A nice application of the Krein-Milman theorem

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Suppose on every lattice point in \mathbb{Z}^2 we write a real number in such a way that each value is the average of its four neighbors. Assume furthermore the set of all these values is bounded. Claim: All the values are equal.

I learned about this problem when taking the Cambridge Part III Functional Analysis examination¹. It seems surprising that Functional Analysis could help here, but it turns out that a short proof of this is possible using the Krein-Milman theorem about convex subsets of locally convex spaces. As I liked this application of that theorem very much I decided to write it up in this short note. The proof also highlights a certain strategy which turns up in various places: Instead of taking just one such configuration of values and working with it, we will look at the set of *all* possible configurations and exploit some of its geometric characteristics, in particular its extreme points.

Let's call a function $f : \mathbb{Z}^2 \to \mathbb{R}$ harmonic if

$$f(m,n) = \frac{1}{4}(f(m,n+1) + f(m,n-1) + f(m+1,n) + f(m-1,n))$$

holds for all $(m, n) \in \mathbb{Z}^2$, in other words each value is the average of its four neighbors. Note that this is analogous to the mean value property of harmonic functions on \mathbb{R}^n , hence the name. We now rephrase the above problem:

Theorem 1. Let $f : \mathbb{Z}^2 \to \mathbb{R}$ be a bounded harmonic function. Then f is constant.

Of course, there is no loss in generality in assuming that f takes values in, say, [-1, 1].

We recall the notion of *extreme points* in order to state the Krein-Milman theorem. Let V be a real or complex vector space and $C \subseteq V$. A point $x \in C$ is called an *extreme point* of C if whenever x = ty + (1 - t)z with $y, z \in C$ and $t \in [0, 1]$, we must have y = z = x. The set of all extreme points of C is denoted by Ext C. The Krein-Milman theorem states that under suitable conditions one can reconstruct C from its set of extreme points.

Theorem (Krein-Milman). Let V be locally convex space and $C \subseteq V$ a compact, convex subset. Then $C = \overline{\text{conv}} \operatorname{Ext} C.$

Here conv X denotes the convex hull of a subset $X \subseteq V$. The proof uses a clever application of Zorn's lemma and the Hahn-Banach separation theorem, see [Rud91, Theorem 3.23] or [Con85, Theorem V 7.4].

We can now prove Theorem 1. Let C be the set of harmonic functions $f : \mathbb{Z}^2 \to [-1, 1]$. Note that C is a convex subset of $V := \ell^{\infty}(\mathbb{Z}^2)$. The idea is to show that the extreme points of C are constant functions and then apply the Krein-Milman theorem.

¹see here, Problem 3

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Lemma. The extreme points of C are the constant functions -1 and 1.

Proof. Suppose $f \in C$ is an extreme point of C. The definition of extreme points easily extends inductively to convex combinations of more than two elements, i.e. if $f = \sum_i t_i f_i$ with $t_i \ge 0, \sum_i t_i = 1$ and $f_i \in C$, then $f_i = f$ for all i. In this case the definition of harmonic functions writes f as a convex combination of the shifts $(m, n) \mapsto f(m, n \pm 1), f(m \pm 1, n)$. Clearly, these shifts are again in C, hence $f(m, n) = f(m \pm 1, n) = f(m, n \pm 1)$ for all $(m, n) \in \mathbb{Z}^2$, and so f is constant. Finally, it is clear that among the the constant functions only those with values -1, 1 can be extreme points and conversely these are indeed extreme points of C.

To apply the Krein-Milman theorem, we need a topology on V which makes it into a locally convex vector space and for which C is compact. The topology we choose is the weak-* topology we obtain when viewing V as the dual of $\ell^1(\mathbb{Z}^2)$. By the Banach-Alaoglu theorem ([Con85, Theorem V.3.1]), the closed unit ball B of V is w^* -compact and clearly $C \subseteq B$, so it suffices to prove that C is a closed subset in the w^* topology. Given $(m, n) \in \mathbb{Z}^2$, let $\delta_{(m,n)} : V \to \mathbb{R}$ be defined by $\delta_{(m,n)}(f) = f(m, n)$. This is (by definition) continuous for the w^* topology. We can now write

$$C = B \cap \bigcap_{(m,n) \in \mathbb{Z}^2} \left(\delta_{(m,n)} - \frac{1}{4} \left(\delta_{(m,n+1)} + \delta_{(m,n-1)} + \delta_{(m+1,n)} + \delta_{(m-1,n)} \right) \right)^{-1}(0)$$

showing that C is indeed closed. Thus, the conditions in the Krein-Milman theorem are satisfied and we obtain $C = \overline{\text{conv}}^{w^*} \{-1, 1\}$. Note that $\text{conv}\{-1, 1\}$ is the set of all constant functions on \mathbb{Z}^2 with values in [-1, 1], so from

$$\operatorname{conv}\{-1,1\} = B \cap \bigcap_{(m,n) \in \mathbb{Z}^2} (\delta_{(m,n)} - \delta_{(0,0)})^{-1}(0)$$

we see that it is already w^* -closed. In particular, C consists of constant functions only.

References

[Con85] J. B. Conway. A Course in Functional Analysis. Graduate Texts in Mathematics 96. 1985.
[Rud91] W. Rudin. Functional analysis. Second. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991.

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