Abelian Varieties

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Contents

1 Motivation: Curves and the Abel-Jacobi Map

Let X be a smooth irreducible projective curve over $\mathbb{C}$, equivalently a compact connected Riemann surface. Let g be its genus.

We recall some basic algebraic geometric notions:

Definition. *The* Divisor group of X *is*

$$\mathrm{Div}(X) = \mathbb{Z}[X] = \{\textit{finite sums} \sum_{P \in X} m_P P,\ m_P \in \mathbb{Z}\}.$$

The degree-map *is*

$$\begin{aligned} \deg : \mathrm{Div}(X) &\longrightarrow \mathbb{Z}, \\ \sum_{P\in X} m_P P &\longmapsto \sum_{P \in X} m_P. \end{aligned}$$

Its kernel is denoted $\mathrm{Div}^0(X) := \ker \deg$.

The function field $k(X)$ *of* X *is the set of rational, equivalently meromorphic, functions on* X*. To* $0 \neq f \in k(X)$ *we associate the* principal divisor

$$\mathrm{div}(f) = \sum_{P \in X} \mathrm{ord}_P(f) P \in \mathrm{Div}^0(X).$$

The class group of X *is*

$$\mathrm{Cl}(X) = \frac{\mathrm{Div}(X)}{\{\mathrm{div}(f) \mid f \in k(X)^*\}}$$

We also let $\mathrm{Cl}^0(X) := \ker(\deg : \mathrm{Cl}(X) \to \mathbb{Z})$.

Another interpretation of $\mathrm{Cl}(X)$ is given by invertible sheaves: A divisor $D \in \mathrm{Div}(X)$ gives rise to an invertible sheaf $\mathcal{O}_X(D)$. Then D is principal if and only if $\mathcal{O}_X(D)$ is trivial. This induces an isomorphism

$$\mathrm{Cl}(X) \simeq \{\text{isomorphism classes of invertible sheaves}\} =: \mathrm{Pic}(X).$$

The set of holomorphic differentials on X is written $H^0(X, \Omega_X)$. It is a complex vector space of dimension g, so $H^0(X, \Omega_X) = \bigoplus_{1 \leq i \leq g} \mathbb{C}\omega_i$, for some holomorphic differentials $\omega_1, \dots, \omega_g$.

Let $\gamma : [0,1] \to X$ be a piecewise C^1 curve. Then we get a g-tuple of complex numbers $(\int_\gamma \omega_i)_{1 \leq i \leq g} \in \mathbb{C}^g$. Better: It is an element of the dual space of $H^0(X, \Omega_X)$.

If γ, γ' are homologous with same endpoints, then $\int_\gamma = \int_{\gamma'}$. In particular, if we take γ to be a closed path, this gives a map

$$\alpha : H_1(X, \mathbb{Z}) \to \mathbb{C}^g,$$
$$\gamma \longrightarrow \Big(\int_\gamma \omega_i \Big)_i.$$

It is called the *period homomorphism.*

Theorem 1.1 (Riemann)**.** *The map α is injective, and its image is a* lattice *in $\mathbb{C}^g$. Moreover, $\mathbb{C}^g / \operatorname{im} \alpha$ is the set of complex points of a smooth algebraic variety over $\mathbb{C}$, called the* Jacobian variety *$J(X)$ of X. The group law on $\mathbb{C}/\operatorname{im}\alpha$ is given by a morphism $J(X) \times J(X) \to J(X)$.*

Recall that a *lattice* in $\mathbb{C}^g$ is a subgroup generated by $2g$ $\mathbb{R}$-linearly independent vectors.

If A is an irreducible *projective* variety over $\mathbb{C}$, together with a morphism $m : A \times A \to A$ such that $m(\mathbb{C}) : A(\mathbb{C}) \times A(\mathbb{C}) \to A(\mathbb{C})$ makes $A(\mathbb{C})$ into a group, we say A is an *abelian variety.*

Fix a point $P_0 \in X$. If $P \in X$, let γ_P be a path from P_0 to P. Any two such paths differ by a closed path, so $(\int_{\gamma_P} \omega_i)_{1 \le i \le g}$ is well-defined modulo $\Lambda := \operatorname{im}(\alpha)$, giving a map

$$X \longrightarrow \mathbb{C}^g/\Lambda = J(X),$$
$$P \longmapsto \Big(\int_{\gamma_P} \omega_i \Big) \bmod \Lambda.$$

This extends to a homomorphism

$$\mathrm{AJ}_{P_0} : \mathrm{Div}(X) \to \mathbb{C}^g/\Lambda,$$

the *Abel-Jacobi map.*

Let $P_0' \in X$ be another point, δ a path from P_0' to P_0. Then

$$\mathrm{AJ}_{P_0'}(P) = \mathrm{AJ}_{P_0}(P) + \Big(\int_\delta \omega_i \Big)_i.$$

More generally, if $D \in \mathrm{Div}(X)$, then

$$\mathrm{AJ}_{P_0'}(D) = AJ_{P_0}(D) + (\deg D) \Big(\int_\delta \omega_i \Big)_i.$$

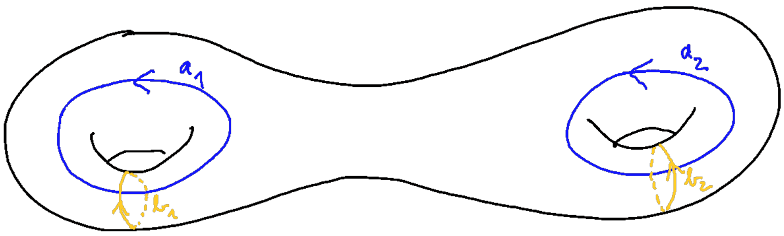
So $\mathrm{AJ} = \mathrm{AJ}_{P_0} : \mathrm{Div}^0(X) \to \mathbb{C}^g/\Lambda$ is independent of P_0.

Theorem 1.2 (Abel-Jacobi Theorem)**.** $\mathrm{AJ} : \mathrm{Div}^0(X) \to \mathbb{C}^g/\Lambda$ *is surjective and its kernel is the set of principal divisors. In other words,* AJ *induces an isomorphism*

$$\mathrm{Cl}^0(X) \xrightarrow{\simeq} \mathbb{C}^g/\Lambda.$$

2 Homology of Riemann Surfaces

Let X be as before. Then $H_1(X, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ is generated by simple closed curves a_j, b_j $(1 \leq j \leq g)$ disjoint except for a_j meeting b_j transversally in one point, with the same orientation.



genus 2 Riemann surface and the generators a_1, a_2, b_1, b_2 of $H_1(X, \mathbb{Z})$

Let $A_{ij} = \int_{a_j} \omega_i, B_{ij} = \int_{b_j} \omega_i$. So $\Lambda = \mathrm{im}(\alpha)$ is span of the columns of the $g \times 2g$-matrix $(A \mid B)$.

To prove Theorem 1.1 we need some special properties of this matrix:

Theorem 2.1 (Riemann period relations)**.**

(a) AB^t *is symmetric.*

(b) The Hermitian matrix $\frac{1}{i}(B\overline{A}^t - A\overline{B}^t)$ *is positive definite.*

These properties can be restated as follows:

(a) $\Leftrightarrow \sum_j (A_{ij}B_{i'j} - B_{ij}A_{i'j}) = 0$ for all i, i'.

(b) $\Leftrightarrow \mathrm{Im}\left(\sum_j \int_{a_j} \overline{\omega} \int_{b_j} \omega\right) > 0$ for all $0 \neq \omega \in H^0(X, \Omega_X)$.

From this, it follows easily that A, B are invertible and that the columns of $(A \mid B)$ linearly independent over $\mathbb{R}$, so Λ is a lattice. Later we will see that (b) is precisely the condition that $\mathbb{C}^g/\Lambda$ is a *projective variety*.

Lemma 2.2. *Let* ω, η *be closed* ($d\omega = 0 = d\eta$) 1*-forms on* X *(not necessarily holomorphic). Then*

$$\int_X \omega \wedge \eta = \sum_j \int_{a_j} \omega \int_{b_j} \eta - \int_{b_j} \omega \int_{a_j} \eta.$$

Assume this for the moment. Take $(\omega, \eta) = (\omega_i, \omega_{i'})$ where $(\omega_i)_i$ is our fixed basis for holomorphic 1-forms. As $dz \wedge dz = 0$, the the left side vanishes, and the first Riemann period relation follows. For the second take $\omega \in H^0(X, \Omega_X)$ and consider $(\overline{\omega}, \omega)$. Locally

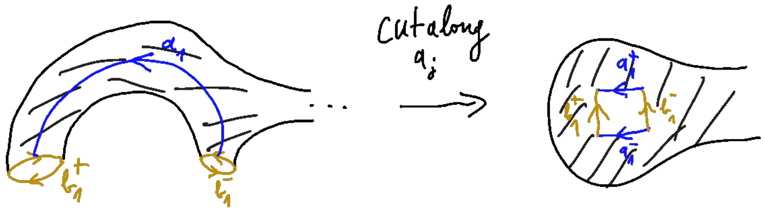
$\omega = f(z)dz$ with holomorphic f, so

$$\overline{\omega} \wedge \omega = f\overline{f} d\overline{z} \wedge dz = 2i|f|^2 dx \wedge dy.$$

So if $\omega \neq 0$, we get

$$0 < \frac{1}{i}\int_X \overline{\omega} \wedge \omega = \sum_{j=1}^{g} \frac{1}{i}\left[\int_{a_j} \overline{\omega} \int_{b_j} \omega - \int_{b_j} \overline{\omega} \int_{a_j} \omega\right] = 2\sum_j \operatorname{Im} \int_{a_j} \overline{\omega} \int_{b_j} \omega$$

Proof of the Lemma. Cut X along the curves a_j, b_j; Let X^* be the resulting surface with boundary. It is a sphere with g holes. The gluing map $\pi : X^* \to X$ induces the zero



Cutting X along a_j, b_j

map $0 = \pi_* : H_1(X^*, \mathbb{Z}) \to H_1(X, \mathbb{Z})$ since $H_1(X^*, \mathbb{Z})$ is generated by the elements $a_i^+ - b_i^+ - a_i^- + b_i^-$. So on X^* there exists a single valued f such that $\omega = df$.[1] If p^+, p^- are points on a_j^+, a_j^- with same image in X, then $f(p^+) - f(p^-) = \int_{p^-}^{p^+} df = \int_{b_j} \omega$. Similarly for points $q^\pm$ on $b_j^\pm$. The oriented boundary of X^* is $\bigcup_j b_j^+ - b_j^- - a_j^+ + a_j^-$. So, by Stokes' Theorem we get

$$\begin{aligned}
\int_X \omega \wedge \eta &= \int_{X^*} \pi^*(\omega \wedge \eta) = \int_{X^*} d(f\eta) \\
&= \int_{\partial X^*} f\eta = \sum_j \Big(\int_{b_j^+} - \int_{b_j^-} - \int_{a_j^+} + \int_{a_j^-} \Big) f\eta \\
&= \sum_j f(q_j^+) \int_{b_j^+} \eta - f(q_j^-) \int_{b_j^-} \eta - f(p_j^+) \int_{a_j^+} \eta + f(p_j^-) \int_{a_j^-} \eta \\
&= \sum_j \int_{a_j} \omega \int_{b_j} \eta - \int_{b_j} \omega \int_{a_j} \eta
\end{aligned}$$

□

[1] Remark by L.T.: This can be seen as follows. For a smooth manifold X, let $I : H^*_{\mathrm{dR}}(X) \to H^*(X, \mathbb{R}) = \mathrm{Hom}(H_*(X), \mathbb{R})$ be the integration map. By naturality of I, we have $I[\pi^*\omega] = \pi^* I[\omega] = 0$ as π^* dual to the zero map $\pi_* = 0$ on H_1. Since I is injective, in fact an isomorphism by the de Rham Theorem, $[\pi^*\omega] = 0$, i.e. ω is exact.

Remark. What this actually says is that the intersection pairing $H_1(X,\mathbb{Z}) \times H_1(X,\mathbb{Z}) \to \mathbb{Z}$ is dual to pairing on closed 1-forms given by $(\omega, \eta) \mapsto \int_X \omega \wedge \eta$.

Let $J = \begin{pmatrix} 0_g & I_g \\ -I_g & 0_g \end{pmatrix}$ which is the intersection matrix for $a_1, \dots, a_g, b_1, \dots, b_g$, i.e. $a_j \frown b_j = -b_j \frown a_j = \delta_{ij}$ (this could be seen as the formal definition of the a_j, b_j). Let $P = (A \mid B)$. Then we can rewrite the Riemann relations as

(a) $\Leftrightarrow PJ^{-1}P^t = 0$,

(b) $\Leftrightarrow Q := \frac{1}{i} PJ^{-1}\overline{P}^t > 0$.

If $0 \neq \lambda \in \mathbb{C}^g$, then $0 < \lambda^t Q \overline{\lambda} = 2\,\mathrm{Im}(\lambda^t B \overline{A}^t \overline{\lambda})$, so A, B are invertible. Then we see that:

Corollary 2.3.

(i) There exists a basis $(\omega_1, \dots, \omega_g)$ such that $\int_{a_j} \omega_i = \delta_{ij}$ (i.e. $A = I_g$) and then B is symmetric and $\mathrm{Im}\, B$ positive definite.

(ii) The columns of $(A \mid B)$ are linearly independent over $\mathbb{R}$, so $\alpha : H_1(X,\mathbb{Z}) \to \mathbb{C}^g$ is injective, and the image $\Lambda = \mathrm{im}\,\alpha$ is a lattice.

How to prove the Abel-Jacobi theorem, i.e. $\mathrm{AJ} : \mathrm{Cl}^0(X) \xrightarrow{\simeq} \mathbb{C}^g/\Lambda = J(X)$?

One way is by using cohomology: The *exponential sequence* (on any complex analytic manifold X) is the short exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{f \mapsto \exp 2\pi i f} \mathcal{O}_X^* \to 0.$$

Here $\mathbb{Z}$ is the constant sheaf and $\mathcal{O}_X$ the sheaf of holomorphic functions. From this we get the long exact sequence in cohomology which breaks up into two sequences:

$$0 \to \underbrace{H^0(X,\mathbb{Z})}_{=\mathbb{Z}} \to \underbrace{H^0(X,\mathcal{O}_X)}_{=\mathbb{C}} \to \underbrace{H^0(X,\mathcal{O}_X^*)}_{=\mathbb{C}^*} \to 0$$

$$0 \to \underbrace{H^1(X,\mathbb{Z})}_{\mathrm{Hom}(H_1(X,\mathbb{Z}),\mathbb{Z})} \to H^1(X,\mathcal{O}_X) \to \underbrace{H^1(X,\mathcal{O}_X^*)}_{=\mathrm{Pic}\,X} \to \underbrace{H^2(X,\mathbb{Z})}_{\simeq \mathbb{Z} \text{ for surface}}$$

This holds for any compact connected $\mathbb{C}$ manifold (except the last isomorphism). For a Riemann surface, $\mathrm{Pic}(X) \simeq \mathrm{Cl}(X)$ and $\mathrm{Cl}(X) \to H^2(X,\mathbb{Z}) \simeq \mathbb{Z}$ is the degree map. So $\mathrm{Cl}^0(X) \simeq \frac{H^1(X,\mathcal{O}_X)}{H^1(X,\mathbb{Z})}$. We have a diagram:

$$\begin{array}{ccccc} \mathrm{Div}^0(X) & \longrightarrow & \mathrm{Cl}^0(X) & \hookrightarrow & \mathrm{Pic}(X) \\ \downarrow \mathrm{AJ} & & \uparrow \exp(2\pi i \cdot)\ \simeq & & \\ J(X) = \frac{H^0(X,\Omega^1)^\vee}{\alpha(H_1(X,\mathbb{Z}))} & \overset{\simeq}{\underset{\exists S}{\dashleftarrow}} & \frac{H^1(X,\mathcal{O}_X)}{H^1(X,\mathbb{Z})} & & \end{array}$$

Serre duality says that there exists an isomorphism $S : H^1(X,\mathcal{O}_X) \xrightarrow{\simeq} H^0(X,\Omega^1)^\vee$ which takes $H^1(X,\mathbb{Z})$ to $H_1(X,\mathbb{Z})$. It is also a nontrivial fact that this diagram commutes.

It follows that AJ induces an isomorphism as claimed. For details, see the handout on Moodle.

3 Complex Tori

Recall: If $w_1, w_2 \in \mathbb{C} \setminus \{0\}, w_2/w_1 \notin \mathbb{R}$, then $\mathbb{C}/(\mathbb{Z}w_1 + \mathbb{Z}w_2)$ is an *elliptic curve* over $\mathbb{C}$, embeddable in $\mathbb{P}^2_{\mathbb{C}}$ by Weierstrass $\wp$-function and its derivative. This gives a bijection

$$\{\text{lattices in } \mathbb{C}, \text{ up to homothety}\} \longleftrightarrow \{\text{iso. classes of elliptic curves}\}$$

The higher dimensional case is more complicated. For more complete treatment, see [Mum70, Chapter 1], [BL04, Chapters 1-4] or [Swi74, Chapters 1-4].

Let V be a finite-dimensional real vector space and $\Gamma \subseteq V$ a lattice. Then V/Γ is a commutative, compact and connected Lie group; also called a *real torus*. By a change of basis we get a real analytic isomorphism $V/\Gamma \simeq \mathbb{R}^n/\mathbb{Z}^n \cong (S^1)^n$.

Now let V be a finite dimensional complex vector space. We call $X = V/\Gamma$ a *complex torus*.

- X is a complex manifold: If $\pi : V \to X$ is the quotient map and $v \in V$, then there exists an open neighborhood $v \in U \subseteq V$ such that $\pi : V \to \pi(V)$ is a homeomorphism and this defines a structure of complex manifold on X.
- Addition/subtraction maps $X \times X \to X$ are holomorphic, so X is a complex Lie group, compact and connected.

Proposition 3.1. *Any compact connected complex Lie group is a complex torus (hence is commutative).*

Proof. See e.g. [Mum70, p. 1] or [BL04, Lemma 1.1.1.]. □

Notice: For any (real or complex) torus $X = V/\Gamma$, the map $\pi : V \to X$ is a connected covering space. As V is simply connected, this means that V is the universal covering space of X (with basepoints (say) $0 \in X, 0 \in V$), and $\Gamma \simeq \pi_1(X, 0) \simeq H_1(X, \mathbb{Z})$ (by Hurewicz isomorphism).

Let $X = V/\Gamma, X' = V'/\Gamma'$ be complex tori. Let $\varphi : V \to V'$ be a linear map such that $\varphi(\Gamma) \subseteq \Gamma'$. It induces a holomorphic map $X \to X'$ which is a homomorphism. Conversely:

Proposition 3.2. *Let $f : X \to X'$ be a holomorphic map.*

(i) If $f(0) = 0$, then there exists a linear $\widetilde{f} : V \to V'$ with $\widetilde{\Gamma} \subseteq \Gamma'$ that induces f. In particular, f is a homomorphism.

(ii) In general, $f(x) = f_0(x) + y$ with $y = f(0) \in X'$ and f_0 is a homomorphism.

Proof. (ii) is clear from (i). As V is simply connected, we can lift f to a continuous $\widetilde{f} : V \to V'$ such that $\widetilde{f}(0) = 0$. Since π, π' are local isomorphisms, $\widetilde{f}$ is holomorphic. For all $v \in V, \gamma \in \Gamma$, $\widetilde{f}(v + \gamma) = \widetilde{f}(v) + g_\gamma(v)$ with $g_\gamma(v) \in \Gamma'$. So $g_\gamma : V \to \Gamma' \subseteq V'$ is holomorphic, so is constant. So the partial derivatives of $\widetilde{f}$ are Γ-invariant, i.e. are holomorphic functions $V/\Gamma \to V'$, hence constant as V/Γ is compact. Thus $\widetilde{f}$ has constant derivative and $\widetilde{f}(0) = 0$, so $\widetilde{f}$ is a linear map. $\square$

Corollary 3.3. *Complex tori $V/\Gamma, V'/\Gamma'$ are isomorphic as complex manifolds iff there exists a $\mathbb{C}$-linear isomorphism $\varphi : V \to V'$ with $\varphi(\Gamma) = \Gamma'$.*

So any complex torus of dimension g is isomorphic to $\mathbb{C}^g/\Pi\mathbb{Z}^{2g}$ where $\Pi \in \mathbb{C}^{g\times 2g}$ is a matrix whose columns are $\mathbb{R}$-linearly independent and Π, Π' give isomorphic tori iff there exist $A \in \mathrm{GL}_g(\mathbb{C}), B \in \mathrm{GL}_{2g}(\mathbb{Z})$ with $\Pi' = A\Pi B$. As the columns of Π span $\mathbb{C}^g$ over $\mathbb{R}$, some subset of them is a $\mathbb{C}$-basis. Hence

Proposition 3.4. *Every complex torus of dimension g is isomorphic to $\mathbb{C}^g/(\mathbb{Z}^g \oplus \Omega\mathbb{Z}^g)$ where Ω is a $g \times g$ complex matrix such that the columns of $\mathrm{Im}(\Omega)$ are linearly independent over $\mathbb{R}$.*

E.g. if $g = 1$, then any complex torus of dimension 1 (i.e. any elliptic curve) is isomorphic to a torus of the forem $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ where $\tau \in \mathbb{C} \setminus \mathbb{R}$.

Proposition 3.5. *If $X = V/\Gamma$ is a real torus of dimension $d \geq 1$, then*

$$H^1(X, \mathbb{Z}) = \mathrm{Hom}(\Gamma, \mathbb{Z}) \simeq \mathbb{Z}^d$$

and for $0 \leq n \leq d$,

$$H^n(X, \mathbb{Z}) = \bigwedge^n H^1(X, \mathbb{Z}) \simeq \mathbb{Z}^{\binom{n}{d}}.$$

Proof. For $n = 1$ we have $H^1(X, \mathbb{Z}) = \mathrm{Hom}(H_1(X, \mathbb{Z}), \mathbb{Z}) = \mathrm{Hom}(\Gamma, \mathbb{Z})$.

We induct on d. If $d = 1$, we are done. Otherwise $\Gamma = \Gamma_1 \oplus \Gamma_2$, with $\Gamma_i \neq 0$, so $X = X_1 \times X_2$ where $X_i = V_i/\Gamma_i, V_i = \mathbb{R}\Gamma_i$. Since $\dim X_i < d$, by induction $H^*(X_i, \mathbb{Z}) \cong \bigwedge^*_{\mathbb{Z}} \mathrm{Hom}(\Gamma_i, \mathbb{Z})$. So by the Künneth formula:

$$\begin{aligned} H^n(X, \mathbb{Z}) &= \bigoplus_{p+q=n} H^p(X_1, \mathbb{Z}) \otimes H^q(X_2, \mathbb{Z}) = \bigoplus_{p+q=n} \bigwedge^p (\mathrm{Hom}(\Gamma_1, \mathbb{Z})) \otimes \bigwedge^q (\mathrm{Hom}(\Gamma_2, \mathbb{Z})) \\ &= \bigwedge^n (\mathrm{Hom}(\Gamma_1, \mathbb{Z}) \oplus \mathrm{Hom}(\Gamma_2, \mathbb{Z})) \\ &= \bigwedge^n \mathrm{Hom}(\Gamma, \mathbb{Z})\,. \end{aligned}$$

$\square$

Remark. H^* has ring structure $H^p \times H^q \to H^{p+q}$ given by the cup-product $\smile$. This isomorphism is compatible with products ($\wedge^p \times \wedge^q \to \wedge^{p+q}$), since in in the Künneth formula, the isomorphism is given by

$$H^p(X_1) \times H^q(X_2) \xrightarrow{(\mathrm{pr}_1^*, \mathrm{pr}_2^*)} H^p(X_1 \times X_2) \times H^q(X_1 \times X_2) \xrightarrow{-\smile-} H^{p+q}(X_1 \times X_2).$$

For $\mathbb{K} = \mathbb{R}, \mathbb{C}$ we have $H^*(X, \mathbb{K}) = H^*(K, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{K}$. Another description is given by differential forms.

Let X be a (C^∞-)manifold. Let $A^n(X) = \{C^\infty$ real-valued n-forms$\}$. The exterior derivative is defined by:

$$d : A^n(X) \to A^{n+1}(X), f dx_{i_1} \wedge \cdots \wedge dx_{i_n} \mapsto df \wedge dx_{i_1} \wedge \cdots = \sum_j \frac{\partial f}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \ldots$$

Then $d^2 = 0$. *De Rham cohomology of* X is

$$H^n_{\mathrm{dR}}(X, \mathbb{R}) = A^n(X)^{d=0}/dA^{n-1}(X).$$

We can do the same with $\mathbb{C}$ coefficients $A^n_{\mathbb{C}}(X) = A^n(X) \otimes_{\mathbb{R}} \mathbb{C}$. Then $H^n_{\mathrm{dR}}(X, \mathbb{C}) = A^n_{\mathbb{C}}(X)^{d=0}/\operatorname{im} d \simeq H^n_{\mathrm{dR}}(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$.

Theorem (De Rham Theorem)**.** *The integration pairing* $H_n(X, \mathbb{Z}) \times H^n_{\mathrm{dR}}(X, \mathbb{R}) \to \mathbb{R}$ *gives an isomorphism* $H^n(X, \mathbb{R}) = \operatorname{Hom}(H_n(X, \mathbb{Z}), \mathbb{R}) \simeq H^n_{\mathrm{dR}}(X, \mathbb{R})$ *and this is compatible with products.*

Back to tori. Let X be a (real or complex) torus. Say $\omega \in A^n(X)$ is *invariant* if for all $y \in X$, $T_y^*\omega = \omega$, where $T_y : X \to X, x \mapsto x + y$. Let

$$A^n(X)^{\mathrm{inv}} = \{\text{invariant } n\text{-forms}\} \subseteq A^n(X)$$

Note that $A^0(X)^{\mathrm{inv}} = \mathbb{R}$.

Proposition 3.6. *If* $\varphi : V \to \mathbb{R}$ *is linear, then* $d\varphi \in A^1(X)^{\mathrm{inv}}$. *This induces isomorphisms* $\bigwedge^n \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R}) \simeq A^n(X)^{\mathrm{inv}}$ *for all* $n \geq 0$.

Proof. Clearly $d\varphi$ defines an invariant 1-form on $A^1(X)$. Pick coordinates x_i (i.e. a basis of V), so $(x_i)_i$ is a basis for $\operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R})$. Then $\omega = \sum f_I dx_I \in A^n(X)$ is invariant iff each f_I is invariant, i.e. constant, so $(dx_I)_I$ is a basis for $A^n(X)^{\mathrm{inv}}$, hence the map $\bigwedge^n \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R}) \to A^n(X)^{\mathrm{inv}}$ is an isomorphism. □

Theorem 3.7. *We have* $A^n(X)^{\mathrm{inv}} \subseteq A^n(X)^{d=0}$, *and the map* $A^n(X)^{\mathrm{inv}} \to H^n_{\mathrm{dR}}(X, \mathbb{R})$ *is an isomorphism. Furthermore, the composite isomorphism* $\bigwedge^n \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R}) \simeq A^n(X)^{\mathrm{inv}} \simeq H^n(X, \mathbb{R}) \simeq \bigwedge^n \operatorname{Hom}(\Gamma, \mathbb{R})$ *is the* $\bigwedge^n$ *of the restriction map* $\operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R}) \xrightarrow{\simeq} \operatorname{Hom}(\Gamma, \mathbb{R})$.

Proof. By the proposition, $A^n(X)^{\mathrm{inv}}$ is spanned by elements of the form $d\varphi_1 \wedge \cdots \wedge d\varphi_n$, $\varphi_i \in \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R})$, and they are closed. Now consider the commutative diagram:

$$\begin{array}{ccccc}
\operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R}) & \xrightarrow[\simeq]{\varphi \mapsto d\varphi} & A^1(X)^{\mathrm{inv}} & \hookrightarrow & A^1(X)^{d=0} \\
\big\downarrow (*) & & & & \big\downarrow \\
\operatorname{Hom}(\Gamma, \mathbb{R}) & = & \operatorname{Hom}(H_1(X, \mathbb{Z}), \mathbb{R}) & \xleftarrow[\simeq]{\int} & H^1_{\mathrm{dR}}(X, \mathbb{R})
\end{array}$$

The map $(*)$ maps φ to $\Gamma \ni \gamma \mapsto \int_{\gamma \in H_1} d\varphi = \int_0^\gamma d\varphi = \varphi(\gamma)$. So $(*)$ is the restriction map which is an isomorphism, so $A^1(X)^{\mathrm{inv}} \to H^1(X, \mathbb{R})$ is an isomorphism. Taking $\bigwedge^n$ gives isomorphism in all degrees. $\square$

Addendum: The same works with complex coefficients: If we $\otimes_{\mathbb{R}} \mathbb{C}$ this, we get:

$$\bigwedge\nolimits_{\mathbb{C}}^n \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \simeq A^n_{\mathbb{C}}(X)^{\mathrm{inv}} \simeq H^n(X, \mathbb{C}) \simeq \bigwedge\nolimits_{\mathbb{C}}^n \mathrm{Hom}(\Gamma, \mathbb{C}).$$

Now suppose $X = V/\Gamma$ is a *complex* torus (so V is a complex vector space). Then

$$\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{R}} V, \mathbb{C}) \xleftarrow{\simeq} \mathrm{Hom}_{\mathbb{C}}(V \oplus \overline{V}, \mathbb{C}) = V^* \oplus \overline{V}^*$$

Then

$$V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}), \overline{V}^* = \mathrm{Hom}_{\text{anti-linear}}(V, \mathbb{C}) \hookrightarrow \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}).$$

In other words, we have an isomorphism $V^* \oplus \overline{V}^* \xrightarrow{\simeq} A^1_{\mathbb{C}}(X)^{\mathrm{inv}} \simeq H^1(X, \mathbb{C}), (\varphi, \psi) \mapsto d\varphi + d\psi$.

In higher degrees, we deduce

$$H^n(X, \mathbb{C}) = \bigwedge\nolimits_{\mathbb{C}}^n (V^* \oplus \overline{V}^*) = \bigoplus_{p+q=n} \bigwedge\nolimits_{\mathbb{C}}^p V^* \otimes \bigwedge\nolimits_{\mathbb{C}}^q \overline{V}^*.$$

Definition. *Let X be any complex manifold. A form $\omega \in A^n_{\mathbb{C}}(X)$ is of* Hodge type (p, q) *if locally*

$$\omega = \sum_{I,J} f_{I,J} dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}$$

where z_i are local holomorphic coordinates on X. We let

$$A^{p,q}(X) = \{\omega \in A^{p+q}_{\mathbb{C}}(X) \text{ of Hodge type } (p, q)\}.$$

Clearly, we have $A^n_{\mathbb{C}}(X) = \bigoplus_{p+q=n} A^{p,q}(X)$. But it is not obvious (and not true for arbitrary complex manifolds X) that this decomposition passes to cohomology:

Theorem 3.8 (Hodge decomposition)**.** *Let $X = V/\Gamma$ be a complex torus. Then for all $n \geq 0$,*

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X, \mathbb{C})$$

where $H^{p,q}(X) \simeq A^{p,q}(X)^{\mathrm{inv}} \cong \bigwedge^p V^* \otimes \bigwedge^q \overline{V}^*$. Also $H^{q,p}(X) = \overline{H^{p,q}(X)}$ inside $H^n(X, \mathbb{C})$.

(For general compact X, with a *Kähler metric*, there is a similar decomposition, replacing "invariant" with "harmonic". This uses PDE theory, in particular the regularity properties of elliptic operators. In our case, it was just easy linear algebra!)

Let $X = V/\Gamma$ be a complex torus. What we have so far:

$$H^1(X, \mathbb{R}) \cong \mathrm{Hom}(\Gamma, \mathbb{R}) = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}),$$

$$
\begin{aligned}
H^1(X,\mathbb{C}) &= \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C} = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\otimes_{\mathbb{R}} V,\mathbb{C}) = V^*\oplus \overline{V}^*,\\
H^n(X,\mathbb{C}) &= \bigwedge\nolimits_{\mathbb{C}}^{n} H^1(X,\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X),\\
H^{p,q}(X) &= \bigwedge\nolimits_{\mathbb{C}}^{p} V^* \otimes_{\mathbb{C}} \bigwedge\nolimits_{\mathbb{C}}^{q} \overline{V}^* = A^{p,q}(X)^{\mathrm{inv}}.
\end{aligned}
$$

Concrete: If $V=\mathbb{C}^g$, then $\mathbb{C}^g = V^* \ni (a_i)\mapsto \sum a_i dz_i \in H^1$ and $(b_i)\in \overline{V}^* \cong \mathbb{C}^g \mapsto \sum_{b_i} d\overline{z_i}$.

Individual pieces:

Proposition 3.9. *Let $H^0(X,\Omega_X^n)=\{$holomorphic n-forms$\}$. Then*

$$H^0(X,\Omega_X) = A^{n,0}(X)^{\mathrm{inv}} \simeq \bigwedge\nolimits_{\mathbb{C}}^{n} V^* = H^{n,0}(X).$$

Proof. Pick basis $\mathbb{C}^g\simeq V$. We know that $A^{n,0}(X)^{\mathrm{inv}}$ has basis $\{dz_I = dz_{i_1}\wedge\cdots\wedge dz_{i_n} \mid I=(i_1<\cdots<i_n)\}$ and $H^0(X,\Omega_X^n)=\{\omega=\sum_I f_I dz_I \mid f_I \text{ holomorphic and } \Gamma\text{-invariant}\}$. By Liouville, these f_I are constant, hence $H^0(X,\Omega_X^n)=A^{n,0}(X)^{\mathrm{inv}}$. □

Theorem 3.10 (Dolbeault isomorphism)**.** *There is a canonical isomorphism*

$$H^{p,q}(X)\simeq H^q(X,\Omega_X^p).$$

It called the Dolbeault isomorphism*.*

We prove it by reducing to the special case $p=0$. We know that $\Omega_X^p=\bigoplus_I \mathcal{O}_X dz_I$ is free, in coordinate-free words: $H^0(X,\Omega_X^p)\otimes_{\mathbb{C}}\mathcal{O}_X\simeq\Omega_X^p$. Thus we get an isomorphism $H^0(X,\Omega_X^p)\otimes_{\mathbb{C}} H^q(X,\mathcal{O}_X)\xrightarrow{\sim} H^q(X,\Omega_X^p)$. We know that $H^0(X,\Omega_X^p)\simeq\bigwedge^p V^*$, so it is enough to show that $H^q(X,\mathcal{O}_X)\simeq\bigwedge_{\mathbb{C}}^q \overline{V}^*$. More precisely:

Theorem 3.11. *The map $H^n(X,\mathbb{C})\to H^n(X,\mathcal{O}_X)$ factors as*

$$H^n(X,\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X) \to H^{0,n}(X)\simeq H^n(X,\mathcal{O}_X).$$

Proof sketch (Almost complete in $g=1$). Fact: $A^0_{\mathbb{C}}(X)=\{C^\infty\text{-functions}\}$ is given by Fourier series (note $X\simeq(\mathbb{R}/\mathbb{Z})^{2g}$). Now suppose that $g=1$, $X=\mathbb{C}/\Gamma$ where $\Gamma=\mathbb{Z}\gamma_1\oplus\mathbb{Z}\gamma_2$ with $\mathrm{Im}(\gamma_2/\gamma_1)>0$. Write $z=x_1\gamma_1+x_2\gamma_2$ with $x_1,x_2\in\mathbb{R}$. For $f\in A^0_{\mathbb{C}}(X)$ we get the Fourier series expansion:

$$f(z)=\sum_{m_1,m_2\in\mathbb{Z}} c_m e^{2\pi i(m_1x_2-m_2x_1)} = \sum_{\gamma\in\Gamma} c_\gamma e^{\pi(\overline{\gamma}z-\gamma\overline{z})/A}$$

where A is the area of the fundamental parallelogram and $|c_\gamma||\gamma|^N\to 0$ for all N as $|\gamma|\to\infty$.

Let $\mathcal{A}_X^{p,q}$ be the sheaf of C^∞ (p,q) forms. By the Cauchy-Riemann equations we have

$$\mathcal{O}_X = \ker\left(\mathcal{A}^0_{X,\mathbb{C}} = \mathcal{A}_X^{0,0} \xrightarrow{\overline{\partial}} \mathcal{A}_X^{0,1}\right)..$$

Now $\overline{\partial}$ is surjective as a map of sheaves: If $\omega = f d\overline{z} \in \mathcal{A}_X^{0,1}(U)$, then (possibly shrinking U a bit) we can find $g \in A^0_{\mathbb{C}}(X)$ such that $g|_U = f$ (using bump functions), with $\int_{\mathbb{C}/\Gamma} g = 0$. So g has Fourier series with $c_0 = 0$; then

$$g d\overline{z} = \overline{\partial} \sum_{\gamma \neq 0} -\frac{A}{\pi\gamma} c_\gamma e^{\pi(\overline{\gamma}z - \gamma\overline{z})/A} \in \overline{\partial}(A^0(X)).$$

So there is a SES:

$$0 \to \mathcal{O}_X \to \mathcal{A}_X^{0,0} \xrightarrow{\overline{\partial}} \mathcal{A}_X^{0,1} \to 0$$

The sheaves $\mathcal{A}_X^{p,q}$ are acyclic, i.e. $H^i(X, \mathcal{A}_X^{p,q}) = 0$ for $i > 0$. This is because they are *fine sheaves* (partition of unity argument). Therefore we can calculate $H^*(X, \mathcal{O}_X)$ using this resolution of $\mathcal{O}_X$, so $H^1(X, \mathcal{O}_X) = \operatorname{coker}(\overline{\partial} : A^{0,0}(X) \to A^{0,1}(X)) = A^{0,1}(X)/\overline{\partial}A^{0,0}(X)$. We just saw: $\omega \in A^{0,1}(X)$ lies in $\operatorname{im}(\overline{\partial})$ iff its 0th Fourier coefficient is 0 and so $A^{0,1}(X) = \operatorname{im}(\overline{\partial}) \oplus \mathbb{C}d\overline{z}$ and $\mathbb{C}d\overline{z} = A^{0,1}(X)^{\text{inv}}$. So $H^1(X, \mathcal{O}_X) = A^{0,1}(X)^{\text{inv}} = H^{0,1}(X)$.

In the general case,

$$0 \to \mathcal{O}_X \to \mathcal{A}_X^{0,0} \xrightarrow{\overline{\partial}} \mathcal{A}_X^{0,1} \xrightarrow{\overline{\partial}} \cdots \to \mathcal{A}_X^{0,g} \to 0$$

is exact ($\overline{\partial}$-Poincare lemma) and $A^{0,q}(X)^{\overline{\partial}=0} = \overline{\partial}A^{0,q-1}(X) \oplus A^{0,q}(X)^{\text{inv}}$. See [Mum70, Chapter 1], [BL04, Section 1.4]. □

4 Pic of Complex Tori

Let X be a complex manifold.

Recall that the *Picard-group*

$$\operatorname{Pic}(X) := \{\text{invertible } \mathcal{O}_X\text{-modules}\}/\text{isomorphism}$$

is a group under $\otimes$.

It is a basic fact that $\operatorname{Pic}(X) \simeq H^1(X, \mathcal{O}_X^*)$. We describe the isomorphism. Given an invertible sheaf $\mathcal{L}$ with trivialization (s_i) on the open cover (U_i), let

$$c_{ij} = s_j^{-1} s_i|_{U_i \cap U_j} \in \mathcal{O}_X^*(U_i \cap U_j).$$

Then $c_{ij}c_{jk} = c_{ik}$ on $U_i \cap U_j \cap U_k$. Thus $(c_{ij})_{ij}$ is a 1-Čech cocycle with values in $\mathcal{O}_X^*$, so it defines an element of $H^1(X, \mathcal{O}_X^*)$. If (s_i') is another trivialization, then $t_i = s_i'(1)/s_i(1) \in \mathcal{O}_X^*(U_i)$, and $c_{ij}' = (s_j'^{-1} s_i')|_{U_i \cap U_j} = c_{ij} t_i / t_j$ and $(i,j) \mapsto t_i/t_j$ is a coboundary. Hence the two trivializations give the same element in $H^1(X, \mathcal{O}_X^*)$. Similarly, one checks that it is independent of the cover U_i, so we get a well-defined map $\operatorname{Pic}(X) \to H^1(X, \mathcal{O}_X^*)$ which in fact is an isomorphism.

Recall the *exponential sequence*:

$$0 \to \underbrace{2\pi i \mathbb{Z}}_{=:\mathbb{Z}(1)} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0.$$

Suppose X is compact and connected, then H^0 of this is

$$0 \to \mathbb{Z}(1) \to \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \to 0$$

and H^1, H^2 terms:

$$0 \to H^1(X, \mathbb{Z}(1)) \xrightarrow{j} H^1(X, \mathcal{O}_X) \to \underbrace{H^1(X, \mathcal{O}_X^*)}_{\operatorname{Pic}(X)} \xrightarrow{c_1} H^2(X, \mathbb{Z}(1)) \to H^2(X, \mathcal{O}_X).$$

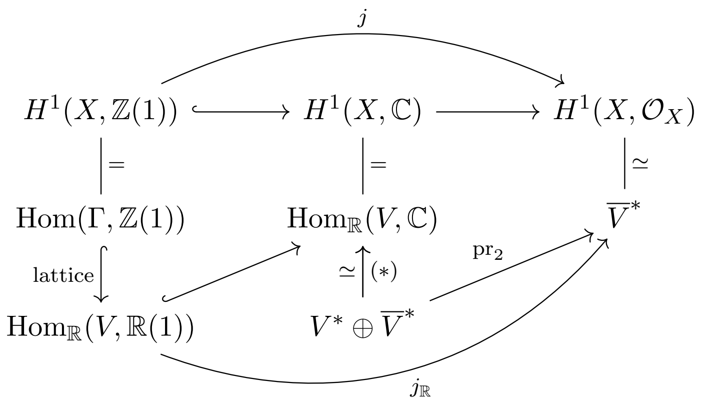
So $\operatorname{Pic}(X)$ contains a subgroup $\operatorname{Pic}^0(X) := \operatorname{coker} j = \ker c_1$. The quotient

$$\operatorname{Pic}(X)/\operatorname{Pic}^0(X) =: \operatorname{NS}(X)$$

is the *Neron-Severi group* of X. Via c_1 it is isomorphic to $\ker(H^2(X, \mathbb{Z}(1)) \to H^2(X, \mathcal{O}_X))$. $H^2(X, \mathbb{Z}(1))$ is finitely generated, hence so is $\operatorname{NS}(X)$.

Now suppose $X = V/\Gamma$ is a complex torus. We inspect $\operatorname{Pic}^0(X)$ and $\operatorname{NS}(X)$.

(1) $\operatorname{Pic}^0(X) = H^1(X, \mathcal{O}_X)/\operatorname{im} j$. We have a commutative diagram:



The right isomorphism and commutativity is essentially Theorem 3.11 for $n = 1$. $(*)$ is given by inclusions $V^*, \overline{V}^* \subseteq \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{C})$. The inverse of $(*)$ is given by $\operatorname{Hom}_{\mathbb{R}}(V, \mathbb{C}) \ni l \mapsto (\lambda, \mu) \in V^* \oplus \overline{V}^*$ where

$$\lambda(v) = \frac{1}{2}(l(v) - il(v)),$$
$$\mu(v) = \frac{1}{2}(l(v) + il(iv)).$$

So $j_{\mathbb{R}}$, the $\mathbb{R}$-linear extension of $j : \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{Z}(1)) \to \overline{V}^*$, is given by $j_{\mathbb{R}}(l)(v) = \mu(v) = \frac{1}{2}(l(v) + il(iv))$ and so $j_{\mathbb{R}}$ is an isomorphism, with inverse $\mu \mapsto \mu - \overline{\mu}$ (since l is purely imaginary). Therefore $j(H^1(X, \mathbb{Z}(1))) \subseteq \overline{V}^*$ is a lattice.

Theorem 4.1. *$\widehat{X} := \operatorname{Pic}^0(X) \simeq \overline{V}^*/\operatorname{im}(j)$ is a complex torus (the* dual *of X) and there are isomorphisms*

$$\widehat{X} \xrightarrow[\simeq]{j_{\mathbb{R}}^{-1}} \frac{\operatorname{Hom}(\Gamma, \mathbb{R}(1))}{\operatorname{Hom}(\Gamma, \mathbb{Z}(1))} \xrightarrow[\simeq]{\exp} \operatorname{Hom}(\Gamma, U(1))$$

where $U(1) = S^1 \subseteq \mathbb{C}^*$.

(2) $\operatorname{NS}(X)$.

Definition. *A* Riemann form *for X is a Hermitian form $H : V \times V \to \mathbb{C}$ for which the alternating form $E = \operatorname{Im} H : V \times V \to \mathbb{R}$ is integer-valued on $\Gamma \times \Gamma$, i.e. $E \in \operatorname{Alt}^2_{\mathbb{Z}}(\Gamma)$.*

From Exercise Sheet 1: To give a Riemann form H is equivalent to giving an alternating map $E : \Gamma \times \Gamma \to \mathbb{Z}$ such that its $\mathbb{C}$-bilinear extension

$$E_{\mathbb{C}} : (\mathbb{C} \otimes \Gamma) \times (\mathbb{C} \otimes \Gamma) = (V \oplus \overline{V}) \times (V \oplus \overline{V}) \to \mathbb{C}$$

satisfies $E_{\mathbb{C}}(V, V) = 0$ (equivalently $E_{\mathbb{C}}(\overline{V}, \overline{V}) = 0$). The correspondence is $H \mapsto E = \operatorname{Im} H$ and $E \mapsto (H : (u, v) \mapsto 2iE_{\mathbb{C}}((u, 0), (0, \overline{v})))$.

Theorem 4.2. $\mathrm{NS}(X) \simeq \{$*Riemann forms on* $X\}$.

Proof.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{NS}(X) & \hookrightarrow & H^2(X,\mathbb{Z}(1)) & \longrightarrow & H^2(X,\mathcal{O}_X) \\ & & & \searrow & \simeq \uparrow 2\pi i & & \simeq \uparrow 2\pi i \\ & & & & H^2(X,\mathbb{Z}) & \xrightarrow{(**)} & H^2(X,\mathcal{O}_X) \end{array}$$

Note that

$$H^2(X,\mathbb{Z}) = \bigwedge^2 \mathrm{Hom}(\Gamma,\mathbb{Z}) = \mathrm{Alt}^2_{\mathbb{Z}}(\Gamma) = \{\text{alternating bilinear } E : \Gamma \times \Gamma \to \mathbb{Z}\}$$

and $H^2(X,\mathcal{O}_X) = \bigwedge^2 \overline{V}^* = \mathrm{Alt}^2_{\mathbb{C}}(\overline{V})$.

Claim: $(**)$ takes $E \in \mathrm{Alt}^2_{\mathbb{Z}}(\Gamma)$ to $E_{\mathbb{C}}|_{\overline{V}\times\overline{V}} \in \mathrm{Alt}^2_{\mathbb{C}}(\overline{V})$.

If so, we get

$$\mathrm{NS}(X) \xrightarrow{\simeq} \{E \in \mathrm{Alt}^2_{\mathbb{Z}}(\Gamma) \mid E_{\mathbb{C}}|_{\overline{V}\times\overline{V}} = 0\} = \{\text{Riemann forms}\}$$

Hence the theorem. Proof of claim:

$$\begin{array}{ccccc} H^2(X,\mathbb{Z}) & \hookrightarrow & H^2(X,\mathbb{C}) & \twoheadrightarrow & H^2(X,\mathcal{O}_X) \\ \| & & \| & & \| \\ \mathrm{Alt}^2_{\mathbb{Z}}(\Gamma) & \longrightarrow & \underbrace{\mathrm{Alt}^2_{\mathbb{C}}(\mathbb{C}\otimes_{\mathbb{R}} V)}_{=\mathrm{Alt}^2_{\mathbb{C}}(V\oplus\overline{V})} & \longrightarrow & \mathrm{Alt}^2_{\mathbb{C}}(\overline{V}) \end{array}$$

The first map in the bottom line is given by $E \mapsto E_{\mathbb{C}}$. By Theorem 3.11, the second map is given by restriction to $\overline{V} \times \overline{V} \subseteq (V \oplus \overline{V} \times V \oplus \overline{V})$. □

Remark. $c_1 : \mathrm{Pic}(X) \to H^2(X,\mathbb{Z}(1))$ is the *first Chern class homomorphism.* It classifies *topological line bundles*, i.e. $c_1(R) = 0$ iff the corresponding C^∞-line bundle is trivial.

So

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 0$$

and $\mathrm{Pic}^0(X) \simeq \mathrm{Hom}(\Gamma, U(1))$ and $\mathrm{NS}(X) \simeq \{\text{Riemann forms}\}$ is free abelian. As $\mathrm{NS}(X)$ is free, this splits (although not canoncially).

Definition.

$$P(X) := \left\{ (H,\alpha) \middle| \begin{array}{l} H \text{ is a Riemann form, } \alpha : \Gamma \to U(1) \text{ s.t.} \\ \alpha(\gamma+\delta) = \alpha(\gamma)\alpha(\delta)e^{\pi i E(\gamma,\delta)},\ E = \mathrm{Im}\, H \end{array} \right\}$$

There is an exact sequence:

$$0 \to \mathrm{Hom}(\Gamma, U(1)) \to P(X) \to \{\text{Riemann forms}\}.$$
$$\alpha \mapsto (0,\alpha)$$

Lemma 4.3. *This is exact on the right, i.e. for all H, there exists an $\alpha : \Gamma \to U(1)$ such that $(H, \alpha) \in P(X)$.*

Theorem 4.4 (Appell-Humbert)**.** *There is an isomorphism $P(X) \simeq \operatorname{Pic}(X)$ such that*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Hom}(\Gamma, U(1)) & \longrightarrow & P(X) & \longrightarrow & \{\textit{Riemann forms}\} & \longrightarrow & 0 \\ & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \\ 0 & \longrightarrow & \operatorname{Pic}^0(X) & \longrightarrow & \operatorname{Pic}(X) & \longrightarrow & \operatorname{NS}(X) & \longrightarrow & 0 \end{array}$$

commutes.

Proof. We will explicitly construct an invertible sheaf $\mathcal{L}(H, \alpha) \in \operatorname{Pic}(X)$ for each $(H, \alpha) \in P(X)$ so that this map makes the diagram commute. It is then clear that it must be an isomorphism by the Five Lemma.

Let $\pi : V \to X = V/\Gamma$ be the quotient map. Idea: We will write down $\mathcal{L}$ with $\pi^*\mathcal{L} \simeq \mathcal{O}_V$ (in fact every invertible $\mathcal{O}_V$-module is trivial). By adjunction, we find a subsheaf $\mathcal{L} \subseteq \pi_*\mathcal{O}_V$.

We say that a connected open subset $U \subseteq X$ is *small* if $U = \pi(U')$, $U' \subseteq V$ open, such that the translates $\overline{U'} + \gamma$, $\gamma \in \Gamma$, are disjoint. If so, then

- $\pi^{-1}(U) = \coprod\{\text{opens } U' \subseteq V \text{ such that } \pi : U' \xrightarrow{\simeq} U\}$,
- Γ permutes $\{U'\}$ simply transitively,
- $\pi_*\mathcal{O}_V(U) = \mathcal{O}_V(\pi^{-1}U) = \prod_{U'} \mathcal{O}_V(U')$.

Every open subset of X is a union of small opens, so to define a sheaf on X, it is enough to define it on the set of small opens.

We want $\mathcal{L}(U) \cong \mathcal{O}_X(U)$ for small U, so let

$$\mathcal{L}(U) = \Big\{(s_{U'}) \in \prod_{\pi:U' \xrightarrow{\simeq} U} \mathcal{O}_V(U') \big| \, \forall\gamma \in \Gamma, z \in U', s_{U'+\gamma}(z + \gamma) = s_{U'}(z)c_\gamma(z) \quad (*)\Big\}$$

for some family (c_γ) with $c_\gamma : V \to \mathbb{C}^\times$ holomorphic, to be be determined. For example, if we let $c_\gamma = 1$ for all γ, we get $\mathcal{L} \simeq \mathcal{O}_X$.

The condition $(*)$ implies that $\mathcal{L}(U) \hookrightarrow \mathcal{O}_V(U')$ for each U'. If $\gamma, \delta \in \Gamma$, then by $(*)$,

$$c_{\gamma+\delta}(z)s_{U'}(z) = s_{U'+\gamma+\delta}(z + \gamma + \delta) = c_\delta(z + \gamma)s_{U'\gamma}(z + \gamma) = c_\delta(z + \gamma)c_\gamma(z)s_{U'}(z)$$

So if $\mathcal{L}(U) \neq 0$, then (c_γ) satisfies the cocycle condition $c_{\gamma+\delta}(z) = c_\gamma(z)c_\delta(z + \gamma)$.

Conversely, provided (c_γ) satisfies the cocycle condition, $\mathcal{L}(U) \xrightarrow{\simeq} \mathcal{O}_V(U')$ for every U'.

Observe that if $g : V \to \mathbb{C}^*$ is holomorphic, and $(c_\gamma)_{\gamma\in\Gamma}$ satisfies the cocycle condition, so does $c'_\gamma(z) = c_\gamma(z)g(z + \gamma)/g(z)$ and defines an isomorphic invertible sheaf $\mathcal{L}'$ (multiply $s_{U'+\gamma}$ by $g(z + \gamma)$).

Now we construct (c_γ) starting from $(H, \alpha) \in P(X)$, thus defining the sheaf $\mathcal{L}(H, \alpha)$.

Define

$$c_\gamma(z) = \alpha(\gamma) \exp\Big(\pi(H(z,\gamma) + \frac{1}{2} H(\gamma,\gamma))\Big)$$

For each γ, $c_\gamma : V \to \mathbb{C}^*$ is holomorphic. We claim that it satisfies the cocycle relation:

$$\begin{aligned}
c_\gamma(z) c_\delta(z+\gamma) &= \alpha(\gamma)\alpha(\delta) \exp \pi\Big(H(z,\gamma) + \frac{1}{2}H(\gamma,\gamma) + H(z,\delta) + H(\gamma,\delta) + \frac{1}{2}H(\delta,\delta)\Big) \\
&= \alpha(\gamma+\delta) \exp \pi\Big(H(z,\gamma+\delta) + \frac{1}{2}\big(H(\gamma+\delta,\gamma+\delta) + H(\gamma,\delta) - H(\delta,\gamma)\big) - iE(\gamma,\delta)\Big) \\
&= c_{\gamma+\delta}(z)
\end{aligned}$$

For the last equality note that $H(\gamma,\delta) - H(\delta,\gamma) = H(\gamma,\delta) - \overline{H(\gamma,\delta)} = 2iE(\gamma,\delta)$.

Now let $\mathcal{L}(H,\alpha)$ be the invertible $\mathcal{O}_X$-module given by $(c_\gamma)_\gamma$. If $(H,\alpha),(H',\alpha') \in P(X)$ give cocycles $(c_\gamma),(c'_\gamma)$, then

$$(H + H', \alpha + \alpha') \longmapsto \text{cocycle } (c_\gamma c'_\gamma)_\gamma$$

So $\mathcal{L}(H+H', \alpha\alpha') \simeq \mathcal{L}(H,\alpha) \otimes \mathcal{L}(H',\alpha')$. Hence we obtain a homomorphism

$$P(X) \to \operatorname{Pic}(X),\ (H,\alpha) \mapsto (\text{isomorphism class of } \mathcal{L}(H,\alpha))$$

A (non-trivial) computation shows that this is compatible with the other vertical maps in the diagram. □

Let $\mathcal{L} \in \operatorname{Pic}(X)$, $x \in X$. Let $T_x : X \to X$ be translation by x. Then $T_x^*\mathcal{L}$ and $\mathcal{L}$ have the same image in $\operatorname{NS}(X)$. Indeed, $\operatorname{NS}(X) \subseteq H^2(X,\mathbb{C}) \simeq A^2_{\mathbb{C}}(X)^{\mathrm{inv}}$, is invariant under T_x^*. So $\varphi_{\mathcal{L}}(x) := T_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$ lies in $\operatorname{Pic}^0(X)$.

Proposition 4.5. *$\varphi_{\mathcal{L}} : X \to \operatorname{Pic}^0(X) = \widehat{X}$ is a homomorphism of complex tori, i.e. it is holomorphic and a group homomorphism.*

Proof. See Sheet 2, Exercise 1. □

Theorem 4.6. *Let $\mathcal{L} = \mathcal{L}(H,\alpha)$. The following are equivalent:*

(i) H is positive definite.

(ii) $H^0(X,\mathcal{L}) \neq 0$ and $\varphi_{\mathcal{L}}$ is an isogeny, i.e. $\ker \varphi_{\mathcal{L}}$ is finite (as $\dim X = \dim \widehat{X}$).

(iii) $\mathcal{L}$ is ample.

Meaning of (iii). Let $n \geq 1$, $d = d_n = \dim H^0(X, \mathcal{L}^{\otimes n})$. Let $f_0, \ldots, f_{d-1}$ be a basis for $H^0(X,\mathcal{L}^n)$. Then $\mathcal{L}$ is ample iff for some $n \geq 1$, $f = (f_0 : \cdots : f_{d-1} : X \to \mathbb{P}^{d-1}(\mathbb{C})$ is well-defined and gives an isomorphism between X and a subvariety of $\mathbb{P}^{d-1}$. If so, then $\mathcal{L}^n \simeq f^*\mathcal{O}_{\mathbb{P}}(1)$. Note that while the f_i themselves are not functions on X, their ratios are (as $\mathcal{L}^{\otimes n}$ is of rank 1), so f makes sense (where not all f_i vanish).

Definition. *A* polarisation *on X is a positive definite Riemann form H.*

By the theorem, X is a projective variety iff X has a polarisation.

5 Group Schemes over Fields

Let k be a field (often algebraically closed). In the following all schemes will be k-schemes. The category of k-schemes (resp. affine schemes) will be denoted by $\mathbf{Sch}/k$ (resp. $\mathbf{Aff}/k$).

Recall that if X, S are k-schemes, then we write $X(S) := \operatorname{Mor}_k(S, X)$ for the set of *S-valued points* of X. If R is a k-algebra, we just write $X(R) := X(\operatorname{Spec} R)$.

In this course, a (k-)variety is a separated k-scheme of finite type over k which is geometrically integral.

Definition. *A* group scheme *(over k) is a k-scheme G, together with a morphism $m : G \times G \to G$ such that for all k-algebras R, $m_R : G(R) \times G(R) \to G(R)$ makes $G(R)$ into a group.*

Examples.

- Additive group: $\mathbb{G}_a = \operatorname{Spec} k[t] = \mathbb{A}^1_k$ and $m : \mathbb{G}_a \times \mathbb{G}_a = \operatorname{Spec} k[t_1, t_2] \to \operatorname{Spec} k[t] = \mathbb{G}_a$ is given by $t \mapsto t_1 + t_2$. Then $\mathbb{G}_a(R) = R$, with group operation $+$.
- Multiplicative group: $\mathbb{G}_m = \operatorname{Spec} k[t, 1/t] = \mathbb{A}^1_k \setminus \{0\}$ and $m : \mathbb{G}_m \times \mathbb{G}_m = \operatorname{Spec} k[t_1, t_2, 1/(t_1 t_2)] \to \operatorname{Spec} k[t, 1/t]$ is given by $t \mapsto t_1 \cdot t_2$. Then $\mathbb{G}_m(R) = (R^\times, \times)$.
- Linear groups: $\mathrm{GL}_n = \operatorname{Spec} k[(t_{ij})_{ij}, \frac{1}{\det(t_{ij})}]$. Then
$$m : \mathrm{GL}_n \times \mathrm{GL}_n = \operatorname{Spec}[(u_{ij}), (v_{ij}), \frac{1}{\det(u_{ij}) \det(v_{ij})}] \longrightarrow \mathrm{GL}_n$$
is given by $t_{ij} \mapsto \sum_{l=1}^n u_{il} v_{lj}$. Then $\mathrm{GL}_n(R)$ is what you think it is.

Recall the Yoneda Lemma:

Lemma 5.1 (Yoneda Lemma)**.** *Let $\mathcal{C}$ be a category, $X, Y \in \operatorname{ob} \mathcal{C}$. Then there is a bijection*
$$\operatorname{Mor}(X, Y) \longleftrightarrow \left\{ \begin{array}{c} \textit{natural transformations } X(-) \to Y(-), \textit{ i.e.} \\ \textit{families } (f_S : X(S) \to Y(S))_{S \in \operatorname{ob} \mathcal{C}} \textit{ such that} \\ f_S(x) \circ g = f_{S'}(x \circ g) \textit{ for all } g : S' \to S, x \in X(S) \end{array} \right\}$$
where $f : X \to Y$ induces the natural transformation $X(-) \to Y(-)$ given by $f_S : X(S) \to Y(S), g \mapsto f \circ g$ where $S \in \operatorname{ob} \mathcal{C}$. Conversely, given a natural transformation $(f_S)_S$, we get a morphism $f : X \to Y$ where $f = f_X(\mathrm{id}_X)$.

In the case of $\mathcal{C} = \mathbf{Sch}/k$, we may restrict ourselves to affine S:

Lemma 5.2 (Yoneda for schemes)**.** *Let X, Y be k-schemes. The usual Yoneda correspondence remains true if we restrict ourselves to S-valued points with S affine, i.e. there is a bijection*
$$\{\textit{Morphisms } X \to Y\} \longleftrightarrow \left\{ \begin{array}{c} \textit{families } X(S) \xrightarrow{f_S} Y(S) \textit{ with } S \textit{ affine such that} \\ f_S(x) \circ g = f_{S'}(x \circ g)\, \forall g : S' \to S,\ S, S' \textit{ affine} \end{array} \right\}$$

Proof. Cover $X = \bigcup_{\alpha \in I} U_\alpha$, where U_α are open affines with inclusions j_α into X, so $j_\alpha \in X(U_\alpha)$. Then given $(f_S)_{S \in (\mathbf{Aff}/k)}$, get $f_{U_\alpha}(j_\alpha) \in Y(U_\alpha) = \mathrm{Mor}_k(U_\alpha, Y)$. If $V \subseteq U_\alpha \cap U_\beta$ is any open affine, then $f_{U_\alpha}(j_\alpha)$ and $f_{U_\beta}(j_\beta)$ restrict to the same element of $Y(V)$. So they glue to give a morphism $f : X \to Y$. □

Proposition 5.3. *Let G be a group scheme. Then*

(i) For all $S \in (\mathbf{Sch}/k)$, $G(S)$ is a group where the group law is given by m_S.

(ii) For all $S' \xrightarrow{f} S$, $G(S) \xrightarrow{-\circ f} G(S')$ is a homomorphism.

Proof. Suppose $S' = \mathrm{Spec}\, R' \xrightarrow{f} \mathrm{Spec}\, R = S$ are affine. For (ii) we have to check that

$$\begin{array}{ccccc} (G \times G)(S) & \overset{=}{\longrightarrow} & G(S) \times G(S) & \xrightarrow{m_S} & G(S) \\ \downarrow & & \downarrow & & \downarrow \\ (G \times G)(S') & \overset{=}{\longrightarrow} & G(S') \times G(S') & \xrightarrow{m_{S'}} & G(S') \end{array}$$

commutes. This is clear. Hence (ii) holds for S, S' affine.

For (i) let $(U_i)_{i \in I}$ be an affine cover of S. Write $U_i \cap U_j = \bigcup_k U_{ij}^k$ with affine U_{ij}^k. Then for all X,

$$X(S) = \{(x_i) \in \prod_i X(U_i) \mid \forall i, j, k : x_i|_{U_{ij}^k} = x_j|_{U_{ij}^k}\}. \qquad (*)$$

Apply this to G and $G \times G$. We check:

- $m_S : G(S) \times G(S)$ is associative: Since $G(S) \hookrightarrow \prod_i G(U_i)$ preserves the multiplication m and $\prod_i G(U_i)$ is a group, multiplication on $G(S)$ is associative. This argument also shows that (ii) holds for any schemes S, S'.
- The two maps $G(S) \times G(S) \to G(S) \times G(S), (x, y) \mapsto (xy, y), (yx, y)$ are bijections. Apply $(*)$: The claim follows from the fact that $G(U_i)$ and $G(U_{ij}^k)$ are groups.

□

Corollary 5.4. *There exist $e \in G(k)$, $i : G \to G$ such that for all S, $e \mapsto$ (identity of $G(S)$), and $i_S : G(S) \to G(S)$ is the inverse map.*

Proof. Let e be the identity of $G(k)$, by (ii) it is the identity of $G(S)$ for all S. Define $i \in G(G)$ to be the inverse (for the group law) of $\mathrm{id}_G : G \to G$. □

Example. Let Γ be any (abstract) group. The constant group scheme is $G = \coprod_{\gamma \in \Gamma} \mathrm{Spec}\, k$. G is affine iff Γ is finite.

Remark. Alternative way to define a group scheme: It is a triple $(G, m : G \times G \to G, e \in G(k), i : G \to G)$ satisfying certain axioms. For example, associativity is expressed by the commutativity of the following diagram:

$$\begin{array}{ccccc} (G \times G) \times G & \xrightarrow{m \times \mathrm{id}_G} & G \times G & & \\ \downarrow{\scriptstyle \simeq} & & & \searrow^{m} & \\ & & & & G \\ & & & \nearrow_{m} & \\ G \times (G \times G) & \xrightarrow{\mathrm{id}_G \times m} & G \times G & & \end{array}$$

The other properties (commutativity, identity, inverses) are similar. I.e. G is a group object in $\mathbf{Sch}/k$.

Definition. *A* homomorphism of group schemes *is a morphism $G \xrightarrow{f} G'$ such that for all k-algebras R (equivalently for all $S \in \mathbf{Sch}/k$), $G(R) \to G'(R)$ (or $G(S) \to G'(S)$) is a homomorphism.*

Exercise. $f : G \to G'$ is a homomorphism iff the diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{f \times f} & G' \times G' \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m'} \\ G & \xrightarrow{\quad f \quad} & G' \end{array}$$

commutes.

Definition. *A* closed subgroup scheme *of G is a closed subscheme $H \subseteq G$ such that for all R (or equivalently for all S), $H(R) \subseteq G(R)$ (or $H(S) \subseteq G(S)$) is a subgroup.*

If so, H is a group scheme, and the inclusion $i : H \hookrightarrow G$ is a homomorphism:

$$\begin{array}{ccc} (H \times H)(S) & \hookrightarrow & (G \times G)(S) \\ \vdots & & \downarrow{\scriptstyle m'} \\ H(S) & \hookrightarrow & G(S) \end{array}$$

The dotted arrow exists as $H(S)$ is a subgroup. And the image of $\mathrm{id}_{H \times H} \in (H \times H)(H \times H)$ in $H(H \times H)$ is the desired morphism $H \times H \to H$.

Examples.

(i) $\operatorname{Spec} k \overset{e}{\hookrightarrow} G$ is a closed subgroup scheme.

(ii) Kernels: Let $f : G \to G'$ be a homomorphism. Define $\ker f$ to be the fibre of f at $e' \in G'(k)$, i.e. there is a pullback square:

$$\begin{array}{ccc} \ker f & \longrightarrow & G \\ \downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} k & \overset{e'}{\hookrightarrow} & G' \end{array}$$

Since e' is a closed immersion, $\ker(f)$ is a closed subscheme of G and $\ker(f)(S) = \ker(f_S : G(S) \to G'(S))$.

(iii) Let $G = \mathrm{GL}_n$, $G' = \mathbb{G}_m$. For all R, have $\det_R : \mathrm{GL}_n(R) \to R^* = \mathbb{G}_m(R)$. So by Yoneda, get a homomorphism $\det : \mathrm{GL}_n \to \mathbb{G}_m$. Its kernel is $\ker\det =: \mathrm{SL}_n$ which is the closed subscheme given by $\det(x_{ij}) = 1$ of $\mathrm{GL}_n = \operatorname{Spec} k[(x_{ij}), (\det(x_{ij}))^{-1}]$.

Remark. Quotients are more subtle.

Let G be a group scheme, $x \in G(k)$. The (left) translation by x is the unique morphism $T_x : G \to G$ such that for all $y \in G(S)$, $T_x(y) = xy$, i.e. T_x is the composite $G = \operatorname{Spec} k \times G \xrightarrow{x \times \mathrm{id}_G} G \times G \xrightarrow{m} G$. Then $T_e = \mathrm{id}_G$ and $T_{xy} = T_x \circ T_y$.

Let X be a variety. Since we assume X to be geometrically integral, k is algebraically closed in $k(X)$[1], the function field of X. We say X is *complete*, if X is proper over k.

Definition. *A* group variety *(or [connected] algebraic group) is a group scheme which is a variety. An* abelian variety *is a complete group variety.*

Examples. $\mathbb{G}_m$, $\mathbb{G}_a$, GL_n are affine group varieties.

The simplest nontrivial example of an abelian variety is an elliptic curve E/k, e.g. given as a nonsignular cubic $E \subseteq \mathbb{P}^2_k$ with a given point $e \in E(k)$).

Completeness has strong implications (e.g. commutativity).

Theorem 5.5 (Mumford's Rigidity Lemma)**.** *Let X, Y, Z be varieties with X complete, $y_0 \in Y$, $f : X \times Y \to Z$ a morphism. If $f(X \times \{y_0\})$ is a single point, then there exists $g : Y \to Z$ such that f factors as $f = g \circ \mathrm{pr}_2$. In particular, for all $y \in Y$, $f(X \times \{y\})$ is a single point.*

Remarks. Here $X \times \{y_0\}$ means $X \times \operatorname{Spec} k(y_0) \hookrightarrow X \times Y$, fibre of $\mathrm{pr}_2 : X \times Y \to Y$ at $y_0 \in Y$. In general, it is not the set-theoretic product of X with $\{y_0\}$. It is if $y_0 \in Y(k)$.

$\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1, (x, y) \mapsto xy$, so $f(\mathbb{A}^1 \times \{0\}) = \{0\}$, but $f|_{\mathbb{A}^1 \times \{1\}}$ is an isomorphism. So completeness of X is essential!

Corollary 5.6. *Let X be an abelian variety, G a group variety, $f : X \to G$ a morphism of schemes. Then if $g = f(e)$, $T_{g^{-1}} \circ f$ is a homomorphism.*

So taking $G = X$, we see that any isomorphism of schemes $X \xrightarrow{\simeq} X$ which takes e to e is an isomorphism of group schemes.

Proof. It suffices to prove that if $f(e) = e$, then f is a homomorphism. Consider $p : X \times X \to G$ such that for all $x, y \in X(S)$, $p(x, y) = f(x)f(y)f(xy)^{-1}$. Then $p(X \times \{e\}) = p(\{e\} \times X) = \{e\}$. So by rigidity, p factors through $(x, y) \mapsto y$ and also through $(x, y) \mapsto x$, so $p(x, y) = p(x, e) = p(e, e) = e$ for all x, y, so f is a homomorphism. □

[1] Proof sketch: Let $\operatorname{Spec} A \subseteq X$ be an affine open. If $k \subseteq k' \subseteq A$ is a finite extension, then $k' \otimes_k k^{\mathrm{alg}} \subseteq A \otimes_k k^{\mathrm{alg}}$ is not an integral domain, unless $k' = k$.

Corollary 5.7. *Abelian varieties are commutative.*

Proof. Apply the previous corollary to $i : X \to X$. Since $i(e) = e$, i is a homomorphism. But a group is commutative iff $i : g \mapsto g^{-1}$ is a homomorphism. So $X(S)$ is commutative for all S. $\square$

In general, we will state things for arbitrary k, but often give a proof only for k algebraically closed.

Proof of Theorem 5.5. Suppose first $k = k^{\mathrm{alg}}$ is algebraically closed, and let $x_0 \in X(k)$. Define $g : Y \to Z$ by $g(y) = f(x_0, y)$, i.e.

$$\begin{array}{ccccc} \operatorname{Spec} k \times Y & \xrightarrow{x_0 \times \mathrm{id}_Y} & X \times Y & \xrightarrow{f} & Z \\ \Big\| {=} & & & \nearrow_{g} & \\ Y & & & & \end{array}$$

commutes. We need to show that $g \circ \mathrm{pr}_2 = f$. As everything is a variety, so separated, it is enough to show this for a dense open subset of $X \times Y$.

Let z_0 be the point in $f(X \times \{y_0\})$ and $W \subseteq Z$ be an open affine neighborhood of it. Set $S = Z - W$, it is a closed subset. Then $f^{-1}(S) \subseteq X \times Y$ is closed, so $\mathrm{pr}_2(f^{-1}(S)) \subseteq Y$ is closed since $X \to \operatorname{Spec} k$ is proper. Then $V := Y \setminus \mathrm{pr}_2(f^{-1}(S)) \subseteq Y$ is open, and $f(X \times V) \subseteq W$. So for all $y \in V(k)$, $f : X \times \{y\} \to W$. As X is complete and W is affine, $f|_{X \times \{y\}}$ is constant, its image is $\{f(x_0, y)\} = \{g(y)\}$. So for all $y \in V(k)$, $f|_{X \times \{y\}} = g \circ \mathrm{pr}_2|_{X \times \{y\}}$, hence $f|_{X \times V} = g \circ \mathrm{pr}_2|_{X \times V}$. Also V is non-empty, as $z_0 \notin S$, so $X \times \{y_0\} \cap f^{-1}(S) = \emptyset$, so $y_0 \notin \mathrm{pr}_2(f^{-1}(S))$, so $y_0 \in V$, hence $V \neq \emptyset$.

Now suppose k is arbitrary, i.e. not necessarily algebraically closed. f factors through pr_2 iff for affine opens $U \subseteq X$, $V \subseteq Y$, $f(U \times V) \subseteq W \subseteq Z$ the map $\mathcal{O}_Z(W) \to \mathcal{O}_{X \times Y}(U \times V) = \mathcal{O}_X(U) \otimes_k \mathcal{O}_Y(V)$ factors through $k \otimes_k \mathcal{O}_Y(V)$. We can check this after replacing k with k^{alg}, since $k \otimes_k \mathcal{O}_Y(V) = \mathcal{O}_X(U) \otimes_k \mathcal{O}_Y(V) \cap k^{\mathrm{alg}} \otimes_k \mathcal{O}_Y(V)$. $\square$

6 Seesaw and Cube

Let $f : X \to Y$ be a morphism, $\mathcal{L}$ an invertible sheaf on X (or coherent sheaf). Then for all $y \in Y$, let X_y be the fibre over y and $\mathcal{L}_y = i_y^*\mathcal{L}$ where $i_y : X_y \hookrightarrow X$ is the inclusion.

Common questions:

(1) How does $H^0(X_y, \mathcal{L}_y)$ vary with y? (or more generally H^i)

(2) What conditions ensure that there exists $\mathcal{M}$ on Y with $\mathcal{L} \cong f^*\mathcal{M}$?

(e.g. if $\mathcal{L} \cong f^*\mathcal{M}$, then all $\mathcal{L}_y \cong \mathcal{O}_{X_y}$ are trivial. Converse?)

Examples.

(1) Let C be a complete nonsingular curve over k, D a divisor on C. Then $H^0(C, \mathcal{O}_C(D)) = L(D)$ and Riemann-Roch gives an estimate for this. How does this vary as you vary D? (We will use this later in construction of the Jacobian of C)

(2) Let Y be a quadric cone in $\mathbb{A}^3$, say $Y = \operatorname{Spec} k[u, v, w]/(uv - w^2)$ and $\operatorname{char} k \neq 2$. Let $X = Y \setminus \{0\} \overset{f}{\hookrightarrow} Y$. Let L be the line $v = w = 0$ through 0. Let $\mathcal{L} = \mathcal{O}_X(L \cap X)$. Obviously, as fibres of f are points (or empty), all $\mathcal{L}_y$ are trivial. But there does not exist an invertible module $\mathcal{M}$ on Y such that $f^*\mathcal{M} \cong \mathcal{L}$ (because $L \subseteq Y$ is not a Cartier divisor, not locally principal).

Theorem 6.1 ("Seesaw Theorem")**.** *Let X, Y be varieties, X complete, $\mathcal{L}$ an invertible $\mathcal{O}_{X\times Y}$-module. Then:*

(i) $F = \{y \in Y \mid \mathcal{L}|_{X\times\{y\}}$ is trivial$\}$ is closed in Y.

(ii) If $F = Y$, then there exists a invertible sheaf $\mathcal{M}$ on Y such that $\mathcal{L} \simeq \operatorname{pr}_2^*\mathcal{M}$.

The proof uses:

Theorem 6.2. *Let X be complete, $S = \operatorname{Spec} A$, A any noetherian k-algebra, $\mathcal{L}$ invertible sheaf on $X \times S$. Then:*

(i) $H^0(X \times S, \mathcal{L})$ is a finite (= finitely generated) A-module.

(ii) There exists a morphism $\alpha : K^0 \to K^1$ of finite free A-modules such for all A-algebras B, there are isomorphisms

$$H^0(X \times \operatorname{Spec} B, \mathcal{L}_B) \simeq \ker(\alpha_B = \alpha \otimes_A \mathrm{id}_B : K^0 \otimes_A B \to K^1 \otimes_A B),$$

functorial for $B \to B'$. Here $\mathcal{L}_B$ is the pullback of $\mathcal{L}$ along $X \times \operatorname{Spec} B \to X \times \operatorname{Spec} A$.

See [Mum70, Chapter 2 §5], or [Har77, Chapter III §12], but still check out Mumford's Corollary 2. The theorem holds for all H^i (with a complex $K^0 \to K^1 \to \dots$ of finite free A-modules), and in fact we need this to prove the $i = 0$ case.

Corollary 6.3. *Same hypotheses as in the previous theorem. There exists a finite A-module M such that for all A-algebras B,*

$$H^0(X \times \operatorname{Spec} B, \mathcal{L}_B) \cong \operatorname{Hom}_A(M, B) = \operatorname{Hom}_B(M \otimes_A B, B).$$

Proof. Let $M = \operatorname{coker}(\alpha^t)$, so

$$(K^1)^\vee \xrightarrow{\alpha^t} (K^0)^\vee \to M \to 0$$

is exact where $(K^i)^\vee = \operatorname{Hom}_A(K^i, A)$. The K^i are finite free, so $\operatorname{Hom}_A((K^i)^\vee, B) = K^i \otimes_A B$. Then $0 \to \operatorname{Hom}_A(M, B) \to K^0 \otimes_A B \xrightarrow{\alpha_B} K^1 \otimes_A B$. ☐

Corollary 6.4. *Under the same hypotheses, for every $d \geq 0$,*

$$Z_d = \{s \in S \mid \dim_{k(s)} H^0(X \times \operatorname{Spec} k(s), \mathcal{L}_s) \geq d\} \subseteq S$$

is a closed subset.

This is the Semicontinuity theorem for H^0, it is true for all H^i.]

Proof. Let $K^0 \simeq A^m, K^1 \simeq A^n$, so α^t is represented by an $(m \times n)$-matrix C. Then

$$\begin{aligned} Z_d &= \{s \in S \mid \operatorname{rank}(\alpha^t \otimes \operatorname{id}_{k(s)}) \leq m - d\} \\ &= \{s \in S \mid \text{all } (m-d+1) \text{ minors of } C \text{ vanish in } k(s)\} \end{aligned}$$

which is closed. ☐

Lemma 6.5. *Let V be a complete K-variety, $\mathcal{L}$ an invertible $\mathcal{O}_V$-module. Then $\mathcal{L} \simeq \mathcal{O}_V$ iff both $H^0(V, \mathcal{L})$ and $H^0(V, \mathcal{L}^\vee)$ are non-zero.*

Proof. Exercise: Use $\operatorname{Hom}_{\mathcal{O}_V}(\mathcal{L}, \mathcal{L}) = \operatorname{Hom}_{\mathcal{O}_V}(\mathcal{O}_V, \mathcal{O}_V) = K$ as V is a complete variety and $\operatorname{Hom}(\mathcal{O}_V, \mathcal{L}) = H^0(\mathcal{L})$. ☐

Proof of the Seesaw theorem.

(i) We may assume that $Y = \operatorname{Spec} A$ is affine. We have

$$\begin{aligned} F &= \{y \in Y \mid \mathcal{L}|_{X \times \{y\}} \text{ is trivial}\} \\ &= \{y \in Y \mid H^0(X \times \{y\}, \mathcal{L}_y) \neq 0 \neq H^0(X \times \{y\}, \mathcal{L}_y^\vee)\}. \end{aligned}$$

This is closed by the above corollary.

Also, if $y \in F$, then $\dim_{k(y)} M \otimes k(y) = \dim_{k(y)} H^0(\mathcal{L}_y) = 1$. So as M is a finite A-module, for any generator $m \otimes 1$ of $M \otimes k(y)$, m generates M in a neighborhood of y by Nakayama's Lemma. So M is cyclic in a neighborhood of y.

(ii) Suppose $F = Y$. We want to show that $\mathcal{L} \cong \mathrm{pr}_2^* \mathcal{M}$ for some $\mathcal{M}$ on Y. We will show that if $\mathcal{M} = \mathrm{pr}_{2*} \mathcal{L}$, then $\mathcal{M}$ is an invertible $\mathcal{O}_Y$-module and the adjunction map $\mathrm{pr}_2^* \mathcal{M} \to \mathcal{L}$ is an isomorphism. This statement is local on Y. So it is enough to show that for all $y \in Y$, there exists an open affine $U \ni y$ such that $\mathcal{L}|_{X \times U}$ is trivial. So we can assume $Y = \operatorname{Spec} A$ is affine. By the above, for all $y \in Y$ (with M as before) $\dim_{k(y)} M \otimes_A k(y) = 1$ since $\mathcal{L}_y \cong \mathcal{O}$. Then by Nakayama again, M is locally free of rank 1. Replacing Y by an affine neighborhood of y, may assume $M = mA$ is free, then $\mathrm{Hom}_{\mathcal{O}_{X \times Y}}(\mathcal{O}_{X \times Y}, \mathcal{L}) = H^0(X \times Y, \mathcal{L}) = \mathrm{Hom}_A(M, A) = m^\vee A$. So $m^\vee$ gives a map $\mathcal{O}_{X \times Y} \to \mathcal{L}$ whose restriction to each $X \times \{y\}$ is the isomorphism $\mathcal{O}_{X \times \{y\}} \xrightarrow{m \otimes \mathrm{id}} \mathcal{L}_y$, similarly for $\mathcal{L}^\vee$. Then $m^\vee : \mathcal{O}_{X \times Y} \to \mathcal{L}$ is an isomorphism.

□

Remark. Proof gives something a bit stronger than (i): There exists a maximal closed subscheme $Z \subseteq Y$ such that $\mathcal{L}|_{X \times Z} \simeq \mathrm{pr}_2^* \mathcal{M}$ for some $\mathcal{M}$ on Z. If Y is affine, and M is cyclic, then $Z = \operatorname{Spec} A/I$ where $I = \mathrm{Ann}_A M$.

Particular case of Seesaw: Suppose $\mathcal{L}$ is an invertible sheaf on $X \times Y$, $\mathcal{L}|_{X \times \{y\}}$ is trivial for all $y \in Y$, and there exists $x_0 \in X(k)$ such that $\mathcal{L}|_{\{x_0\} \times Y}$ is trivial. Then $\mathcal{L} \cong \mathrm{pr}_2^* \mathcal{M}$, so $\mathcal{O}_Y \simeq (\mathrm{pr}_2^* \mathcal{M})|_{\{x_0\} \times Y} = \mathcal{M}$, i.e. $\mathcal{L}$ is trivial.

One can easily find non-trivial $\mathcal{L}$ on $X \times Y$ (e.g. $X = Y =$ elliptic curve) such that for some $x_0 \in X(k), y_0 \in Y(k)$, $\mathcal{L}|_{\{x_0\} \times Y}$ and $\mathcal{L}|_{X \times \{y_0\}}$ are trivial.

For a product of three varieties, we however have:

Theorem 6.6 (Theorem of the cube)**.** *Let X, Y, Z be varieties, X, Y complete. Let x, y, z be k-points of X, Y, Z, $\mathcal{L}$ an invertible sheaf on $X \times Y \times Z$. Suppose the restriction of $\mathcal{L}$ to each of $\{x\} \times Y \times Z$, $X \times \{y\} \times Z$, $X \times Y \times \{z\}$ is trivial. Then $\mathcal{L}$ is trivial.*

Corollary 6.7. *Let X be an abelian variety, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. For any variety Y and $f, g, h : Y \to X$:*

$$(f+g+h)^*\mathcal{L} \cong (f+g)^*\mathcal{L} \otimes (f+h)^*\mathcal{L} \otimes (g+h)^*\mathcal{L} \otimes f^*\mathcal{L}^\vee \otimes g^*\mathcal{L}^\vee \otimes h^*\mathcal{L}^\vee.$$

Here $f + g : Y \to X$ is the composite $Y \xrightarrow{(f,g)} X \times X \xrightarrow{m} X$, etc.

Proof. Let $\mathrm{pr}_i^3 : X \times X \times X \to X$, $i = 1, 2, 3$, and $\mathrm{pr}_i^2 : X \times X \to X$, $i = 1, 2$, be the projections.

First consider the case $Y = X \times X \times X$, $(f, g, h) = (\mathrm{pr}_i^3)_{i=1,2,3}$. Let $q : X \times X \to X \times X \times X$, $(x, y) \mapsto (x, y, e)$. Then

$$\begin{aligned}(\mathrm{pr}_1^3 + \mathrm{pr}_2^3 + \mathrm{pr}_3^3) \circ q = (\mathrm{pr}_1^2 + \mathrm{pr}_2^2) \circ q = m : (x, y) \mapsto x + y,\\ (\mathrm{pr}_1^3 + \mathrm{pr}_3^3) \circ q = \mathrm{pr}_1^3 \circ q = \mathrm{pr}_1^2,\\ (\mathrm{pr}_2^3 + \mathrm{pr}_3^3) \circ q = \mathrm{pr}_2^3 \circ q = \mathrm{pr}_2^2,\end{aligned}$$

$$\mathrm{pr}_3^3 \circ q = e.$$

So if $\mathcal{M} = (\mathrm{LHS}) \otimes (\mathrm{RHS})^\vee = (\mathrm{pr}_1^3 + \mathrm{pr}_2^3 + \mathrm{pr}_3^3)^*\mathcal{L} \otimes (\mathrm{pr}_1^3 + \mathrm{pr}_2^3)^*\mathcal{L}^\vee \otimes \dots$, then

$\mathcal{M}|_{X\times X\times\{e\}} = q^*\mathcal{M} = m^*\mathcal{L} \otimes m^*\mathcal{L}^\vee \otimes \mathrm{pr}_1^{2*}\mathcal{L}^\vee \otimes \mathrm{pr}_2^{2*}\mathcal{L}^\vee \otimes \mathrm{pr}_1^{2*}\mathcal{L} \otimes \mathrm{pr}_2^{2*}\mathcal{L} \otimes \mathcal{O}_{X\times X} \cong \mathcal{O}_{X\times X}$

same for $X \times \{e\} \times X$ and $\{e\} \times X \times X$. Then $\mathcal{L}$ is trivial by the theorem of the cube.

In the general case consider $Y \xrightarrow{(f,g,h)} X \times X \times X \xrightarrow{\mathrm{pr}_1,\mathrm{pr}_2,\mathrm{pr}_3} X$. Then $\mathcal{M}_{f,g,h} = (f,g,h)^*\mathcal{M}_{\mathrm{pr}_1,\mathrm{pr}_2,\mathrm{pr}_3}$, so it is trivial. □

Corollary 6.8 (Theorem of the Square)**.** *Let X be an abelian variety, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. Then for all $x, y \in X(k)$, $T_{x+y}^*\mathcal{L} = T_x^*\mathcal{L} \otimes T_y^*\mathcal{L} \otimes \mathcal{L}^\vee$*

Proof. Take f to be the constant morphism x, i.e. the composite $X \to \operatorname{Spec} k \xrightarrow{x} X$, g the constant morphism y and $h = \mathrm{id}_X$. Then $f + h = T_x$, $g + h = T_y$, $f + g + h = T_{x+y}$, and $f+g$ is the constant morphism $x+y$. So $\mathcal{M}_{f,g,h} = T_{x+y}^*\mathcal{L} \otimes T_x^*\mathcal{L}^\vee \otimes T_y^*\mathcal{L}^\vee \otimes \mathcal{L} \otimes \mathcal{O}_X \simeq \mathcal{O}_X$, hence the claim. □

Corollary 6.9. *Let X be an abelian variety, $\mathcal{L}$ an invertible sheaf on X, $n \in \mathbb{Z}$, $[n] : X \to X$ multiplication by n. Then $[n]^*\mathcal{L} \cong \mathcal{L}^{\otimes n(n+1)/2} \otimes (i^*\mathcal{L})^{\otimes n(n-1)/2}$ where $i : X \to X$, $x \mapsto -x$.*

Proof. $n = 0$ or 1 is trivial. Induction on $n \geq 2$. Take $f = [n-1]$, $g = \mathrm{id}_X = [1]$, $h = [-1] = i$. Then $\mathcal{M}_{f,g,h} \simeq \mathcal{O}_X$ tells us that

$$[n-1]^*\mathcal{L} \simeq [n]^*\mathcal{L} \otimes [n-2]^*\mathcal{L} \otimes [0]^*\mathcal{L} \otimes [n-1]^*\mathcal{L}^\vee \otimes \mathcal{L}^\vee \otimes i^*\mathcal{L}^\vee,$$

i.e.

$$\begin{aligned} [n]^*\mathcal{L} &\simeq [n-1]^*\mathcal{L}^{\otimes 2} \otimes [n-2]^*\mathcal{L}^\vee \otimes \mathcal{L} \otimes i^*\mathcal{L} \\ &\simeq \mathcal{L}^{\otimes[n(n-1)-\frac{1}{2}(n-1)(n-2)+1]} \otimes (i^*\mathcal{L})^{\otimes[(n-1)(n-2)-\frac{1}{2}(n-2)(n-3)+1]} \\ &\simeq \mathcal{L}^{\otimes\frac{1}{2}n(n+1)} \otimes (i^*\mathcal{L})^{\otimes\frac{1}{2}n(n-1)} \end{aligned}$$

The result then follows for $n \geq 0$.

For $n < 0$ note that $[-n]^*\mathcal{L} = i^*[n]^*\mathcal{L}$, it follows from the $n > 0$ case. □

7 Pic of an Abelian Variety and Projectivity

Proposition 7.1. *Let G/k be any group variety. Then G is non-singular.*

Proof. Assume $k = k^{\mathrm{alg}}$. The set of nonsingular closed points is dense (as G is a variety). Take $y \in G(k)$ to be nonsingular. Then for every $x \in G(k)$, $T_{xy^{-1}} : G \to G$ is an automorphism taking y to x, hence also x is nonsingular. □

Definition. *Let X an abelian variety over k, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module.*

(i) Define $\varphi_{\mathcal{L}} : X(k^{\mathrm{alg}}) \to \mathrm{Pic}(X_{k^{\mathrm{alg}}})$ by

$$\varphi_{\mathcal{L}}(x) = T_x^*\mathcal{L} \otimes \mathcal{L}^\vee \in \mathrm{Pic}(X_{k^{\mathrm{alg}}})$$

for $x \in X(k)$. By the theorem of the square, $\varphi_{\mathcal{L}} : X(k^{\mathrm{alg}}) \to \mathrm{Pic}(X_{k^{\mathrm{alg}}})$ is a homomorphism of groups.

(ii) $K(\mathcal{L}) := \ker \varphi_{\mathcal{L}} \subseteq X(k^{\mathrm{alg}})$ is a subgroup. $\mathrm{Pic}^0(X) := \{\mathcal{L} \in \mathrm{Pic}(X) \mid \varphi_{\mathcal{L}} = 0\}$. Let $\mathrm{NS}(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$.

Remark. By definition, $x \in K(\mathcal{L})$ iff $T_x^*\mathcal{L} \otimes \mathcal{L}^\vee$ is trivial. By Seesaw part (i), this implies that $K(\mathcal{L})$ is the set of k^{alg}-points of a closed subscheme of X.

Proposition 7.2. *Let $\mathcal{M}(\mathcal{L}) = m^*\mathcal{L} \otimes \mathrm{pr}_1^*\mathcal{L}^\vee \otimes \mathrm{pr}_2^*\mathcal{L}^\vee$ on $X \times X$ ("Mumford line bundle"). Then $\mathcal{L} \in \mathrm{Pic}^0(X)$ iff $M(\mathcal{L}) \simeq \mathcal{O}_{X\times X}$.*

Proof. Assume $k = k^{\mathrm{alg}}$. Let $x \in X(k)$. Then since

$$\begin{aligned} m \circ (\mathrm{id}_X, x) &= T_x, \\ \mathrm{pr}_1 \circ (\mathrm{id}_X, x) &= \mathrm{id}_X, \\ \mathrm{pr}_2 \circ (\mathrm{id}_X, x) &= \text{constant } x : X \to X, \end{aligned}$$

we have $\mathcal{M}|_{X\times\{x\}} \simeq T_x^*\mathcal{L} \otimes \mathcal{L}^\vee$ and similarly $\mathcal{M}|_{\{e\}\times X} \simeq \mathcal{O}_X$. So by Seesaw (ii), $\mathcal{M}(\mathcal{L}) \simeq \mathcal{O}_{X\times X}$ iff for all x, $T_x^*\mathcal{L} \otimes \mathcal{L}^\vee \simeq \mathcal{O}_X$ i.e. $\mathcal{L} \in \mathrm{Pic}^0(X)$. □

This is one of a number of different characterizations of Pic^0.

Let D be an effective divisor on X, i.e. $D = \sum_i n_i D_i$, $D_i \subseteq X$ integral closed subscheme of codimension 1. Note that Weil divisors are the same as Cartier divisors as X is nonsingular. Define $H(D) = \{x \in X(k^{\mathrm{alg}}) \mid T_x D = D\}$. As $\mathcal{O}_X(T_x D) = T_{-x}^*\mathcal{O}_X(D)$[1], $H(D) \subseteq K(\mathcal{O}_X(D))$ is a subgroup.

Remark. $H(D)$ is the set of k^{alg}-points of a closed subscheme of X, but for much more obvious reasons than for $K(\mathcal{L})$. Indeed, if $Y \subseteq X$ is closed, then $T_x Y = Y$ iff $\{x\} \times Y \subseteq m^{-1}(Y) \subseteq X \times X$ iff $x \in \bigcap_{y\in Y}\{x \in X \mid (x,y) \in m^{-1}(Y)\} = \bigcap_{y\in Y} \mathrm{pr}_1(X \times \{y\} \cap m^{-1}(Y))$ which is closed since pr_1 is proper.

[1]Suppose $\mathrm{div}(f) = D$ locally, then as $(T_x^* f)(y) = f(x+y)$, we have $\mathrm{div}(T_x^* f) = D - x = T_{-x}D$

Theorem 7.3. *Let $\mathcal{L} = \mathcal{O}_X(D)$, D an effective divisor. TFAE:*

(i) $\mathcal{L}$ is ample, i.e. $H^0(X, \mathcal{L}^{\otimes m})$ for sufficiently large m gives an embedding $X \hookrightarrow \mathbb{P}^N_k$.

(ii) $K(\mathcal{L})$ is finite.

(iii) $H(D)$ is finite.

Proof. "(*ii*) $\Rightarrow$ (*iii*)" is obvious. Assume $k = k^{\mathrm{alg}}$. "(*i*) $\Rightarrow$ (*ii*)" Assume $\mathcal{L}$ is ample, but $K(\mathcal{L})$ is infinite. By a previous remark, $K(\mathcal{L})$ is the set of k-points of some reduced closed subscheme, necessarily a group scheme. Looking at the irreducible component containing e we get that $K(\mathcal{L})$ contains an abelian subvariety Y of positive dimension. The restriction of $\mathcal{L}$ to Y is ample. So replacing X by Y we may assume $K(\mathcal{L}) = X(k)$, i.e. $\varphi_{\mathcal{L}} = 0$ and $\dim X > 0$. Then for all $x \in X(k)$, $T_x^*\mathcal{L} \simeq \mathcal{L}$, so $m^*\mathcal{L} \simeq \mathrm{pr}_1^*\mathcal{L} \otimes \mathrm{pr}_2^*\mathcal{L}$ on $X \times X$ by Proposition 7.2 as $\mathcal{L} \in \mathrm{Pic}^0(X)$. Pullback via $d : X \to X \times X$, $d(x) = (x, -x)$. Then $m \circ d$ is the constant morphism e, $\mathrm{pr}_1 \circ d = \mathrm{id}_X$ and $\mathrm{pr}_2 \circ d = i = [-1]$. So $\mathcal{O}_X \simeq \mathcal{L} \otimes i^*\mathcal{L}$. $\mathcal{L}$ is ample, so $i^*\mathcal{L}$ is ample as i is an automorphism, hence $\mathcal{O}_X$ is ample which is not possible as $\dim X > 0$.

"(*iii*) $\Rightarrow$ (*i*)" Consider $\mathcal{O}_X(2D) = \mathcal{L}^{\otimes 2} \simeq T_x^*\mathcal{L} \otimes T_{-x}^*\mathcal{L} = \mathcal{O}_X(T_xD + T_{-x}D)$ (Theorem of the Square), i.e. for all $x \in X(k)$, there exists $s_x \in H^0(X, \mathcal{O}_X(2D))$ with $\mathrm{div}(s_x) = T_xD + T_{-x}D - 2D$. If $y \in X(k)$, then $y \in T_xD \cup T_{-x}D$ iff one of $y \pm x$ is in D. So given y, there exists x such that $y \notin T_xD \cup T_{-x}D = \{\text{zero set of } s_x\}$. So the map $X \xrightarrow{f} \mathbb{P}^N$, where $N = \dim H^0(X, \mathcal{O}(2D)) - 1$, given by sections of $\mathcal{O}_X(2D)$ is a morphism, i.e. defined everywhere. Claim: The fibres of f are finite. If so, then $\mathcal{O}_X(2D) = f^*\mathcal{O}_{\mathbb{P}^n}(1)$ is ample, hence so is $\mathcal{L}$, because of the following general fact: If $f : X \to Y$ is a morphism of complete varieties with finite fibres, and $\mathcal{M}$ on Y is ample, then $f^*\mathcal{M}$ is ample on X [Har77, Chapter III, Exercise 5.7].

If some fibre of f is infinite, then it contains a curve C. Let $y \in C(k)$. Then by above there exists $x \in X(k)$ such that $y \notin$ zero set of $s_x = T_xD \cup T_{-x}D$. Then as $f(C)$ consists of only a single point, for this x, $C \cap (T_xD \cap T_{-x}D) = \emptyset$.

Lemma 7.4. *($k = k^{\mathrm{alg}}$). Let $C \subseteq X$ be any curve, $Y \subseteq X$ an irreducible divisor with $C \cap Y = \emptyset$. Then for all $y_1, y_2 \in C$, $T_{y_1 - y_2}Y = Y$.*

Assume the lemma, and apply it to each irreducible component Y of T_xD. So for all $y_1, y_2 \in C(k)$, $T_{y_1-y_2}$ maps T_xD to itself, so it maps D to itself. Since $C(k)$ is infinite, $H(D)$ is infinite. □

Proof of the lemma. Let $U = \{x \in X(k) \mid T_xY \not\supseteq C, \text{ i.e. } T_xY \cap Y \text{ is finite}\}$. We know $Y \cap C = \emptyset$. Then for all $x \in U$, $T_{-x}Y \cap C = \emptyset = Y \cap T_xC$ (because the "degree of divisor on a curve is constant in a family", see next section). Let $y_1, y_2 \in C(k)$, $z \in Y(k)$. Then $z \in T_{z-y_2}C \cap Y \neq \emptyset$, so $Y \supseteq T_{z-y_2}C$, hence $z - y_2 + y_1 = T_{z-y_2}(y_1) \in Y$, i.e. $T_{y_1-y_2}Y = Y$. □

Corollary 7.5. *Abelian varieties are projective.*

Proof. Assume $k = k^{\text{alg}}$. We need to find an ample line bundle $\mathcal{L}$ on X. Let $U \subseteq X$ be any nonempty open affine. Then $D = X \setminus U$ with the reduced subscheme structure is a reduced divisor (see Example Sheet 3, Exercise 6). Let $x \in H(D) = \{x \in X(k) \mid T_x D = D\}$. Assume $e \in U$. Then $T_x U = U$, so $x \in U(k)$, i.e. $H(D) \subseteq U(k)$. But U is affine, and $H(D)$ is the set of k-points of some closed subscheme of X, which is complete. So $H(D)$ is a complete subvariety of the affine scheme U, hence $H(D)$ is finite and thus $\mathcal{O}(D)$ is ample by the theorem. □

So in theory one could write down equation for abelian varieties embedded in $\mathbb{P}^n$, but this is complicated, unless perhaps we are in the case of elliptic curves. See e.g. [Mum66; Mum67a; Mum67b].

Corollary 7.6. *For all $n \geq 1$, $\ker([n] : X(k^{\text{alg}}) \to X(k^{\text{alg}}))$ is finite, and $[n] : X \to X$ is surjective. In particular, $X(k^{\text{alg}})$ is a divisible group.*

Proof. The first statement implies the second by dimension reasons since X is complete. Assume $k = k^{\text{alg}}$. Suppose $\ker[n]$ is infinite. Then $\ker[n] \supseteq V$ for some variety V of dimension > 0. Let $\mathcal{L}$ be any ample invertible sheaf on X (exists by the previous corollary). Then $[n]^*\mathcal{L}$ is trivial on the fibres of $[n]$, so in particular $[n]^*\mathcal{L}|_V$ is trivial. But $[n]^*\mathcal{L} = \mathcal{L}^{\otimes n(n+1)/2} \otimes i^*\mathcal{L}^{\otimes n(n-1)/2}$. As $\mathcal{L}$ is ample, so is $i^*\mathcal{L}$, hence so is $[n]^*\mathcal{L}$. So $[n]^*\mathcal{L}|_V$ is ample, contradicting $\dim V > 0$. □

Remark. One can show more precisely: If $\operatorname{char} k \nmid n$, then $\ker[n] \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$, if $\operatorname{char} k = p \mid n$, one always has $\# \ker[n] < n^{2g}$. Here $g = \dim X$.

Theorem 7.7. *There exists a* dual abelian variety $\widehat{X}$ *to X, $\dim \widehat{X} = \dim X$, together with an isomorphism $\psi : \widehat{X}(k^{\text{alg}}) \xrightarrow{\sim} \operatorname{Pic}^0(X_{k^{\text{alg}}})$. Moreover, for all ample $\mathcal{L}$ on X, there exists a unique surjective homomorphism $X \xrightarrow{\lambda_{\mathcal{L}}} \widehat{X}$ such that the composition $X(k^{\text{alg}}) \xrightarrow{\lambda_{\mathcal{L}}} \widehat{X}(k^{\text{alg}}) \simeq \operatorname{Pic}^0(X^{\text{alg}})$ is just $\varphi_{\mathcal{L}}$.*

In fact, $\widehat{X}$ parameterizes *families* of invertible sheaves: There exists an invertible sheaf $\mathcal{P}$ on $X \times \widehat{X}$, with the following property: Let S be any k-scheme. We let

$$\operatorname{Pic}(X \times S)^0 = \{\mathcal{L} \in \operatorname{Pic}(X \times S) \mid \forall s \in S,\ \mathcal{L}|_{X \times \{s\}} \in \operatorname{Pic}^0(X \times \{s\})\}.$$

Then:

(i) If $\mathcal{L} \in \operatorname{Pic}(X \times S)^0$, then there exists a unique $f : S \to \widehat{X}$ such that

$$\mathcal{L} \simeq (\operatorname{id}_X \times f)^*\mathcal{P} \otimes \operatorname{pr}_2^* \mathcal{M},$$

for some $\mathcal{M} \in \operatorname{Pic}(S)$.

(ii) This gives a (functorial in S) bijection

$$\widehat{X}(S) \xrightarrow{\sim} \frac{\operatorname{Pic}(X \times S)^0}{\operatorname{pr}_2^* \operatorname{Pic}(S)} \cong \{\mathcal{L} \in \operatorname{Pic}(X \times S)^0 \mid \mathcal{L}|_{e \times S} \cong \mathcal{O}_S\}.$$

Note that if we take $S = \operatorname{Spec} k^{\text{alg}}$, we recover $\widehat{X}(k^{\text{alg}}) \simeq \operatorname{Pic}^0(X_{k^{\text{alg}}})$.

Idea of proof:

(1) Show that if $\mathcal{L}$ is ample, $\varphi_{\mathcal{L}} : X(k^{\text{alg}}) \twoheadrightarrow \operatorname{Pic}^0(X_{k^{\text{alg}}})$. It is not difficult to show that $\operatorname{im}(\varphi_{\mathcal{L}}) \subseteq \operatorname{Pic}^0$, see Example Sheet 3, Question 2.

(2) Define $\widehat{X}$ to be the quotient of X by $\ker(\varphi_{\mathcal{L}})$.

- If $\operatorname{char} k = 0$, we just take the quotient of X by the finite group $K(\mathcal{L})$ of automorphisms of $X_{k^{\text{alg}}}$.
- If $\operatorname{char} k = p > 0$, have to work not with $K(\mathcal{L})$, but the largest closed subscheme $\underline{K}(\mathcal{L})$ such that $\mathcal{M}(\mathcal{L})|_{X \times \underline{K}(\mathcal{L})}$ is trivial, see [Mum70, Chapter III] for details.

Definition. *A* polarisation *of an abelian variety X is an isogeny (i.e. a surjective homomorphism) $\lambda : X \to \widehat{X}$ such that for some ample $\mathcal{L} \in \operatorname{Pic}(X_{k^{\text{alg}}})$, $\psi \circ \lambda = \varphi_{\mathcal{L}}$.*

8 Jacobians of Curves

Throughout let X/k be a curve (i.e. nonsingular complete variety of dimension 1), $g = \dim H^0(X, \Omega_{X/k}) = \dim H^1(X, \mathcal{O}_X)$ the genus of X

$\mathrm{div}(X)$ is the free abelian group on closed points of X. There is a degree homomorphism $\deg : \mathrm{div}(X) \to \mathbb{Z}$, $\sum n_i P_i \mapsto \sum n_i [k(P_i) : k]$. The divisor class group is $\mathrm{Cl}(X) = \mathrm{Div}(X)/\{\mathrm{div}(f) \mid f \in k(X)^*\}$. And $\mathrm{Cl}^0(X) = \ker(\deg : \mathrm{Cl}(X) \to \mathbb{Z})$.

Theorem 8.1. *There exists an abelian variety $J = J(X)$, the* Jacobian *of X, over k of dimension g with an isomorphism $J(k^{\mathrm{alg}}) \simeq \mathrm{Cl}^0(X_{k^{\mathrm{alg}}})$.*

Recall: (see e.g. [Har77, Chapter IV §1]) To a divisor D we associate the sheaf $\mathcal{O}_X(D)$ with

$$\mathcal{O}_X(D)(U) = \{f \in k(X) \mid \mathrm{div}(f) + D \geq 0 \text{ on } U\}$$

for open subsets $U \subseteq X$. Then $\mathcal{O}_X(D) \simeq \mathcal{O}_X(D')$ iff there exists a function f with $\mathrm{div}(f) = D' - D$. This gives an isomorphism $\mathrm{Cl}(X) \simeq \mathrm{Pic}(X)$. Let $L(D) = \{f \in k(X) \mid \mathrm{div}(f) + D \geq 0\} = H^0(X, \mathcal{O}_X(D))$ and $\ell(D) = \dim L(D)$. For $\mathcal{L} \in \mathrm{Pic}(X)$, define $\deg \mathcal{L} = \deg D$ where D is a divisor with $\mathcal{L} \simeq \mathcal{O}_X(D)$. Then $\mathrm{Pic}^0(X) := \{\mathcal{L} \in \mathrm{Pic}(X) \mid \deg \mathcal{L} = 0\}$.

The *canonical divisor class* K_X is such that $\mathcal{O}_X(K_X) \simeq \Omega^1_{X/k}$, it has degree $\deg K_X = 2g - 2$.

Theorem (Riemann-Roch Theorem)**.**

Divisor version: $\ell(D) - \ell(K_X - D) = 1 - g + \deg D$.

Sheaf version:

- $h^0(\mathcal{L}) - h^1(\mathcal{L}) = 1 - g + \deg \mathcal{L}$ *for all* $\mathcal{L} \in \mathrm{Pic}(X)$*. (easy part)*
- *(Serre duality)* $H^1(X, \mathcal{L}) \simeq H^0(X, \Omega_{X/k} \otimes \mathcal{L}^\vee)^\vee$ *(not so easy)*

  So in particular $h^1(\mathcal{O}_X(D)) = h^0(\mathcal{O}_X(K_X - D))$.

Proposition 8.2. *Let V be a quasiprojective variety over k, $G \subseteq \mathrm{Aut}(V)$ a finite subgroup. Then there exists a unique variety $V' = V/G$ and a proper morphism with finite fibres $\varphi : V \to V'$ such that*

(i) For all $\gamma \in G$, $\varphi \circ \gamma = \varphi$.

(ii) φ induces a bijection $V(k^{\mathrm{alg}})/G \xrightarrow{\simeq} V'(k^{\mathrm{alg}})$ and an isomorphism on function fields $k(V') \xrightarrow{\simeq} k(V)^G$.

(iii) ("categorical quotient") For all $\psi : V \to W$, morphism of k-schemes such that

$\psi \circ \gamma = \psi$ for all $\gamma \in G$, there is a unique $\theta : V' \to W$ such that $\theta \circ \varphi = \psi$.

$$\begin{array}{ccc} V & \xrightarrow{\quad\psi\quad} & W \\ & {\scriptstyle\varphi}\searrow \quad \nearrow{\scriptstyle\theta} & \\ & V/G = V' & \end{array}$$

Sketch of proof. (See e.g. [Mum70, Chapter III])

(1) $V = \operatorname{Spec} A$ is affine. Then $B = A^G$ is a k-algebra of finite type, and A is a finite B-module. Then $V' = \operatorname{Spec} B$ satisfies the properties.

(2) V arbitrary quasi-projective. Let $x \in V$ be a closed point. Then there exists an open affine $U \subseteq V$ containing the orbit xG (Take the complement of a hypersurface not containing any elements of the finite set xG, e.g. take union of of hyperplane over some k'/k missing xG and its conjugates).

So $\bigcap_{\gamma \in G} U\gamma$ is an open affine (since V is separated) containing xG, i.e. V can be covered by G-equivariant open affines. Then use (1) and glue.

□

Remark. The first step in (2), every Gx is contained in an open affine, is the key hypothesis. There exists a proper V (3-fold in characteristic 0) and free $\mathbb{Z}/2$-action such that V/G does not exist as a scheme. It is proper but not projective, V is Hironaka's famous counterexample, see [Har77].

Remark. Proper + finite fibres ⇔ finite morphism.

Back to the curve X/k (smooth, projective). Recall $\operatorname{Cl}(X) \xrightarrow{\simeq} \operatorname{Pic}(X) \xrightarrow{\deg} \mathbb{Z}$.

Proposition 8.3. *Let S be any connected k-scheme, $\mathcal{L} \in \operatorname{Pic}(X \times S)$. Then*

(i) $\deg \mathcal{L}_{X \times \{s\}}$ is independent of $s \in S$.

(ii) For all $m \geq 0$, $\{s \in S \mid \dim_{k(s)} H^0(X \times \{s\}, \mathcal{L}|_{X \times \{s\}}) \geq m\}$ is closed.

Proof. (ii) follows from Seesaw, Corollary 6.4. (i) holds because the Euler characteristic $h^0 - h^1 = 1 - g + \deg D$ is constant in flat connected families, see [Har77, Chapter III §9]. □

So $\operatorname{Pic}(X \times S) = \coprod_{n \in \mathbb{Z}} \operatorname{Pic}^n(X \times S)$ if S is connected, where

$$\operatorname{Pic}^n(X \times S) = \{\mathcal{L} \in \operatorname{Pic}(X \times S) \mid \forall s \in S, \deg \mathcal{L}|_{X \times \{s\}} = n\}.$$

And for all $\mathcal{G} \in \operatorname{Pic}^n(X)$,

$$\begin{aligned} \operatorname{Pic}^0(X \times S) &\xrightarrow{\sim} \operatorname{Pic}^n(X \times S) \\ \mathcal{L} &\mapsto \mathcal{L} \otimes \operatorname{pr}_1^* \mathcal{G} \end{aligned}$$

In particular, if say $X(k) \neq \emptyset$, then $\mathrm{Pic}^0 \simeq \mathrm{Pic}^n$ for all n, and $\mathrm{Pic}(X\times S) \cong \mathrm{Pic}^0(X\times S)\times\mathbb{Z}$.

From now on assume $k = k^{\mathrm{alg}}$. Notation:

- $D, D', \dots$ will be divisors of some degree (usually g).
- $E, \dots$ divisor of degree 0.

Proposition 8.4.

(i) If $\deg(D) = g$, *then* $\ell(D) = h^0(X, \mathcal{O}_X(D)) \geq 1$.

(ii) There exists D_0 *of degree* g, *with* $D_0 \geq 0$ *and* $\ell(D_0) = 1$.

Proof.

(i) By Riemann Roch, $h^0(\mathcal{O}_X(D)) = h^1(\mathcal{O}_X(D)) + 1 \geq 1$ if $\deg D = g$.

(ii) Let $\mathcal{L} \in \mathrm{Pic}(X)$ with $\deg \mathcal{L} = d \geq 2g+1$. Then $h^1(\mathcal{L}) = h^0(\mathcal{L}^\vee \otimes \Omega) = 0$. Then $h^0(\mathcal{L}) = d + 1 - g$. Also recall (e.g. [Har77, Chapter IV, Corollary 3.2(b)]) that $d \geq 2g+1$ implies: Sections of $\mathcal{L}$ give a closed immersion $X \hookrightarrow \mathbb{P}^{d-g}_k$ (i.e. $\mathcal{L}$ is very ample), and the image is not contained in any hyperplane[1].

Since $k = k^{\mathrm{alg}}$ is infinite, there exist $P_1, \dots, P_{d-g} \in X(k) \subseteq \mathbb{P}^{d-g}(k)$ not lying on any codimension 2 linear subspace. Then

$$H^0(X, \mathcal{L} \otimes \mathcal{O}(-\sum P_i)) = \{s \in H^0(X, \mathcal{L}) \mid s(P_1) = \dots = s(P_{d-g}) = 0\}$$

has dimension $H^0(\mathcal{L}) - (d-g) = 1$, so $\mathcal{L} \otimes \mathcal{O}_X(-\sum P_i) \cong \mathcal{O}_X(D_0)$ for some $D_0 \geq 0$, $\deg(D) = g$, $\ell(D_0) = 1$.

□

Now fix a diviros D_0 with $D_0 \geq 0$, $\deg D_0 = g$ and $\ell(D_0) = 1$. Then for all $E \in \mathrm{Div}^0(X)$, there exists $D' = P_1 + \dots + P_g$ (say) with $\mathcal{O}(D') \cong \mathcal{O}(D_0 + E)$. So the map

$$\begin{aligned} \pi_k : \{D' \geq 0 \text{ of degree } g\} &\longrightarrow \mathrm{Cl}^0(X), \\ D' &\longmapsto \mathcal{O}_X(D' - D_0) \end{aligned}$$

is surjective. Note that

$$\begin{aligned} \{D' \geq 0 \text{ of degree } g\} &= \{\text{unordered } g\text{-tuples of elements of } X(k)\} \\ &= X(k)^g / \mathrm{Sym}(g) = (X^g / \mathrm{Sym}(g))(k) \end{aligned}$$

$X^{(g)} := X^g / \mathrm{Sym}(g)$ is a first approximation to the Jacobian J which we will construct together with morphism $\pi : X^{(k)} \to J$. [N.B. "most" of the fibres of π_k have just one element]

[1] The map is defined by taking a basis of $H^0(X, \mathcal{L})$ and take these basis elements as coordinates in $\mathbb{P}^{d-g}_k$. They are linearly independent, so no linear form can vanish everywhere on the image

Actually, $X^{(g)}$ is nonsingular (essential case is $\mathbb{A}^g/\operatorname{Sym}(g) = \operatorname{Spec} k[t_1,\dots,t_g]^{\operatorname{Sym}(g)} = \operatorname{Spec} k[S_1,\dots,S_g]$ where $S_1,\dots,S_g$ are the elementary symmetric polynomials).

We use this to construct J with $J(k) \cong \operatorname{Pic}^0(X)$. Precisely: Fix $x_0 \in X(k)$.

Theorem 8.1 (souped-up). *There exists an abelian variety J/k, and $\mathcal{P} \in \operatorname{Pic}^0(X \times J)$ with $\mathcal{P}|_{\{x_0\}\times J} \cong \mathcal{O}_J$, such that for all k-schemes S:*

$$\begin{aligned} J(S) &\xrightarrow{\simeq} \{\text{isomorphism classes of } \mathcal{L} \in \operatorname{Pic}^0(X \times S) \text{ with } \mathcal{L}|_{\{x_0\}\times S} \cong S\}, \\ (f : S \to J) &\longmapsto (\operatorname{id}_X \times f)^*\mathcal{P} \end{aligned}$$

In particular, $J(k) \simeq \operatorname{Pic}^0(X)$.

Remark. If $\mathcal{L} \in \operatorname{Pic}^0(X \times S)$, for any $\mathcal{M} \in \operatorname{Pic}(S)$ let $\mathcal{L}' = \mathcal{L} \otimes \operatorname{pr}_2^* \mathcal{M} \in \operatorname{Pic}^0(X \times S)$. Then for all $s \in S$, $\mathcal{L}|_{X\times\{s\}} \simeq \mathcal{L}'|_{X\times\{s\}}$, hence $\mathcal{L}$ and $\mathcal{L}'$ should correspond to the same element of $J(S)$. But $\mathcal{L}' \otimes \mathcal{L}^\vee|_{\{x_0\}\times S} = \mathcal{M}$. So by fixing $\mathcal{L}|_{\{x_0\}\times S} \simeq \mathcal{O}_S$, we get rid of this ambiguity.

Lemma 8.5 (Version 0). *There exists a variety U_0 (ultimately a dense open in J) and $\mathcal{P}_0 \in \operatorname{Pic}^0(X \times U_0)$, with $\mathcal{P}_0|_{\{x_0\}\times U_0} \simeq \mathcal{O}_{U_0}$ such that for all varieties S,*

$$U_0(S) \xrightarrow{\simeq} \left\{ \text{iso. classes } \mathcal{L} \in \operatorname{Pic}^0(X \times S) \;\middle|\; \begin{array}{c} \mathcal{L}|_{\{x_0\}\times S} \cong \mathcal{O}_S \text{ and for all } s \in S, \\ h^0(\mathcal{L}|_{X\times\{s\}} \otimes \mathcal{O}_X(D_0)) = 1 \end{array} \right\}$$

via $(f : S \to U_0) \mapsto (\operatorname{id}_X \times f)^*\mathcal{P}_0$.

Note that always $h^0(\mathcal{L}_{X\times\{s\}} \otimes \mathcal{O}_X(D_0)) \geq 1$ by Proposition 8.4.

Proof. Construct U_0 as an open subset of $X^{(g)}$. There is $\mathcal{M} \in \operatorname{Pic}(X \times X^{(g)})$ with $\mathcal{M}|_{X\times\{D'\}} \simeq \mathcal{O}_X(D')$ for all $D' \in X^{(g)}(k)$ and $\mathcal{M}|_{\{x_0\}\times X^{(g)}} \simeq \mathcal{O}_{X^{(g)}}$ which we construct as follows:

$$\begin{array}{ccc} X \times X^g & \supseteq & \Delta_X \times X^{g-1} = \{(x_1, x_1, \dots, x_g)\} \\ {\scriptstyle\text{quotient}}\big\downarrow{\scriptstyle \operatorname{id}_X \times \varphi} & & \big\downarrow \\ X \times X^{(g)} & \supseteq & Y = (\operatorname{id}_X \times \varphi)(\Delta_X \times X^{g-1}) \end{array}$$

Then for all $D' \in X^{(g)}(k)$, $Y|_{X\times\{D'\}} = D'$. Let $\mathcal{M}' = \mathcal{O}_{X\times X^{(g)}}(Y)$. Then

$$\mathcal{M} = \mathcal{M}' \otimes \operatorname{pr}_2^* \mathcal{M}'|^\vee_{\{x_0\}\times X^{(g)}}$$

satisfies the conditions.

Let $W = \{s \in X^{(g)} \mid h^0(\mathcal{M}|_{X\times\{s\}}) = 1\}$. This is open in $X^{(g)}$ by semicontinuity, and is nonempty, as $D_0 \in W(k)$ by definition of D_0. Then take $(U_0, \mathcal{P}_0) = (W, \mathcal{M}|_W \otimes \operatorname{pr}_1^* \mathcal{O}_X(-D_0))$. If $f : S \to U_0$ is any morphism, then $\mathcal{L} = (\operatorname{id}_X \times f)^*(\mathcal{M} \otimes \mathcal{O}_X(-D_0)) \in \operatorname{Pic}^0(X\times S)$ is trivial on $\{x_0\}\times S$ and $h^0(\mathcal{L}|_{X\times\{s\}} \otimes \mathcal{O}(D_0)) = 1$ for all $s \in S$ by construction. We want every $\mathcal{L}$ to arise in this way. Let $\mathcal{L} \in \operatorname{Pic}^0(X \times S)$ and consider $\mathcal{L} \otimes \operatorname{pr}_1^* \mathcal{O}_X(D_0) =$

$\mathcal{Q}$. Then $h^0(\mathcal{Q}|_{X\times\{s\}}) = 1$ for all $s \in S$. As in the proof of seesaw, locally on S, $\mathcal{L}$ has a section, unique up to unit in $\mathcal{O}_S$, whose restriction to each fibre $X \times \{s\}$ is nonzero. The zero-set of these sections the glue to give family of divisors of degree g in $X \times S$ which determines a morphism $S \to X^{(g)}$ and its image is in U_0. □

Having constructed U_0, we just need to glue together some copies (translates!) to cover J.

Let $D_1, D_2, \ldots$ be some divisors ≥ 0 of degree g, but we no longer assume $\ell(D_i) = 1$.

We modify the lemma by replacing 0 by $i \geq 1$:

Lemma 8.5 (Version 1)**.** *There exists a variety U_i (ultimately a dense open in J) and $\mathcal{P}_i \in \operatorname{Pic}^0(X \times U_i)$, with $\mathcal{P}_i|_{\{x_0\}\times U_i} \simeq \mathcal{O}_{U_i}$ such that for all varieties S,*

$$U_i(S) \xrightarrow{\simeq} \left\{ \text{iso. classes } \mathcal{L} \in \operatorname{Pic}^0(X \times S) \;\middle|\; \begin{array}{c} \mathcal{L}|_{\{x_0\}\times S} \cong \mathcal{O}_S \text{ and for all } s \in S, \\ h^0(\mathcal{L}|_{X\times\{s\}} \otimes \mathcal{O}_X(D_i)) = 1 \end{array} \right\}$$

via $(f : S \to U_i) \mapsto (\mathrm{id}_X \times f)^* \mathcal{P}_i$.

For the proof, just take $(U_i, \mathcal{P}_i) = (W, \mathcal{M}|_W \otimes \mathcal{O}_X(-D_i))$.

Now glue: Let $U_{ij} \subseteq U_i, U_j$ be the open subscheme whose S-points are

$$\left\{ \mathcal{L} \in \operatorname{Pic}^0(X \times S) \;\middle|\; \begin{array}{c} \mathcal{L}|_{\{x_0\}\times S} \cong \mathcal{O}_S \text{ and for all } s \in S, \\ h^0(\mathcal{L}|_{X\times\{s\}} \otimes \mathcal{O}_X(D_i)) = 1 = h^0(\mathcal{L}|_{X\times\{s\}} \otimes \mathcal{O}_X(D_j)) \end{array} \right\}$$

The U_{ij} are compatible for U_i, U_j, U_l. This defines a scheme $J = \bigcup_i U_i$ by gluing, once we have chosen the D_i's. Go back to $X^{(g)} \xrightarrow{\pi} J$ defined locally as follows: $W_0 = W \xrightarrow{\simeq} U_0$ and $\pi_i : W_i \to U_i$ where $W_i = \{s \in X^{(g)} \mid h^0(\mathcal{M}|_{X\times\{s\}} \otimes \mathcal{O}_X(D_i - D_0)) = 1\}$ is open in $X^{(g)}$ and contains a point corresponding to $D' \in [2D_0 - D_i]$ since $\ell(D_0) = 1$. By the lemma, $\pi_i \in U_i(W_i)$ corresponds to some $\mathcal{L}_i$ on W_i. Take this $\mathcal{L}_i$ to be $\mathcal{M} \otimes \mathcal{O}(-D_0)$.

Every $D \in X^{(g)}(k)$ lies in W_i for some D_i ($D_i \in [2D_0 - D]$ will do). So $X^{(g)}$ being quasi-compact is a finite union of W_i, for a suitable finite family $(D_i)_{0\leq i\leq n}$. The π_i are surjective, so $J = \bigcup_{i=0}^n U_i$.

Now define the group law $m : J \times J \to J$. Define it on the open subsets $U_i \times U_j$ as follows: Let $(x, y) \in U_i(k) \times U_j(k)$ correspond to $\mathcal{P}_{i,x}, \mathcal{P}_{j,y} \in \operatorname{Pic}^0(X)$. Then $\mathcal{P}_{i,x} \otimes \mathcal{P}_{j,y}$ corresponds to some $z \in U_l(k)$ for some l. Take this to be the image of (x, y) under m. Note that $\mathcal{P}_{i,x} \otimes \mathcal{P}_{j,y}$ is the fibre of $\mathcal{L} = \mathrm{pr}_1^* \mathcal{P}_i \otimes \mathrm{pr}_2^* \mathcal{P}_j$ on $U_i \times U_j$ above (x, y) and $h^0(\mathcal{L}|_{(x,y)} \otimes \mathcal{O}(D_l)) = 1$. Then there is a neighborhood V of $(x, y) \in U_i \times U_j$ on which h^0 of $\mathcal{L} \otimes \mathcal{O}(D_l) = 1$. Hence this gives a morphism $V \to U_l$ and this is our m (locally).

Then one needs to check that this defines a morphism $J \times J \to J$, that J becomes a group variety in this way, and that $\pi : X^{(g)} \to J$ is surjective, thus proving that J is projective, hence an abelian variety.

9 Extra Lecture: Proof of Cube

Recall:

Theorem (Theorem of the cube)**.** *Let X, Y, Z be varieties, X, Y complete. Let x, y, z be k-points of X, Y, Z, $\mathcal{L}$ an invertible sheaf on $X \times Y \times Z$. Suppose the restriction of $\mathcal{L}$ to each of $\{x\} \times Y \times Z$, $X \times \{y\} \times Z$, $X \times Y \times \{z\}$ is trivial. Then $\mathcal{L}$ is trivial.*

Remark. This implies that

$$\operatorname{Pic}(X \times Y) \oplus \operatorname{Pic}(X \times Z) \oplus \operatorname{Pic}(Y \times Z) \xrightarrow{\text{projections}^*} \operatorname{Pic}(X \times Y \times Z)$$

is surjective.

Proof. We will prove a slightly more general statement.

(a) First replace Z by $\operatorname{Spec} A$, A a finite local k-algebra, e.g. $k[t]/(t^n)$. As $z \in Z(k)$, $Z = \{z\}$, and $A/\mathfrak{m}_A = k(z) = k$. We induct on $\dim_k A$. If the dimension is 1, then $Z = \operatorname{Spec} k$, so we are done as $\mathcal{L} = \mathcal{L}|_{X \times Y \times \{z\}} \simeq \mathcal{O}$.

   Now suppose $\dim_k A > 1$. Then there is an ideal $I \subseteq A$ with $\dim_k I = 1$ (take any minimal non-zero ideal). Let $Z_1 = \operatorname{Spec} A/I \hookrightarrow Z$.

   Lemma 9.1. *Let V be a complete variety. Then $H^0(V \times \operatorname{Spec} B, \mathcal{O}) = B$ for any k-algebra B.*

   This is the special case $A = k$ of Corollary 6.3

   Lemma 9.2. *Let V be a complete variety. There is an exact sequence (functorial in V)*

$$0 \to H^1(V, \mathcal{O}_V) \to \operatorname{Pic}(V \times Z) \to \operatorname{Pic}(V \times Z_1)$$

   A particular case of this is $A = k[t]/(t^2)$, $Z_1 = \operatorname{Spec} k$, $I = (t)$. Then

$$H^1(\mathcal{O}) = \ker(\operatorname{Pic}(V \times \operatorname{Spec} k[t]/(t^2)) \to \operatorname{Pic} V) = \text{“tangent space to Pic”}.$$

   Proof.$I = (t) = kt, t^2 = 0$, so $(1 + a)(1 + b) = 1 + (a + b)$ for all $a, b \in I$. Then $0 \to I \to A^\times \to (A/I)^\times \to 0$ is exact where the first map is given by $a \mapsto 1 + a$. We globalise this and get an exact sequence $0 \to I\mathcal{O}_{V \times S} \to \mathcal{O}_{V \times Z}^\times \to \mathcal{O}_{V \times Z_1}^\times \to 0$ of abelian group sheaves on the topological space of $V \approx V \times Z$.

   Also $\mathcal{O}_V \overset{t}{\simeq} I\mathcal{O}_{V \times Z}$. Note that $H^0(V \times Z, \mathcal{O})^\times = A^\times \to H^0(V \times Z_1, \mathcal{O})^\times = (A/I)^\times$ is still surjective, so the long exact sequence in cohomology becomes

$$0 \to H^1(V, \mathcal{O}_V) \to \operatorname{Pic}(V \times Z) \to \operatorname{Pic}(V \times Z_1).$$

∎

Back to cube, $Z = \operatorname{Spec} A$. By induction, we may assume $\mathcal{L}|_{X\times Y\times Z_1}$ is trivial. By Lemma 9.2 applied to X, Y and $X \times Y$ we get the following diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & H^1(X\times Y, \mathcal{O}) & \longrightarrow & \operatorname{Pic}(X\times Y\times Z) & \xrightarrow{\;c\;} & \operatorname{Pic}(X\times Y\times Z_1) \\
& \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow \\
0 \to & H^1(X,\mathcal{O})\oplus H^1(Y,\mathcal{O}) & \to & \operatorname{Pic}(X\times Z)\oplus \operatorname{Pic}(Y, Z) & \to & \operatorname{Pic}(X\times Z_1)\oplus \operatorname{Pic}(Y, Z_1)
\end{array}$$

The vertical maps are (y^*, x^*) where $Y \overset{x}{\hookrightarrow} X\times Y \overset{y}{\hookleftarrow} X$.

Then $\mathcal{L} \in \ker b \cap \ker c \simeq \ker a$.

Lemma 9.3. *a is an isomorphism.*

This then implies $\mathcal{L} \simeq \mathcal{O}$, so we are done.

Lemma 9.3 is a special case of:

Theorem (Künneth formula)**.** *Let X, Y be varieties over k, $\mathcal{F}$ (resp. $\mathcal{G}$) a quasi-coherent $\mathcal{O}_X$-module (resp. $\mathcal{O}_Y$-module). Let $\mathcal{H} = \operatorname{pr}_1^*\mathcal{F}\otimes \operatorname{pr}_2^*\mathcal{G}$. Then:*

$$H^n(X\times Y, \mathcal{H}) = \bigoplus_{p+q=n} H^p(X,\mathcal{F})\otimes H^q(Y,\mathcal{G}).$$

In our case take $\mathcal{F} = \mathcal{O}_X, \mathcal{G} = \mathcal{O}_Y$. Then $H^1(X\times Y, \mathcal{O}) = H^0(X,\mathcal{O})\otimes H^1(Y,\mathcal{O}) \oplus H^1(X,\mathcal{O})\otimes H^0(Y,\mathcal{O})$ and $H^0(X,\mathcal{O}) = k = H^0(Y,\mathcal{O})$ as X, Y are complete.

Idea of proof of the Künneth formula: Let $X = \bigcup U_i, Y = \bigcup V_j$ be open affine coverings. Then $X\times Y = \bigcup_{i,j} U_i\times V_j$. Now compare $\check{C}^\bullet(\{U_i\times V_j\}, \mathcal{H})$ and $\check{C}^\bullet(\{U_i\},\mathcal{F})\otimes_k \check{C}^\bullet(\{V_j\},\mathcal{G})$. See Stacks, 0BED.

(b) $Z = \operatorname{Spec} A$, A a local noetherian k-algebra, $z \in Z$ the closed point. Let $Z_n = \operatorname{Spec} A/\mathfrak{m}_A^n$ for $n \geq 1$. By (a), $\mathcal{L}|_{X\times Y\times Z_n}$ is trivial for all n. Recall from the seesaw proof that there exist finite cyclic A-modules M, M' such that for all k-algebra homomorphisms $A \to B$, $H^0(X\times Y\times \operatorname{Spec} B, \mathcal{L}_B) = \operatorname{Hom}_A(M, B)$ and same with $\mathcal{L}_B^\vee, M'$. Since $\mathcal{L}|_{X\times Y\times Z_n}$ is trivial, Lemma 9.1 gives $M\otimes A/\mathfrak{m}^n \cong A/\mathfrak{m}^n$[1]. Therefore $\operatorname{Ann}_A(M) \subseteq \bigcap_{n\geq 1}\mathfrak{m}^n = \{0\}$. So $M \simeq A \simeq M'$. Then $\mathcal{L}_B$ and $\mathcal{L}_B^\vee$ both have non-zero H^0, so $\mathcal{L}\simeq\mathcal{O}$.

This is a scheme-theoretic version of semicontinuity: there is a maximal closed subscheme $Z^* \subseteq Z$ such that $\mathcal{L}|_{V\times Z^*} \simeq \mathcal{O}$ where $V = X\times Y$. As $Z^* \supseteq Z_n$ for all n we get $Z^* = Z$.

(c) Now let Z be a variety. Then $\mathcal{L}|_{X\times Y\times \operatorname{Spec}\mathcal{O}_{Z,z}} \simeq \mathcal{O}$ by part (b), so $F = \{z' \in Z \mid \mathcal{L}|_{X\times Y\times\{z'\}}$ is trivial$\}$ is closed (by seesaw) and contains the generic point of Z as it

[1] L.T.: How do we get this from the Lemma? We get $\operatorname{Hom}_A(M, A/\mathfrak{m}^n) \simeq A/\mathfrak{m}^n$, but how do we get $M\otimes A/\mathfrak{m}^n$ from this? Anyway, it is also clear from $\operatorname{Hom}_A(M, A/\mathfrak{m}^n)\simeq A/\mathfrak{m}^n$ that $\operatorname{Ann}_A M \subseteq \mathfrak{m}^n$.

is also the generic point of $\operatorname{Spec} \mathcal{O}_{Z,z}$. Then $F = Z$, hence $\mathcal{L} = \mathrm{pr}_3^* \mathcal{M}$ by seesaw for some $\mathcal{M}$ on Z. Then $\mathcal{O}_Z \simeq \mathcal{L}|_{\{x\}\times\{y\}\times Z} \simeq \mathcal{M}$. Then $\mathcal{L}$ is trivial.

□

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Index