathbb{C} \subseteq \mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$. Thus $\mathbb{C}$ is a maximal subfield of $\mathbb{H}$ and there isa basis labelled by the elements of G, say $e_1 = 1, e_\sigma = j$. Note that $e_\sigma x e_\sigma^{-1} = \sigma(x)$ for all $x \in \mathbb{C}$.

Define $\phi : G \times G \to L^\times$ via $e_\sigma e_\tau = \phi(\sigma, \tau) e_{\sigma\tau}$ where $\phi(\sigma, \tau) \in L^\times$ and $\sigma, \tau \in G$. We are thinking of an extension of G by $L^\times$ as a subgroup of the group of units in our algebra. The algebra is associative if and only if ϕ is a 2-cocycle. Note that if we take $e_1 = 1$, then the 2-cocycle is normalised.

Now let L/k be any finite Galois extension with Galois group $G = \mathrm{Gal}(L/k)$. Let $\phi : G \times G \to L^\times$ be a normalised 2-cocycle. We define an algebra $A = A(L, G, \phi)$ as follows: It is the L-vector space on the basis $\{e_\sigma \mid \sigma \in G\}$ with symbols e_σ. Define multiplication on the basis by

$$e_\sigma e_\tau = \phi(\sigma, \tau) e_{\sigma\tau} \text{ and } (\sigma a) e_\sigma = e_\sigma a.$$

Since ϕ is a 2-cocycle, this extends to give an associative multiplication. e_1 is the multiplicative identity since ϕ is normalised. We identity L with $Le_1 \subseteq A$. The centre of $A(L, G, \phi)$ is k. Indeed, assume $x = \sum_{\sigma \in G} \lambda_\sigma e_\sigma \in Z(A(L, G, \phi))$ with $\lambda_\sigma \in L$. Then for $\beta \in L$ we have

$$\sum_{\sigma \in G} \lambda_\sigma \beta e_\sigma = \beta\big(\sum \lambda_\sigma e_\sigma\big) = \beta x = x\beta = \big(\sum \lambda_\sigma e_\sigma\big)\beta = \sum \lambda_\sigma \sigma(\beta) e_\sigma.$$

So $\sigma(\beta) = \beta$ if $\lambda_\sigma \neq 0$. However, if $\sigma \neq 1$, we can choose β such that $\sigma(\beta) \neq \beta$, so $\lambda_\sigma = 0$ for $\sigma \neq 1$. Then $x = \lambda_1 e_1$ Now $xe_\tau = e_\tau x$ for all τ, so $\tau(\lambda_1) = \lambda_1$ for any τ and hence $\lambda \in L^G = K$. Thus $Z(A(L, G, \phi)) = \{\lambda e_1 \mid \lambda \in k\}$.

Next we show that A is simple. Let $I \neq 0$ be a two-sided ideal and $x = \lambda_{\sigma_1} e_{\sigma_1} + \cdots + \lambda_{\sigma_m} e_{\sigma_m}$ be a non-zero element in I with $\lambda_{\sigma_i} \in L^\times$ and m minimal. If $m > 1$, we can find $\beta \in L^\times$ such that $\sigma_m(\beta) \neq \sigma_{m-1}(\beta)$. Then $y = x - \sigma_m(\beta) x \beta^{-1} \in I$ and the coefficient of e_{σ_m} in y is zero. Hence we conclude that $m = 1$, so $x = \lambda e_\sigma$ with $\lambda \in L^\times$. This is a unit with inverse $x^{-1} = \sigma^{-1}(\lambda^{-1}) e_{\sigma^{-1}}$, so $I = A$ and A is simple.

Note that $\dim_K A(L, G, \phi) = (\dim_K L)^2$.

Definition. *The central simple k-algebra $A(L, G, \phi)$ is the* crossed product *of L/k by the Galois group* $\mathrm{Gal}(L/k)$ *with the given normalised 2-cocycle $\phi : G \times G \to L^\times$.*

Now suppose $\phi' : G \times G \to L^\times$ is another normalised 2-cocycle such that $[\phi] = [\phi']$, in other words ϕ and ϕ' differ by a coboundary, i.e.

$$\phi'(\sigma, \tau) = \phi(\sigma, \tau) \sigma(u_\tau) u_{\sigma\tau}^{-1} u_\sigma$$

for some 1-cochain $u : G \to L^\times$. Define an L-linear map

$$\begin{aligned} F : A(L, G, \phi') &\longrightarrow A(L, G, \phi) \\ e'_\sigma &\longmapsto u_\sigma e_\sigma \end{aligned}$$

Then one checks that F is a homomorphism. By simplicity and dimension reasons, it is an isomorphism.

Proposition 4.5. *The map*

$$\begin{aligned} H^2(G, L^\times) &\longrightarrow \mathrm{Br}(k), \\ [\phi] &\longmapsto [A(L, G, \phi)] \end{aligned}$$

is a homomorphism of abelian groups.

Proof. Let ϕ and ϕ' be 2-cocycles. We have to show that

$$A(L, G, \phi + \phi') \sim A(L, G, \phi) \otimes A(L, G, \phi').$$

Let $A = A(L, G, \phi)$, $B = A(L, G, \phi')$, $C = A(L, G, \phi + \phi')$. Regard A and B as L-vector spaces. Define

$$V = A \otimes_L B = \frac{A \otimes_k B}{\langle la \otimes b - a \otimes lb \mid a \in A, b \in B, l \in L \rangle}.$$

V has a unique right $A \otimes_k B$-module structure given by

$$(a' \otimes_L b')(a \otimes_k b) = a'a \otimes_L b'b$$

for $a', a \in A$, $b', b \in B$. Also V has a unique left C-structure given by

$$(le''_\sigma)(a \otimes_L b) = le_\sigma a \otimes e'_\sigma b$$

for $l \in L$, $\sigma \in G$, $a \in A$, $b \in B$. Here we denote the basis elements of A, B, C by $e_\sigma, e'_\sigma, e''_\sigma$.

The two actions are compatible and so the right action of $A \otimes_k B$ on V defines a homomorphism

$$f : (A \otimes_k B)^{\mathrm{op}} \to \mathrm{End}_C(V)$$

which is injective because $A \otimes_k B$ is simple (to be proved later). Now $(A \otimes_k B)^{\mathrm{op}}$ and $\mathrm{End}_C(V)$[1] have the same dimension n^4 where $n = [L : K] = \#G$, so f is an isomorphism. When we prove Artin-Wedderburn we will see that $\mathrm{End}_C(V) \cong M_r(D)^{\mathrm{op}}$ for some division algebra D which is the endomorphism algebra of a simple C-module. Also $[C] = [D]$. From this we get $(A \otimes_k B)^{\mathrm{op}} \cong M_n(D)^{\mathrm{op}}$, $A \otimes_k B \cong M_n(D)$ and so

$$[A \otimes_k B] = [D] = [C]$$

in $\mathrm{Br}(k)$. □

Remarks.

1. The map in Proposition 4.5 is injective. We can see by counting dimensions that $[A(L, G, \phi)] = [A(L, G, \phi')]$ if and only if $A(L, G, \phi) \cong A(L, G\phi')$.
2. The image of the map is in fact $\mathrm{Br}(L/k)$.

4.1 Some proofs

Now we return to fill in the remaining proofs. First we have the following lemma:

Lemma 4.6 (Schur's lemma)**.** *If M is a simple module over some ring A, then $\mathrm{End}_A(M)$ is a division algebra.*

Proof. Immediate, by simplicity any endomorphism $M \to M$ is either 0 or an isomorphism. □

Proof of Artin-Wedderburn, Theorem 4.1. Consider a minimal non-zero right A-submodule of A_A (A regarded as a right A-module). Thus M has only the submodules 0 and M, i.e. M is a simple right A-module. Then consider $\sum_{a \in A} aM$. This is a two-sided ideal in A, hence it is A by simplicity.

Now consider $\theta_a : M \to aM$ given by multiplication by $a \in A$ on the left. This is a right A-module map. By looking at $\ker \theta_a$, θ_a is either the zero map or an isomorphism. Thus

[1] That $\mathrm{End}_C(V)$ indeed has dimension n^4 follows from Theorem 4.12 (ii) applied to $A = \mathrm{End}_k(V), B = C$ or directly from the discussion following that theorem. Note that $\dim C = n^2, \dim V = n^3, \dim \mathrm{End}_k(V) = n^6$.

$\sum_a aM$ is a sum of copies of M. An easy induction shows that a finite sum of simple modules is a *direct* sum, possibly after ignoring multiple occurrences of the same module.

Now consider $\mathrm{End}_A(M) =: D$. By Schur's lemma this is a division algebra. But $A_A = \bigoplus_{i=1}^r M_i$ where the M_i are simple right A-modules all isomorphic to each other. Consider $\mathrm{End}_A(A_A)$. We have a map

$$\begin{aligned} A &\longrightarrow \mathrm{End}_A(A_A), \\ a &\longmapsto \text{multiplication on the left by } a \end{aligned}$$

An endomorphism is determined by the image of the generator 1, so this map is an isomorphism, so $A \cong \mathrm{End}_A(A_A)$. But $\mathrm{End}_A(\bigoplus_{i=1}^r M_i) \cong M_r(D)$. Hence $A \cong M_r(D)$. □

Corollary 4.7. *With the notation as in the proof, every finitely generated right A-module V is isomorphic to a direct sum of finitely many copies of M. Any two submodules of the same dimension over k are isomorphic and* $\mathrm{End}_A(V) \cong M_r(D)$ *where r is the number of copies of M in the direct sum.*

Proof. A_A is a sum of copies of M. If V is finitely generated by $v_1, \ldots, v_r$ as an A-module, then the surjective map $A^r \to V$, $(a_1, \ldots, a_r) \mapsto \sum a_i v_i$ shows that V is a quotient of a sum of copies of A_A and hence a quotient of a sum of finitely many copies of M. An easy induction shows that this is in fact a direct sum of copies of M. □

We still have to show that the tensor product of two central simple k-algebras is again such an algebra. We start with some easy linear algebra.

Definition. *Let V be a finite-dimensional k-vector space and $\{e_i\}$ a fixed basis. For $v = \sum a_i e_i \in V$ we let $J(v) = \{i \in I \mid a_i \neq 0\}$ be the support of v w.r.t. the basis $\{e_i\}$. For a subspace $W \subseteq V$, a non-zero element $\sum a_i e_i = w \in W$ is* primordial *w.r.t. the basis $\{e_i\}$ if $J(w)$ is minimal among the sets $J(w')$ with $w' \in W, w' \neq 0$, and $a_i = 1$ for some i.*

Lemma 4.8.

(i) For $0 \neq w, w' \in W$ with $J(w)$ minimal, then $J(w') \subseteq J(w)$ if and only if $w' = cw$ for some $c \in k$.

(ii) The primordial elements span W.

Proof.

(i) Is clear.

(ii) Induction on $\#J(w)$. Let $0 \neq w = \sum a_i e_i \in W$. Among the non-zero elements w' of W with $J(w') \subseteq J(w)$ we can choose one with $\#J(w')$ minimal. Then $w_0 = cw'$ will be primordial for some $c \in k^\times$. Now $w_0 = \sum b_i e_i$ with $b_j = 1$ say. Then $w = a_j w_0 + (w - a_j w_0)$ and $\#J(w - a_j w_0) < \#J(w)$, hence by induction we see that w is a linear combination of primordial elements.

□

Remark. The same is true for D-vector spaces for a division ring D.

Lemma 4.9. *Let A be a k-algebra, D a central division algebra. Then every two-sided ideal I in $A \otimes_k D$ is generated as a left D-module by $J = I \cap (A \otimes 1)$.*

Note that $I \cap (A \otimes 1)$ is an ideal of A.

Proof. There is a left D-module structure on $A \otimes_k D$ given by $\delta(a \otimes \delta') = a \otimes \delta\delta'$. The ideal I is a D-submodule of $A \otimes_k D$. Let $\{e_i\}$ be a basis for A as a k-vector space. Then $\{(e_i \otimes 1)\}$ is a basis for $A \otimes D$ as a left D-module. Let r be primordial w.r.t. this basis. Then $r = \sum_{i \in J(r)} \delta_i(e_i \otimes 1) = \sum e_i \otimes \delta_I$ with $\delta_i \in D$. Then for any non-zero $\delta \in D$, $r\delta \in I$ and $r\delta = \sum \delta_i \delta(e_i \otimes 1)$. In particular, $J(r\delta) = J(r)$ and so $r\delta = \delta' r$ for some $\delta' \in D$ by the lemma. As some $\delta_j = 1$ (since R is primordial) this implies $\delta = \delta'$ and so each δ_i commutes with every $\delta \in D$, thus $\delta_i \in Z(D) = k$. So $r \in A \otimes 1$. Hence every primordial element of I is in $A \otimes_k 1$. The claim then follows from the previous lemma. □

Proposition 4.10. *The tensor product of two (finite-dimensional) simple k-algebras, at least one of which is central, is again simple.*

Proof. By Artin-Wedderburn we may assume that one of the algebras is $M_n(D)$ for some division ring with centre k. Let A be the other algebra. By Lemma 4.9 $A \otimes_k D$ is simple, hence by Artin-Wedderburn again $A \otimes_k D \cong M_m(D')$ for some division algebra D'. Thus

$$A \otimes M_n(D) \cong M_n(A \otimes D) \cong M_n(M_m(D')) \cong M_{nm}(D')$$

is simple. □

Corollary 4.11. *The tensor product of two central simple k-algebras is again central simple.*

Proof. By Proposition 4.10, the tensor product is again simple. Since $Z(A \otimes_k B) = Z(A) \otimes_k Z(B)$, it also follows that it is central. □

Thus the product in the Brauer group is defined.

Next we consider inverses in $\mathrm{Br}(k)$. Given a central simple k-algebra A, let V be the underlying vector space and consider the map

$$\begin{aligned} A \otimes A^{\mathrm{op}} &\longrightarrow \mathrm{End}_k(V), \\ a \otimes a' &\longmapsto (v \mapsto ava') \end{aligned}$$

It is a ring homomorphism. The map is injective since $A \otimes A^{\mathrm{op}}$ is simple by Proposition 4.10 and the kernel does not contain $1 \otimes 1$. So the map is an isomorphism since both sides have the same dimension n^2 where $n = \dim_K A$. Hence we proved $A \otimes_k A^{\mathrm{op}} \cong M_n(k)$ and so $[A] \cdot [A^{\mathrm{op}}] = [1]$.

Theorem 4.12 (Double Centraliser Theorem). *Let A be a central simple k-algebra with simple subalgebra B. Then*

(i) The centraliser $C_A(B)$ is simple.

(ii) $\dim B \cdot \dim C_A(B) = \dim A$.

(iii) $C_A(C_A(B)) = B$.

(iv) If B is central simple, then $C_A(B)$ is also central simple.

Proof. Exercise. □

Direct proof of (ii) in a special case: Let C be a central simple k-algebra, V a left C-module. We regard V as a right C^{op}-module. By Corollary 4.7, $V \cong M^{\oplus r}$ where M is a simple C^{op}-module. Then $\text{End}_C(V) \cong \text{End}_{C^{\text{op}}}(V) \cong M_r(D^{\text{op}})$ where $D^{\text{op}} = \text{End}_{C^{\text{op}}}(M)$. But $C^{\text{op}} = M^{\oplus m}$ for some m and $C^{\text{op}} = M_m(D^{\text{op}})$. Now consider dimensions:

$$\begin{aligned} \dim V &= r \dim M \\ \dim C &= \dim C^{\text{op}} = m^2 \dim D = m \dim M \\ \dim \text{End}_C(V) &= r^2 \dim M \\ \dim \text{End}_C(V) \dim C &= r^2 \dim D \cdot m \dim M = (\dim V)^2 \end{aligned}$$

Remarks. We established the map

$$H^2(\text{Gal}(L/k), L^\times) \to \text{Br}(k).$$

The image is $\text{Br}(L/k)$. For the converse we have to establish that given a central simple algebra we can produce a 2-cocycle. In a central simple algebra A we consider maximal subfields L (equivalently maximal commutative subalgebras). From the double centraliser theorem we deduce $\dim A = (\dim_k L)^2$. Take an L-basis for A and consider multiplication of two basis elements and we get a 2-cocycle. We also need to see that within A, L is invariant under conjugation and the action is the Galois action.

Final remarks.

1. For a finite field k, $\text{Br}(k) = 0$ (Theorem by Wedderburn: finite division algebras over fields).
2. For a non-archimedean local field k, $\text{Br}(k) \cong \mathbb{Q}/\mathbb{Z}$.
3. For a number field k there is a short exact sequence:

$$0 \to \text{Br}(k) \to \bigoplus_v \text{Br}(k_v) \xrightarrow{\sum} \mathbb{Q}/\mathbb{Z} \to 0$$

where the sum runs through all the places v of k.

5 Lyndon-Hochschild-Serre spectral sequence

The aim is to link the cohomology of a group G with that of a normal subgroup H with that of the quotient $Q = G/H$. We already saw this for low degree cohomology when we met the five term exact sequence.

We consider a double cochain complex A. It consists of abelian groups $A^{p,q}$, indexed by $p, q \in \mathbb{Z}$, and maps d', d'' of bidegree $(1,0)$ resp. $(0,1)$ such that $d'^2 = 0$, $d''^2 = 0$, $d'd'' + d''d' = 0$.

We let $A^n = \bigoplus_{p+q=n} A^{p,q}$ and $d = d' + d''$. Then $((A^n), d)$ is a single chain complex, called the *total complex*. The (total) cohomology $H^*(A)$ is the cohomology of the total complex.

In our context we are going to have $A^{p,q} = 0$ for all p, q not in the first quadrant. In our case let $X^\bullet$ be a $\mathbb{Z}G$-projective resolution of the trivial module $\mathbb{Z}$ and $Y^\bullet$ a $\mathbb{Z}(G/H)$-projective resolution of $\mathbb{Z}$. Note that $X^\bullet$ is also a $\mathbb{Z}H$-projective resolution. Let M be a $\mathbb{Z}G$-module. Then G acts on $\mathrm{Hom}_H(X^\bullet, M)$ by $(gf)(x) = g(f(g^{-1}x))$. Since H then acts trivially, we may view $\mathrm{Hom}_H(X^\bullet, M)$ as a $\mathbb{Z}Q$-module.

Then we form the double complex $\mathcal{A} = \mathrm{Hom}_{G/H}(Y^\bullet, \mathrm{Hom}_H(X^\bullet, M))$. We let $d' = \mathrm{Hom}_{G/H}(d_Y, \mathrm{id})$ and $d'' = \mathrm{Hom}_{G/H}(\mathrm{id}, d_X^*)$.

Warning: There is an alternating sign suppressed in the definition of d''. People have different conventions. Cartan-Eilenberg put in $(-1)^p$ where p denotes the degree w.r.t. the grading of X.

The cohomology of the total complex A can be approximated in different ways.

Aim. Filter the double complex in order to filter the cohomology spectral sequences to get information about the associated graded version of $H^*(A)$ w.r.t. this filtration.

First calculate the cohomology $H''(A)$ w.r.t. d''. Since $d'd'' = -d''d'$, the horizontal differential d' induces a differential on $H''(A)$. We may then calculate $H'(H''(A))$. (Alternatively we could have looked at $H''(H'(A))$.)

This gives the E_2-page - there is a cochain map we will define on $H'(H''(A))$ and then we repeat to get $E_3, \ldots$ etc.

Consider how $H'H''(A)$ is computed. Start in position (p, q). Let $a^{p,q} \in A^{p,q}$ be a vertical cocycle, i.e. $d''a^{p,q} = 0$. It defines a class in $H''(\mathcal{A})$. For $a^{p,q}$ to represent a horizontal cocycle in $H''(A)$ under d' it must be true that $d'a^{p,q}$ (which has position $(p+1, q)$) is the image under d'' of an element $a^{p+1,q-1}$ in the position $(p+1, q-1)$. Thus $d(a^{p,q} - a^{p+1,q-1}) = -d'a^{p+1,q-1} \in A^{p+2,q-1}$. So $a^{p,q} - a^{p+1,q-1}$ is a cocycle modulo everything two steps to the right of the (p,q)-th position. Similarly, $a^{p,q}$ represents a coboundary in $H''(A)$ under d' if there are two elements $b^{p-1,q}$ and $b^{p,q-1}$ such that $d''b^{p-1,q} = 0$ and $d'b^{p-1,q} = d''b^{p,q-1} + a^{p,q}$. Thus $d(b^{p-1,q} - b^{p,q-1}) = a^{p,q}$ modulo everything two steps to the right of $(p-1, q)$.

This motivates the idea that filtrations of the complex will be useful. Let F^pA be the double subcomplex where components to the left of the p-th column are zero. So the total complex of F^pA is given by

$$(F^pA)^n = \bigoplus_{\substack{p'+q=n \\ p'\geq p}} A^{p',q}.$$

Note that $(F^0A)^n = A^n$ and $(F^pA)^n = 0$ for $p > n$. This gives a decreasing filtration of $A^\bullet$.

Let $C_r^{p,q}$ be the set of elements in $(F^pA)^{p+q}$ whose image under d is in $(F^{p+r}A)^{p+q+1}$. Each such element is a sum of components along the line $p + q = n$, starting at the (p,q)-th position, such that the vertical and horizontal maps cancel within the range $p \leq p' < p + r$. Note that the image under d of such an element lies in $(F^{p+r}A)^{n+1}$, i.e. it starts at coordinates $(p + r, q - r + 1)$. Define

$$E_r^{p,q} = \frac{C_r^{p,q} + (F^{p+1}A)^{p+q}}{d(C_{r-1}^{p-r+1,q+r-2}) + (F^{p+1}A)^{p+q}}.$$

Then d induces maps $d_r^{p,q} : E_r^{p,q} \to E^{p+r,q-r+1}$ satisfying $d_r^2 = 0$.

If we compute the cohomology of the resulting complex, we get

$$H(E_r, d_r) = E_{r+1},$$

i.e.

$$E_{r+1}^{p,q} = \frac{\ker d_r^{p,q}}{\operatorname{im} d_r^{p-r,q+r-1}}.$$

A representative of an element a in $E_r^{p,q}$ defines an element in a subquotient of $A^{p,q}$ at its upper left (p,q), but its extended structure to the right is crucial in calculating dr. In particular $da \in F^{p+1}A$ represents d_r of the element represented by a. For each fixed (p,q) the differential $d_r^{p,q}$ which starts there and differential $d_r^{p-r,q+r-1}$ which ends there must vanish for r sufficiently large (all our terms are in the top right quadrant). It follows that each $E_r^{p,q}$ eventually stabilises at a common value, denoted by $E_\infty^{p,q}$ (but the r for which $E_r^{p,q} = E_\infty^{p,q}$ may depend on p,q).

Suppose that $a \in A^n$ is a cocycle starting at $A^{p,q}$ where $p + q = n$, i.e. $a \in (F^pA)^n \setminus (F^{p+q}A)^n$ and $da = 0$. So a determines an element of $E_\infty^{p,q}$ since it determines an element of $E_r^{p,q}$ for all $r \geq 1$ and d_r is zero on that element.

In other words, we have a map

$$F^pH^{p+q}(A) := \operatorname{im}\Big(H^{p+q}(F^pA) \to H^{p+q}(A)\Big) \to E_\infty^{p,q}.$$

In fact, it is surjective and the kernel is $F^{p+1}H^{p+q}(A)$. Thus the filtration of the double complex A induces a descending filtration of $H^n(A)$ for each n and

$$\frac{F^pH^{p+q}(A)}{F^{p+1}H^{p,q}(A)} \cong E_\infty^{p,q}.$$

Note that the spectral sequence E_r determines those factors and so determines the associated graded version $\operatorname{gr} H^*(A)$. When calculating we may be left with the extension problem of how to fit these factors together to give $H^*(A)$.

Back to our complex arising from $G, H \trianglelefteq G, G/H$ and $\mathbb{Z}G$-module M. We can take two spectral sequences arising from $H'H''(A)$ as E_2-page and from $H''H'(A)$ as E_2-page. We will find that the second one shows relatively easily that the total cohomology $H^*(A)$ of the complex is just $H^*(G,M)$. Then we can use the first sequence to calculate what this cohomology is from knowledge of cohomology of H and G/H. Recall that

$$\begin{aligned} A^{\bullet,\bullet} &= \operatorname{Hom}_{G/H}(Y^\bullet, \operatorname{Hom}_H(X^\bullet, M)) \\ d' &= \operatorname{Hom}_{G/H}(d_Y, \operatorname{id}) \\ d'' &= \operatorname{Hom}_{G/H}(\operatorname{id}, d_X^\bullet) \text{ (with sign actually)} \end{aligned}$$

The first spectral sequence: Calculate $H'H''(A)$ to give E_2-page of spectral sequence. We have

$$H''(\operatorname{Hom}_{G/H}(Y^\bullet, \operatorname{Hom}_H(X^\bullet, M))) = \operatorname{Hom}_{G/H}(Y^\bullet, H^*(\operatorname{Hom}_H(X^\bullet, M)))$$

since the terms of $Y^\bullet$ are all $\mathbb{Z}G/H$-projective and so $\operatorname{Hom}_{G/H}(Y^\bullet, -)$ preserves exactness and therefore homology groups. Thus

$$\begin{aligned} E_2 &= H'H''(A) \\ &= H * (\operatorname{Hom}_{G/H}(Y^\bullet, H^*(X^\bullet, M))) \\ &= H^*(G/H, H^*(H, M)) \end{aligned}$$

The second spectral sequence: We have

$$H'(\operatorname{Hom}_{G/H}(Y^\bullet, \operatorname{Hom}_H(X^\bullet, M))) = H^*(G/H, \operatorname{Hom}_H(X^\bullet, M)).$$

Lemma 5.1. *$H^p(G/H, \operatorname{Hom}_H(X^\bullet, M)) = 0$ for $p > 0$.*

Proof. Since each X_q is $\mathbb{Z}G$-projective and hence a direct summand of a free module, it suffices to prove this for $X = \mathbb{Z}G$. Let $\widetilde{M}$ be the trivial $\mathbb{Z}G$-module with the same underlying additive group as M. Claim: There is a $\mathbb{Z}G$-isomorphism

$$\operatorname{Hom}_H(\mathbb{Z}G, M) \cong \operatorname{Hom}_H(\mathbb{Z}G, \widetilde{H})$$

when G acts on the left hand side by $(gf)(x) = gf(g^{-1}x)$ but on the right hand side we have the action as an coinduced module. [Proof of claim: For $f \in \mathrm{Hom}_H(\mathbb{Z}G, M)$ define $f' \in \mathrm{Hom}_H(\mathbb{Z}G, \widetilde{M})$ via $f'(x) = xf(x^{-1})$ for $x \in G$. Check this is indeed in $\mathrm{Hom}_H(\mathbb{Z}G, \widetilde{M})$. Observe that $(f')' = f$. Also check $f \mapsto f'$ gives a $\mathbb{Z}G$-isomorphism.]

This isomorphism allows us to use Shapiro's lemma. Also note that $\mathrm{Hom}_H(\mathbb{Z}G, \widetilde{M}) = \mathrm{Hom}(\mathbb{Z}(G/H), \widetilde{M})$. Since H acts trivially on $\widetilde{M}$,

$$\begin{aligned} H^p(G/H, \mathrm{Hom}_H(\mathbb{Z}G, M)) &\cong H^p(G/H, \mathrm{Hom}(\mathbb{Z}(G/H), \widetilde{M})) \\ &\cong H^p(1, \widetilde{M}) \\ &= 0 \end{aligned}$$

if $p > 0$. □

Thus $H'(A)$ is concentrated on the line $p = 0$, i.e. all other terms are 0. We have

$$\begin{aligned} H^0(G/H, \mathrm{Hom}_H(X^\bullet, M)) &= \mathrm{Hom}_H(X^\bullet, M)^{G/H} \\ &= \mathrm{Hom}_G(X^\bullet, M). \end{aligned}$$

Then

$$\begin{aligned} H''H'(A) &= H^*(\mathrm{Hom}_G(X^\bullet, M)) \\ &= H^*(G, M). \end{aligned}$$

Thus the E_2-page gives $H^*(G, M)$. Since the E_2-page is concentrated in one line, it follows that $E_r = E_\infty$ for $r \geq 2$ and thus E_∞ is concentrated on the line $p = 0$. Hence the filtration of $H^n(A)$ has only one non-trivial factor. So

$$H^n(A) = H^n(G, M).$$

5.1 Example: Cohomology of S_3

Let $G = S_3$. Consider $1 \to C_3 \to G \to C_2 \to 1$.

The first spectral sequence: $H^p(C_2, H^q(C_3, \mathbb{Z}))$ will give the E_2-page. Here the action of C_2 on $H^q(C_3, \mathbb{Z})$ is induced by conjugation, $(12)(123)(12)^{-1} = (132)$. So the non-trivial element of C_2 acts on C_3 via the inversion map which is a group homomorphism as C_3 is abelian. The induced map is a ring homomorphism of the cohomology ring $H^*(C_3, \mathbb{Z})$. The underlying groups are given by

$$\begin{aligned} H^0(C_3, \mathbb{Z}) &\cong \mathbb{Z} \\ H^{2k}(C_3, \mathbb{Z}) &\cong \mathbb{Z}/3\mathbb{Z}, k > 0 \\ H^{2k+1}(C_3, \mathbb{Z}) &= 0 \end{aligned}$$

(see example sheet) In fact, $H^*(C_3, \mathbb{Z}) \cong \mathbb{Z}[c]/(3c)$ where c is of degree 2. What is the action of C_2? The action on $H^2(C_3, \mathbb{Z})$ is given by multiplication by -1 (to check this, consider find a 2-cocyle representing the given cohomology class and use the definition of the action of C_2 on cocycles). Thus the action on $H^{4k}(C_3, \mathbb{Z})$ is trivial and on $H^{4k+2}(C_3, \mathbb{Z})$ it is multiplication by -1.

So

$$\begin{aligned} H^0(C_2, H^{4k+2}(C_3, \mathbb{Z})) &= 0 \\ H^0(C_2, H^{4k}(C_3, \mathbb{Z})) &= \mathbb{Z}/3\mathbb{Z} \end{aligned}$$

We know from Example Sheet 1 that $H^p(C_2, \mathbb{Z}/3\mathbb{Z}) = 0$ if $p \geq 1$. So the E_2-page is

$$\begin{array}{cccccc} 0 & & & & & \\ \mathbb{Z}/3\mathbb{Z} & 0 & & & & \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & \\ \mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 0 \end{array}$$

d_2

Note that all differentials start or finish at 0, and so $E_2 = E_\infty$. Also notice that there are no extension problems, e.g.

$$0 \to \mathbb{Z}/2\mathbb{Z} \to H^4(A) \to \mathbb{Z}/3\mathbb{Z} \to 0$$

and then necessarily $H^4(A) \cong \mathbb{Z}/6\mathbb{Z}$. Then

$$H^n(S_3, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0, \\ 0 & n \text{ odd}, \\ \mathbb{Z}/2\mathbb{Z} & n \equiv 2 \bmod 4, \\ \mathbb{Z}/6\mathbb{Z} & n \equiv 0 \bmod 4, n \neq 0. \end{cases}$$