Functional Analysis

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Contents

1 Hahn-Banach extension theorems

Let X, Y be normed spaces. Notation:

1. $X \sim Y$ means that X and Y are isomorphic, i.e. there exists a linear bijection $T : X \to Y$ such that T and T^{-1} are continuous.
2. $X \cong Y$ means that X and Y are isometrically isomorphic, i.e. there exists a linear surjection $T : X \to Y$ such that for all $x \in X$: $\|Tx\| = \|x\|$. (Then T is injective and T^{-1} is also isometric)
3. For $x \in X$, $f \in X^*$, then write $\langle x, f \rangle = f(x)$. When X is a Hilbert space and X^* is identified with X, then $\langle \cdot, \cdot \rangle$ is the inner product.
4. S_X denotes the unit sphere and B_X denotes the closed unit ball in X.

Definition. *Let X be a real vector space. A functional $p : X \to \mathbb{R}$ is called*

- positive homogeneous *if $p(tx) = tp(x)$ for all $t \geq 0, x \in X$.*
- subadditive *if $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.*

Theorem 1.1 (Hahn-Banach)**.** *Let X be a real vector space and p be a positive homogeneous subadditive functional on X. Let Y be a subspace of X and $g : Y \to \mathbb{R}$ be a linear map such that for all $y \in Y$: $g(y) \leq p(y)$. Then there exists a linear $f : X \to \mathbb{R}$ such that $f|_Y = g$ and for all $x \in X$: $f(x) \leq p(x)$.*

Proof. By Zorn's lemma there exists a maximal extension $h : Z \to \mathbb{R}$ of g that is still dominated by p. If $Z = X$, we are done. Assume that $Z \neq X$. Fix $z_1 \in X \setminus Z$ and $\alpha \in \mathbb{R}$. Let $Z_1 = Z + \mathbb{R}z_1$ and $h_1 : Z_1 \to \mathbb{R}$, $h_1(z + \lambda z_1) = h(z) + \lambda\alpha$ where $\lambda \in \mathbb{R}, z \in Z$. Clearly h_1 is linear and extends h. We show that there exists a choice of α such that $h_1 \leq p|_{Z_1}$. This will then give a contradiction.

We need $h_1(z+\lambda z_1) = h(z)+\lambda\alpha \leq p(z+\lambda z_1)$ for all $z \in Z, \lambda \in \mathbb{R}$. By positive homogeneity of p, this is equialent to

$$\begin{aligned} h_1(z + z_1) = h(z) + \alpha &\leq p(z + z_1) \\ \text{and } h_1(z - z_1) = h(z) - \alpha &\leq p(z - z_1) \end{aligned}$$

for all $z \in Z$. This happens iff

$$h(w) - p(w - z_1) \leq \alpha \leq p(z + z_1) - h(z) \quad \forall z, w \in Z.$$

Such an α exists iff $h(w) - p(w - z_1) \le p(z + z_1) - h(z)$ for all $z, w \in Z$

This *is* true since for all $z, w \in Z$:

$$h(w) + h(z) = h(w + z) \le p(w + z) = p(w - z_1 + z + z_1) \le p(w - z_1) + p(z + z_1).$$

□

Definition. *Let X be a real or complex vector space. A* seminorm *on X is a function $p : X \to \mathbb{R}$ such that*

- *$p(x) \ge 0$ for all $x \in X$.*
- *$p(\lambda x) = |\lambda| p(x)$ for all scalars λ and $x \in X$.*
- *$p(x + y) \le p(x) + p(y)$ for all $x, y \in X$.*

Note that

norm $\Longrightarrow$ seminorm $\Longrightarrow$ pos. hom. and subadditive

Theorem 1.2. *Let X be a real or complex vector space and P be a seminorm on X. Let Y be a subspace of X, $g : Y \to \mathbb{K}$ be linear such that for all $y \in Y$: $|g(y)| \le p(y)$. Then there exists a linear $f : X \to \mathbb{K}$ such that $f|_Y = g$ and for all $x \in X$: $|f(x)| \le p(x)$.*

Proof. The real case: For all $y \in Y$, $g(y) \le |g(y)| \le p(y)$. So by the first theorem there exists a linear $f : X \to \mathbb{R}$ such that $f|_Y = g$ and $f \le p$. Then for $x \in X$ we also have $-f(x) = f(-x) \le p(-x) = p(x)$, so $|f(x)| \le p(x)$.

Complex case: $\operatorname{Re} g : Y \to \mathbb{R}, y \mapsto \operatorname{Re}(g(y))$ is real linear and $|\operatorname{Re} g(y)| \le |g(y)| \le p(y)$ for $y \in Y$. So by the real case there exists a real-linear $h : X \to \mathbb{R}$ such that $h|_Y = \operatorname{Re} g$. Next we show that there exists a unique complex linear $f : X \to \mathbb{C}$ such that $\operatorname{Re} f = h$. *Uniqueness:* For $x \in X$, $f(x) = h(x) + i \operatorname{Im} f(x) = h(x) + i \operatorname{Im}(-if(ix)) = h(x) - ih(ix)$. *Existence:* Define $f : X \to \mathbb{C}$ by $f(x) = h(x) - ih(ix)$. This is real linear and $f(ix) = h(ix) - ih(-x) = h(ix) + ih(x) = i(h(x) - ih(ix)) = if(x)$. So f is complex linear and $h = \operatorname{Re} f$. Now $\operatorname{Re} f|_Y = h|_Y = \operatorname{Re} g$, so by uniqueness $f|_Y = g$. Finally, given $x \in X$, choose $\lambda \in \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ such that $|f(x)| = \lambda f(x) = f(\lambda x) = h(\lambda x) \le p(\lambda x) = |\lambda| p(x) = p(x)$. □

Remark: For a complex vector space V, let $V_{\mathbb{R}}$ be V viewed as a real vector space. Then the proof above shows that given a complex normed space X, the map $f \mapsto \operatorname{Re} f : (X^*)_{\mathbb{R}} \to (X_{\mathbb{R}})^*$ is an isometric isomorphism.

Corollary 1.3. *Let X be a real or complex vector space and p be a seminorm on X. Then for any $x_0 \in X$ there exists a linear $f : X \to \mathbb{K}$ such that $f(x_0) = p(x_0)$ and $|f(x)| \le |p(x)| \le p(x)$ for all $x \in X$.*

Proof. Let $Y = \mathbb{K} x_0$. Apply the theorem to $g : Y \to \mathbb{K}$, $g(\lambda x_0) = \lambda p(x_0)$. □

Theorem 1.4. *Let X be a real or complex normed space. Then*

(i) Given a subspace Y of X and $g \in Y^*$, there exists $X \in X^*$ such that $f|_Y = g$ and $\|f\| = \|g\|$.

(ii) Given $x_0 \in X \setminus \{0\}$, there exists $f \in S_{X^*}$ such that $f(x_0) = \|x_0\|$.

Proof. Easy consequence of the previous results. □

Remarks:

1. Part (i) is a sort of linear version of Tietze's extension theorem.
2. Part (ii) says that X^* separates points of X: For all $x \neq y \in X$ there exists $f \in X^*$ such that $f(x) \neq f(y)$.
3. The f in (ii) is called a *norming functional for* x_0. We have

$$\|x_0\| = \max\{|g(x_0)| \mid g \in B_{X^*}\}.$$

   f is also called a *support functional at* x_0: Assume X is real and $\|x_0\| = 1$. Then $\{x \in X \mid f(x) \leq 1\} \supseteq B_X$ and so the hyperplane $\{x \in X \mid f(x) = 1\}$ can be thought of as a tangent plane to B_X at x_0.

1.1 Bidual

Let X be a normed space. Then $X^{**} = (X^*)^*$ is the *bidual* or *second dual* of X. For $x \in X$ define $\widehat{x} : X^* \to \mathbb{K}$ by $\widehat{x}(f) = f(x)$. This map $\widehat{x}$ is linear and for all $f \in X^*$: $|\widehat{x}(f)| = |f(x)| \leq \|f\| \, \|x\|$. So $\widehat{x} \in X^{**}$ and $\|\widehat{x}\| \leq \|x\|$. The map $x \mapsto \widehat{x} : X \to X^{**}$ is the *canonical embedding* of X into X^{**}.

Theorem 1.5. *The canonical embedding is an isometric isomorphism of X into X^{**}.*

Proof. Follows from Theorem 1.4 (ii). □

Remarks:

1. In bracket notation $\langle f, \widehat{x} \rangle = \langle x, f \rangle$ for $x \in X, f \in X^*$.
2. Let $\widehat{X} = \{\widehat{x} \mid x \in X\}$ be the image of X in X^{**}. Then $\widehat{X}$ is closed in X^{**} iff X is complete.
3. In general, the closure of $\widehat{X}$ in X^{**} is a Banach space, containing a dense isometric copy of X, so every normed space has a completion.

Definition. *A normed space X is* reflexive *if the canonical embedding $X \to X^{**}$ is surjective.*

Note: reflexive $\implies$ complete

Examples.

1. ℓ_p for $1 < p < \infty$, Hilbert spaces, finite-dimensional spaces are reflexive.
2. $c_0, \ell_1, L_1[0,1]$ are not reflexive.

Remark: there exist Banach spaces X such that $X \cong X^{**}$, but that are not reflexive.

1.2 Dual operators

Let X, Y be normed spaces. Recall that

$$\mathcal{B}(X,Y) = \{T : X \to Y \mid T \text{ is linear and bounded}\}$$

is a normed space in the operator norm. If Y is complete, so is $\mathcal{B}(X,Y)$.

Let $T \in \mathcal{B}(X,Y)$. The *dual operator* of T is the map $T^* : Y^* \to X^*$ given by $T^*(g) = g \circ T$ where $g \in Y^*$. In the bracket notation $\langle x, T^* g\rangle = \langle Tx,\ g\rangle$ where $x \in X, g \in Y^*$. T^* is bounded and $\|T^*\| = \|T\|$. Indeed,

$$\sup_{g \in B_{Y^*}} \|T^* g\| = \sup_{g \in B_{Y^*}} \sup_{x \in B_X} |\langle x, T^* g\rangle| = \sup_{x \in B_X} \sup_{g \in B_{Y^*}} |\langle Tx, g\rangle| \overset{1.4(ii)}{=} \sup_{x \in B_X} \|Tx\| = \|T\|\,.$$

Remark: If X, Y are Hilbert spaces and we identify X^*, Y^* with X, Y resp. in the usual way, then $T^* : Y \to X$ is the adjoint of T.

Example. Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. We use the canonical identification $\ell_p^* \cong \ell_q$. If $R : \ell_p \to \ell_p$ is the right shift, then $R^* : \ell_q \to \ell_q$ is the left shift.

Properties:

1. $(\mathrm{Id}_X)^* = \mathrm{Id}_{X^*}$.
2. $(\lambda S + \mu T)^* = \lambda S^* + \mu T^*$ ($S, T \in \mathcal{B}(X,Y), \lambda, \mu$ scalars)
3. $(ST)^* = T^* S^*$ ($T \in \mathcal{B}(X,Y), S \in \mathcal{B}(Y,Z)$)
4. $T \mapsto T^* : \mathcal{B}(X,Y) \to \mathcal{B}(Y^*, X^*)$ is an into isometric isomorphism.
5. The following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \downarrow & & \downarrow \\ X^{**} & \xrightarrow{T^{**}} & Y^{**} \end{array}$$

   Here the vertical arrows are the canonical embeddings. Let $x \in X$. We need $T^{**}\widehat{x} = \widehat{Tx}$. For $g \in Y^*$:

$$\langle g, T^{**}\widehat{x}\rangle = \langle T^* g, \widehat{x}\rangle = \langle x, T^* g\rangle = \langle Tx, g\rangle = \langle g, \widehat{Tx}\rangle.$$

From the above properties, if $X \sim Y$, then $X^* \sim Y^*$.

1.3 Quotient spaces

Let X be a normed space and Y be a closed subspace of X. The quotient space X/Y becomes a normed space in the *quotient norm:*

$$\|x+Y\| = \inf\{\|x+y\| \mid y \in Y\} = d(x,Y).$$

The quotient map $q : X \to X/Y$ is linear, onto, and bounded with $\|q\| \leq 1$.

Let $D_X = \{x \in X \mid \|x\| < 1\}$. Since $\|q\| \leq 1$, $q(D_X) \subseteq D(X/Y)$. In fact, $q(D_X) = D_{X/Y}$. Indeed, given $x+Y \in D_{X/Y}$, $\|x+Y\| < 1$, so there exists $y \in Y$ such that $\|x+y\| < 1$. So $x+y \in D_X$ and $q(x+y) = q(x) = x+Y$. So $\|q\| = \sup_{x \in D_X} \|q(x)\| = \sup_{z \in D_{X/Y}} \|z\| = 1$ if $Y \neq X$. Moreover, q is an open map.

Given another normed space Z and $T : X \to Z$ linear, bounded such that $Y \subseteq \ker T$, there exists a unique map $\widetilde{T}$ such that $T = \widetilde{T} \circ q$. Moreover $\widetilde{T}$ is linear and bounded with $\|\widetilde{T}\| = \|T\|$. Indeed, $\widetilde{T}(D_{X/Y}) = \widetilde{T}(q(D_X)) = T(D_X)$, so $\|\widetilde{T}\| = \|T\|$.

Theorem 1.6. *Let X be a normed space. If X^* is separable, then so is X.*

Proof. Let $\{f_n \mid n \in \mathbb{N}\}$ be dense in S_{X^*}. For all n choose $x_n \in B_X$ such that $f_n(x_n) > \frac{1}{2}$. Let $Y = \overline{\operatorname{span}\{x_n \mid n \in \mathbb{N}\}}$. Then Y is separable, so enough to show that $Y = X$. If $Y \neq X$, then can pick $h \in S_{(X/Y)^*}$. Set $f = h \circ q$ where $q : X \to X/Y$ is the quotient map. Then $\|f\| = \|h\| = 1$, i.e. $f \in S_{X^*}$. Now for all $n \in \mathbb{N}$, $\|f_n - f\| \geq |(f_n - f)(x_n)| > \frac{1}{2}$ since $f|_Y = 0$. This is a contradiction since the $\{f_n\}$ were assumed to be dense in S_{X^*}. □

Remark: The converse is false, e.g. $X = \ell_1$ is separable, but $X^* \cong \ell_\infty$ is not.

Theorem 1.7. *Every separable normed space X embeds isometrically into ℓ_∞.*

Proof. Let $\{x_n \mid n \in \mathbb{N}\}$ be dense in X. For all n there exists $f_n \in S_{X^*}$ such that $f_n(x_n) = \|x_n\|$. For $x \in X$ and for all $n \in \mathbb{N}$, $|f_n(x)| \leq \|x\|$, so $(f_n(x))_{n=1}^\infty \in \ell_\infty$. Define $T : X \to \ell_\infty$ by $Tx = (f_n(x))_{n=1}^\infty$. This is well-defined, linear and bounded (by above $\|Tx\| \leq \|x\|$). For all n, $\|Tx_n\|_\infty \geq |f_n(x_n)| = \|x_n\|$, so $\|Tx_n\|_\infty = \|x_n\|$ for all n. By dense, T is isometric. □

Remarks:

1. This says that ℓ_∞ is *isometrically universal* for the class $\mathcal{SB}$ of separable Banach space.
2. A dual version of the theorem says that every separable Banach space is a quotient of ℓ_1 (exercise).

Theorem 1.8 (Vector-valued Liouville)**.** *Let X be a complex Banach space and $f : \mathbb{C} \to X$ be holomorphic*[1] *and bounded. Then f is constant.*

[1] $f : \mathbb{C} \to X$ is holomorphic if the limit $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists for all $z_0 \in \mathbb{C}$.

Proof. Fix $w \in \mathbb{C}$. We show that $f(w) = f(0)$. Let $\varphi \in X^*$ and consider $\varphi \circ f : \mathbb{C} \to \mathbb{C}$. Then $\varphi \circ f$ is bounded and holomorphic, hence constant by the ordinary Liouville theorem, so $\varphi(f(w)) = \varphi(f(0))$. Since X^* separates points in X, $f(w) = f(0)$. $\square$

1.4 Locally convex spaces

Definition. *A* locally convex space *(LCS) is a pair $(X, \mathcal{P})$ where X is a real or complex vector space and $\mathcal{P}$ is a family of seminorms on X that separates the points of X in the sense that for every $x \in X \setminus \{0\}$ there exists $p \in \mathcal{P}$ such that $p(x) \neq 0$.*

The family $\mathcal{P}$ induces a topology on X: A subset $U \subseteq X$ is open iff for every $x \in U$ there exist $n \in \mathbb{N}, p_1, \ldots, p_n \in \mathcal{P}, \varepsilon > 0$ such that $\{y \in X \mid p_k(y - x) < \varepsilon,\ 1 \leq k \leq n\} \subseteq U$.

Remarks:

1. Addition and scalar multiplication are continuous.
2. The topology is Hausdorff.
3. We have $x_n \to x$ iff for every $p \in \mathcal{P}$, $p(x_n - x) \to 0$ (also true for nets).
4. Let Y be a subspace of X and $\mathcal{P}_Y = \{p|_Y \mid p \in \mathcal{P}\}$. Then $(Y, \mathcal{P}_Y)$ is a LCS and the corresponding topology is the subspace topology induced by X.
5. Given families $\mathcal{P}, \mathcal{Q}$ of seminorms on X both separating the points of X, they are called *equivalent* (written $\mathcal{P} \sim \mathcal{Q}$) if they induce the same topology on X.

   Fact: A LCS $(X, \mathcal{P})$ is metrizable iff there exists a countable $\mathcal{Q} \sim \mathcal{P}$.

Definition. *A Fréchet space is a complete metrizable LCS.*

Examples.

1. Every normed space $(X, \|\cdot\|)$ is a LCS with $\mathcal{P} = \{\|\cdot\|\}$.
2. Let U be a non-empty open subset of $\mathbb{C}$. Let $\mathcal{O}(U)$ be the set of holomorphic functions on U. For a compact set $K \subseteq U$, define $p_K(f) := \sup\{|f(z)| : z \in K\}$ for all $f \in \mathcal{O}(U)$. Let $\mathcal{P} = \{p_K \mid K \subseteq U, K \text{ compact}\}$. Then $(\mathcal{O}(U), \mathcal{P})$ is a LCS whose topology is the topology of local uniform convergence. There exist compact sets K_n such that $K_n \subseteq \operatorname{Int} K_{n+1}$ and $\bigcup K_n = U$. One can check that $\{p_{K_n} \mid n \in \mathbb{N}\} \sim \mathcal{P}$. So $(\mathcal{O}(U), \mathcal{P})$ is metrizable and in fact a Fréchet space. It is not *normable*, i.e. its topology is not induced by a norm. This follows from Montel's theorem: If $(f_n) \in \mathcal{O}(U)$ is such that for every compact $K \subseteq U$, $\{f_n|_K \mid n \in \mathbb{N}\}$ is bounded in $(C(K), \|\cdot\|_\infty)$, then (f_n) has a convergent subsequence.
3. Fix $d \in \mathbb{N}$ and let Ω be a non-empty open subset of $\mathbb{R}^d$. Let $C^\infty(\Omega)$ be the space of all smooth functions $\Omega \to \mathbb{R}$. For a multiindex $\alpha = (\alpha_1, \ldots, \alpha_d) \in (\mathbb{Z}_{\geq 0})^d$ we define

$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d}$. For $\alpha \in (\mathbb{Z}_{\geq 0})^d$, compact $K \subseteq \Omega$, and $f \in C^\infty(\Omega)$ let

$$p_{K,\alpha}(f) = \sup\{|D^\alpha f(x)| : x \in K\}.$$

Let $\mathcal{P} = \{p_{K,\alpha} \mid K \subseteq \Omega \text{ compact}, \alpha \in \mathbb{Z}^d_{\geq 0}\}$. Then $(C^\infty(\Omega), \mathcal{P})$ is a LCS. It is a Fréchet space and is not normable.

4. Weak and Weak* topology - see Chapter 3.

Lemma 1.9. *Let $(X, \mathcal{P})$ and $(Y, \mathcal{Q})$ be LCSs and $T : X \to Y$ be linear. Then TFAE:*

(i) T is continuous.

(ii) T is continuous at 0.

(iii) For all $q \in \mathcal{Q}$ there exists $n \in \mathbb{N}$, $p_1, \ldots, p_n \in \mathcal{P}$, $C \geq 0$ such that for all $x \in X$, $q(Tx) \leq C \max_{1 \leq k \leq n} p_k(x)$.

Proof. "$(i) \Leftrightarrow (ii)$" is clear. For "$(ii) \Rightarrow (iii)$" let $q \in \mathcal{Q}$ and $V = \{y \in Y \mid q(y) \leq 1\}$. Then V is a neighborhood of 0 in Y, so there exists a neighborhood U of 0 in X such that $T(U) \subseteq V$. WLOG $U = \{x \in X \mid p_k(x) \leq \varepsilon \text{ for } 1 \leq k \leq n\}$ for some $n \in \mathbb{N}, p_1, \ldots, p_n \in \mathcal{P}, \varepsilon > 0$. Let $x \in X$ and $t = \max_{1 \leq k \leq n} p_k(x)$. We show $q(Tx) \leq \frac{1}{\varepsilon} t$. If $t > 0$, then $p_k(\frac{\varepsilon x}{t}) \leq \varepsilon$ for $1 \leq k \leq n$, so $\frac{\varepsilon x}{t} \in U$ and $q(T(\frac{\varepsilon x}{t})) \leq 1$, i.e. $q(Tx) \leq \frac{t}{\varepsilon}$. If $t = 0$, then for all scalars λ, $p_k(\lambda x) = 0$ for all $1 \leq k \leq n$, so $\lambda x \in U$ and $q(T(\lambda x)) \leq 1$. So $q(Tx) = 0$.

Conversely, "$(iii) \Rightarrow (ii)$". Let V be a neighborhood of 0 in Y. We seek a neighborhood U of 0 in X such that $T(W) \subseteq V$. WLOG, $V = \{y \in Y \mid q_k(y) \leq \varepsilon \text{ for } 1 \leq k \leq n\}$ for some $n \in \mathbb{N}, q_1, \ldots, q_n \in \mathcal{Q}, \varepsilon > 0$. By (iii) for each $k = 1, \ldots, n$ there exists $m_k \in \mathbb{N}, p_{k1}, \ldots, p_{km_k} \in \mathcal{P}$ and $C_k \geq 0$ such that $q_k(Tx) \leq C_k \max_{1 \leq j \leq m_k} p_{kj}(x)$. Let $U = \{x \in X \mid p_{kj}(x) \leq \frac{\varepsilon}{C_k + 1}, 1 \leq j \leq m_k, 1 \leq k \leq n\}$. This is a neighborhood of 0 in X and $T(U) \leq V$. ☐

Definition. *Let $(X, \mathcal{P})$ be a LCS. The dual space of $(X, \mathcal{P})$ is the space X^* of all continuous linear functionals on X, i.e. all linear maps $X \to \mathbb{K}$ which are continuous in the topology of $(X, \mathcal{P})$.*

Lemma 1.10. *Let $(X, \mathcal{P})$ be a LCS, $f : X \to \mathbb{K}$ be linear. Then $f \in X^* \Leftrightarrow \ker f$ is closed.*

Proof. "$\Rightarrow$" is clear. For "$\Leftarrow$" we may assume that $\ker f \neq X$. Fix $x_0 \in X \setminus \ker f$. Then there exists a neighborhood U of 0 such that $x_0 + U \subseteq X \setminus \ker f$. WLOG $U = \{x \in X \mid p_k(x) \leq \varepsilon, 1 \leq k \leq n\}$ for some $n \in \mathbb{N}, p_1, \ldots, p_n \in \mathcal{P}, \varepsilon > 0$. Note that U is convex and balanced. Since f is linear, the same is true for $f(U)$. So either $f(U)$ is bounded or $f(U) = \mathbb{K}$. In the latter case, $f(x_0 + U) = f(x_0) + f(U) = \mathbb{K}$, contradicting $x_0 \notin \ker f$. So there exists $M \geq 0$ such that $|f(x)| \leq M$ for all $x \in U$, i.e. $f(U) \subseteq \{\lambda \mid |\lambda| \leq M\}$. Hence for all $\varepsilon > 0$, $f(\frac{\varepsilon}{M} U) \subseteq \{\lambda \mid |\lambda| \leq \varepsilon\}$. So f is continuous at 0 and hence $f \in X^*$. ☐

Theorem 1.11 (Hahn-Banach)**.** *Let $(X, \mathcal{P})$ be a LCS. Then*

(i) Given a subspace Y of X and $g \in Y^*$, there exists $f \in X^*$ such that $f|_Y = g$.

(ii) Given a closed subspace Y of X and $x_0 \in X \setminus Y$, there exists $f \in X^*$ such that $f|_Y = 0, f(x_0) \neq 0$

Proof.

(i) By the characterization of continuous linear maps between LCSs there exists $n \in \mathbb{N}, p_1, \ldots, p_n \in \mathcal{P}, C \geq 0$ such that for all $y \in Y$, $|g(y)| \leq C \max_{1 \leq k \leq n} p_k(y)$. Let $p(x) = C \max_{1 \leq k \leq n} p_k(x)$ for $x \in X$. Then p is a seminorm on X and for all $y \in Y$, $|g(y)| \leq p(y)$. By the seminorm version of Hahn-Banach there exists a linear $f : X \to \mathbb{K}$ such that $f|_Y = g$ and for all $x \in X$, $|f(x)| \leq p(x)$. Then f is continuous.

(ii) Let $Z = \operatorname{span} Y \cup \{x_0\}$ and define $g : Z \to \mathbb{K}$ by $g(y + \lambda x_0) = \lambda$ where $y \in Y, \lambda \in \mathbb{K}$. Then g is linear and $\ker g = Y$, so by the previous Lemma, $g \in Z^*$. Then extend g to X by part (i). □

2 The dual spaces of $L_p(\mu)$ and $C(K)$

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. For $1 \le p < \infty$ we have

$$L_p(\mu) = \Big\{ f : \Omega \to \mathbb{K} \mid f \text{ is measurable and } \int_\Omega |f|^p d\mu < \infty \Big\}.$$

This is a (semi-)normed space in the L_p-norm: $\|f\|_p = \left(\int_\Omega |f|^p d\mu\right)^{1/p}$.

For $p = \infty$ we have

$$L_\infty(\mu) = \{ f : \Omega \to \mathbb{K} \mid f \text{ is measurable and essentially bounded}\}.$$

This is a (semi-)normed space in the L_∞-norm:

$$\|f\|_\infty = \operatorname{ess\,sup} |f| = \inf\{\sup_{\Omega \setminus N} |f| : N \in \mathcal{F}, \mu(N) = 0\}.$$

The essential sup is attained: There exists $N \in \mathcal{F}$ such that $\mu(N) = 0$ and $\sup_{\Omega \setminus N} |f| = \|f\|_\infty$.

Remark: Technically, for $1 \le p \le \infty$, the L_p-norm is only a seminorm. In general, if $\|\cdot\|$ is a seminorm on a real or complex vector space X, then $N = \{x \in X \mid \|x\| = 0\}$ is a subspace and $\|x + N\| = \|x\|$ defines a norm on X/N. So for us equality in L_p will mean a.e. equality.

We also recall:

Theorem 2.1. *$L_p(\mu)$ is a Banach space for $1 \le p \le \infty$.*

2.1 Dual space of L_p

2.1.1 Complex measures

Let Ω be a set and $\mathcal{F}$ a σ-field on Ω. A *complex measure* on $\mathcal{F}$ is a countably additive set function $\nu : \mathcal{F} \to \mathbb{C}$, i.e. $\mu(\emptyset) = 0$ and $\nu(\cup A_n) = \sum \nu(A_n)$ for countably many pairwise disjoint $A_n \in \mathcal{F}$.

The *total variation measure* $|\nu|$ of ν is defined as follows: For $A \in \mathcal{F}$,

$$|\nu|(A) = \sup \left\{ \sum_{k=1}^n |\nu(A_k)| : A = \bigcup_{k=1}^n A_k \text{ is a measurable partition of } A \right\}.$$

Note that $|\nu| : \mathcal{F} \to [0, \infty]$ is a positive measure (i.e. a measure). $|\nu|$ is the smallest positive measure dominating ν (i.e. for all $A \in \mathcal{F}$, $|\nu(A)| \leq |\nu|(A)$ and if μ is a positive measure such that for all $A \in \mathcal{F}, |\nu(A)| \leq \mu(A)$, then $|\nu| \leq \mu$).

In fact, $|\nu|$ is a finite measure (see Remark 3 below). The *total variation* $\|\nu\|_1$ of ν is defined by $\|\nu\|_1 = |\nu|(\Omega)$.

Remark: Any complex measure ν is continuous (from below and above).

A *signed measure* on $\mathcal{F}$ is a countably additive set function $\nu : \mathcal{F} \to \mathbb{R}$ (i.e. a complex measure that only takes on real values).

Theorem 2.2. *Let Ω be a set, $\mathcal{F}$ a σ-field on Ω and $\nu : \mathcal{F} \to \mathbb{R}$ a signed measure. Then there exists a measurable partition $\Omega = P \cup N$ of Ω such that for all $A \in \mathcal{F}$, $A \subseteq P \implies \nu(A) \geq 0$ and $A \subseteq N \implies \nu(A) \leq 0$.*

Remarks:

1. The partition $\Omega = P \cup N$ is the *Hahn decomposition* of Ω (or of ν).
2. Define $\nu^+(A) = \nu(A \cap P)$ and $\nu^-(A) = -\nu(A \cap N)$ for $A \in \mathcal{F}$. Then ν^+ and ν^- are finite positive measures such that $\nu = \nu^+ - \nu^-$ and $|\nu| = \nu^+ + \nu^-$. These properties determine ν^+, ν^- uniquely. This is called the *Jordan decomposition* of ν.
3. If $\nu : \mathcal{F} \to \mathbb{C}$ is a complex measure, then $\operatorname{Re}\nu$, $\operatorname{Im}\nu$ are signed measures with Jordan decomposition $\operatorname{Re}\nu = \nu_1 - \nu_2$ and $\operatorname{Im}\nu = \nu_3 - \nu_4$. Hence $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$ (the *Jordan decomposition* of ν). It follows that $\nu_k \leq |\nu|$ for $k = 1, 2, 3, 4$ and $|\nu| \leq \nu_1 + \nu_2 + \nu_3 + \nu_4$, hence $|\nu|(\Omega) < \infty$.
4. Let $\nu = \nu^+ - \nu^-$ as in 2. For $A, B \in \mathcal{F}$ if $B \subseteq A$, then $\nu(B) = \nu^+(B) - \nu^-(B) \leq \nu^+(B) \leq \nu^+(A)$. Also, $P \cap A \subseteq A$ and $\nu(P \cap A) = \nu^+(A)$, so $\nu^+(A) = \sup\{\nu(B) \mid B \in \mathcal{F}, B \subseteq A\}$ for any $A \in \mathcal{F}$. This will be the idea of the proof.

Proof of Theorem 2.2. Define $\nu^+(A) := \sup\{\nu(B) \mid B \in \mathcal{F}, B \subseteq A\}$ for $A \in \mathcal{F}$. Then $\nu^+(\emptyset) = 0$ and ν^+ is finitely additive and positive.

Claim: $\nu^+(\Omega) < \infty$: Assume $\nu^+(\Omega) = \infty$. Inductively construct $(A_n)_{n=1}^{\infty}$ and $(B_n)_{n=0}^{\infty}$ in $\mathcal{F}$ such that $B_0 = \Omega$ and for all $n \in \mathbb{N}$, $\nu^+(B_{n-1}) = \infty$, $A_n \subseteq B_{n-1}$, $\nu(A_n) > n$ and $B_n = A_n$ or $B_{n-1} \setminus A_n$ (such that $\nu^+(B_n) = \infty$). Then either there exists N such that for all $n \geq N$, $A_n \supseteq A_{n+1}$. Then $\nu(\cap A_n) = \lim \nu(A_n)$, a contradiction. Or there exist $k_1 \subseteq k_2 \subseteq \dots$ such that for $m \neq n$, $A_{k_m} \cap A_{k_n} = \emptyset$. So $\nu(\cup A_{k_n}) = \sum \nu(A_{k_n})$, a contradiction.

Claim: There exists $P \in \mathcal{F}$ such that $\nu^+(\Omega) = \nu(P)$. For all n there exists $A_n \in \mathcal{F}$ such that $\nu(A_n) > \nu^+(\Omega) - 2^{-n}$. For $m \neq n$, $\nu(A_m \cap A_n) = \nu(A_m) + \nu(A_n) - \nu(A_m \cup A_n) > \nu^+(\Omega) - 2^{-m} - 2^{-n}$. Let $P = \bigcup_n \bigcap_{m \geq n} A_m$. Then $\nu^+(\Omega) \geq \nu(P) = \lim_{m\to\infty} \lim_{k\to\infty} \nu(A_m \cap A_{m+1} \cap \dots \cap A_{m+k}) \geq \nu^+(\Omega)$. Then let $N = \Omega \setminus P$. This works. $\square$

2.1.2 Absolute continuity

Throughout $(\Omega, \mathcal{F}, \mu)$ is a measure space.

Definition. *A complex measure $\nu : \mathcal{F} \to \mathbb{C}$ is* absolutely continuous *w.r.t. μ if for all $A \in \mathcal{F}$, $\mu(A) = 0$ implies $\nu(A) = 0$. We denote this by $\nu \ll \mu$.*

Remarks:

1. If $\nu \ll \mu$, then $|\nu| \ll \mu$. In this case, if $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$ is the Jordan decomposition of ν, then $\nu_k \ll \mu$ for all k.
2. If $\nu \ll \mu$, then $\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall A \in \mathcal{F} : \mu(A) < \delta \implies |\nu(A)| < \varepsilon$.

Example. Let $f \in L_1(\mu)$. Then $\nu(A) = \int_A f d\mu$, $A \in \mathcal{F}$, defines a complex measure and $\nu \ll \mu$.

Theorem 2.3 (Radon-Nikodym)**.** *Let μ be σ-finite and $\nu : \mathcal{F} \to \mathbb{C}$ be a complex measure such that $\nu \ll \mu$. Then there exists a unique $f \in L_1(\mu)$ such that $\nu(A) = \int_A f d\mu$ for all $A \in \mathcal{F}$. Moreover f takes values in $\mathbb{C}/\mathbb{R}/\mathbb{R}^+$ according to whether ν is a complex/signed/positive measure, respectively.*

Proof. Uniqueness is clear from basic measure theory. Existence: wlog ν is a positive measure (take Jordan decomposition). Wlog μ is finite. Let

$$\mathcal{H} = \Big\{ h : \Omega \to \mathbb{R}^+ \mid h \text{ is measurable and } \forall A \in \mathcal{F} : \int_A h d\mu \leq \nu(A) \Big\}.$$

Note: $0 \in \mathcal{H}$. If $h_1, h_2 \in \mathcal{H}$, then also $h_1 \vee h_2 = \max(h_1, h_2) \in \mathcal{H}$. If $h_n \in \mathcal{H}$ for all n and $h_n \nearrow h$, then $h \in \mathcal{H}$.

Let $\alpha = \sup\{\int_\Omega h d\mu \mid h \in \mathcal{H}\}$. Note $0 \leq \alpha \leq \nu(\Omega)$. Choose $h_n \in \mathcal{H}$ such that $\int_\Omega h_n d\mu \to \alpha$. Wlog $h_n \leq h_{n+1}$ for all n (replace h_n by $h_1 \vee h_2 \vee \cdots \vee h_n$). Then $h_n \nearrow f \in \mathcal{H}$ and $\int_\Omega f d\mu = \alpha$ by monotone convergence. So we have $f \geq 0$ measurable, such that for all $A \in \mathcal{F} : \int_A f d\mu \leq \nu(A)$.

For $n \in \mathbb{N}$ and $A \in \mathcal{F}$ define

$$\nu_n(A) = \int_A \left(f + \frac{1}{n} \right) d\mu - \nu(A) = \int_A f d\mu + \frac{1}{n}\mu(A) - \nu(A).$$

ν_n has Hahn-decomposition $\Omega = P_n \cup N_n$. For $A \subseteq N_n$ measurable, we have $0 \geq \nu_n(A) = \int_A (f + \frac{1}{n}) d\mu - \nu(A)$, so $\int_A (f + \frac{1}{n}) d\mu \leq \nu(A)$. Therefore $f + \frac{1}{n} 1_{N_n} \in \mathcal{H}$, and then

$$\alpha \geq \int_\Omega \left(f + \frac{1}{n} 1_{N_n} \right) d\mu = \alpha + \frac{1}{n}\mu(N_n),$$

so $\mu(N_n) = 0$. Let $N = \bigcup_n N_n$, $P = \bigcap_n P_n$. Then $\Omega = P \cup N$, $P \cap N = \emptyset$, $\mu(N) = 0 = \nu(N)$ (as $\nu \ll \mu$).

For $A \in \mathcal{F}$, $n \in \mathbb{N}$ we have

$$\nu(A) = \nu(A \cap P) = \int_{A\cap P} f d\mu + \frac{1}{n}\mu(A \cap P) - \nu_n(A \cap P) \leq \int_A f d\mu + \frac{1}{n}\mu(P).$$

Now let $n \to \infty$ and we are done. □

Remarks:

1. The proof shows that any complex measure ν can be written as $\nu = \nu_1 + \nu_2$ where $\nu_1 \ll \mu$ and $\nu_2 \perp \mu$ (i.e. there exists $N \in \mathcal{F}$ such that $\mu(N) = 0, |\nu_2|(\Omega \setminus N) = 0$). This is the *Lebesgue decomposition* of ν.
2. The unique f in the theorem is called the *Radon-Nikodym derivative* of ν w.r.t. μ, denoted $\frac{d\nu}{d\mu}$. One can prove that for measurable $g : \Omega \to \mathbb{C}$, g is ν-integrable iff $g\frac{d\nu}{d\mu}$ is μ-integrable, and then
$$\int_\Omega g d\nu = \int_\Omega g \frac{d\nu}{d\mu} d\mu.$$

2.1.3 Dual Space of L_p

We fix a measure space $(\Omega, \mathcal{F}, \mu)$ throughout. Let $1 \leq p < \infty$ and let q be the conjugate index of p.

For $g \in L_q = L_q(\mu)$ we define $\varphi_g : L_p \to \mathbb{K}$ by $\varphi_g(f) = \int_\Omega g f d\mu$. By Hölder this is well-defined and $|\varphi_g(f)| \leq \|g\|_q \|f\|_p$, so $\varphi_g \in L_p^*$ and $\|\varphi_g\| \leq \|g\|_q$. So we have a linear map $\varphi : L_q \to L_p^*, g \mapsto \varphi_g$.

Theorem 2.4. *Let* $(\Omega, \mathcal{F}, \mu)$, p, q, φ *be as above.*

(i) If $1 < p < \infty$, *then* φ *is an isometric isomorphism, so* $L_p^* \cong L_q$.

(ii) If $p = 1$ *and* μ *is* σ*-finite, then* φ *is an isometric isomorphism, so* $L_1^* \cong L_\infty$.

Proof.

(i) φ isometric: Let $g \in L_q$. We have seen $\|\varphi_g\| \leq \|g\|_q$. Let

$$f = \begin{cases} |g|^q/g & \text{if } g \neq 0, \\ 0 & \text{if } g = 0 \end{cases}$$

Then
$$\int_\Omega |f|^p d\mu = \int_\Omega |g|^{(q-1)p} d\mu = \int_\Omega |g|^q d\mu$$

So $f \in L_p$ and $\|f\|_p^p = \|g\|_q^q$. Thus $\|\varphi_g\| \cdot \|f\|_p \geq |\varphi_g(f)| = \int_\Omega |g|^q d\mu = \|g\|_q^q$. Hence $\|\varphi_g\| \geq \|g\|_q^{q-\frac{q}{p}} = \|g\|_q$.

φ onto:

- Case 1: μ is finite. Fix $\psi \in L_p^*$. Seek $g \in L_q$ such that $\psi(f) = \int_\Omega gf d\mu$ for all $f \in L_p$. Define $\nu(A) = \psi(1_A)$ (note that $1_A \in L_p$ since μ is finite) for $A \in \mathcal{F}$. Then $\nu(\emptyset) = 0$ and if $A = \bigcup_{n=1}^\infty A_n$ is a measurable partition, then

$$\Big|\nu(A) - \sum_{n=1}^N \nu(A_n)\Big| = \Big|\psi\big(1_{A\setminus\bigcup_{n=1}^N A_n}\big)\Big|$$
$$\leq \|\psi\| \, \|1_{A\setminus\bigcup_{n=1}^N A_n}\|_p = \|\psi\| \, \mu\Big(A \setminus \bigcup_{n=1}^N A_n\Big)^{1/p} \to 0.$$

So ν is countably additive and if $\mu(A) = 0$, then $\nu(A) = \psi(1_A) = 0$, so $\nu \ll \mu$.

By the Radon-Nikodym theorem, there exists $g \in L_1(\mu)$ such that $\nu(A) = \int_A g d\mu$ for all $A \in \mathcal{F}$. So $\psi(1_A) = \int_A g 1_A d\mu$ for all $A \in \mathcal{F}$, and hence $\psi(f) = \int_g f d\mu$ for all simple functions f. Given $f \in L_\infty \subseteq L_p$ there exists simple functions f_n such that for all $n \in \mathbb{N}$, $|f_n| \leq |f|$ and $f_n \to f$ a.e. Then $f_n \to f$ in L_p and $gf_n \to gf$ in L_1 by dominated convergence. So $\int_\Omega gf d\mu = \lim_{n\to\infty} \int_\Omega gf_n d\mu = \lim_{n\to\infty} \psi(f_n) = \psi(f)$ as ψ is continuous. For $n \in \mathbb{N}$ let $A_n = \{0 < |g| \leq n\}$. Then $f = \frac{|g|^q}{g} 1_{A_n} \in L_\infty$, so

$$\int_\Omega gf d\mu = \int_{A_n} |g|^q d\mu = \psi(f) \leq \|\psi\| \, \|f\|_p = \|\psi\| \cdot \Big(\int_{A_n} |g|^q d\mu\Big)^{1/p}$$

So $\left(\int_{A_n} |g|^q\right)^{1/q} \leq \|\psi\|$, so by monotone convergence $g \in L_q$ and $\|g\|_q \leq \|\psi\|$. Now $\varphi_g, \psi \in L_p^*$ and φ_g, ψ agree on the dense subspace L_∞, so $\varphi_g = \psi$.

For the other cases we introduce some notation. For $B \in \mathcal{F}$, let $\mathcal{F}_B = \{A \in \mathcal{F} \mid A \subseteq B\}$ and $\mu_B = \mu|_{\mathcal{F}_B}$. Then $(B, \mathcal{F}_B, \mu_B)$ is a measure space and $L_p(\mu_B) \subseteq L_p(\mu)$. Given $\psi \in L_p(\mu)^*$, let $\psi_B = \psi|_{L_p(\mu_B)}$. Then $\psi_B \in L_p(\mu_B)^*$ and $\|\psi_B\| \leq \|\psi\|$.

Claim: Let $B, C \in \mathcal{F}$ with $B \cap C = \emptyset$. Then $\|\psi_{B\cup C}\| = (\|\psi_B\|^q + \|\psi_C\|^q)^{1/q}$. Proof: Given $f \in L_p(\mu_{B\cup C})$, we have

$$\begin{aligned} |\psi_{B\cup C}(f)| &\leq |\psi_B(f|_B)| + |\psi_C(f|_C)| \leq \|\psi_B\| \, \|f|_B\|_p + \|\psi_C\| \, \|f|_C\|_p \\ &\leq (\|\psi_B\|^q + \|\psi_C\|^q)^{1/q} (\|f|_B\|_p^p + \|f|_C\|_p^p)^{1/p} \\ &= (\|\psi_B\|^q + \|\psi_C\|^q)^{1/q} \, \|f\|_p \, . \end{aligned}$$

So $\|\psi_{B\cup C}\| \leq (\|\psi_B\|^q + \|\psi_C\|^q)^{1/q}$. Fix $a, b \geq 0$ with $a^p + b^p = 1$ and $a\|\psi_B\| + b\|\psi_C\| = (\|\psi_B\|^q + \|\psi_C\|^q)^{1/q}$ (use $\ell_q^* \cong \ell_p$). Let $f \in L_p(\mu_B)$, $g \in L_p(\mu_C)$ such that $\|f\|_p \leq 1, \|g\|_p \leq 1$. Fix scalars λ, μ such that $|\lambda| = |\mu| = 1$ and $\lambda\psi_B(f) = |\psi_B(f)|$ and $\mu\psi_C(g) = |\psi_C(g)|$. Then

$$a|\psi_B(f)| + b|\psi_C(g)| = \psi_{B\cup C}(a\lambda f + b\mu g) \leq \|\psi_{B\cup C}\| \, \|a\lambda f + b\mu g\|_p \leq \|\psi_{B\cup C}\| \, .$$

Taking sup over f, g we get $a\|\psi_B\| + b\|\psi_C\| \leq \|\psi_{B\cup C}\|$.

- Case 2: μ is σ-finite. So there exists a measurable partition $\Omega = \bigcup_{n\in\mathbb{N}} A_n$ of Ω such that $\mu(A_n) < \infty$. By case 1, for every n, there exists $g_n \in L_q(\mu_{A_n})$ such that $\psi_{A_n}(f) = \int_{A_n} gfd\mu_{A_n}$ for all $f \in L_p(\mu A_n)$. Since φ is isometric, $\|\psi_{A_n}\| = \|g_n\|_q$. Let $g = g_n$ on A_n for all n. Then
$$\sum_{n=1}^N \|g_n\|_q^q = \sum_{n=1}^N \|\psi_{A_n}\|^q = \left\|\psi_{\bigcup_{n=1}^N A_n}\right\|^q \le \|\psi\|^q .$$
So by monotone convergence $g \in L_q(\mu)$, we have $\varphi_g = \psi$ on $L_p(\mu_{A_n})$ for every n. Since $\bigcup_n L_p(\mu_{A_n})$ has dense linear span, $\varphi_g = \psi$.

- General case. First recall that for $f \in L_p(\mu)$, $\{f \neq 0\} = \bigcup_{n\in\mathbb{N}}\{|f| > \frac{1}{n}\}$ is σ-finite since $\mu(\{|f| > \frac{1}{n}\}) \le n^p \|f\|_p^p < \infty$ (Markov). Let $\psi \in L_p(\mu)^*$. There exists a sequence (f_n) in $L_p(\mu)$ such that $\|f_n\|_p \le 1$ and $\psi(f_n) \to \|\psi\|$. Then $B = \bigcup_{n\in\mathbb{N}}\{f_n \neq 0\}$ is σ-finite and $\|\psi_B\| = \|\psi\|$. By the claim, $\|\psi\|^q = \|\psi_B\|^q + \left\|\psi_{\Omega\setminus B}\right\|^q$, so $\psi_{\Omega\setminus B} = 0$. So we are done by case 2.

(ii) φ isometric: Let $g \in L_\infty(\mu)$. We already have $\|\varphi_g\| \le \|g\|_\infty$. For the reverse, wlog $g \neq 0$. Fix $0 < s < \|g\|_\infty$. Let $A = \{|g| > s\}$. Then $\mu(A) > 0$. Then $\mu(A) > 0$. Since μ is σ-finite, there exists $B \subseteq A$, $0 < \mu(B) < \infty$. Let $f = \frac{|g|}{g} 1_B$. Then $f \in L_1$ and
$$s\mu(B) \le \varphi_g(f) = \int_B |g| d\mu \le \|\varphi_g\| \|f\|_1 = \|\varphi_g\| \mu(B).$$
Then $s \le \|\varphi_g\|$, so $\|g\|_\infty \le \|\varphi\|_g$.

φ onto:

- Case 1: μ is finite. Let $\psi \in L_1^*$ and proceed as in (i): Define $\nu(A) = \psi(1_A)$. As before, ν is a complex measure and $\nu \ll \mu$, so by the Radon-Nikodym theorem there exists $g \in L_1$ such that $\nu(A) = \int_A gd\mu$ for all $A \in \mathcal{F}$. Thus $\int_\Omega g1_A d\mu = \psi(1_A)$ for all $A \in \mathcal{F}$. As before, $\int_A gfd\mu = \psi(f)$ for all $f \in L_\infty$ ($L_\infty \subseteq L_1$ since μ is finite).

  Next we show that $g \in L_\infty$. Fix $t > \|\psi\|$ and let $A = \{|g| > t\}$ and $f = |g|/g1_A$. Then $f \in L_\infty$ and so
$$t\mu(A) \le \int_A |g| d\mu = \int_\Omega gfd\mu = \psi(f) \le \|\psi\| \|f\|_1 = \|\psi\| \mu(A)$$
  Hence $\mu(A) = 0$ and $g \in L_\infty$.

  Now $\varphi_g = \psi$ on L_∞, L_∞ dense in L_1 and so $\varphi_g = \psi$.

- Case 2: μ is σ-finite. So there exists a measurable partition $\Omega = \bigcup_n A_n$ of Ω such that $\mu(A_n) < \infty$ for every n. Let $\psi \in L_1(\mu)^*$. By case 1, for every n there exists $g_n \in L_\infty(\mu_{A_n})$ such that $\psi_{A_n}(f) = \int_{A_n} g_n fd\mu_{A_n}$ for all $f \in L_1(\mu_{A_n})$.

  φ is isometric, so $\|g_n\|_\infty = \|\psi_{A_n}\| \le \|\psi\|$. Let $g = g_n$ on A_n for all n. Then $g \in L_\infty(\mu)$. Have $\varphi_g = \psi$ on $L_1(\mu_{A_n})$ for all n. By density $\varphi_g = \psi$.

$\square$

Corollary 2.5. *For any measure space $(\Omega, \mathcal{F}, \mu)$ and $1 < p < \infty$, the Banach space $(L_p(\mu), \|\cdot\|_p)$ is reflexive.*

Proof. By the theorem we have an isometric isomorphism $\varphi : L_q \to L_p^*$, $\langle f, \varphi(g)\rangle = \int_\Omega gf d\mu$ for $f \in L_p,\ g \in L_q$. This induces an isometric isomorphism $\varphi^* : L_p^{**} \to L_q^*$. Also there is an isometric isomorphism $\psi : L_p \to L_q^*$ given by $\langle g, \psi f\rangle = \int_\Omega fg d\mu$.

Hence we get an isometric isomorphism $(\varphi^*)^{-1} \circ \psi : L_p \to L_p^{**}$. For $f \in L_p,\ g \in L_q$ we have

$$\langle g, \varphi^*(\widehat{f})\rangle = \langle \varphi(g), \widehat{f}\rangle = \langle f, \varphi g\rangle = \int_\Omega gf d\mu = \langle g, \psi(f)\rangle$$

So $\varphi^*(\widehat{f}) = \psi(f)$, i.e. $(\varphi^*)^{-1}\psi(f) = \widehat{f}$. $\square$

2.2 Dual space of $C(K)$

Throughout K is a compact Hausdorff space. Some notation:

$$\begin{aligned} C(K) &= \{f : K \to \mathbb{C} \mid f \text{ continuous}\} \\ C^{\mathbb{R}}(K) &= \{f : K \to \mathbb{R} \mid f \text{ continuous}\} \\ C^+(K) &= \{f \in C^{\mathbb{R}}(K) \mid f \geq 0\} \end{aligned}$$

$C(K),\ C^{\mathbb{R}}(K)$ are complex resp. real Banach spaces in the sup norm $\|\cdot\|_\infty$. We also let

$$\begin{aligned} M(K) = C(K)^* &= \{\varphi : C(K) \to \mathbb{C} \mid \varphi \text{ linear, bounded}\} \\ M^{\mathbb{R}}(K) &= \{\varphi \in M(K) \mid \varphi(f) \in \mathbb{R} \text{ for all } f \in C^{\mathbb{R}}(K)\} \\ M^+(K) &= \{\varphi : C(K) \to \mathbb{C} \mid \varphi \text{ linear}, \varphi(f) \geq 0 \text{ for all } f \in C^+(K)\} \end{aligned}$$

Elements of $M^+(K)$ are called *positive linear functionals.*

Aim: Describe $M(K)$ and $C^{\mathbb{R}}(K)^*$. It is enough to consider $M^+(K)$:

Lemma 2.6.

(i) *For all $\varphi \in M(K)$ there exist unique $\varphi_1, \varphi_2 \in M^{\mathbb{R}}(K)$ such that $\varphi = \varphi_1 + i\varphi_2$.*

(ii) *$\varphi \mapsto \varphi|_{C^{\mathbb{R}}(K)} : M^{\mathbb{R}}(K) \to C^{\mathbb{R}}(K)^*$ is an isometric isomorphism.*

(iii) *$M^+(K) \subseteq M(K)$ and $M^+(K) = \{\varphi \in M(K) \mid \|\varphi\| = \varphi(1_K)\}$.*

(iv) *For all $\varphi \in M^{\mathbb{R}}(K)$ there exist unique $\varphi^+, \varphi^- \in M^+(K)$ such that $\varphi = \varphi^+ - \varphi^-$ and $\|\varphi\| = \|\varphi^+\| + \|\varphi^-\|$.*

Proof.

(i) Let $\varphi \in M(K)$. Uniqueness: Assume $\varphi = \varphi_1 + i\varphi_2$ with $\varphi_1, \varphi_2 \in M^{\mathbb{R}}(K)$. For $f \in C^{\mathbb{R}}(K)$ we have $\varphi(f) = \varphi_1(f) + i\varphi_2(f)$ and $\overline{\varphi(f)} = \varphi_1(f) - i\varphi_2(f)$, so $\varphi_1(f) = \frac{\varphi(f)+\overline{\varphi(f)}}{2}, \varphi_2(f) = \frac{\varphi(f)-\overline{\varphi(f)}}{2i}$. So φ_1, φ_2 are determined by φ on $C^{\mathbb{R}}(K)$ and hence on $C(K) = C^{\mathbb{R}}(K) + iC^{\mathbb{R}}(K)$.

   Existence: Define $\varphi_1(f) = \frac{\varphi(f)+\overline{\varphi(\overline{f})}}{2}$, $\varphi_2(f) = \frac{\varphi(f)-\overline{\varphi(\overline{f})}}{2i}$ for $f \in C(K)$. This works.

(ii) If $\varphi \in M^{\mathbb{R}}(K)$, then $\varphi|_{C^{\mathbb{R}}(K)}$ is real-linear and continuous.

   Isometric: We have $\|\varphi|_{C^{\mathbb{R}}(K)}\| \le \|\varphi\|$. Given $f \in C(K)$, there is $\lambda \in \mathbb{C}, |\lambda| = 1$ such that $|\varphi(f)| = \lambda\varphi(f) = \varphi(\lambda f) = \varphi(\operatorname{Re}\lambda f) + i\varphi(\operatorname{Im}\lambda f) = \varphi(\operatorname{Re}(\lambda f)) \le \|\varphi|_{C^{\mathbb{R}}(K)}\| \|\operatorname{Re}(\lambda f)\|_\infty \le \|\varphi_{C^{\mathbb{R}}(K)}\| \, \|f\|_\infty$, so $\|\varphi\| \le \|\varphi_{C^{\mathbb{R}}}(K)\|$.

   Onto: Given $\psi \in C^{\mathbb{R}}(K)^*$, define $\varphi(f) = \psi(\operatorname{Re} f) + i\psi(\operatorname{Im} f)$ for $f \in C(K)$, so φ is continuous, real-linear and $\varphi(if) = i\varphi(f)$ for all $f \in C(K)$. So $\varphi \in M(K)$ and $\varphi_{C^{\mathbb{R}}(K)} = \psi$.

(iii) Let $\varphi \in M^+(K)$ and $f \in C^{\mathbb{R}}(K)$ with $\|f\|_\infty \le 1$. Then $1_K \pm f \ge 0$, so $\varphi(1_K) \pm \varphi(f) = \varphi(1_K \pm f) \ge 0$, so $\varphi(f) \in \mathbb{R}$ and $|\varphi(f)| \le \varphi(1_K)$, so $\|\varphi|_{C^{\mathbb{R}}(K)}\| = \varphi(1_K)$. By (ii), $\varphi \in M^{\mathbb{R}}(K)$ and $\|\varphi\| = \|\varphi|_{C^{\mathbb{R}}(K)}\| = \varphi(1_K)$.

   Now assume $\varphi \in M(K)$ and $\|\varphi\| = \varphi(1_K)$. Aim: $\varphi \in M^+(K)$. WLOG $\|\varphi\| = \varphi(1_K) = 1$. Let $f \in C^{\mathbb{R}}(K)$, $\|f\|_\infty \le 1$. Let $\varphi(f) = a + ib$ with $a, b \in \mathbb{R}$. For $t \in \mathbb{R}$, $|\varphi(f + it1_K)|^2 = |a + i(b+t)|^2 = a^2 + b^2 + 2bt + t^2$. It is also $\le \|\varphi\|^2 \|f + it1_K\|_\infty^2 \le 1 + t^2$. So $a^2 + b^2 + 2bt \le 1$ for all $t \in \mathbb{R}$. Hence $b = 0$. So $\varphi(f) \in \mathbb{R}$ and $\varphi \in M^{\mathbb{R}}(K)$. Let $f \in C^+(K)$, $\|f\|_\infty \le 1$, so $0 \le f \le 1$. Then $-1_K \le 1_K - 2f \le 1_K$ and so $\|1_K - 2f\|_\infty \le 1$. Hence $\varphi(1_K - 2f) = 1 - 2\varphi(f) \le 1$ and hence $\varphi(f) \ge 0$. Thus $\varphi \in M^+(K)$.

(iv) Let $\varphi \in M^{\mathbb{R}}(K)$. Existence: [Idea: If $0 \le g \le f$, then $\varphi(g) = \varphi^+(g) - \varphi^-(g) \le \varphi^+(g) \le \varphi^+(f)$].

   Define φ^+ on $C^+(K)$: For $f \in C^+(K)$, $\varphi^+(f) = \sup\{\varphi(g) : g \in C^+(K), 0 \le g \le f\}$. Note $\varphi^+(f) \ge \varphi(0) = 0$ and $\varphi^+(f) \ge \varphi(f)$. Then φ^+ is positive homogeneous and additive: Let $f_1, f_2 \in C^+(K)$. Given $0 \le g_1 \le f_1, 0 \le g_2 \le f_2$, we have $0 \le g_1 + g_2 \le f_1 + f_2$, so $\varphi^+(f_1 + f_2) \ge \varphi(g_1 + g_2) = \varphi(g_1) + \varphi(g_2)$, so $\varphi^+(f_1 + f_2) \ge \varphi^+(f_1) + \varphi^+(f_2)$. Conversely, given $0 \le g \le f_1 + f_2$, $\varphi(g) = \varphi(g \wedge f_1) + \varphi(g - (g \wedge f_1)) \le \varphi^+(f_1) + \varphi^+(f_2)$. Thus $\varphi^+(f_1 + f_2) \le \varphi^+(f_1) + \varphi^+(f_2)$.

   Now define φ^+ on $C^{\mathbb{R}}(K)$: Given $f \in C^{\mathbb{R}}(K)$, write $f = f_1 - f_2$ for $f_1, f_2 \in C^+(K)$ (e.g. $f_1 = f \vee 0, f_2 = (-f) \vee 0$) Define $\varphi^+(f) = \varphi^+(f_1) - \varphi^+(f_2)$. By properties of φ^+ on $C^+(K)$, φ^+ is well-defined and real linear on $C^{\mathbb{R}}(K)$. Finally, define $\varphi^+(f) = \varphi^+(\operatorname{Re} f) + i\varphi^+(\operatorname{Im} f)$ for $f \in C(K)$. Then φ^+ is complex linear on $C(K)$. From above $\varphi^+ \in M^+(K)$. Then define $\varphi^- = \varphi^+ - \varphi$. For $f \in C^+(K)$, then $\varphi^-(f) = \varphi^+(f) - \varphi(f) \ge \varphi(f) - \varphi(f) = 0$. So $\varphi^- \in M^+(K)$ and $\varphi = \varphi^+ - \varphi^-$.

   Further $\|\varphi\| \le \|\varphi^+\| + \|\varphi^-\| = \varphi^+(1_K) + \varphi^-(1_K) = 2\varphi^+(1_K) - \varphi(1_K)$. If $0 \le f \le 1_K$, then $-1_K \le 2f - 1_K \le 1_K$, so $\|2f - 1_K\|_\infty \le 1$, so $2\varphi(f) - \varphi(1_K) = \varphi(2f - 1_K) \le \|\varphi\|$. Hence $2\varphi^+(1_K) - \varphi(1_K) \le \|\varphi\|$.

Uniqueness: Assume $\varphi = \psi_1 - \psi_2$ where $\psi_1, \psi_2 \in M^+(K)$ and $\|\varphi\| = \|\psi_1\| + \|\psi_2\|$. If $0 \le g \le f$, then $\varphi(g) = \psi_1(g) - \psi_2(g) \le \psi_1(g) \le \psi_1(f)$. Sup over g gives us $\varphi^+(f) \le \psi_1(f)$. So $\psi_1 - \varphi^+ \in M^+(K)$. Hence $\psi_2 - \varphi^- = \psi_1 - \varphi^+ \in M^+(K)$. Thus $\|\psi_1 - \varphi^+\| + \|\psi_2 - \varphi^-\| = (\psi_1 - \varphi^+)(1_K) + (\psi_2 - \varphi^-)(1_K) = (\psi_1(1_K) + \psi_2(1_K)) - (\varphi^+(1_K) + \varphi^-(1_K)) = \|\varphi\| - \|\varphi\| = 0$. Thus $\psi_1 = \varphi^+, \psi_2 = \varphi^-$.

□

2.2.1 Topological Preliminaries

Recall K is normal: for disjoint closed subsets E, F of K there exist disjoint open subsets U, V of K such that $E \subseteq U,\ F \subseteq V$. Equivalently, if $E \subseteq U \subseteq K$ with E closed, U open, there exists an open V such that $E \subseteq V \subseteq \overline{V} \subseteq U$.

Lemma (Urysohn's Lemma)**.** *Given disjoint closed subsets E, F of K there exists a continuous function $f : K \to [0,1]$ such that $f = 0$ on E, $f = 1$ on F.*

Notation: $f \prec U$ means $U \subseteq K$, U open, $f : K \to [0,1]$ continuous and $\operatorname{supp} f \subseteq U$.

$E \prec f$ means $E \subseteq K$, E closed, $f : K \to [0,1]$ continuous and $f = 1$ on E.

Urysohn says: If $E \subseteq U \subseteq K$ with E closed, U open, then there exists f such that $E \prec f \prec U$. [Choose open V such that $E \subseteq V \subseteq \overline{V} \subseteq U$ and apply Urysohn to E, $K \setminus V$.]

Lemma 2.7. *Let $E \subseteq K$ be closed, $n \in \mathbb{N}$, $U_j \subseteq K$ open, $1 \le j \le n$ and $E \subseteq \bigcup_{j=1}^n U_j$.*

(i) There exist open sets V_j with $\overline{V_j} \subseteq U_j$, $1 \le j \le n$ such that $E \subseteq \bigcup_{j=1}^n V_j$.

(ii) There exist $f_j \prec U_j$, $1 \le j \le n$ such that $\sum_{j=1}^n f_j \le 1$ on K and $\sum_{j=1}^n f_j = 1$ on E.

Proof.

(i) By induction on n. $E \setminus U_n$ is closed and covered by $\bigcup_{j<n} U_j$, so by induction there exist open V_j with $\overline{V_j} \subseteq U_j$ such that $E \setminus U_n \subseteq \bigcup_{j<n} V_j$. Then $E \setminus \bigcup_{j<n} V_j \subseteq U_n$, so by normality there exists open V_n such that $E \setminus \bigcup_{j<n} V_j \subseteq V_n \subseteq \overline{V_n} \subseteq U_n$.

(ii) Choose open sets V_j as in (i). By Urysohn there exist functions g_j such that $\overline{V_j} \prec g_j \prec U_j$ and g_0 such that $K \setminus \bigcup_{j=1}^n V_j \prec g_0 \prec K \setminus E$. Let $g = \sum_{j=0}^n g_j$. Then g is continuous, $g \ge 1$ on K. Set $f_j = g_j/g$ for $1 \le j \le n$. Then $f_j : K \to [0,1]$ is continuous for all j and $\sum_{j=1}^n f_j \le \sum_{j=0}^n g_j/g = 1$. On E we have $g_0 = 0$, so $\sum_{j=1}^n f_j = \sum_{j=0}^n g_j/g = 1$.

□

2.2.2 Borel Measures

Let X be a Hausdorff topological space. Let $\mathcal{G}$ be the set of open subsets of X (i.e. the topology). The *Borel σ-field* $\mathcal{B} = \sigma(\mathcal{G})$ is the σ-field on X generated by $\mathcal{G}$. Members of $\mathcal{B}$ are the *Borel sets.*

A *Borel measure on X* is a measure μ on $\mathcal{B}$. We say μ is *regular* if it satisfies the following properties:

(i) For all compact $E \subseteq X$, $\mu(E) < \infty$.

(ii) For every $A \in \mathcal{B}$, $\mu(A) = \inf\{\mu(U) \mid A \subseteq U \in \mathcal{G}\}$ ("outer regularity").

(iii) For every $U \in \mathcal{G}$, $\mu(U) = \sup\{\mu(E) \mid E \subseteq U, E \text{ compact}\}$ ("inner regularity").

A complex Borel measure ν on X is *regular* if $|\nu|$ is regular.

Note that if X is compact and Hausdorff, then a Borel measure μ is regular iff $\mu(X) < \infty$ and (ii) holds, iff $\mu(X) < \infty$ and $\forall A \in \mathcal{B} : \mu(A) = \sup\{\mu(E) \mid E \subseteq A, E \text{ closed}\}$.

2.2.3 Integration w.r.t. a Complex Measure

Let Ω be a set, $\mathcal{F}$ a σ-field on Ω and ν a complex measure on $\mathcal{F}$. A measurable $f : \Omega \to \mathbb{C}$ is ν-integrable if f is $|\nu|$-integrable, i.e. $\int_\Omega |f| d|\nu| < \infty$. Then we define

$$\int_\Omega f d\nu = \int_\Omega f d\nu_1 - \int_\Omega f d\nu_2 + i \int_\Omega f d\nu_3 - i \int_\Omega f d\nu_4$$

where $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$ is the Jordan decomposition of ν. Recall $\nu_k \leq |\nu| \leq \nu_1 + \nu_2 + \nu_3 + \nu_4$, so f is $|\nu|$-integrable iff f is ν_k-integrable for all k.

Properties:

1. For $A \in \mathcal{F}$, $\int_\Omega 1_A d\nu = \nu(A)$.
2. $\int_\Omega f d\nu$ is linear in f.
3. Dominated convergence (D.C.) holds: Given measurable $(f_n)_{n\in\mathbb{N}}, f, g$ such that $|f_n| \leq g$ for all n, $\int_\Omega |g| d|\nu| < \infty$, $f_n \to f$ a.e., then f_n, f are ν-integrable and $\int_\Omega f_n d\nu \to \int_\Omega f d\nu$.
4. If f is ν-integrable, then $\left|\int_\Omega f d\nu\right| \leq \int_\Omega |f| d|\nu|$. Proof: This holds for simple functions by 1, 2 and then for all functions by 3.

2.2.4 Riesz Representation Theorem

Let ν be a complex Borel measure on K. For $f \in C(K)$, f is Borel measurable and $\int_K |f| d|\nu| \leq \|f\|_\infty |\nu|(K) < \infty$. Define $\varphi : C(K) \to \mathbb{C}$ by $\varphi(f) = \int_K f d\nu$. This is linear and $|\varphi(f)| \leq \|f\|_\infty \|\nu\|_1$, so $\varphi \in M(K) = C(K)^*$ and $\|\varphi\| \leq \|\nu\|_1$.

If ν is a signed measure, then $\varphi \in M^{\mathbb{R}}(K) \cong C^{\mathbb{R}}(K)^*$. If ν is a positive measure, then $\varphi \in M^+(K)$.

Theorem 2.8 (Riesz Representation Theorem)**.** *Let $\varphi \in M^+(K)$. Then there exists a unique regular Borel measure μ on K that represents φ, i.e.*

$$\int_K f d\mu = \varphi(f) \quad \forall f \in C(K).$$

Moreover $\|\varphi\| = \mu(K) = \|\mu\|_1$.

Proof. Uniqueness: Suppose μ_1, μ_2 both represent φ. For $E \subseteq U \subseteq K$, E closed, U open, there exists f, $E \prec f \prec U$. Then

$$\mu_1(E) \leq \int_K f d\mu_1 = \varphi(f) = \int_K f d\mu_2 \leq \mu_2(U)$$

Since μ_2 is regular, $\mu_1(E) \leq \mu_2(E)$. By symmetry, we get equality, so $\mu_1 = \mu_2$ on closed sets.

Existence: [Want: Let $\mu(A) = \varphi(1_A)$ but 1_A need not be continuous.]

We will construct an outer measure μ^*. For $U \in \mathcal{G}$ let $\mu^*(U) = \sup\{\varphi(f) \mid f \prec U\}$. We have $f \leq 1_K$, so $\varphi(f) \leq \varphi(1_K)$.

Note that $\mu^*(\emptyset) = 0$ and $\mu^*(K) = \varphi(1_K) = \|\varphi\|$ (Lemma 2.6).

μ^* is subadditive on $\mathcal{G}$: Assume $U \subseteq \bigcup_{n=1}^\infty U_n$ ($U \in \mathcal{G}, U_n \in \mathcal{G}$ for all n). Given $f \prec U$, for some $n \in \mathbb{N}$, $\operatorname{supp} f \subseteq \bigcup_{j=1}^n U_j$ by compactness. By Lemma 7 there exist $h_j \prec U_j$ such that $\sum h_j \leq 1$ on K, $\sum h_j = 1$ on $\operatorname{supp} f$. So

$$\varphi(f) = \varphi\Big(\sum_{k=1}^n h_j f\Big) = \sum_{j=1}^n \varphi(h_j f) \leq \sum_{j=1}^\infty \mu^*(U_j).$$

Taking sup over all $f \prec U$, gives $\mu^*(U) \leq \sum_{n=1}^\infty \mu(U_n)$.

Clearly, for $U, V \in \mathcal{G}, U \subseteq V$, we have $\mu^*(U) \leq \mu^*(V)$. So $\mu^*(U) = \inf\{\mu^*(V) \mid U \subseteq V \in \mathcal{G}\}$. We can extend μ^* to $\mathcal{P}(K)$: $\mu^*(A) = \inf\{\mu^*(U) \mid A \subseteq U \in \mathcal{G}\}, A \subseteq K$. Have $\mu^*(\emptyset) = 0, \mu^*(K) = \varphi(1_K)$.

μ^* is subadditive on $\mathcal{P}(K)$: Assume $A \subseteq \bigcup_{n=1}^\infty A_n$, fix $\varepsilon > 0$ and for all n choose $U_n \in \mathcal{G}$ such that $A_n \subseteq U_n$, $\mu^*(U_n) \subseteq \mu^*(A_n) + \varepsilon 2^{-n}$. Then $A \subseteq \bigcup_{n=1}^\infty U_n$, so $\mu^*(A) \leq \mu^*(\bigcup_{n=1}^\infty U_n) \leq \sum_n \mu^*(U_n) \leq \sum_n \mu^*(A) + \varepsilon$. Hence $\mu^*(A) \leq \sum_n \mu^*(A_n)$. So μ^* is an outer measure on K.

So the set $\mathcal{M}$ of μ^*-measurable sets is a σ-field and $\mu^*|_{\mathcal{M}}$ is a measure.

We show that $\mathcal{G} \subseteq \mathcal{M}$: Fix $U \in \mathcal{G}$. Need: For all $A \subseteq K : \mu^*(A) \geq \mu^*(A \cap U) + \mu^*(A \setminus U)$. Proof: First assume $A = V \in \mathcal{G}$. Let $f \prec V \cap U$, $g \prec V \setminus \operatorname{supp} f$. Then $f + g \prec V$, so

$\mu^*(V) \geq \varphi(f+g) = \varphi(f)+\varphi(g)$. Taking sup over g gives $\mu^*(V) \geq \varphi(f)+\mu^*(V \setminus \operatorname{supp} f) \geq \varphi(f) + \mu^*(V \setminus U)$, so taking sup over f: $\mu^*(V) \geq \mu^*(V \cap U) + \mu^*(V \setminus U)$.

General $A \subseteq K$. Let $V \in \mathcal{G}$, $V \supseteq A$. Then $\mu^*(V) \geq \mu^*(V \cap U) + \mu^*(V \setminus U) \geq \mu^*(A \cap U) + \mu^*(A \setminus U)$. Taking inf over V gives $\mu^*(A) \geq \mu^*(A \cap U) + \mu^*(A \setminus U)$.

It follows that $\mathcal{B} \subseteq \mathcal{M}$ and $\mu = \mu^*|_{\mathcal{B}}$ is a Borel measure on K. Recall: $\mu(K) = \varphi(1_K) = \|\varphi\| < \infty$ and μ is regular by definition.

It remains to show that $\varphi(f) = \int_K f d\mu$ for all $f \in C(K)$. It is enough to show this for all $\varphi \in C^{\mathbb{R}}(K)$. Furthermore, it is enough to show that $\varphi(f) \leq \int_K f d\mu$ for all $f \in C^{\mathbb{R}}(K)$: Replace f by $-f$ to get the other inequality.

Let $f \in C^{\mathbb{R}}(K)$ and choose $a < b$ in $\mathbb{R}$ such that $f(K) \subseteq [a, b]$. Wlog $a > 0$ (since we know that our desired equality holds for constant functions). Fix $\varepsilon > 0$ and choose $0 < y_0 < a < y_1 < \cdots < y_n = b$ such that $y_j - y_{j-1} < \varepsilon$ for all j. Let $A_j = f^{-1}((y_{j-1}, y_j]))$ for $j = 1, \ldots, n$.

So $K = \bigcup_{j=1}^n A_j$ is a Borel partition of K. For each j choose $U_j \in \mathcal{G}$, $A_j \subseteq U_j$ with $\mu(U_j) < \mu(A_j) + \frac{\varepsilon}{n}$ and $U_j \subseteq f^{-1}((y_{j-1}, y_j + \varepsilon))$. Then by Lemma 2.7 there exist $h_j \prec U_j$ such that $\sum_{j=1}^n h_j = 1_K$. Then

$$\begin{aligned}
\varphi(f) = \sum_j \varphi(f h_j) \leq \sum_j \varphi((y_j + \varepsilon) h_j) = \sum_{j=1}^n (y_j + \varepsilon)\varphi(h_j) &\leq \sum_{j=1}^n (y_j + \varepsilon)\mu(U_j) \\
&\leq \sum_{j=1}^n (y_{j-1} + 2\varepsilon)(\mu(A_j) + \frac{\varepsilon}{n}) \\
&= \int_K \sum_{j=1}^n y_{j-1} 1_{A_j} d\mu + 2\varepsilon\mu(K) + (b + 2\varepsilon)\varepsilon \\
&\leq \int_K f d\mu + \varepsilon(2\mu(K) + b + 2\varepsilon)
\end{aligned}$$

Hence $\varphi(f) \leq \int_K f d\mu$. □

Corollary 2.9. *For every $\varphi \in M(K)$ there exists a unique regular complex Borel measure ν on K such that $\varphi(f) = \int_K f d\nu$ for all $f \in C(K)$.*

Moreover, $\|\varphi\| = \|\nu\|_1$ and if $\varphi \in M^{\mathbb{R}}(K)$, then ν is a signed measure.

Proof. Existence: Lemma 2.6 and the theorem.

Uniqueness: Follows from $\|\varphi\| = \|\nu\|_1$.

Proof of $\|\varphi\| = \|\nu\|_1$: We have seen $\|\varphi\| \leq \|\nu\|_1$. Recall

$$\|\nu\|_1 = |\nu|(K) = \sup\{\sum_{j=1}^n |\nu(A_j)| : K = \bigcup_{j=1}^n A_j \text{ is a Borel partition of } K\}.$$

Fix a Borel partition $K = \bigcup_{j=1}^n A_j$ of K. For each j choose a closed set $E_j \subseteq A_j$ such that $|\nu|(A_j \setminus E_j) < \frac{\varepsilon}{n}$ (regularity). Note that $E_j \subseteq K \setminus \bigcup_{\substack{l=1 \\ l \neq j}}^n E_l$. So there exist open sets U_j such that $E_j \subseteq U_j \subseteq K \setminus \bigcup_{\substack{l=1 \\ l \neq j}}^n E_l$ and $|\nu|(U_j \setminus E_j) < \frac{\varepsilon}{n}$. Set $E = \bigcup_{j=1}^n E_j \subseteq \bigcup U_j$. By Lemma 2.6 there exist $h_j \prec U_j$ such that $\sum_{j=1}^n h_j \leq 1$ on K and $\sum h_j = 1$ on E. Note that $h_j = 1$ on E_j for all j.

Choose $\lambda_j \in \mathbb{C}$ with $|\lambda_j| = 1$ and $|\nu(E_j)| = \lambda_j \nu(E_j)$. Then

$$\left| \sum_{j=1}^n |\nu(E_j)| - \varphi(\sum_{j=1}^n \lambda_j h_j) \right| = \left| \sum_{j=1}^n \lambda_j \int_K (1_{E_j} - h_j) d\nu \right| \leq \sum_{j=1}^n \int_K |1_{E_j} - h_j| d|\nu|$$
$$\leq \sum_{j=1}^n |\nu|(U_j \setminus E_j) < \varepsilon$$

So

$$\sum_{j=1}^n |\nu(A_j)| \leq \sum_{j=1}^n |\nu(E_j)| + \varepsilon \leq |\varphi(\sum \lambda_j h_j)| + 2\varepsilon \leq \|\varphi\| \left\| \sum \lambda_j h_j \right\|_\infty + 2\varepsilon \leq \|\varphi\| + 2\varepsilon$$

This holds for all $\varepsilon > 0$ and for all Borel partitions $K = \bigcup_{j=1}^n A_j$, so $\|\nu\|_1 \leq \|\varphi\|$. □

Corollary 2.10. *The space of regular complex (resp. signed) Borel measures on K is a complex (resp. real) Banach space in $\|\cdot\|_1$ and it is isomorphic to $M(K) = C(K)^*$ (resp. $M^{\mathbb{R}}(K) \cong C^{\mathbb{R}}(K)^*$)*

3 Weak topologies

3.1 Weak topologies in general

Let X be a set and $\mathcal{F}$ a collection of functions such that each $f \in \mathcal{F}$ is a function $f : X \to Y_f$ where Y_f is a topological space. The *weak topology* $\sigma(X, \mathcal{F})$ on X generated by $\mathcal{F}$ is the smallest topology on X such that all $f \in \mathcal{F}$ are continuous.

Remarks:

1. $\mathcal{S} = \{f^{-1}(U) \mid f \in \mathcal{F},\ U \text{ is open in } Y_f\}$ generates $\sigma(X, \mathcal{F})$, i.e. is a subbase for it. So $\sigma(X, \mathcal{F})$ consists of arbitrary unions of finite intersections of members of $\mathcal{S}$. So $V \subseteq X$ is open in $\sigma(X, \mathcal{F})$ iff

$$\forall x \in V\, \exists n \in \mathbb{N}, f_1, \dots, f_n \in \mathcal{F}, \text{ open } U_j \in Y_{f_j} \text{ s.t. } x \in \bigcap_{j=1}^{n} f_j^{-1}(U_j) \subseteq V.$$

   Equivalently

$$\forall x \in V\, \exists n \in \mathbb{N}, f_1, \dots, f_n \in \mathcal{F}, \text{ open neighborhoods } U_j \text{ of } f_j(x) \text{ in } Y_{f_j}$$
$$\text{s.t. } \bigcap_{j=1}^{n} f_j^{-1}(U_j) \subseteq V.$$

2. If $\mathcal{S}_f$ is a subbase for the topology of Y_f $(f \in \mathcal{F})$, then $\{f^{-1}(U) \mid f \in \mathcal{F}, U \in \mathcal{S}_f\}$ is a subbase for $\sigma(X, \mathcal{F})$.
3. If Y_f is Hausdorff for all $f \in \mathcal{F}$ and $\mathcal{F}$ separates points of X (i.e. $x \neq y$ in X $\implies$ $\exists f \in \mathcal{F} : f(x) \neq f(y)$), then $\sigma(X, \mathcal{F})$ is Hausdorff.
4. If $Y \subseteq X$, then let $\mathcal{F}_Y = \{f|_Y : f \in \mathcal{F}\}$. Then $\sigma(X, \mathcal{F})|_Y = \sigma(Y, \mathcal{F}_Y)$.
5. Universal property: Let Z be a topological space and $g : Z \to X$ a function. Then g is continuous iff for all $f \in \mathcal{F}$, $f \circ g : Z \to Y_f$ is continuous.

Examples.

1. Let X be a topological space, $Y \subseteq X$, $\iota : Y \to X$ the inclusion map. Then $\sigma(Y, \{\iota\})$ is the subspace topology on Y.

2. Let Γ be a set and for each $\gamma \in \Gamma$, X_γ be a topological space. Let

$$X = \prod_{\gamma \in \Gamma} X_\gamma = \{x \mid x \text{ is a function on } \Gamma \text{ with } x(\gamma) \in X_\gamma \text{ for all } \gamma \in \Gamma\}.$$

For $\gamma \in \Gamma$ let $\pi_\gamma : X \to X_\gamma$ be the projection onto X_γ.

The product topology on X is the weak topology $\sigma(X, \{\pi_\gamma \mid \gamma \in \Gamma\})$. So $V \subseteq X$ is open iff for all $x \in V$ there exist $n \in \mathbb{N}$, $\gamma_1, \dots, \gamma_n \in \Gamma$ and open neighborhoods U_i of x_{γ_i} in X_{γ_i} for $1 \le i \le n$, such that $\{y = (y_\gamma)_{\gamma\in\Gamma} \in X \mid y_{\gamma_i} \in U_i, 1 \le i \le n\} \subseteq V$.

Proposition 3.1. *Let X be a set and for each $n \in \mathbb{N}$ let (Y_n, d_n) be a metric space and $f_n : X \to Y_n$ be a function. Assume $\mathcal{F} = \{f_n \mid n \in \mathbb{N}\}$ separates the points of X. Then $\sigma(X, \mathcal{F})$ is metrizable.*

Proof. WLOG $d_n \le 1$ for every n (replace d_n with the equivalent metric $\min(d_n, 1)$ or $\frac{d_n}{d_n+1}$). Define for $x, y \in X$,

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} d_n(f_n(x), f_n(y)).$$

It is easy to check that d is a metric.

Note that each f_n is Lipschitz as a map $(X, d) \to (Y, d_n)$ and hence continuous. Thus, $\sigma = \sigma(X, \mathcal{F})$ is contained in the metric topology of (X, d). Conversely, if each f_n is σ-continuous, then $(x, y) \mapsto d_n(f_n(x), f_n(y))$ is also σ-continuous. So by the M-test, $d : (X, \sigma) \times (X, \sigma) \to \mathbb{R}$ is continuous. So for $x \in X, \varepsilon > 0$, the ball $\{y \in X \mid d(y, x) < \varepsilon\}$ is σ-open. Hence the metric topology of (X, d) is contained in $\sigma(X, \mathcal{F})$. □

Theorem 3.2 (Tychonov)**.** *The product of compact topological spaces is compact in the product topology.*

Proof. Let Γ be a set, for each $\gamma \in \Gamma$, let X_γ be a compact space and let $X = \prod_{\gamma\in\Gamma} X_\gamma$ with the product topology.

Let $\mathcal{A}$ be a non-empty family of closed subsets of X with the finite intersection property (f.i.p.), i.e. for every $n \in \mathbb{N}$ and $A_1, \dots, A_n \in \mathcal{A}$, $\bigcap_{i=1}^n A_i \ne \emptyset$. We need to show that $\bigcap_{A\in\mathcal{A}} A \ne \emptyset$.

By Zorn's Lemma there exists a maximal (w.r.t. inclusion) family $\mathcal{B}$ of (not necessarily closed) subsets of X such that $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B}$ has f.i.p. Then $\bigcap_{A\in\mathcal{A}} A \supseteq \bigcap_{B\in\mathcal{B}} \overline{B}$. So it is enough to shows that $\bigcap_{B\in\mathcal{B}} \overline{B} \ne \emptyset$.

Observe if $A \subseteq X$ and for all $B \in \mathcal{B}$, $A \cap B \ne \emptyset$, then $A \in \mathcal{B}$. Indeed, if $B_1, \dots, B_n \in \mathcal{B}$, then $\mathcal{B} \cup \{\bigcap_{i=1}^n B_i\}$ has f.i.p. So by maximality $\bigcap_{i=1}^n B_i \in \mathcal{B}$, and so $A \cap \bigcap_{i=1}^n B_i \ne \emptyset$. So $\mathcal{B} \cup \{A\}$ has f.i.p., so again by maximality $A \in \mathcal{B}$.

Fix $\gamma \in \Gamma$. $\{\pi_\gamma(B) \mid B \in \mathcal{B}\}$ has f.i.p. As X_γ is compact, $\bigcap_{B\in\mathcal{B}} \overline{\pi_\gamma(B)} \neq \emptyset$. Choose $x_\gamma \in \bigcap_{B\in\mathcal{B}} \overline{\pi_\gamma(B)}$. Do this for every $\gamma \in \Gamma$ to obtain $x = (x_\gamma)_{\gamma\in\Gamma} \in X$. We show that $x \in \bigcap_{B\in\mathcal{B}} \overline{B}$.

Let V be an open neighborhood of x. We need $V \cap B \neq \emptyset$ for all $B \in \mathcal{B}$. WLOG $V = \bigcap_{i=1}^n \pi_{\gamma_i}^{-1}(U_i)$ where $n \in \mathbb{N}$, $\gamma_1, \dots, \gamma_n \in \Gamma$ and U_i is an open neighborhood of x_{γ_i} in X_{γ_i} $(1 \leq i \leq n)$. Since $x_{\gamma_i} \in \bigcap_{B\in\mathcal{B}} \overline{\pi_{\gamma_i}(B)}$, $U_i \cap \pi_{\gamma_i}(B) \neq \emptyset$ for all $B \in \mathcal{B}$, so $\pi_{\gamma_i}^{-1}(U_i) \cap B \neq \emptyset$ for all $B \in \mathcal{B}$.

So by the observation above, $\pi_{\gamma_i}^{-1}(U_i) \in \mathcal{B}$. Hence $V = \bigcap_{i=1}^n \pi_{\gamma_i}^{-1}(U_i) \in \mathcal{B}$. Thus $V \cap B \neq \emptyset$ for all $B \in \mathcal{B}$. It follows that $x \in \overline{B}$ for every B. □

3.2 Weak topologies on vector spaces

Let E be a real or complex vector space and F be a subspace of the space of all linear functionals on E that separates the points of E, i.e. for all $x \neq 0$ in E there exists $f \in F$ such that $f(x) \neq 0$. We consider the weak topology $\sigma(E, F)$. So $U \subseteq E$ is open iff for every $x \in U$ there exist $n \in \mathbb{N}, f_1, \dots, f_n \in F$, $\varepsilon > 0$ such that $\{y \in E \mid |f_i(y) - f_i(x)| < \varepsilon, 1 \leq i \leq n\} \subseteq U$.

For $f \in F$ define $p_f : E \to \mathbb{R}$, $p_f(x) = |f(x)|$. Let $\mathcal{P} = \{p_f \mid f \in F\}$. Then $\mathcal{P}$ is a family of seminorms on E that separates points on E. The topology of the LCS $(E, \mathcal{P})$ is exactly $\sigma(E, F)$.

Lemma 3.3. *Let E be as above. Let $f, g_1, \dots, g_n$ be linear functionals on E such that $\bigcap_{i=1}^n \ker g_i \subseteq \ker f$. Then $f \in \operatorname{span}\{g_1, \dots, g_n\}$.*

Proof. Define $T : E \to \mathbb{F}^n$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) by $Tx = (g_i x)_{i=1,\dots,n}$. Then $\ker T = \bigcap_{i=1}^n \ker g_i \subseteq \ker f$, so there exists a linear $h : \operatorname{im} T \to \mathbb{F}$ such that $f = h \circ T$. We can extend this to $h : \mathbb{F}^n \to \mathbb{F}$. There exist $a_1, \dots, a_n \in \mathbb{F}$ such that $h(y) = \sum_{i=1}^n a_i y_i$ for all $y = (y_i)_{i=1,\dots,n} \in \mathbb{F}^n$. So for all $x \in E$, $f(x) = hTx = \sum_{i=1}^n a_i g_i(x)$. □

Proposition 3.4. *Let E, F be as above. A linear functional f on E is continuous w.r.t. $\sigma(E, F)$ iff $f \in F$, i.e. $(E, \sigma(E, F))^* = F$.*

Proof. "$\Leftarrow$" By definition of $\sigma(E, F)$.

"$\Rightarrow$" If f is continuous, then $V = \{x \in E \mid |f(x)| < 1\}$ is an open neighborhood of 0. So there exist $n \in \mathbb{N}$, $g_1, \dots, g_n \in F$, $\varepsilon > 0$ such that $U = \{y \in E \mid |g_i(y)| < \varepsilon, 1 \leq i \leq n\} \subseteq V$. If $x \in \bigcap_{i=1}^n \ker g_i$, then for all scalars λ, $\lambda x \in U \subseteq V$, so $|f(\lambda x)| = |\lambda||f(x)| < 1$. So $f(x) = 0$. So by the previous Lemma, $f \in \operatorname{span}\{g_1, \dots, g_n\} \subseteq F$. □

Examples.

1. Let X be a normed space. The *weak topology on* X is the weak topology $\sigma(X, X^*)$. (By Hahn-Banach, X^* separates the points of X).

   The weak topology on X is sometimes written w-topology and write $(X, w) = (X, \sigma(X, X^*))$.

   An open set in $\sigma(X, X^*)$ is called *weak-open or w-open.*

   So $U \subseteq X$ is w-open iff for every $x \in U$ there exist $n \in \mathbb{N}$, $f_1, \ldots, f_n \in X^*$, $\varepsilon > 0$ such that $\{y \in X \mid |f_i(y - x)| < \varepsilon, 1 \leq i \leq n\} \subseteq U$.

2. The *weak-star topology* or w^*-topology on X^* is the weak topology $\sigma(X^*, X)$ where we identify X with its image in X^{**} under the canonical embedding $X \to X^{**}$. Open sets of X^* in the w^*-topology are called w^*-open. $U \subseteq X^*$ is w^* open iff for all $f \in U$, there exist $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$, $\varepsilon > 0$ such that $\{g \in X^* \mid |(g - f)(x_i)| < \varepsilon, 1 \leq i \leq n\} \subseteq U$.

Properties:

1. (X, w) and (X^*, w^*) are LCSs. So they are Hausdorff and addition and scalar multiplication are continuous.

2. $\sigma(X, X^*) \subseteq$ norm-topology and equality holds iff $\dim X < \infty$.

3. $\sigma(X^*, X) \subseteq \sigma(X^*, X^{**}) \subseteq$ norm-topology. Equality in the first place holds iff X is reflexive, equality in the second place holds iff $\dim X < \infty$.

4. If Y is a subspace of X, then $\sigma(X, X^*)|_Y = \sigma(Y, \{f|_Y \mid f \in X^*\}) = \sigma(Y, Y^*)$.

   Similarly $\sigma(X^{**}, X^*)|_X = \sigma(X, X^*)$. So the canonical embedding $X \to X^{**}$ is a w-to-w^*-homeomorphism from onto $\widehat{X}$.

Proposition 3.5. *Let X be a normed space.*

(i) A linear functional f on X is w-continuous iff $f \in X^*$, i.e. $(X, w)^* = X^*$.

(ii) A linear functional φ on X^* is w^*-continuous iff $\varphi \in X$, i.e. there exists $x \in X$ such that $\varphi = \widehat{x}$. So $(X^*, w^*)^* = X$.

It follows that $\sigma(X^*, X) = \sigma(X^*, X^{**})$ iff X is reflexive.

Proof. (i) and (ii) are immediate from Proposition 3.4. For the last statement: "⇐" is clear. "⇒" Given $\varphi \in X^{**}$, φ is w-continuous, so w^* continuous, so by (ii) there exists $x \in X$ such that $\varphi = \widehat{x}$. □

Definition. *Let X be a normed space. $A \subseteq X$ is* weakly bounded *if $\{f(x) \mid x \in A\}$ is bounded for all $f \in X^*$.*[1] *Similarly, $B \subseteq X^*$ is* w^*-bounded *if $\{f(x) \mid f \in B\}$ is bounded for all $x \in X$.*[2]

[1] ⇔ $\forall$ w-neighbhorhoods U of 0 there exists $\lambda > 0$ such that $A \subseteq \lambda U$.
[2] ⇔ $\forall$ w^*-neighborhoods U of 0 there exists $\lambda > 0$ such that $B \subseteq \lambda U$.

Recall:

Lemma (Principle of Uniform Boundedness (PUB)). *Let X be a Banach space, Y a normed space, $\mathcal{T} \subseteq \mathcal{B}(X,Y)$. If $\mathcal{T}$ is pointwise bounded, i.e.* $\sup_{T\in\mathcal{T}} \|Tx\| < \infty$ *for every* $x \in X$, *then $\mathcal{T}$ is bounded, i.e.* $\sup_{T\in\mathcal{T}} \|T\| < \infty$.

Proposition 3.6. *Let X be a normed space.*

(a) If $A \subseteq X$ is weakly bounded, then A is $\|\cdot\|$-bounded.

(b) If X is complete and $B \subseteq X^*$ is w^*-bounded, then B is $\|\cdot\|$-bounded.

Proof.

(a) $\widehat{A} := \{\widehat{x} \mid x \in A\} \subseteq X^{**} = \mathcal{B}(X^*, \mathbb{F})$. As A is w-bounded, $\widehat{A}$ is pointwise bounded and hence $\|\cdot\|$-bounded by PUB. Thus A is $\|\cdot\|$-bounded since for all $x \in X$, $\|\widehat{x}\| = \|x\|$.

(b) $B \subseteq X^* = \mathcal{B}(X, \mathbb{F})$. If B is w^*-bounded, it is pointwise bounded, so it is bounded by PUB.

□

Notation: Let X be a normed space.

1. If a sequence $(x_n)_n$ in X converges to $x \in X$ in the weak topology, then we write $x_n \xrightarrow{w} X$ and say that (x_n) *weakly converges* to x.

   This happens iff $\langle x_n, f\rangle \to \langle x, f\rangle$ for all $f \in X^*$ iff $\widehat{x}_n \to \widehat{x}$ pointwise.

2. If a sequence $(f_n)_n$ in X^* converges to $f \in X^*$ in the w^*-topology, then we write $f_n \xrightarrow{w^*} f$ and say that (x_n) *w^*-converges to f*.

   This happens iff $\langle x, f_n\rangle \to \langle x, f\rangle$ for all $x \in X$, i.e. iff $f_n \to f$ pointwise.

Recall:

Lemma (Consequence of PUB). *Let X be a Banach space, Y a normed space and (T_n) a sequence in $\mathcal{B}(X,Y)$. If $T_n \to T$ pointwise on X for some function $T : X \to Y$, then* $T \in \mathcal{B}(X,Y)$ *and* $\|T\| \le \liminf_{n\to\infty} \|T_n\| \le \sup_n \|T_n\| < \infty$.

Proposition 3.7. *Let X be a normed space.*

(i) If $x_n \xrightarrow{w} x$ in X, then $\sup \|x_n\| < \infty$ *and* $\|x\| \le \liminf \|x_n\|$.

(ii) If X is complete and $f_n \xrightarrow{w^*} f$ in X^*, then $\sup_n \|f_n\| < \infty$ *and* $\|f\| \le \liminf \|f_n\|$.

3.3 Hahn-Banach separation theorems

Let $(X, \mathcal{P})$ be a LCS. Let C be a convex subset of X with $0 \in \operatorname{Int} C$. Define $\mu_C : X \to \mathbb{R}$, $\mu_C(x) = \inf\{t > 0 \mid x \in tC\}$. Given $x \in X$, $x/n \to 0$ as $n \to \infty$, so there exists $n \in \mathbb{N}$ such that $x/n \in C$, i.e. $x \in nC$, so μ_C is well-defined. μ_C is called the *Minkowski functional* (or *gauge functional*) of C.

Example. If X is a normed space and $C = B_X$, then $\mu_C = \|\cdot\|$.

Lemma 3.8. *μ_C is positive homogeneous and subadditive. Also*

$$\{x \in X \mid \mu_C(x) < 1\} \subseteq C \subseteq \{x \in X \mid \mu_C(x) \leq 1\}$$

If C is open (resp. closed), then the first (resp. second) inclusion is an equality.

Proof. Homogeneity is obvious.

Observation: If $t > \mu_C(x)$, then $x \in tC$. Indeed, if $t > \mu_C(x)$, then there exists $s < t$ such that $x \in sC$. Then $\frac{x}{t} = \frac{s}{t}\frac{x}{s} + \left(1 - \frac{s}{t}\right) \cdot 0 \in C$ by convexity. So $x \in tC$. Now given $x, y \in X$, for $s > \mu_C(x), t > \mu_C(y)$, we have $x \in sC, y \in tC$. So $\frac{x+y}{s+t} = \frac{s}{s+t}\frac{x}{s} + \frac{t}{s+t}\frac{y}{t} \in C$ by convexity, so $\mu_C(x + y) \leq s + t$. Taking inf over s, t gives $\mu_C(x + y) \leq \mu_C(x) + \mu_C(y)$.

If $\mu_C(x) < 1$, then $x \in C$ by the observation. Assume C is open and $x \in C$. We have $\left(1 + \frac{1}{n}\right)x \to x \in C$, C open, so there exists $n \in \mathbb{N}$ such that $\left(1 + \frac{1}{n}\right)x \in C$, so $\mu_C(x) < 1$.

If $x \in C$, then by definition $\mu_C(x) \leq 1$. Assume C is closed and $\mu_C(x) \leq 1$. Then $\left(1 - \frac{1}{n}\right)x \to x$ and $\left(1 - \frac{1}{n}\right)x \in C$, so $x \in C$ as C is closed. □

Remark: If C is symmetric in the real case (i.e. $x \in C \implies -x \in C$) or balanced in the complex case ($x \in C, \alpha \in \mathbb{C}, |\alpha| = 1 \implies \alpha x \in C$), then μ_C is a seminorm. If in addition, C is bounded (i.e. $\forall$ Neighborhoods U of 0 $\exists t > 0 : C \subseteq tU$, equivalently every $p \in \mathcal{P}$ is bounded on C), then μ_C is a norm.

Theorem 3.9 (Hahn-Banach Separation Theorem)**.** *Let $(X, \mathcal{P})$ be a LCS, C an open, convex subset of X with $0 \in C$ and let $x_0 \in X \setminus C$. Then there exists $f \in X^*$ such that for every $x \in C$, $f(x) < f(x_0)$ (real case) or $\operatorname{Re} f(x) < \operatorname{Re} f(x_0)$ (in the complex case).*

Proof. WLOG the scalar field is $\mathbb{R}$. Indeed, in the complex case for all real-linear $f : X \to \mathbb{R}$ there exists a unique complex-linear $g : X \to \mathbb{C}$ such that $f = \operatorname{Re} g$.

Let $Y = \operatorname{span} x_0$ and $g : Y \to \mathbb{R}$, $g(\lambda x_0) = \lambda \mu_C(x_0)$. Then g is linear and for all $\lambda \geq 0$, $g(\lambda x_0) = \mu_C(\lambda x_0)$ and for all $\lambda < 0$, $g(\lambda x_0) = \lambda \mu_C(x_0) \leq 0 \leq \mu_C(\lambda x_0)$. So for all $y \in Y$, $g(y) \leq \mu_C(y)$. By the first version of Hahn-Banach there exists a linear $f : X \to \mathbb{R}$ such that $f|_Y = g$ and $f \leq \mu_C$ on X. Then for every $x \in C \cap (-C)$, $f(x) \leq \mu_C(x) < 1$ and $-f(x) = f(-x) \leq \mu_C(-x) < 1$. So $|f(x)| < 1$. So for $\varepsilon > 0$, $|f| < \varepsilon$ on the open neighborhood $\varepsilon(C \cap (-C))$ of 0. So f is continuous at 0, hence $f \in X^*$.

For all $x \in C$, $f(x) \leq \mu_C(x) < 1 \leq \mu_C(x_0) = f(x_0)$. □

Theorem 3.10 (Hahn-Banach Separation Theorem)**.** *Let $(X, \mathcal{P})$ be a LCS, A, B non-empty disjoint, convex subsets of X.*

(i) If A is open, then there exist $f \in X^*, \alpha \in \mathbb{R}$ such that for all $x \in A, y \in B$, $f(x) < \alpha \leq f(y)$.

(ii) If A is compact and B is closed, then there exists $f \in X^*$ such that $\sup_A f < \inf_B f$.

Proof.

(i) Fix $a \in A, b \in B$. Let $C = A - B + b - a, x_0 = b - a$. Then C is convex and $C = \bigcup_{y \in B}(A - y + b - a)$ is open, $0 \in C$ and $x_0 \notin C$ as $A \cap B = \emptyset$. So there exists $f \in X^*$ such that for all $z \in C$, $f(z) < f(x_0)$. So for every $x \in A, y \in B$, $f(x) < f(y)$. Let $\alpha = \inf_B f$. This exists and for all $x \in A, y \in B$, $f(x) \leq \alpha \leq f(y)$. Since $f \neq 0$, there exists $u \in X$, $f(u) > 0$. Given $x \in A$, $x + \frac{1}{n}u \to x$ and A is open, so there exists $n \in \mathbb{N}$ with $x + \frac{1}{n}u \in A$ and hence $f(x) < f(x + \frac{1}{n}u) \leq \alpha$.

(ii) Claim: There exists an open, convex neighborhood U of 0 such that $(A+U) \cap B \neq \emptyset$. Proof of claim: For all $x \in A$ there exists an open neighborhood V_x of 0 such that $(x + V_x) \cap B = \emptyset$ as B is closed. Since addition is continuous, there exists an open neighborhood W_x of 0 such that $W_x + W_x \subseteq V_x$. Since A is compact, there are finitely many points $x_1, \ldots, x_n \in A$ such that $A \subseteq \bigcup_{i=1}^n (x_i + W_{x_i})$. Since $\bigcap_{i=1}^n W_{x_i}$ is an open neighborhood of 0, there exist $m \in \mathbb{N}, p_1, \ldots, p_m \in \mathcal{P}, \varepsilon > 0$ such that $U = \{x \in X \mid p_i(x) < \varepsilon, 1 \leq i \leq n\} \subseteq \bigcap_{i=1}^n W_{x_i}$. Then U is an open, convex neighborhood of 0. We show $(A + U) \cap B = \emptyset$. Given $x \in A$, there exists i such that $x \in x_i + W_{x_i}$. Hence $x + U \subseteq x_i + W_{x_i} + U \subseteq x_i + W_{x_i} + W_{x_i} \subseteq x_i + V_{x_i}$. So $(x + U) \cap B = \emptyset$ and thus $(A + U) \cap B = \emptyset$.

Then $A + U$ is open and convex, so by (i) there exists $f \in X^*$, $\alpha \in \mathbb{R}$ such that for all $x \in A + U, y \in B, f(x) < \alpha \leq f(y)$. As f is continuous, $\sup_A f$ is attained, so $\sup_A f < \alpha \leq \inf_B f$.

□

Remark: The way the theorem is stated is for real spaces. For the complex case replace f in the inequalities by $\operatorname{Re} f$.

3.4 Consequences

Theorem 3.11 (Mazur's theorem)**.** *Let X be a normed space and C be a convex subset. Then $\overline{C}^{\|\cdot\|} = \overline{C}^{w}$. In particular C is $\|\cdot\|$-closed iff C is w-closed.*

Proof. Since the w-topology is weaker than the $\|\cdot\|$-topology, $\overline{C}^{\|\cdot\|} \subseteq \overline{C}^{w}$. For the converse, fix $x \in X \setminus \overline{C}^{\|\cdot\|}$. Apply Theorem 3.10 (ii) to $A = \{x\}$, $B = \overline{C}^{\|\cdot\|}$ in the LCS X to get

$f \in X^*$ such that $f(x) < \inf_{\overline{C}^{\|\cdot\|}} f = \alpha$. The set $\{y \in X \mid f(y) < \alpha\}$ is a w-neighborhood of x disjoint from C. So $\overline{C}^{\|\cdot\|}$ is w-closed, hence $\overline{C}^{w} = \overline{C}^{\|\cdot\|}$. □

Corollary 3.12. *Assume $x_n \xrightarrow{w} 0$ in a normed space X. Then for all $\varepsilon > 0$ there exists $x \in \operatorname{conv}\{x_n \mid n \in \mathbb{N}\}$ such that $\|x\| < \varepsilon$.*

Proof. Let $C = \operatorname{conv}\{x_n \mid n \in \mathbb{N}\}$, so by Mazur's theorem $\overline{C}^{\|\cdot\|} = \overline{C}^{w}$, so $0 \in \overline{C}^{\|\cdot\|}$. □

Remark: So there exist $p_1 < q_1 < p_2 < q_2 < \dots$ in $\mathbb{N}$, convex combinations $z_n = \sum_{i=p_n}^{q_n} t_i x_i$ such that $z_n \to 0$ in $\|\cdot\|$.

Theorem 3.13 (Banach-Alaoglu)**.** *For any normed space X, the dual ball B_{X^*} is w^*-compact.*

Proof. For $x \in X$, let $K_x = \{\lambda \in \mathbb{F} \mid |\lambda| \leq \|x\|\}$. Let $K = \prod_{x \in X} K_x$ with the product topology, which is compact by Tychonov's theorem. We can view $K = \{f : X \to \mathbb{R} \mid f(x) \in K_x \text{ for all } x \in X\}$. Then $B_{X^*} = \{f \in K \mid f \text{ linear}\}$.

Let $\pi_x : K \to K_x$ be the projection onto K_x, i.e. $\pi_x(f) = f(x)$. So $\pi_x|_{B_{X^*}} = \widehat{x}|_{B_{X^*}}$ ($\widehat{x} \in X^{**}$). Then $\sigma(K, \{\pi_x \mid x \in X\})|_{B_{X^*}} = \sigma(B_{X^*}, \{\pi_x|_{B_{X^*}} \mid x \in X\}) = (B_{X^*}, w^*)$.

So it is enough to check that B_{X^*} is closed in K.

$$\begin{aligned} B_{X^*} &= \{f \in K \mid \pi_{\lambda x + \mu y}(f) - \lambda \pi_x(f) - \mu \pi_y(f) = 0, \forall \lambda, \mu \in \mathbb{F}, x, y \in X\} \\ &= \bigcap_{\lambda,\mu,x,y} (\pi_{\lambda x + \mu y} - \lambda \pi_x - \mu \pi_y)^{-1}(\{0\}) \quad \text{is closed} \end{aligned}$$

□

Proposition 3.14. *Let X be a normed space and K a compact Hausdorff space. Then*

(i) X is separable iff (B_{X^*}, w^*) is metrizable.

(ii) $C(K)$ is separable iff K is metrizable.

Proof.

(i) "⇒" Let $\{x_n \mid n \in \mathbb{N}\}$ be dense in X. Let $f_n : B_{X^*} \to \mathbb{F}$, $f_n(\varphi) = \varphi(x_n)$, i.e. $f_n = \widehat{x}_n|_{B_X^*}$ for $n \in \mathbb{N}$. Let $\mathcal{F} = \{f_n \mid n \in \mathbb{N}\}$. Note that if $\varphi, \psi \in B_{X^*}$ and $f_n(\varphi) = f_n(\psi)$ for all n, then $\varphi(x_n) = \psi(x_n)$ for all n, then $\varphi = \psi$ by density, so $\mathcal{F}$ separates points. By Proposition 3.1, the weak topology $\sigma = \sigma(B_{X^*}, \mathcal{F})$ is metrizable. Since σ is weaker than the w^*-topology, $\mathrm{id} : (B_{X^*}, w^*) \to (B_{X^*}, \sigma)$ is a continuous bijection. (B_{X^*}, w^*) is compact by Banach-Alaoglu and (B_{X^*}, σ) is Hausdorff, so id is a homeomorphism, so σ is the w^*-topology.

(ii) "⇒" Let $X = C(K)$. By (i) "⇒" (B_{X^*}, w^*) is metrizable. Define $\delta : K \to (B_{X^*}, \mathrm{w}^*)$, $k \mapsto \delta_k$ where $\delta_k(f) = f(k), f \in X = C(K)$. δ is injective by Urysohns's Lemma.

δ is continuous: Let $f \in X$. Consider $\widehat{f} \circ \delta$. For $k \in K$, $(\widehat{f} \circ \delta)(k) = f(k)$, so $\widehat{f} \circ \delta = f$ is continuous for all $f \in X$, so δ is continuous by the universal property of the weak topology.

So $\delta : K \to (B_{X^*}, w^*)$ is a continuous injection; K is compact and (B_{X^*}, w^*) is Hausdorff, so it is a homeomorphism onto its image and thus K is metrizable.

"⇐" Let d be a metric on K that induces the topology of K. Since (K, d) is a compact metric space, it is separable, so there exists a dense set $\{k_n \mid n \in \mathbb{N}\}$. Let $f_n(k) = d(k, k_n)$ for every $n \in \mathbb{N}, k \in K$. These separate the points of K as the k_n are dense in K. Let A be the subalgebra of $C(K)$ generated by the f_n. This is a subalgebra of $C(K)$ that separates points of K, contains 1_K, and in the complex case, closed under conjugation. So by the Stone-Weierstraß theorem $\overline{A} = C(K)$. Since A is separable, so is $C(K)$.

(i) "⇐" Assume that $K = (B_{X^*}, w^*)$ is metrizable. So by (ii), $C(K)$ is separable. Define $T : X \to C(K)$ by $(Tx)(f) = f(x)$ for $x \in X, f \in K$, i.e. $Tx = \widehat{x}|_{B_{X^*}}$. Then T is linear and $\|Tx\|_\infty = \|x\|$. So $X \cong T(X)$, so X is separable.

□

Remarks:

1. If X is separable, then (B_{X^*}, w^*) is compact, metrizable, so sequentially compact.
2. X separable $\implies$ X^* w^*-separable (note that $X^* = \bigcup_{n\in\mathbb{N}} nB_{X^*}$).

   By Mazur, X is separable iff X is w-separable.

   If X is w-separable, then X^* is w^*-separable. The converse is false, e.g. $X = \ell_\infty$.
3. If K is compact Hausdorff, then K is a subspace (i.e. homeomorphic to a subset) of $(B_{C(K)^*}, w^*)$.
4. Any normed space X embeds isometrically into $C(K)$ for some compact Hausdorff K. If X is separable, then can take K to be a compact metrizable space, e.g. $K = (B_{X^*}, w^*)$.

Proposition 3.15. *X^* is separable iff (B_X, w) is metrizable.*

Proof. "⇒" If X^* is separable, then $(B_{X^{**}}, w^*)$ is metrizable. Since (B_X, w) is a subspace of $(B_{X^{**}}, w^*)$ (under the canonical embedding), we are done.

"⇐" If (B_X, w) is metrizable, then there exists a sequence $(V_n)_n$ of w-neighborhoods of 0 in B_X such that every w-neighborhood U of 0 in B_X contains one of the V_n. WLOG for all $n \in \mathbb{N}$, there exist a finite set $F_n \subseteq X^*, \varepsilon_n > 0$ such that $V_n = \{x \in B_X \mid f \in F_n : |f(x)| < \varepsilon_n\}$. We show that $\overline{\operatorname{span} \bigcup F_n} = X^*$, then we are done. Let $g \in X^*$, $\varepsilon > 0$.

Then $U = \{x \in B_X \mid |g(x)| < \varepsilon\}$ is a w-neighborhood of 0, so there exists $n \in \mathbb{N}$ with $V_n \subseteq U$. Then on $\bigcap_{f\in F_n} \ker f \cap B_X$ we have $|g| < \varepsilon$, i.e. $\|g|_{\bigcap_{f\in F_n} \ker f}\| < \varepsilon$. By Hahn-Banach there exists $h \in X^*$ such that $h|_{\bigcap_{f\in F_n} \ker f} = g|_{\bigcap_{f\in F_n} \ker f}$ and $\|h\| < \varepsilon$. Then $\bigcap_{f\in F_n} \ker f \subseteq \ker(g-h)$, so by Lemma 3.3, $g - h \in \operatorname{span} F_n$. □

Theorem 3.16 (Goldstine's Theorem)**.** *For any normed space X, $\overline{B_X}^{w^*} = B_{X^{**}}$.*

Proof. $(B_{X^{**}}, w^*)$ is compact by Banach-Alaoglu and hence closed in X^{**}. Hence $\overline{B_X}^{w^*} \subseteq B_{X^{**}}$. Now let $\varphi \in X^{**} \setminus \overline{B_X}^{w^*}$. We need: $\|\varphi\| > 1$, then we are done. Let $A = \{\varphi\}$, $B = \overline{B}_X^{w^*}$. Then A, B are non-empty, disjoint convex sets. A is compact, B is closed. By Theorem 3.10 (ii) there exists[3] $f \in X^*$ such that $\widehat{f}(\varphi) = \varphi(f) > \sup_B \widehat{f} \geq \sup_{B_X} \widehat{f} = \sup_{B_X} f = \|f\|$. Since $|\varphi(f)| \leq \|\varphi\| \cdot \|f\|$, we have $\|\varphi\| > 1$. □

Remark: So if X is separable, then $X^{**} = \bigcup_n n\overline{B_X}^{w^*}$ is w^*-separable. So $\ell_\infty^* = \ell_1^{**}$ is w^*-separable.

Theorem 3.17. *Let X be a Banach space. Then TFAE*

(i) X is reflexive.

(ii) (B_X, w) is compact.

(iii) X^* is reflexive.

Proof. "$(i) \Rightarrow (ii)$" Since X is reflexive, $(B_X, w) = (B_{X^{**}}, w^*)$ is compact (by Banach Alaoglu).

"$(ii) \Rightarrow (i)$" (B_X, w) is a compact subset of $(B_{X^{**}}, w^*)$ and hence w^*-closed. But by Goldstine $B_{X^{**}} = \overline{B_X}^{w^*} = B_X$, so $X^{**} = X$.

"$(i) \Leftrightarrow (iii)$" has been proved on sheet 1. Alternative proof: "$(i) \Rightarrow (iii)$": If X is reflexive, then on X^* the w-topology is the same as the w^*-topology. So $(B_{X^*}, w) = (B_{X^*}, w^*)$ which is compact by Banach-Alaoglu. By "$(ii) \Rightarrow (i)$", X^* is reflexive. "$(iii) \Rightarrow (ii)$" If X^* is reflexive, then on X^{**} the w-topology and w^*-topology are the same. So $B_{X^{**}}$ is w-compact by Banach-Alaoglu. $B_X \subseteq B_{X^{**}}$ and B_X is convex, $\|\cdot\|$-closed (as X is complete) and hence w-closed by Mazur. Hence B_X is w-compact. □

Remark: If X is a separable, reflexive space, then (B_X, w) is compact and metrizable, and hence sequentially compact.

Lemma 3.18. *Let (K, d) be a non-empty, compact metric space. Then there exists a continuous surjection $\varphi : \Delta \to K$ where $\Delta = \{0,1\}^{\mathbb{N}}$ with the product topology.*

[3] Note that $(X^{**}, \sigma(X^{**}, X^*))^* = X^*$.

Proof. For each $\varepsilon \in \Sigma = \bigcup_{n=0}^{\infty} \{0,1\}^n$ we define a non-empty closed subset K_ε of K such that

- $K_\emptyset = K$.
- $K_\varepsilon = K_{\varepsilon,0} \cup K_{\varepsilon,1}$
- $\max_{\varepsilon \in \{0,1\}^n} \operatorname{diam} K_\varepsilon \to 0$ as $n \to \infty$.

This can be done inductively using the following fact: If $A \neq \emptyset$ is a closed subset of K, then A is totally bounded, so for all $\varepsilon > 0$ there exist $n \in \mathbb{N}$, closed $B_i \subseteq A, 1 \leq i \leq n$ such that $A = \bigcup_{i=1}^{n} B_i$ and $\operatorname{diam} B_i < \varepsilon$ for all i.

Let $\varphi : \Delta \to K$ be as follows: $\varphi((\varepsilon_i)_{i=1}^{\infty})$ is the unique point in $L = \bigcap_{n=0}^{\infty} K_{\varepsilon_1,\ldots,\varepsilon_n}$. For all n, $\operatorname{diam} L \leq \operatorname{diam} K_{\varepsilon_1,\ldots,\varepsilon_n} \to 0$, so $\#L \leq 1$, and $L \neq \emptyset$ since $\{K_{\varepsilon_1,\ldots,\varepsilon_n} \mid n \in \mathbb{N}\}$ has the f.i.p.

φ continuous: Given $\varepsilon = (\varepsilon_i)_{i=1}^{\infty}$, $n \in \mathbb{N}$ and $\delta = (\delta_i)_{i=1}^{\infty}$ such that $\delta_i = \varepsilon_i$ for $1 \leq i \leq n$, then $d(\varphi(\delta), \varphi(\varepsilon)) \leq \operatorname{diam} K_{\varepsilon_1,\ldots,\varepsilon_n}$.

φ is onto: Given $x \in K$, construct $\varepsilon_1, \varepsilon_2, \ldots$ such that for all n, $x \in K_{\varepsilon_1,\ldots,\varepsilon_n}$. Then $\varphi((\varepsilon_i)_{i=1}^{\infty}) = x$. ☐

Remark: Δ is homeomorphic to the middel-third Cantor set via $(\varepsilon_i)_{i=1}^{\infty} \mapsto \sum_{i=1}^{\infty} (2\varepsilon_i) 3^{-i}$.

Theorem 3.19. *Every separable normed space X embeds isometrically into $C[0,1]$.*

Proof. Let $K = (B_{X^*}, w^*)$. This is a compact metrizable space. By the proof of Proposition 3.14, X embeds isometrically into $C(K)$. By the previous Lemma there exists a continuous surjection $\varphi : \Delta \to K$. (Here think of Δ as the middle-third Cantor set)

So we get $C(K) \overset{\simeq}{\hookrightarrow} C(\Delta)$, $f \mapsto f \circ \varphi$. Finally, $C(\Delta) \overset{\simeq}{\hookrightarrow} C[0,1]$ by piecewise linear extension $f \mapsto \widetilde{f}$ (use that $[0,1] \setminus \Delta = \bigcup_n (a_n, b_n)$ disjoint union). ☐

Remark: So $C[0,1] \in \mathcal{SB}$ and $C[0,1]$ is isometrically universal for $\mathcal{SB}$.

4 Convexity and the Krein-Milman theorem

Let X be a real (or complex) vector space and let K be a convex subset of X. A point $x \in K$ is an *extreme point* of K if whenever $x = (1-t)y + tz$ with $y, z \in K$ and $t \in (0,1)$, then $y = z = x$. Let $\operatorname{Ext} K$ be the set of extreme points of K.

Examples.

- $\operatorname{Ext} B_{(\mathbb{R}^2, \|\cdot\|_1)} = \{\pm e_1, \pm e_2\}$.
- $\operatorname{Ext} B_{(\mathbb{R}^2, \|\cdot\|_2)} = S_{(\mathbb{R}^2, \|\cdot\|_2)}$.
- $\operatorname{Ext} B_{c_0} = \emptyset$: Given $x = (x_i)_{i=1}^\infty \in B_{c_0}$ choose $n \in \mathbb{N}$ such that $|x_n| < \frac{1}{2}$. Let $y = x + \frac{1}{2} e_n, z = x - \frac{1}{2} e_n$. Then $y, z \in B_{c_0}$ and $x = \frac{1}{2}(y+z)$, $y \neq x, z \neq x$.

Theorem 4.1 (Krein-Milman)**.** *Let $(X, \mathcal{P})$ be a LCS and K a compact convex subset of X. Then $K = \overline{\operatorname{conv}} \operatorname{Ext} K$. So in particular, if $K \neq \emptyset$, then* $\operatorname{Ext} K \neq \emptyset$*.*

Corollary 4.2. *If X is a normed space, then $B_{X^*} = \overline{\operatorname{conv}}^{w^*}(\operatorname{Ext} B_{X^*})$. So* $\operatorname{Ext} B_{X^*} \neq \emptyset$*.*

Remark: So there does not exist a normed space X such that $X^* \cong c_0$.

Definition. *Let $(X, \mathcal{P})$ be a LCS and K a non-empty, compact, convex subset of K. A* face *of K is a non-empty closed, convex subset F of K such that whenever $(1-t)y + tz \in F$ for some $y, z \in K, t \in (0,1)$, then $y, z \in F$.*

Examples.

1. K is a face of K, and for $x \in K$, $\{x\}$ is a face iff it is an extreme point of K.
2. Let $f \in X^*$ and $\alpha = \sup_K f$. Then $E = \{x \in K \mid f(x) = \alpha\}$ is a face of K. Indeed, E is non-empty, compact, convex and if $y, z \in K$, $t \in (0,1)$ and $x = (1-t)y + tz \in E$, then $\alpha = f(x) = (1-t)f(y) + tf(z) \leq (1-t)\alpha + t\alpha = \alpha$, so $f(y) = f(z) = \alpha$, i.e. $y, z \in E$.
3. If F is a face of K, and E is a face of F, then E is a face of K. So if $x \in \operatorname{Ext} F$, then $x \in \operatorname{Ext} K$.

Proof of Theorem 4.1. WLOG $K \neq \emptyset$.

Claim: $\operatorname{Ext} K \neq \emptyset$. Proof of claim: By Zorn's Lemma there exists a minimal face F of K w.r.t. inclusion. Suppose there exist $x \neq y$ in F. Since X^* separates the points of X, there exists $f \in X^*$ such that $f(x) < f(y)$. Let $\alpha = \sup_F f$ and $E = \{z \in F \mid f(z) = \alpha\}$.

Then E is a face of F and hence of K and $E \subsetneq F$ as $x \notin E$. This is a contradiction, so $F = \{w\}$ for some w which means $w \in \operatorname{Ext} K$.

Now let $L = \overline{\operatorname{conv}} \operatorname{Ext} K$. Since K is closed and convex, $L \subseteq K$. Assume there exists $x_0 \in K \setminus L$. Then by Hahn-Banach separation there exists $f \in X^*$ such that $f(x_0) > \sup_L f$. Let $\alpha = \sup_K f$ and $F = \{x \in K \mid f(x) = \alpha\}$. Then F is a face of K, so by the claim there exists $z \in \operatorname{Ext} F \subseteq \operatorname{Ext} K$. But $F \cap L = \emptyset$ since $\alpha \geq f(x_0)$, so $z \notin L$. □

Lemma 4.3. *Let $(X, \mathcal{P})$ be a LCS and K a compact subset of X and $x_0 \in K$. Then for any neighborhood V of x_0 there exist $n \in \mathbb{N}, f_1, \ldots, f_n \in X^*$, $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that $x_0 \in \{x \in X \mid f_i(x) < \alpha_i \text{ for } 1 \leq i \leq n\} \cap K \subseteq V$*

Proof. Let τ be the topology of $(X, \mathcal{P})$ and let $\sigma = \sigma(X, X^*)$ be the weak topology on X generated by $X^* = (X, \tau)^*$. By definition $\sigma \subseteq \tau$. So $\operatorname{id} : (K, \tau) \to (K, \sigma)$ is a continuous bijection from a compact space to a Hausdorff space, hence a homeomorphism. □

Lemma 4.4. *Let $(X, \mathcal{P})$ be a LCS, $K \subseteq X$ non-empty, compact, convex and $x_0 \in \operatorname{Ext}(K)$. Then if V is a neighborhood of x_0, there exist $f \in X^*$, $\alpha \in \mathbb{R}$ such that $x_0 \in \{x \in X \mid f(x) < \alpha\} \cap K \subseteq V$.*

Proof. By Lemma 4.3 there exist $n \in \mathbb{N}, f_1, \ldots, f_n \in X^*, \alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that $x_0 \in \{x \in X \mid f_i(x) < \alpha_i, i = 1, \ldots, n\} \cap K \subseteq V$. Let $K_i = \{x \in K \mid f_i(x) \geq \alpha_i\}$ for $i = 1, \ldots, n$. Let $L = \operatorname{conv} \bigcup_{i=1}^n K_i$. Note that each K_i is convex, compact, $x_0 \notin \bigcup_{i=1}^n K_i$, $K \setminus V \subseteq \bigcup_{i=1}^n K_i$ and $L = \{\sum_{i=1}^n t_i x_i \mid \forall i : x_i \in K_i, t_i \geq 0, \sum_{i=1}^n t_i = 1\}$ (as each K_i is convex). Since $x_0 \in \operatorname{Ext}(K)$, whenever $x = \sum_{i=1}^n t_i y_i$, $y_i \in K$, $t_i > 0$ for all i, $\sum_{i=1}^n t_i = 1$, then $y_1 = \cdots = y_m = x$, so $x_0 \notin L$. Since L is the continuous image of the compact space $K_1 \times K_2 \times \cdots \times K_n \times \{(t_i)_{i=1}^n \in \mathbb{R}^n : t_i \geq 0 \,\forall i, \sum t_i = 1\}$ under the map $(x_1, \ldots, x_n, (t_i)_{i=1}^n) \mapsto \sum t_i x_i$, it follows that L is compact. WLOG $L \neq \emptyset$, otherwise $K \subseteq V$ and the result is clear. By Hahn-Banach there exists $f \in X^*$ such that $f(x_0) < \inf_L f$. Let $\alpha \in \mathbb{R}$ be such that $f(x_0) < \alpha < \inf_L f$. Then $x_0 \in \{x \in X \mid f(x) < \alpha\} \cap K$ and this set is disjoint from L, hence disjoint from $K \setminus V$, so contained in V. □

Theorem 4.5 (Partial converse to Krein-Milman). *Let $(X, \mathcal{P})$ be a LCS, $K \subseteq X$ non-empty, convex, compact and $S \subseteq K$. If $K = \overline{\operatorname{conv}} S$, then $\operatorname{Ext} K \subseteq \overline{S}$.*

Proof. Suppose there exists $x_0 \in \operatorname{Ext} K \setminus \overline{S}$. Then $V = X \setminus \overline{S}$ is a neighborhood of x_0. By Lemma 4.4 there exist $f \in X^*$ and $\alpha \in \mathbb{R}$ such that $x_0 \in \{x \in X \mid f(x) < \alpha\} \cap K \subseteq V$. Let $L = \{x \in K \mid f(x) \geq \alpha\}$. Then L is closed and convex with $L \supseteq \overline{S}$, and hence $L \supseteq \overline{\operatorname{conv}} S = K$, a contradiction to $x_0 \notin L$. □

Example. Let K be a compact Hausdorff space. Then

$$\operatorname{Ext}(B_{C(K)^*}) = \{\lambda \delta_k \mid \lambda \text{ scalar}, |\lambda| = 1, k \in K\}$$

where $\delta_k(f) = f(k)$ for $f \in C(K)$ (see Sheet 3).

Theorem 4.6 (Banach-Stone theorem)**.** *Let K, L be compact Hausdorff spaces. Then K and L are homeomorphic iff $C(K) \cong C(L)$.*

Proof. "$\Rightarrow$" is obvious. "$\Leftarrow$" Let $T : C(K) \to C(L)$ be an isometric isomorphism. Then $T^* : C(L)^* \to C(K)^*$ is an isometric isomorphism. So $T^*(B_{C(L)^*}) = B_{C(K)^*}$ and hence $T^*(\operatorname{Ext} B_{C(L)^*}) = \operatorname{Ext} B_{C(K)^*}$. Thus for all $l \in L$ there exist a scalar $\lambda(l)$, $|\lambda(l)| = 1$, and $\varphi(l) \in K$ such that $T^*(\delta_l) = \lambda(l)\delta_{\varphi(l)}$. Then $\lambda(l) = (T^*(\delta_l))(1_K) = \delta_l(T1_K) = (T1_K)(l)$, i.e. $\lambda = T(1_K) \in C(L)$.

So $\delta_{\varphi(l)} = \overline{\lambda(l)}T^*(\delta_l)$ for $l \in L$. Since $\delta : L \to (B_{C(L)^*}, w^*)$ is continuous (see proof of Proposition 3.14), λ is continuous and T^* is w^*-to-w^*-continuous, it follows that $l \mapsto \delta_{\varphi(l)}$ is continuous and hence φ is continuous as $\delta : K \to (B_{C(K)^*}, w^*)$ is a homeomorphism $K \to \delta(K)$.

φ injective: If $\varphi(l_1) = \varphi(l_2)$, then $T^*(\overline{\lambda(l_1)}\delta_{l_1}) = \delta_{\varphi(l_1)} = \delta_{\varphi(l_2)} = T^*(\overline{\lambda(l_2)}\delta_{l_2})$ and hence $\overline{\lambda(l_1)}\delta_{l_1} = \overline{\lambda(l_2)}\delta_{l_2}$. Evaluate at 1_L to get $\overline{\lambda(l_1)} = \overline{\lambda(l_2)}$, and hence $\delta_{l_1} = \delta_{l_2}$ and hence $l_1 = l_2$.

φ surjective: Given $k \in K$ there exist μ scalar, $|\mu| = 1$, and $l \in L$ such that $T^*(\mu\delta_l) = \delta_k$. So $\mu\lambda(l)\delta_{\varphi(l)} = \delta_k$. Evaluate at 1_K to get $\mu\lambda(l) = 1$, so $\delta_{\varphi(l)} = \delta_k$, i.e. $\varphi(l) = k$.

Now $\varphi : L \to K$ is a continuous bijection, and hence a homeomorphism. $\square$

5 Banach algebras

A real or complex *algebra* is a real or resp. complex vector space A with a bilinear map $A \times A \to A$, $(a,b) \mapsto ab$ such that $a(bc) = (ab)c$ for all $a,b,c \in A$.

A is a *unital algebra* if there is a (necessarily unique) element $1 \in A$ such that $1 \neq 0$ and $1a = a1 = a$ for all $a \in A$. 1 is called the *unit* of A.

An *algebra norm* on an algebra A is a norm $\|\cdot\|$ on A such that $\|ab\| \leq \|a\| \, \|b\|$ for all $a,b \in A$. Thus multiplication is continuous w.r.t. $\|\cdot\|$.

A *normed algebra* is an algebra with an algebra norm. A *Banach algebra* is a complete normed algebra.

A *unital normed algebra* is a unital algebra with an algebra norm such that $\|1\| = 1$.

Note that if A is a unital algebra with an algebra norm $\|\cdot\|$, there exists an equivalent algebra norm $|||\cdot|||$ on A such that $|||1||| = 1$, e.g. $|||a||| = \sup\{\|ab\| : \|b\| \leq 1\}$.

Let A, B be algebras. A *homomorphism* from A to B is a linear map $\theta : A \to B$ such that for all $x, y \in A$, $\theta(xy) = \theta(x)\theta(y)$.

If A, B are unital with units $1_A, 1_B$, resp., then θ is a unital homomorphism if $\theta(1_A) = 1_B$.

Say θ is an isomorphism if θ is a bijective homomorphism.

Note: If A, B are normed algebras, then a homomorphism $A \to B$ is not assumed continuous. But isomorphisms will be assumed to be homeomorphisms.

Note: From now on the scalar field is $\mathbb{C}$.

Examples.

1. Let K be a compact Hausdorff space. Then $C(K)$ is a commutative, unital Banach algebra under pointwise multiplication.
2. Let K be as in 1. A *uniform algebra on K* is a closed subalgebra of $C(K)$ that separates the points of K and contains the constant functions. E.g. the *disc algebra* $A(\Delta) = \{f \in C(\Delta) \mid f \text{ is holomorphic on } \operatorname{Int}\Delta\}$ where $\Delta = \{z \in \mathbb{C} \mid |z| \leq 1\}$. More generally, let $K \subseteq \mathbb{C}$, $K \neq \emptyset$ compact. Then we have the following uniform algebras on K:
$$\mathcal{P}(K) \subseteq \mathcal{R}(K) \subseteq \mathcal{O}(K) \subseteq A(K) \subseteq C(K)$$

where $\mathcal{P}(K), \mathcal{R}(K), \mathcal{O}(K)$ are the closures in $C(K)$ of, respectively, polynomials, rational functions without poles in K, holomorphic functions on some open neighborhood of K, and $A(K) = \{f \in C(K) \mid f \text{ is holomorphic on } \operatorname{Int} K\}$. Later we will see that always $\mathcal{R}(K) = \mathcal{O}(K)$ and $\mathcal{P}(K) = \mathcal{R}(K)$ iff $\mathbb{C} \setminus K$ is connected (this is Runge's Theorem). In general $\mathcal{R}(K) \neq A(K)$. $A(K) = C(K)$ iff $\operatorname{Int} K = \emptyset$.

3. $L_1(\mathbb{R})$ with the L_1-norm and convolution as multiplication is a commutative B.A. without a unit (e.g. by the Riemann-Lebesgue lemma).

4. Let X be a Banach space. Then $\mathcal{B}(X)$ with the operator norm and composition as multiplication is a unital B.A. It is not commutative if $\dim X \geq 2$. Special case: X is a Hilbert space. Then $\mathcal{B}(X)$ is a C^*-algebra (later).

Elementary constructions:

1. **Subalgebras:** Let A be an algebra and B a subalgebra of A. If A is unital with unit 1, we say B is a unital subalgebra if $1 \in B$. If A is a normed algebra, then $\overline{B}$ (closure of B in A) is also a subalgebra.

2. **Unitization:** Let A be an algebra. The *unitization of* A is the vector space $A_+ = A \oplus \mathbb{C}$ with multiplication $(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$. Then A_+ is an algebra with unit $1 = (0, 1)$. The set $\{(a, 0) \mid a \in A\}$ is an ideal of A_+, isomorphic to A. Under this identification, write $A_+ = \{a + \lambda 1 \mid a \in A, \lambda \in \mathbb{C}\}$.

   If A is a normed algebra, then so is A_+ with norm $\|a + \lambda 1\| = \|a\| + |\lambda|$. So A_+ is a unital normed algebra, and A is a closed ideal of A_+. If A is a Banach algebra, then A_+ is a unital Banach algebra.

3. **Ideals:** Let A be a normed algebra. If J is an ideal of A, then so is $\overline{J}$. If J is a closed ideal of A, then A/J is a normed algebra with the quotient norm. If A is unital and J is a proper closed ideal (i.e. $J \neq A$), then A/J is a unital normed algebra with unit $1 + J$.

4. **Completion:** Let A be a normed algebra. Let $\widetilde{A}$ be the Banach space completion of A. Then the multiplication on A extends to $\widetilde{A}$ and $\widetilde{A}$ becomes a Banach algebra that contains A as a dense subalgebra.

5. Let A be a unital Banach algebra. For $a \in A$ define $L_a : A \to A, x \mapsto ax$. Then L_a is a bounded linear map. The map $a \mapsto L_a : A \to \mathcal{B}(A)$ is an isometric homorphism.

   So every Banach algebra is a closed subalgebra of $\mathcal{B}(X)$ for some Banach space X.

Lemma 5.1. *Let A be a unital Banach algebra and $a \in A$. If $\|1 - a\| < 1$, then a is invertible. Moreover, $\|a^{-1}\| \leq \frac{1}{1 - \|1 - a\|}$.*

Proof. Let $h = 1 - a$. Then $\|h\| < 1$ and for all n, $\|h^n\| \leq \|h\|^n$. Hence $b := \sum_{n=0}^{\infty} h^n$ converges absolutely, and so converges. Then b is the inverse of a and $\|b\| \leq \sum_{n=0}^{\infty} \|h\|^n = \frac{1}{1 - \|h\|}$. □

Notation: For a unital algebra A, we let $G(A) = \{a \in A \mid a \text{ is invertible}\}$.

Corollary 5.2. *Let A be a unital Banach algebra.*

(i) $G(A)$ is open.

(ii) $x \mapsto x^{-1} : G(A) \to G(A)$ is continuous.

(iii) If x_n is a sequence in $G(A)$ and $x_n \to x \notin G(A)$, then $\|x_n^{-1}\| \to \infty$.

(iv) If $x \in \partial G(A) = \overline{G(A)} \setminus G(A)$, then there is a sequence (z_n) with $\|z_n\| = 1$ for all n such that $z_n x \to 0$ and $x z_n \to 0$ as $n \to \infty$ (x is a "topological divisor of zero"). It follows that x has no left or right inverse in A or even in a unital Banach algebra B that contains A as a subalgebra (isometrically).

Proof.

(i) Let $x \in G(A), y \in A$, assume $\|y - x\| < \frac{1}{\|x^{-1}\|}$. Then $\left\|1 - x^{-1}y\right\| \leq \left\|x^{-1}\right\| \|x - y\| < 1$, so $x^{-1}y$ is invertible and thus $y = x(x^{-1}y) \in G(A)$.

(ii) Fix $x \in G(A)$. Let $y \in G(A)$. Then $y^{-1} - x^{-1} = y^{-1}(x - y)x^{-1}$. So $\|y^{-1} - x^{-1}\| \leq \|y^{-1}\| \|x - y\| \left\|x^{-1}\right\|$. If $\|x - y\| < \frac{1}{2\|x^{-1}\|}$, then $\|y^{-1}\| - \|x^{-1}\| \leq \|y^{-1} - x^{-1}\| \leq \frac{1}{2}\|y^{-1}\|$ and hence $\|y^{-1}\| \leq 2\|x^{-1}\|$. Thus if $\|x - y\| < \frac{1}{2\|x^{-1}\|}$, then $\|y^{-1} - x^{-1}\| \leq 2\|x^{-1}\|^2\|x - y\| \to 0$ as $y \to 0$.

(iii) From (i), if $y \in A$, $\|y - x_n\| < \frac{1}{\|x_n^{-1}\|}$, then $y \in G(A)$. Hence for all n, $\|x - x_n\| \geq \frac{1}{\|x_n^{-1}\|}$, hence $\left\|x_n^{-1}\right\| \to \infty$.

(iv) Choose (x_n) in $G(A)$ such that $x_n \to x$. let $z_n = \frac{x_n^{-1}}{\|x_n^{-1}\|}$. Then $\|z_n\| = 1$ for all n. So $\|z_n x\| = \|z_n x_n + z_n(x - x_n)\| \leq \left\|\frac{1}{\|x_n^{-1}\|}\right\| + \|z_n\| \|x - x_n\| = \frac{1}{\|x_n^{-1}\|} + \|x - x_n\| \to 0$ as $n \to \infty$ by (iii). Similarly, $x z_n \to 0$.

If $y \in B$ and $yx = 1_B$, then $z_n = yxz_n \to 0$. Similarly, x has no right inverse in B.

□

5.1 Spectrum and Characters

Definition. *Let A be an algebra and $x \in A$. We define the* spectrum $\sigma_A(x)$ *of x in A as follows: If A is unital, then $\sigma_A(x) = \{\lambda \in \mathbb{C} \mid \lambda 1 - x \notin G(A)\}$. If A is not unital, then $\sigma_A(x) = \sigma_{A_+}(x)$.*

Examples:

1. $A = M_n(\mathbb{C})$ the set of $n \times n$ complex matrices, $x \in A$. Then $\sigma_A(x)$ is the set of all eigenvalues of x.

2. $A = C(K)$, K compact Hausdorff, $f \in A$. Then $\sigma_A(f) = f(K)$ as $g \in A$ is invertible iff $0 \notin g(K)$.

3. If X is a Banach space, $A = \mathcal{B}(X)$, $T \in A$, then

$$\sigma_A(T) = \{\lambda \in \mathbb{C} \mid \lambda I - T \text{ is not an isomorphism}\}.$$

Theorem 5.3. *Let A be a Banach algebra, $x \in A$. Then $\sigma_A(x)$ is a non-empty, compact subset of $\{\lambda \in \mathbb{C} \mid |\lambda| \leq \|x\|\}$.*

Proof. WLOG A is unital. The map $\mathbb{C} \to A, \lambda \to \lambda 1 - x$ is continuous and $\sigma_A(x)$ is the inverse image of $A \setminus G(A)$ which is closed by the previous result. So $\sigma_A(x)$ is closed. If $|\lambda| > \|x\|$, then $\left\|\frac{x}{\lambda}\right\| < 1$, so $1 - \frac{x}{\lambda}$ is invertible, hence $\lambda - x$ is invertible, i.e. $\lambda \notin \sigma_A(x)$. Hence $\sigma_A(x) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|x\|\}$. Thus the spectrum is compact.

Suppose $\sigma_A(x) = \emptyset$. Then we can define $R : \mathbb{C} \to G(A) \subseteq A$ by $R(\lambda) = (\lambda 1 - x)^{-1}$. It is continuous. In fact it is holomorphic:

$$R(\lambda) - R(\mu) = R(\lambda)((\mu 1 - x) - (\lambda 1 - x))R(\mu) = (\mu - \lambda)R(\lambda)R(\mu)$$

So $\frac{R(\lambda)-R(\mu)}{\lambda-\mu} = -R(\lambda)R(\mu) \to -R(\mu)^2$ as $\lambda \to \mu$ as R is continuous.

If $|\lambda| > \|x\|$, then $R(\lambda) = \frac{1}{\lambda}(1 - \frac{x}{\lambda})^{-1}$, so $\|R(\lambda)\| \leq \frac{1}{|\lambda|}\frac{1}{1-\left\|\frac{x}{\lambda}\right\|} = \frac{1}{|\lambda|-\|x\|} \to 0$ as $|\lambda| \to \infty$. By vector-valued Liouville (Theorem 1.8), $R \equiv 0$ which is a contradiction. So $\sigma_A(x) \neq \emptyset$. □

Corollary 5.4 (Gelfand-Mazur)**.** *A complex unital normed division algebra A is isometrically isomorphic to $\mathbb{C}$.*

Proof. Define $\theta : \mathbb{C} \to A, \theta(\lambda) = \lambda 1$. Then θ is isometric and a homomorphism. We prove it is surjective. Let B be the completion of A. Given $x \in A$, $\sigma_B(x) \neq \emptyset$ by the theorem. Pick $\lambda \in \sigma_B(x)$. Then $\lambda 1 - x \notin G(B)$ and so $\lambda 1 - x \notin G(A)$. Since A is a division algebra, $\lambda 1 - x = 0$ and so $x = \theta(\lambda)$. □

Definition. *Let A be a Banach algebra and $x \in A$. The* spectral radius of x in A *is* $r_A(x) := \sup\{|\lambda| \mid \lambda \in \sigma_A(x)\}$.

Note that $r_A(x) \leq \|x\|$.

Note: Let A be a unital algebra, $x, y \in A$. Assume $xy = yx$. Then $xy \in G(A)$ iff $x \in G(A)$ and $y \in G(A)$ (obvious).

Lemma 5.5 (Polynomial Spectral Mapping Theorem)**.** *Let A be a unital Banach algebra, $x \in A$. Then for any complex polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$ we have $\sigma_A(p(x)) = p(\sigma_A(x)) = \{p(\lambda) \mid \lambda \in \sigma_A(x)\}$.*

Proof. This is clear for constant polynomials as $\sigma_A(\lambda 1) = \{\lambda\}$. Assume $n \geq 1$ and $a_n \neq 0$. Fix $\mu \in \mathbb{C}$. We write $\mu - p(z) = c\prod_{j=1}^n(\lambda_j - z)$ where $c \neq 0, \lambda_1, \dots, \lambda_n \in \mathbb{C}$. Then $\mu 1 - p(x) = c\prod_{j=1}^n(\lambda_j 1 - x)$. So $\mu \in \sigma_A(p(x))$ iff there exists j such that $\lambda_j \in \sigma_A(x)$ iff there exists $\lambda \in \sigma_A(x)$ such that $\mu = p(\lambda)$ as $p^{-1}(\mu) = \{\lambda_1, \dots, \lambda_n\}$. □

Theorem 5.6 (Beurling-Gelfand Spectral Radius Formula)**.** *Let A be a Banach algebra, $x \in A$. Then $r_A(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|x^n\|^{1/n}$.*

Proof. WLOG A is unital. If $\lambda \in \sigma_A(x)$, then $\lambda^n \in \sigma_A(x^n)$, and hence $|\lambda^n| \leq \|x^n\|$, i.e. $|\lambda| \leq \|x^n\|^{1/n}$. It follows that $r_A(x) \leq \inf_{n\in\mathbb{N}} \|x^n\|^{1/n}$. Consider $R : \{\lambda \in \mathbb{C} \mid |\lambda| > r_A(x)\} \to G(A) \subseteq A$, $R(\lambda) = (\lambda 1 - x)^{-1}$. As in the proof of Theorem 5.3 this is holomorphic. Fix $\varphi \in A^*$. Then $\varphi \circ R : \{\lambda \mid |\lambda| > r_A(x)\} \to \mathbb{C}$ is holomorphic, and hence it has a Laurent expansion. For $|\lambda| > \|x\|$ ($\geq r_A(x)$), $R(\lambda) = \frac{1}{\lambda}(1 - \frac{x}{\lambda})^{-1} = \frac{1}{\lambda}\sum_{n=0}^\infty \frac{x^n}{\lambda^n}$. So $\varphi \circ R(\lambda) = \sum_{n=0}^\infty \varphi(x^n)\frac{1}{\lambda^{n+1}}$. This is the Laurent expansion of $\varphi \circ R$ on $\{\lambda \mid |\lambda| > r_A(x)\}$. Fix $\lambda \in \mathbb{C}$ with $|\lambda| > r_A(x)$. Then $\varphi(x^n/\lambda^n) \to 0$ for every $\varphi \in A^*$. Thus $\{\frac{x^n}{\lambda^n} \mid n \in A\}$ is weakly bounded, and hence norm bounded. Fix $M \geq 0$ such that for all $n \in \mathbb{N}$, $\left\|\frac{x^n}{\lambda^n}\right\| \leq M$, so $\|x^n\|^{1/n} \leq M^{1/n}|\lambda|$. Hence $\limsup \|x^n\|^{1/n} \leq |\lambda|$ for every λ with $|\lambda| > r_A(x)$. □

Theorem 5.7. *Let A be a unital Banach algebra, B a closed unital subalgebra of A, $x \in B$. Then $\sigma_B(x) \supseteq \sigma_A(x)$ and $\partial\sigma_B(x) \subseteq \partial\sigma_A(x)$. It follows that $\sigma_B(x)$ is the union of $\sigma_A(x)$ together with some of the bounded components of $\mathbb{C} \setminus \sigma_A(x)$.*

Proof. $\sigma_B(x) \supseteq \sigma_A(x)$ is trivial as $G(B) \subseteq G(A)$.

Let $\lambda \in \partial\sigma_B(x)$. Choose (λ_n) in $\mathbb{C} \setminus \sigma_B(x)$ such that $\lambda_n \to \lambda$. Then $\lambda_n 1 - x \in G(B)$ for all n and $\lambda_n 1 - x \to \lambda 1 - x \notin G(B)$. So $\lambda 1 - x \in \partial G(B)$. By Corollary 5.2 (iv), $\lambda 1 - x \notin G(A)$. Since $\lambda_n 1 - x \in G(A)$ for all n, it follows that $\lambda \in \partial\sigma_A(x)$. □

Proposition 5.8. *Let A be a unital Banach algebra and C a maximal commutative subalgebra of A. Then C is closed and unital and for every $x \in C$, $\sigma_C(x) = \sigma_A(x)$.*

Proof. $\overline{C}$ is also a commutative subalgebra, so $C = \overline{C}$ by maximality. $C + \mathbb{C}1$ is also a commutative algebra, so again by maximality $1 \in C$. Fix $x \in C$. We know that $\sigma_C(x) \supseteq \sigma_A(x)$. Let $\lambda \notin \sigma_A(x)$. Then there exists $y \in A$ such that $y(\lambda 1 - x) = (\lambda 1 - x)y = 1$. For any $z \in C$, we have $z(\lambda 1 - x) = (\lambda 1 - x)z$, so $yz(\lambda 1 - x)y = y(\lambda 1 - x)zy$, so $yz = zy$. So the subalgebra generated by C and $\{y\}$ is commutative. By maximality $y \in C$ and so $\lambda \notin \sigma_C(x)$. □

Definition. *A* character *on an algebra A is a non-zero homomorphism $A \to \mathbb{C}$. Let Φ_A be the set of all characters of A.*

Note: If A is unital and $\varphi \in \Phi_A$, then $\varphi(1) = 1$.

Lemma 5.9. *Let A be a Banach algebra, $\varphi \in \Phi_A$. Then φ is bounded and $\|\varphi\| \leq 1$. Moreover, if A is unital, then $\|\varphi\| = 1$.*

Proof. WLOG A is unital: define $\varphi_+ : A_+ \to \mathbb{C}$ by $\varphi_+(x+\lambda 1) = \varphi(x)+\lambda$. Then $\varphi_+ \in \Phi_{A_+}$ and $\varphi_+|_A = A$. Given $x \in A$, if $|\varphi(x)| > \|x\|$, then $\varphi(x)1 - x \in G(A)$, so there exists $y \in A$ such that $(\varphi(x)1 - x)y = 1$. Apply φ: Then $0 \cdot \varphi(y) = \varphi(1) = 1$, a contradiction. Hence $|\varphi(x)| \leq \|x\|$. Since $\varphi(1) = 1$, it follows that $\|\varphi\| = 1$ in the unital case. □

Lemma 5.10. *Let A be a unital Banach algebra. If J is a proper ideal of A, then the ideal $\overline{J}$ is also proper. Hence maximal ideals are closed.*

Proof. Since J is proper, $J \cap G(A) = \emptyset$. Since $G(A)$ is open, it follows that $\overline{J} \cap G(A) = \emptyset$, so $\overline{J}$ is proper. If M is a maximal ideal, then $\overline{M}$ is a proper ideal containing M, hence $\overline{M} = M$ by maximality. □

Notation: Let $\mathcal{M}_A$ be the set of all maximal ideals of an algebra A.

Theorem 5.11. *Let A be a commutative unital Banach algebra. Then the map $\varphi \mapsto \ker \varphi$ is a bijection $\Phi_A \to \mathcal{M}_A$.*

Proof. Let $\varphi \in \Phi_A$. Then $\ker \varphi$ is an ideal as φ is a homomorphism. In fact it must be maximal as $A/\ker \varphi \xrightarrow{\sim} \mathbb{C}$ is a field. So the map is well-defined.

Injective: Let $\varphi, \psi \in \Phi_A$ be characters with $\ker \varphi = \ker \psi$. For $x \in A$, have $\varphi(x)1 - x \in \ker \varphi = \ker \psi$, so $0 = \psi(\varphi(x)1 - x) = \varphi(x) - \psi(x)$.

Surjective: Let $M \in \mathcal{M}_A$. Then A/M is a field and a unital Banach algebra. Hence by Gelfand-Mazur $A/M \cong \mathbb{C}$. Then the quotient map $\varphi : A \to A/M \cong \mathbb{C}$ is a character. □

Corollary 5.12. *Let A be a commutative unital Banach algebra, $x \in A$.*

(i) $x \in G(A)$ *iff* $\varphi(x) \neq 0$ *for all* $\varphi \in \Phi_A$.

(ii) $\sigma_A(x) = \{\varphi(x) \mid \varphi \in \Phi_A\}$.

(iii) $r_A(x) = \sup\{|\varphi(x)| \mid \varphi \in \Phi_A\}$.

Proof.

(i) "⇒" is clear. "⇐" Assume $x \notin G(A)$. Then $J = Ax$ is a proper ideal. Hence by Zorn's lemma $J \subseteq M$ for some maximal ideal M which by the previous theorem is $\ker \varphi$ for some $\varphi \in \Phi_A$, so $\varphi(x) = 0$.

(ii) Immediate from (i).

(iii) Immediate from (ii).

□

Corollary 5.13. *Let A be a Banach algebra, $x, y \in A$. Assume $xy = yx$. Then $r_A(x+y) \leq r_A(x) + r_A(y)$, $r_A(xy) \leq r_A(x) r_A(y)$.*

Proof. WLOG A is unital. WLOG A is commutative: Replace A by a maximal commutative subalgebra containing x, y using Proposition 5.8. For $\varphi \in \Phi_A$ we have $|\varphi(x+y)| \leq |\varphi(x)|+|\varphi(y)| \leq r_A(x)+r_A(y)$, so $r_A(x+y) \leq r_A(x)+r_A(y)$ and similarly for $r_A(xy)$. $\square$

Examples.

1. $A = C(K)$ with K compact Hausdorff. Then $\Phi_A = \{\delta_k \mid k \in K\}$. Proof: "$\supseteq$" is obvious. For the reverse inclusion let M be a maximal ideal of A. We have to show that there exists $k \in K$ such that $M = \ker \delta_k$. Suppose not. Then for all $k \in K$ there exists $f_k \in M$ with $f_k(k) \neq 0$ and then there is an open neighborhood U_k of k such that $f_k \neq 0$ on U_k. By compactness there exist $k_1, \dots, k_n \in K$ such that $K = \bigcup_{j=1}^n U_{k_j}$. Then $g = \sum_{j=1}^n |f_{k_j}|^2 > 0$ on K and hence $g \in G(A)$. Also $g = \sum_{j=1}^n f_{k_j} \overline{f}_{k_j}$, so $g \in M$, a contradiction.
2. Let $K \subseteq \mathbb{C}$ be non-empty, compact. Then $\Phi_{\mathcal{R}(K)} = \{\delta_w \mid w \in K\}$.
3. The disc algebra $A(\Delta)$. Then $\Phi_{A(\Delta)} = \{\delta_w \mid w \in \Delta\}$.
4. The *Wiener algebra* is $W = \{f \in C(\mathbb{T}) \mid \sum_{n\in\mathbb{Z}} |\widehat{f}_n| < \infty\}$. Here $\mathbb{T} = S^1 \subseteq \mathbb{C}$ and $\widehat{f}_n = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}d\theta$. W is a commutative unital Banach algebra with pointwise operations and norm $\|f\|_1 = \sum_{n\in\mathbb{Z}} |\widehat{f}_n|$. This is isometrically isomorphic to the commutative unital Banach algebra $\ell_1(\mathbb{Z})$ with convolution as algebra product, i.e. $(a * b)_n = \sum_{j+k=n} a_j b_k$.

   Then $\Phi_W = \{\delta_w \mid w \in \mathbb{T}\}$. So $f \in W$ is invertible in W iff f is non-zero on $\mathbb{T}$ (Wiener's theorem).

Let A be a commutative, unital Banach algebra. Then

$$\begin{aligned}\Phi_A &= \{\varphi \in B_{A^*} \mid \varphi(1) = 1,\, \varphi(xy) = \varphi(x)\varphi(y)\, \forall x, y \in A\} \\ &= B_{A^*} \cap \widehat{1}^{-1}(\{-1\}) \cap \bigcap_{x,y\in A} (\widehat{xy} - \widehat{x}\widehat{y})^{-1}(\{0\})\end{aligned}$$

is a w^*-closed subset of B_{A^*}. So by Banach-Alaoglu Φ_A is a compact, Hausdorff space in the w^*-topology, called the *Gelfand-topology.* Φ_A with the Gelfand-topology is called the *spectrum of A*, the *character space of A* or the *maximal ideal space of A.*

For $x \in A$, its *Gelfand transform* is $\widehat{x} : \Phi_A \to \mathbb{C}$, $\varphi \mapsto \varphi(x)$, i.e. the restriction of $\widehat{x} \in A^{**}$ to Φ_A. Then $\widehat{x} \in C(\Phi_A)$. The map $A \to C(\Phi_A)$, $x \mapsto \widehat{x}$ is the Gelfand map.

Theorem 5.14 (Gelfand Representation Theorem)**.** *The Gelfand map $A \to C(\Phi_A)$ is a continuous, unital homomorphism. For $x \in A$, have*

(1) $\|\widehat{x}\|_\infty = r_A(x) \leq \|x\|$.

(2) $\sigma_{C(\Phi_A)}(\widehat{x}) = \sigma_A(x)$.

(3) $x \in G(A)$ *iff* $\widehat{x} \in G(C(\Phi_A))$.

Proof. Clearly the Gelfand map is a unital homomorphism. Continuity follows from $\|\widehat{x}\|_\infty = \sup\{|\widehat{x}(\varphi)| \mid \varphi \in \Phi_A\} = r_A(x) \leq \|x\|$. For (ii) note that $\sigma_{C(\Phi_A)}(\widehat{x}) = \operatorname{im}\widehat{x} = \{\varphi(x) \mid \varphi \in \Phi_A\} = \sigma_A(x)$. (iii) follows from (ii). □

Remark: In general, the Gelfand map is neither injective, nor surjective. Its kernel is

$$\{x \in A \mid \sigma_A(x) = \{0\}\} = \{x \in A \mid \lim_{n\to\infty} \|x^n\|^{1/n} = 0\} = \bigcap_{\varphi\in\Phi_A} \ker\varphi = \bigcap_{M\in\mathcal{M}_A} M.$$

Elements $x \in A$ with $\lim_{n\to\infty} \|x^n\|^{1/n} = 0$ are called *quasi-nilpotent*. The intersection $\bigcap_{M\in\mathcal{M}_A} M =: J(A)$ is called the *Jacobson radical* of A. We say that A is *semisimple* if $J(A) = \{0\}$.

6 Holomorphic Functional Calculus

Let $U \subseteq \mathbb{C}$ be non-empty and open. Recall $\mathcal{O}(U) = \{f : U \to \mathbb{C} \mid f \text{ is holomorphic}\}$ is a LCS with seminorms $\|f\|_K = \sup_K |f|$ where $f \in \mathcal{O}(U)$ and $\emptyset \neq K \subseteq U$ compact. $\mathcal{O}(U)$ is also an algebra with pointwise multiplication, which is continuous in the topology. $\mathcal{O}(U)$ is a *Fréchet algebra* (we will not go into this).

Notation: Define $e, u \in \mathcal{O}(U)$ by $e(z) = 1$, $u(z) = z$ for all $z \in U$.

$\mathcal{O}(U)$ is unital with unit e.

Theorem 6.1 (Holomorphic Functional Calculus (HFC))**.** *Let A be a commutative, unital Banach algebra, $x \in A$, $U \subseteq \mathbb{C}$ an open set with $\sigma_A(x) \subseteq U$. Then there exists a unique, continuous unital homomorphism $\Theta_x : \mathcal{O}(U) \to A$ such that $\Theta_x(u) = x$.*

Moreover, for all $\varphi \in \Phi_A, f \in \mathcal{O}(U)$, $\varphi(\Theta_x(f)) = f(\varphi(x))$, and for all $f \in \mathcal{O}(U)$, $\sigma_A(\Theta_x(f)) = \{f(\lambda) \mid \lambda \in \sigma_A(x)\}$.

Remark: We think of Θ_x as "evaluation at x" and write $f(x)$ for $\Theta_x(f)$.

Since $e(x) = 1, u(x) = x$ and Θ_x is a homomorphism, if $p(z) = \sum_{k=0}^n a_k z^k$ is a complex polynomial, then $p(x) = \sum_{k=0}^n a_k x^k$. So think of HFC as a generalization of Lemma 5.5

Theorem 6.2 (Runge's approximation theorem)**.** *Let $\emptyset \neq K \subseteq \mathbb{C}$ be compact. Then $\mathcal{O}(K) = \mathcal{R}(K)$, i.e. if f is holomorphic on some open set containing K and $\varepsilon > 0$, then there is a rational function r without poles in K such that $\|f - r\|_K < \varepsilon$. More precisely, given a set Λ containing a point from each bounded component of $\mathbb{C} \setminus K$, we may choose the r such that all its poles lie in Λ.*

Note: If $\mathbb{C} \setminus K$ is connected, we can take $\Lambda = \emptyset$, so we can even choose r to be a polynomial. So $\mathcal{O}(K) = \mathcal{P}(K)$.

6.1 Vector-valued integration

Let $a < b$ be real numbers, X a Banach space and $f : [a, b] \to X$ continuous. We define $\int_a^b f(t)dt$. Take a sequence $\mathcal{D}_n : a = t_0^{(n)} < t_1^{(n)} < \cdots < t_{k_n}^{(n)} = b$, $n \in \mathbb{N}$, of dissections of $[a, b]$ such that

$$|\mathcal{D}_n| := \max_{1 \leq j \leq k_n} |t_j^{(n)} - t_{j-1}^{(n)}| \to 0 \quad \text{as } n \to \infty.$$

Since f is uniformly continuous, the limit

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} f(t_j^{(n)})(t_j^{(n)} - t_{j-1}^{(n)})$$

exists and is independent of $(\mathcal{D}_n)$. We denote this limit by $\int_a^b f(t)dt$.

Note that for $\varphi \in X^*$, $\varphi\big(\int_a^b f(t)dt\big) = \int_a^b \varphi(f(t))dt$. If we now take φ to be a norming functional for $\int_a^b f(t)dt$, we get

$$\left\|\int_a^b f(t)dt\right\| \leq \int_a^b \|f(t)\|\, dt.$$

Next, let $\gamma : [a,b] \to \mathbb{C}$ be a path (here continuously differentiable) and $f : [\gamma] \to X$ be continuous, where $[\gamma]$ is the image of γ. We define

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

For a chain $\Gamma = (\gamma_1, \dots, \gamma_n)$ and a continuous function $f : [\Gamma] = \bigcup_{i=1}^n [\gamma_i] \to X$, we define

$$\int_\Gamma f(z)dz = \sum_{j=1}^n \int_{\gamma_j} f(z)dz$$

From the above:

$$\left\|\int_\Gamma f(z)dz\right\| \leq \ell(\Gamma) \sup_{z\in[\Gamma]} \|f(z)\|\, .$$

Here $\ell(\Gamma) = \sum_j \ell(\gamma_j)$ is the sum of the lengths of the γ_j.

Theorem (Vector-valued Cauchy)**.** *Let $U \subseteq \mathbb{C}$ be open, Γ a cycle in U such that $n(\Gamma, w) = 0$ for all $w \notin U$. Then for a holomorphic function $f : U \to X$, we have*

$$\int_\Gamma f(z)dz = 0.$$

Proof. Indeed, for all $\varphi \in X^*$, $\varphi\big(\int_\Gamma f(z)dz\big) = \int_\Gamma \varphi(f(z))dz = 0$ by the scalar-valued version of Cauchy's theorem. The result follows from Hahn-Banach. □

6.2 Proof of HFC

Lemma 6.3. *Let A, x, U be as in Theorem 6.1. Let $K = \sigma_A(x)$. Fix a cycle Γ in $U \setminus K$ such that*

$$n(\Gamma, w) = \begin{cases} 1 & w \in K, \\ 0 & w \in \mathbb{C} \setminus U \end{cases}$$

Define $\Theta_x : \mathcal{O}(U) \to A$ *by*

$$\Theta_x(f) = \frac{1}{2\pi i}\int_\Gamma f(z)(z1-x)^{-1}dz.$$

Then:

(i) Θ_x *is well-defined, linear and continuous.*

(ii) For a rational function r *without poles in* U, $\Theta_x(r) = r(x)$ *in the usual sense.*

(iii) For all $\varphi \in \Phi_A, f \in \mathcal{O}(U)$, $\varphi(\Theta_x(f)) = f(\varphi(x))$ *and for all* $f \in \mathcal{O}(U)$, $\sigma_A(\Theta_x(f)) = f(\sigma_A(x))$.

Remark: So HFC is a vector-valued Cauchy integral formula. The lemma proves Theorem 6.1 except for multiplicativity and uniqueness of Θ_x.

Proof of Lemma 6.3.

(i) Well-defined: $z \mapsto f(z)(z1-x)^{-1}$ is well-defined on $[\Gamma]$ and continuous by Corollary 5.2 (ii).

Linearity is immediate from linearity of $\int$.

Continuity: $\|\Theta_x(f)\| \le \frac{1}{2\pi}\ell(\Gamma)\sup_{z\in[\Gamma]}|f(z)|\cdot\|(z1-x)^{-1}\|$. The continuous function $(z1-x)^{-1}$ on the compact set $[\Gamma]$ is bounded (independent of f), so there exists $M \ge 0$ such that $\|\Theta_x f\| \le M\,\|f\|_{[\Gamma]}$ for all $f \in \mathcal{O}(U)$. So Θ_x is continuous.

(ii) First, $\Theta_x(e) = 1$: We have $\Theta_x(e) = \frac{1}{2\pi i}\int_\Gamma (z1-x)^{-1}dz = \frac{1}{2\pi i}\int_{|z|=R}(z1-x)^{-1}dx$ since Γ and $|z| = R$ are homologous in $\mathbb{C}\setminus K$ for $R > \|x\|$, so equality follows by vector-valued Cauchy. So

$$\begin{aligned}\Theta_x(e) &= \frac{1}{2\pi i}\int_{|z|=R}\sum_{n=0}^{\infty}\frac{x^n}{z^{n+1}}dz\\ &= \sum_{n=0}^{\infty}\Big(\frac{1}{2\pi i}\int_{|z|=R}\frac{dz}{z^{n+1}}\Big)x^n\\ &= 1\end{aligned}$$

Let r be a rational function without poles in U. Then $r = \frac{p}{q} \in \mathcal{O}(U)$ where p, q are polynomials and q has no zeros in U. By Lemma 5.5, $\sigma_A(q(x)) = \{q(\lambda) \mid \lambda \in \sigma_A(x)\}$, so $0 \notin \sigma_A(q(x))$. So we can define $r(x) = p(x)\cdot q(x)^{-1}$. For $z, w \in \mathbb{C}$, $p(z)q(w) - q(z)p(w) = (z-w)s(z,w)$ where s is a polynomial in z, w. Hence $p(z)q(x) - q(z)p(x) = (z1-x)s(z,x)$, so $r(z)1 - r(x) = (z1-x)s(z,x)q(z)^{-1}q(x)^{-1}$. Then

$$\Theta_x(r) = \frac{1}{2\pi i}\int_\Gamma r(z)(z1-x)^{-1}dz$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\int_\Gamma (z1-x)^{-1}dz \cdot r(x) + \frac{1}{2\pi i}\int_\Gamma s(z,x)q(z)^{-1}dz \cdot q(x)^{-1} \\
&= \Theta_x(e)r(x) + 0 \cdot q(x)^{-1} \\
&= r(x)
\end{aligned}
$$

(iii) For $\varphi \in \Phi_A, f \in \mathcal{O}(U)$, we have

$$
\begin{aligned}
\varphi(\Theta_x(f)) &= \frac{1}{2\pi i}\int_\Gamma \varphi(f(z)(z1-x)^{-1})dz \\
&= \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-\varphi(x)}dz \\
&= n(\Gamma, \varphi(x))f(\varphi(x)) \\
&= f(\varphi(x))
\end{aligned}
$$

Then

$$
\sigma_A(\Theta_x(f)) = \{\varphi(\Theta_x(f)) \mid \varphi \in \Phi_A\} = \{f(\varphi(x)) \mid \varphi \in \Phi_A\} = \{f(\lambda) \mid \lambda \in \sigma_A(x)\}.
$$

□

Proof of Theorem 6.2. Let $U \subseteq \mathbb{C}$ be open such that $U \supseteq K$. Let $A = \mathcal{R}(K)$, $x \in A$ be $x(z) = z$, for $z \in K$. Then $\sigma_A(x) = K \subseteq U$. Let $\Theta_x : \mathcal{O}(U) \to A$ be as in Lemma 6.3. For $f \in \mathcal{O}(U)$, $\Theta_x(f)(z) = \delta_z(\Theta_x(f)) = f(\delta_z(x)) = f(z)$. So $\mathcal{R}(K) \ni \Theta_x(f) = f|_K$.

Next let B the closed subalgebra of A generated by $1, x, (\lambda 1 - x)^{-1}$ for $\lambda \in \Lambda$. So B is the closure in $C(K)$ of the rational functions whose poles lie in Λ. So B is a closed unital subalgebra of A. If B is a bounded component of $\mathbb{C} \setminus \sigma_A(x) = \mathbb{C} \setminus K$, then there exists $\lambda \in \Lambda \cap V$. Then $\lambda 1 - x$ is invertible in B, so $\lambda \notin \sigma_B(x)$. It follows from Theorem 5.7 that $\sigma_B(x) = \sigma_A(x) = K \subseteq U$. The argument above shows that Θ_x actually takes values in B. □

Corollary 6.4. *Let $\emptyset \neq U \subseteq \mathbb{C}$ be open. Then the subalgebra $\mathcal{R}(U)$ of $\mathcal{O}(U)$ consisting of rational functions without poles in U is dense in $\mathcal{O}(U)$.*

Proof. Let $\emptyset \neq K \subseteq U$ be compact. Let $\widehat{K}$ be K together with all bounded components of $\mathbb{C} \setminus K$ that lie in U. Then $\widehat{K}$ is compact and $\widehat{K} \subseteq U$. For every bounded component V of $\mathbb{C} \setminus \widehat{K}$, $V \setminus U \neq \emptyset$, so we can pick $\lambda_V \in V \setminus U$. Let Λ be the set of all such λ_V's. By Runge's theorem, given $f \in \mathcal{O}(U)$ and $\varepsilon > 0$ there exists a rational function r whose poles lie in Λ such that $\|f - r\|_{\widehat{K}} < \varepsilon$. So $r \in \mathcal{R}(U)$ and $\|f - r\|_K < \varepsilon$. The result follows. □

Proof of Theorem 6.1. Let A, x, U be as in the theorem. Let Θ_x be as in Lemma 6.3. For existence, we just need to check that $\Theta_x(fg) = \Theta_x(f)\Theta_x(g)$ for all $f, g \in \mathcal{O}(U)$. By Lemma 6.3 (ii) this holds for all $f, g \in \mathcal{R}(U)$. Since Θ_x is continuous and $\mathcal{R}(U)$ is dense in $\mathcal{O}(U)$, it is true for all $f, g \in \mathcal{O}(U)$. Uniqueness follows similarly from the denseness of $\mathcal{R}(U)$ in $\mathcal{O}(U)$. □

7 C*-algebras

A $*$-*algebra* is a (complex) algebra A with an *involution* $A \to A, x \mapsto x^*$, i.e. a map satisfiying:

(i) $(\lambda x + \mu y)^* = \overline{\lambda} x^* + \overline{\mu} y^*$

(ii) $(xy)^* = y^* x^*$

(iii) $x^{**} = x$

Note that if A is unital, then $1^* = 1$.

A *C*-algebra* is a Banach algebra with an involution such that the *C*-equation* holds:

$$\|x^* x\| = \|x\|^2 \quad \forall x \in A$$

So a C*-algebra is a $*$-algebra with a complete algebra norm satisfying the C*-equation. Such a norm is called a *C*-norm*.

A *Banach $*$-algebra* is a Banach algebra with an involution such that $\|x^*\| = \|x\|$ for all x.

Remarks:

1. In a C*-algebra A, $\|x^*\| = \|x\|$ for all x. Indeed, $\|x\|^2 = \|x^* x\| \leq \|x^*\| \|x\|$, so $\|x\| \leq \|x^*\|$ and doing the same for x^* gives the reverse inequality.

   So the involution is continuous.

2. If A is a C*-algebra with multiplicative identity $1 \neq 0$, then $\|1\| = 1$ since $\|1\|^2 = \|1^* 1\| = \|1\|$.

A $*$-*subalgebra* of a $*$-algebra A is a subalgebra B of A that is such that $x^* \in B$ for all $x \in B$.

A closed $*$-subalgebra (called a *C*-subalgebra*) of a C*-algebra is a C*-algebra. The closure of a $*$-subalgebra of a C*-algebra is a $*$-subalgebra, and hence a C*-subalgebra.

A homomorphism $\theta : A \to B$ between $*$-algebras is called a $*$-*homomorphism* if $\theta(x^*) = \theta(x)^*$ for all $x \in A$. A $*$-isomorphism is a bijective $*$-homomorphism.

Examples.

1. $C(K)$, K a compact Hausdorff space, with involution given by $f^*(z) = \overline{f(z)}$. This is a commutative unital C*-algebra.
2. $\mathcal{B}(H)$, H a Hilbert space, with involution $T \mapsto T^*$, where T^* is the adjoint of T, i.e. $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x, y \in H$.
3. Any C*-subalgebra of $\mathcal{B}(H)$.

Remark: Any C*-algebra is isometrically $*$-isomorphic to a C*-subalgebra of $\mathcal{B}(H)$ for some Hilbert space H. This is the Gelfand-Naimark theorem.

From now on A will always be a C*-algebra.

An element $x \in A$ is said to be

- *hermitian* or *self-adjoint* if $x^* = x$,
- *unitary* if A is unital and $x^*x = 1 = xx^*$,
- *normal* if $x^*x = xx^*$.

Examples.

1. If A is unital, then 1 is hermitian and unitary. In general, hermitian elements and unitary elements are normal.
2. In $C(K)$ a function f is hermitian iff $f(K) \subseteq \mathbb{R}$ and f is unitary iff $f(K) \subseteq \mathbb{T}$.

Remarks:

1. For $x \in A$ there exist unique hermitian $h, k \in A$ such that $x = h + ik$. Indeed, if $x = h + ik$, then $x^* = h - ik$, so $h = \frac{x+x^*}{2}, k = \frac{x-x^*}{2i}$. Note that x is normal iff $hk = kh$.
2. For $x \in A$, A unital, $x \in G(A)$ iff $x^* \in G(A)$. So $\sigma_A(x^*) = \{\overline{\lambda} \mid \lambda \in \sigma_A(x)\}$ and $r_A(x^*) = r_A(x)$.

Lemma 7.1. *If $x \in A$ is normal, then $r_A(x) = \|x\|$.*

Proof. If x is hermitian, then $\|x^2\| = \|x^*x\| = \|x\|^2$, so by induction $\|x^{2^n}\| = \|x\|^{2^n}$ for every n. Then $r_A(x) = \lim_{n\to\infty} \|x^{2^n}\|^{1/2^n} = \|x\|$.

Now assume x is normal. Then x^*x is hermitian and hence

$$\|x\|^2 = \|x^*x\| = r_A(x^*x) \overset{5.13}{\leq} r_A(x^*) r_A(x) \leq r_A(x)\, \|x\|$$

So $\|x\| \leq r_A(x)$, and hence $\|x\| = r_A(x)$. □

Lemma 7.2. *Assume A is unital, $x \in A$, $\varphi \in \Phi_A$. Then $\varphi(x^*) = \overline{\varphi(x)}$.*

Proof. Write $x = h + ik$ with h, k hermitian. Then $\varphi(x) = \varphi(h) + i\varphi(k)$ and $\varphi(x^*) = \varphi(h) - i\varphi(k)$, so the result follows if we show that for hermitian x, $\varphi(x) \in \mathbb{R}$. Let $\varphi(x) = a + ib$ with $a, b \in \mathbb{R}$. For $t \in \mathbb{R}$,

$$|\varphi(x + it1)|^2 = a^2 + (b+t)^2 \leq \|x + it1\|^2 = \|(x + it1)^*(x + it1)\| = \|x^2 + t^2 1\| \leq \|x^2\| + t^2$$

So $a^2 + b^2 + 2bt \leq \|x^2\|$ for all $t \in \mathbb{R}$, so $b = 0$. □

Remark: The assumption that A is unital is not needed, but unitization is tricky (see Sheet 4).

Corollary 7.3. *Assume A is unital.*

(1) If $x \in A$ is hermitian, then $\sigma_A(x) \subseteq \mathbb{R}$.

(2) If $x \in A$ is unitary, then $\sigma_A(x) \subseteq \mathbb{T}$.

If B is a unital C-subalgebra of A and $x \in B$ is normal, then $\sigma_B(x) = \sigma_A(x)$.*

Proof.

(1) WLOG A is commutative (replace A by the closure of the unital subalgebra generated by x, note that the spectrum can only get larger). Then $\sigma_A(x) = \{\varphi(x) \mid \varphi \in \Phi_A\} \subseteq \mathbb{R}$ by the previous lemma.

(2) Again we can assume that A is commutative. For $\varphi(x) \in \Phi_A$, we have $|\varphi(x)|^2 = \overline{\varphi(x)}\varphi(x) = \varphi(x^*)\varphi(x) = \varphi(x^*x) = 1$, so $\varphi(x) \in \mathbb{T}$. So $\sigma_A(x) = \{\varphi(x) \mid \varphi \in \Phi_A\} \subseteq \mathbb{T}$.

For the last part, first assume $x \in B$ is hermitian. By Theorem 5.7, $\sigma_B(x) \supseteq \sigma_A(x)$ and $\partial\sigma_B(x) \subseteq \partial\sigma_A(x)$. By the first part, $\sigma_A(x), \sigma_B(x) \subseteq \mathbb{R}$, so $\sigma_A(x) = \partial\sigma_A(x), \sigma_B(x) = \partial\sigma_B(x)$.

Now assume $x \in B$ is normal and let $\lambda \in \mathbb{C}$. Then

$$\begin{aligned}\lambda 1 - x \text{ invertible in } B &\iff \lambda 1 - x \text{ and } (\lambda 1 - x)^* \text{ invertible in } B \\ &\iff (\bar{\lambda} 1 - x)(\lambda 1 - x) \text{ invertible in } B \\ &\iff (\bar{\lambda} 1 - x)(\lambda 1 - x) \text{ invertible in } A \\ &\iff \lambda 1 - x \text{ invertible in } A\end{aligned}$$

□

Remark: Let H be a Hilbert space and $T \in \mathcal{B}(H)$ be hermitian or unitary. By the corollary, $\sigma(T) = \partial\sigma(T) \subseteq \sigma_{\mathrm{ap}}(T) \subseteq \sigma(T)$. So $\sigma(T) = \sigma_{\mathrm{ap}}(T)$. This also holds for normal operators.

Theorem 7.4 (Commutative Gelfand-Naimark Theorem)**.** *Let A be a commutative, unital C*-algebra. Then there exists a compact Hausdorff space K such that A is isometrically $*$-isomorphic to $C(K)$. More precisely, the Gelfand map $x \mapsto \widehat{x} : A \to C(\Phi_A)$ is an isometric $*$-isomorphism.*

Proof. We already know that the Gelfand map is a unital homomorphism.

- $*$-homomorphism: We have $(\widehat{x})^*(\varphi) = \overline{\widehat{x}(\varphi)} = \overline{\varphi(x)} \overset{7.2}{=} \varphi(x^*) = \widehat{(x^*)}(\varphi)$. So $\widehat{(x^*)} = (\widehat{x})^*$.
- isometric: $\|\widehat{x}\|_\infty = r_A(x) = \|x\|$ (A is commutative, so all $x \in A$ are normal).
- surjective: Since the Gelfand map is an isometric, unital $*$-homomorphism, its image is a closed, unital $*$-subalgebra of $C(K)$ that separates the points of Φ_A. By Stone-Weierstraß the image is $C(K)$.

□

Applications:

1. Let A be a unital C*-algebra.

   $x \in A$ is *positive* if x is hermitian and $\sigma_A(x) \subseteq [0, \infty)$. A positive $x \in A$ has a unique positive square root: a positive y such that $y^2 = x$.

   Existence: Let B be the unital C*-subalgebra generated by x. Then $x \in B$ and $\sigma_B(x) = \sigma_A(x) \subseteq [0, \infty)$. Consider the Gelfand map $z \mapsto \widehat{z} : B \to C(\Phi_B)$. For all $\varphi \in \Phi_B$, $\widehat{x}(\varphi) = \varphi(x) \geq 0$. Then there exists $y \in B$ such that $\widehat{y}(\varphi) = \sqrt{\widehat{x}(\varphi)}$. Then $\widehat{y}$ is a positive square root of $\widehat{x}$, so y is a positive square root of x.

   Uniqueness: Assume $z \in A$ is another positive square root of x. Then $zx = z^3 = xz$, so there exists a commutative unital C*-subalgebra D of A containing x, z. Then consider the Gelfand map $w \mapsto \widehat{w} : D \to C(\Phi_D)$. Note that also $y \in D$. So $\widehat{y}$ and $\widehat{z}$ are both positive square roots of $\widehat{x}$. So $\widehat{y} = \widehat{z}$ and $y = z$.

2. Let $T \in \mathcal{B}(H)$ be invertible where H is a Hilbert space. Then there exist unique $R, U \in \mathcal{B}(H)$ such that R is positive, U is unitary and $T = RU$. TT^* is positive, so let $R = (TT^*)^{1/2}$ and $U = R^{-1}T$. U is invertible and $UU^* = R^{-1}TT^*R^{-1} = R^{-1}R^2R^{-1} = I$ and $T = RU$.

8 Borel Functional Calculus and Spectral Theory

Throughout:

- $H \neq 0$ is a complex Hilbert space, $\mathcal{B}(H)$ is the C^*-algebra of bounded, linear operators on H.
- K is a compact Hausdorff space, $\mathcal{B}$ the Borel σ-field on K.

A *resolution of the identity of H over K* is a map $P : \mathcal{B} \to \mathcal{B}(H)$ such that

(i) $P(\emptyset) = 0, P(K) = I$.

(ii) For every $E \in \mathcal{B}$, $P(E)$ is an orthogonal projection (i.e. $P(E)^2 = P(E)$, $P(E)^* = P(E)$).

(iii) For all $E, F \in \mathcal{B}$, $P(E \cap F) = P(E)P(F)$.

(iv) For all $E, F \in \mathcal{B}$, if $E \cap F = \emptyset$, then $P(E \cup F) = P(E) + P(F)$.

(v) For all $x, y \in H$, the map $P_{x,y} : \mathcal{B} \to \mathbb{C}, E \mapsto \langle P(E)x, y\rangle$ is a bounded regular complex Borel measure.

Example. Let $H = L_2[0,1]$, $K = [0,1]$, $P(E)(f) = f \cdot 1_E$.

Simple properties:

(i) For all $E, F \in \mathcal{B}$, $P(E)P(F) = P(F)P(E)$.

(ii) For all $E, F \in \mathcal{B}$, If $E \cap F = \emptyset$, then $P(E)(H) \perp P(F)(H)$.

(iii) For all $x \in H$, $P_{x,x}$ is a positive measure of total mass $P_{x,x}(K) = \|x\|^2$.

(iv) P is finitely additive, but not countably additive in general. But for every $x \in H$, the function $\mathcal{B} \to H, E \mapsto P(E)(x)$ is countably additive.

(v) If $E_n \in \mathcal{B}$ and $P(E_n) = 0$ for all $n \in \mathbb{N}$, then $P(\bigcup_{n\in\mathbb{N}} E_n) = 0$.

Let P be as above. A Borel function $f : K \to \mathbb{C}$ is *P-essentially bounded* if there exists a set $E \in \mathcal{B}$ such that $P(E) = 0$ and f is bounded on $K \setminus E$. Then we define $\|f\|_\infty = \inf\{\|f\|_{K\setminus E} \mid E \in \mathcal{B}, P(E) = 0\}$. This inf is attained.

Let $L_\infty(P)$ be the set of all P-essentially bounded Borel functions on K. We identify functions f and g if $f = g$ P-almost everywhere, i.e. if there exists $E \in \mathcal{B}$ such that

$P(E) = 0$ and $f = g$ on $K \setminus E$. Then $(L_\infty(P), \|\cdot\|_\infty)$ is a commutative unital C*-algebra with pointwise operations.

Lemma 8.1 (Definition of $\int_K f dP$)**.** *Let P be as above. Then there exists an isometric, unital $*$-homomorphism $\Phi : L_\infty(P) \to \mathcal{B}(H)$ such that*

(i) $\langle \Phi(f)x, y\rangle = \int_K f dP_{x,y}$,

(ii) $\|\Phi(f)(x)\|^2 = \int_K |f|^2 \, dP_{x,x}$,

(iii) $S \in \mathcal{B}(H)$ *commutes with all the* $\Phi(f)$ *iff* S *commutes with all the* $P(E)$

Note: Property (i) defines Φ uniquely. We write $\int_K f dP$ for $\Phi(f)$. So (i) becomes

$$\Big\langle \Big(\int_K f dP\Big)x, y\Big\rangle = \int_K f dP_{x,y}.$$

Proof. Let $s = \sum_{j=1}^m \alpha_j 1_{E_j}$ be a simple function, i.e. $K = \bigcup_{j=1}^m E_j$ is a Borel partition and $\alpha_1, \dots, \alpha_m \in \mathbb{C}$. Let $\Phi(s) = \sum_{j=1}^m \alpha_j P(E_j)$.

Let $t = \sum_{k=1}^n \beta_k 1_{F_k}$ be another simple function. We check Φ is

- well-defined: If $s = t$ P-a.e., then for all j, k either $P(E_j \cap F_k) = 0$ or $\alpha_j = \beta_k$, hence

$$\sum_j \alpha_j P(E_j) = \sum_{j,k} \alpha_j P(E_j \cap F_k) = \sum_{j,k} \beta_k P(E_j \cap F_k) = \sum_k \beta_k P(F_k).$$

- additive: $s + t = \sum_{j,k} (\alpha_j + \beta_k) 1_{E_j \cap F_k}$. Then

$$\Phi(s+t) = \sum_{j,k} (\alpha_j+\beta_k) P(E_j \cap F_k) = \sum_{j,k} \alpha_j P(E_j \cap F_k) + \sum_{j,k} \beta_k P(E_j \cap F_k) = \Phi(s) + \Phi(t).$$

- multiplicative: $st = \sum_{j,k} \alpha_j \beta_k 1_{E_j \cap F_k}$, so

$$\Phi(st) = \sum_{j,k} \alpha_j \beta_k P(E_j \cap F_k) = \sum_{j,k} \alpha_j \beta_k P(E_j) P(F_k) = \Phi(s)\Phi(t).$$

- $*$-homomorphism: $\overline{s} = \sum \overline{\alpha}_j 1_{E_j}$. So $\Phi(\overline{s}) = \sum \overline{\alpha}_j P(E_j) = \Phi(s)^*$.
- unital: $\Phi(1_K) = P(K) = I$.
- isometric: $\langle \Phi(s)x, y\rangle = \sum_j \alpha_j \langle P(E_j)x, y\rangle = \sum_j \alpha_j P_{x,y}(E_j) = \int_K s dP_{x,y}$. Hence

$$\|\Phi(s)x\|^2 = \langle \Phi(s)x, \Phi(s)x\rangle = \langle \Phi(s)^*\Phi(s)x, x\rangle = \langle \Phi(|s|^2)x, x\rangle = \int_K |s|^2 \, dP_{x,x}.$$

  Hence $\|\Phi(s)x\|^2 \le \|s\|_\infty^2 \|x\|^2$, so $\Phi(s) \le \|s\|_\infty$. If $\|s\|_\infty > 0$, then there exists j such that $P(E_j) \neq 0$ and $|\alpha_j| = \|s\|_\infty$. There exists a unit vector $x \in P(E_j)(H)$. Then $\|\Phi(s)\| \ge \|\Phi(s)x\| = |\alpha_j| \, \|P(E_j)x\| = |\alpha_j| = \|s\|_\infty$, so $\|\Phi(s)\| = \|s\|_\infty$.

So Φ is an isometric unital $*$-homomorphism on the $*$-subalgebra of simple functions. Let $f \in L_\infty(P)$. Choose simple functions $s_n \to f$. Then $\|\Phi(s_m) - \Phi(s_n)\| = \|s_m - s_n\|_\infty \to 0$ as $n, m \to \infty$, so $(\Phi(s_n))_n$ is Cauchy in $\mathcal{B}(H)$. Let $\Phi(f) = \lim_{n\to\infty} \Phi(s_n)$. This is is well-defined. By continuity, Φ is an isometric, unital $*$-homomorphism $L_\infty(P) \to \mathcal{B}(H)$ satisfying (i) and (ii).

For (iii): Since $\Phi(1_E) = P(E)$, one direction is clear. Conversely, if S commutes with all $P(E)$, then S commutes with all $\Phi(s)$ with s simple, and then by continuity S commutes with all $\Phi(f)$ with $f \in L_\infty(P)$. □

Let $L_\infty(K)$ be the set of all bounded Borel functions $f : K \to \mathbb{C}$. This is a commutative, unital C*-algebra with pointwise operations and $\|\cdot\|_K$. If P is as above, then the inclusion $L_\infty(K) \subseteq L_\infty(P)$ is a norm-decreasing unital $*$-homomorphism.

Theorem 8.2 (Spectral Theorem for commutative C*-algebras)**.** *Let A be a commutative unital C*-subalgebra of $\mathcal{B}(H)$. Then there exists a unique resolution P of the identity of H over $K = \Phi_A$ such that*

$$\int_K \widehat{T} dP = T$$

for all $T \in A$, where $\widehat{T}$ is the Gelfand transform of T.

Moreover,

(i) If $\emptyset \neq U \subseteq K$ is open, then $P(U) \neq 0$.

(ii) If $S \in \mathcal{B}(H)$, then S commutes with all $T \in A$ iff S commutes with all $P(E)$.

Remark: The inverse Gelfand map $C(K) \to A \subseteq \mathcal{B}(H), \widehat{T} \mapsto T$ is an isometric, unital $*$-homomorphism. So Theorem 8.2 is an operator version of the Riesz Representation Theorem (RRT).

Proof. For $x, y \in H$, $\widehat{T} \mapsto \langle Tx, y\rangle$ is a bounded linear functional on $C(K)$ of norm $\leq \|x\|\,\|y\|$. By RRT there exists a unique bounded regular complex Borel measure $\mu_{x,y}$ on K such that $\langle Tx, y\rangle = \int_K \widehat{T} d\mu_{x,y}$. For real-valued $\widehat{T}$, T is hermitian, so $\int_K \widehat{T} d\mu_{x,y} = \langle Tx, y\rangle = \overline{\langle Ty, x\rangle} = \int_K \widehat{T} d\overline{\mu_{y,x}}$. So $\mu_{x,y} = \overline{\mu_{y,x}}$ by uniqueness in RRT.

Also

$$\int_K \widehat{T} d\mu_{\lambda x+y,z} = \langle T(\lambda x + y), z\rangle = \lambda \int_K \widehat{T} d\mu_{x,z} + \int_K \widehat{T} d\mu_{y,z}$$

So $\mu_{\lambda x+y,z} = \lambda\mu_{x,z} + \mu_{y,z}$.

For $f \in L_\infty(K)$, $(x, y) \mapsto \int_K f d\mu_{x,y}$ is a sesquilinear form of norm $\leq \|f\|_K$ and it is a hermitian form if f is $\mathbb{R}$-valued.

Hence there exists a unique $\psi(f) \in \mathcal{B}(H)$ such that $\int_K f d\mu_{x,y} = \langle \psi(f)x, y\rangle$ for all x, y, $\|\psi(f)\| \leq \|f\|_K$ and $\psi(f)$ is hermitian if f is $\mathbb{R}$-valued. Then

- ψ is linear: by linearity of integration,

- $*$-map: $\psi(\overline{f}) = \psi(f)^*$ since this holds for $\mathbb{R}$-valued f and ψ is linear.
- $\psi(\widehat{T}) = T$: By construction $\langle \psi(\widehat{T})x, y\rangle = \int_K \widehat{T} d\mu_{x,y} = \langle Tx, y\rangle$ for all x, y.
- ψ is multiplicative: For $S, T \in A$, $\widehat{ST} = \widehat{S}\widehat{T}$, so

$$\int_K \widehat{S}\widehat{T} d\mu_{x,y} = \langle STx, y\rangle = \int_K \widehat{S} d\mu_{Tx,y},$$

  so $\widehat{T} d\mu_{x,y} = d\mu_{Tx,y}$. For $f \in L_\infty(K)$,

$$\int_K f\widehat{T} d\mu_{x,y} = \int_K f d\mu_{Tx,y} = \langle \psi(f)(Tx), y\rangle = \langle Tx, \psi(f)^* y\rangle = \int_K \widehat{T} d\mu_{x,\psi(f)^* y},$$

  so $f d\mu_{x,y} = d\mu_{x,\psi(f)^*y}$. For $g \in L_\infty(K)$, $\int_K gf d\mu_{x,y} = \int_K g d\mu_{x,\psi(f)^*y}$, so $\langle \psi(gf)x, y\rangle = \langle \psi(g)x, \psi(f)^*y\rangle = \langle \psi(f)\psi(g)x, y\rangle$, so $\psi(fg) = \psi(f)\psi(g)$.

So $\psi : L_\infty(K) \to \mathcal{B}(H)$ is a norm-decreasing, unital $*$-homomorphism extending the inverse Gelfand map.

Define $P(E) = \psi(1_E)$. It is easy to see that P is a resolution of the identity of H over K. Note $P_{x,y}(E) := \langle P(E)x, y\rangle = \int_K 1_E d\mu_{x,y} = \mu_{x,y}(E)$. So $P_{x,y} = \mu_{x,y}$.

For all $T \in A$, $\langle \Big(\int_K \widehat{T} dP\Big)x, y\rangle = \int_K \widehat{T} dP_{x,y} = \int_K \widehat{T} d\mu_{x,y} = \langle Tx, y\rangle$, so $T = \int_K \widehat{T} dP$.

This shows the existence of P.

Uniqueness: If $T = \int_K \widehat{T} dP$, then $\langle Tx, y\rangle = \int_K \widehat{T} dP_{x,y}$, so this defines $P_{x,y}$ uniquely by RRT, so P is defined uniquely.

Finally we prove the remaining properties of P:

(i) Let $\emptyset \neq U \subseteq K$ be open. By Urysohn there exists a continuous function $f : K \to [0, 1]$ such that $f \neq 0$, $\operatorname{supp} f \subseteq U$. So there exists a positve $T \in A$ such that $\widehat{T}^2 = f$. So $T \neq 0$. Pick $x \in H$ with $Tx \neq 0$. Then $0 < \|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^2x, x\rangle = \int_K f dP_{x,x} \leq P_{x,x}(U) = \langle P(U)x, x\rangle$, so $P(U) \neq 0$.

(ii) Let $S \in \mathcal{B}(H)$. For $T \in A$, $\langle STx, y\rangle = \langle Tx, S^*y\rangle = \int_K \widehat{T} d\mu_{x,S^*y}$ and $\langle TSx, y\rangle = \int_K \widehat{T} d\mu_{Sx,y}$. So T commutes with all $T \in A$ iff $\mu_{x,S^*y} = \mu_{Sx,y}$ for all x, y.

Moreover, $\langle SP(E)x, y\rangle = \langle P(E)x, S^*y\rangle = \mu_{x,S^*y}(E)$ and $\langle P(E)Sx, y\rangle = \mu_{Sx,y}(E)$. The result follows.

□

Note: If A is a unital Banach algebra and $x \in A$, we can define $e^x := \sum_{n=0}^\infty \frac{x^n}{n!}$. For $x, y \in A$ with $xy = yx$ we have $e^{x+y} = e^x e^y$.

Lemma 8.3 (Fuglede-Putnam-Rosenblum)**.** *If A is a unital C^*-algebra, $x, y, z \in A$, x, y normal and $xz = zy$, then $x^*z = zy^*$.*

Proof. Omitted due to time reasons, use the exponential defined above and the vector valued Liouville Theorem. □

Theorem 8.4 (Spectral Theorem for normal operators)**.** *Let $T \in \mathcal{B}(H)$ be normal. Then there exists a unique resolution P of the identity of H over $\sigma(T)$ such that $T = \int_{\sigma(T)} \lambda dP$. Moreover, for $S \in \mathcal{B}(H)$, $ST = TS$ iff S commutes with all $P(E)$.*

Proof. Let A be the unital C*-subalgebra of $\mathcal{B}(H)$ generated by T. Since T is normal, A is commutative. By Corollary 7.3, $\sigma_A(T) = \sigma(T)$. For $\varphi \in \Phi_A$, φ is uniquely determined by $\varphi(T)$ (since $\varphi(T^*) = \overline{\varphi(T)}$), so $\varphi \mapsto \varphi(T) : \Phi_A \to \sigma(T)$ is a continuous bijection and so a homeomorphism (as Φ_A is compact and $\sigma(T)$ Hausdorff). The maps $\widehat{T}$ and $\widehat{T^*}$ in $C(\Phi_A)$ correspond to $\lambda \mapsto \lambda$ and $\lambda \mapsto \overline{\lambda}$ in $C(\sigma(T))$. Existence of P follows from Theorem 8.2.

Uniqueness: If $T = \int_{\sigma(T)} \lambda dP$, then $p(T, T^*) = \int_{\sigma(T)} p(\lambda, \overline{\lambda}) dP$ for all polynomials p. The functions $p(\lambda, \overline{\lambda})$, p polynomial, are dense in $C(\sigma(T))$ by Stone-Weierstraß. So $P_{x,y}$ are uniquely determined, and hence so is P.

For $S \in \mathcal{B}(H)$, we have $ST = TS$ iff S commutes with T and T^* by Lemma 8.3 iff S commutes with all elements of A iff S commutes with all $P(E)$ by Theorem 8.2. □

Theorem 8.5 (Borel Functional Calculus)**.** *Let $T \in \mathcal{B}(H)$ be a normal operator, $K = \sigma(T)$ and P as in Theorem 8.4. The map*

$$L_\infty(K) \to \mathcal{B}(H), f \mapsto f(T) := \int_K f dP$$

satisfies:

(i) It is a unital $*$-homomorphism and $z(T) = T$ where $z(\lambda) = \lambda$ for all $\lambda \in K$.

(ii) $\|f(T)\| \leq \|f\|_K$ with equality for $f \in C(K)$.

(iii) If $S \in \mathcal{B}(H)$ and $ST = TS$, then $Sf(T) = f(T)S$ for all $f \in L_\infty(K)$.

(iv) $\sigma(f(T)) \subseteq \overline{f(K)}$.

Proof. All follow from the previous results.

For (iv), if $\lambda \notin \overline{f(K)}$, then $\lambda 1_K - f \in G(L_\infty(K))$, so $\lambda I - f(T) \in G(\mathcal{B}(H))$, so $\lambda \notin \sigma(f(T))$. □

Applications:

1. T normal, then $T = RU$ where $R = \int_K |\lambda| \, dP$ is hermitian and $U = \int_{\sigma(T)} \frac{\lambda}{|\lambda|} dP$ is unitary.
2. If U is unitary, then $U = e^{iQ}$ for some operator Q (as there is a Borel, bounded function $f : \mathbb{T} \to \mathbb{R}$ with $e^{if(t)} = t$, then let $Q = f(U)$).
3. Let $T \in G(\mathcal{B}(H))$, we can write $T = e^S e^{iQ}$. So $G(\mathcal{B}(H))$ is connected.