A nice application of the Krein-Milman theorem

LEONARD TOMCZAK

Consider the following elementary problem: Suppose on every lattice point in $\mathbb{Z}^2$ we write a real number such that each value is the average of its four neighbors and furthermore the set of all these values is bounded. Then show that all the values must be equal.

I learned about this problem when taking the Cambridge Part III Functional Analysis examination[1]. It seems surprising that Functional Analysis could help here, but it turns out that a short proof of this is possible using the Krein-Milman theorem about convex subsets of locally convex spaces. As I liked this application of that theorem very much I decided to write it up in this short note. The proof also highlights a certain strategy which turns up in various places: Instead of taking just one such configuration of values and working with it, we will look at the set of *all* possible configurations and exploit some of its geometric characteristics, in particular its extreme points.

Let's call a function $f : \mathbb{Z}^2 \to \mathbb{R}$ *harmonic* if

$$f(m,n) = \frac{1}{4}(f(m,n+1) + f(m,n-1) + f(m+1,n) + f(m-1,n))$$

holds for all $(m,n) \in \mathbb{Z}^2$, in other words each value is the average of its four neighbors. Note that this is analogous to the mean value property of harmonic functions on $\mathbb{R}^n$, hence the name. We now rephrase the above problem:

Theorem 1. *Let $f : \mathbb{Z}^2 \to \mathbb{R}$ be a bounded harmonic function. Then f is constant.*

Of course, there is no loss in generality in assuming that f takes values in, say, $[-1, 1]$.

We recall the notion of *extreme points* in order to state the Krein-Milman theorem. Let V be a real or complex vector space and $C \subseteq V$. A point $x \in C$ is called an *extreme point* of C if whenever $x = ty + (1-t)z$ with $y, z \in C$ and $t \in [0, 1]$, we must have $y = z = x$. The set of all extreme points of C is denoted by $\operatorname{Ext} C$. The Krein-Milman theorem states that under suitable conditions one can reconstruct C from its set of extreme points.

Theorem (Krein-Milman)**.** *Let V be locally convex space and $C \subseteq V$ a compact, convex subset. Then*

$$C = \overline{\operatorname{conv}} \operatorname{Ext} C.$$

Here $\operatorname{conv} X$ denotes the convex hull of a subset $X \subseteq V$. The proof uses a clever application of Zorn's lemma and the Hahn-Banach separation theorem, see [Rud91, Theorem 3.23] or [Con85, Theorem V 7.4].

We can now prove Theorem 1. Let C be the set of harmonic functions $f : \mathbb{Z}^2 \to [-1, 1]$. Note that C is a convex subset of $V := \ell^\infty(\mathbb{Z}^2)$. The idea is to show that the extreme points of C are constant functions and then apply the Krein-Milman theorem.

[1] see here, Problem 3

Lemma. *The extreme points of C are the constant functions -1 and 1.*

Proof. Suppose $f \in C$ is an extreme point of C. The definition of extreme points easily extends inductively to convex combinations of more than two elements, i.e. if $f = \sum_i t_i f_i$ with $t_i \geq 0, \sum_i t_i = 1$ and $f_i \in C$, then $f_i = f$ for all i. In this case the definition of harmonic functions writes f as a convex combination of the shifts $(m,n) \mapsto f(m, n \pm 1), f(m \pm 1, n)$. Clearly, these shifts are again in C, hence $f(m,n) = f(m \pm 1, n) = f(m, n \pm 1)$ for all $(m,n) \in \mathbb{Z}^2$, and so f is constant. Finally, it is clear that among the the constant functions only those with values $-1, 1$ can be extreme points and conversely these are indeed extreme points of C. □

To apply the Krein-Milman theorem, we need a topology on V which makes it into a locally convex vector space and for which C is compact. The topology we choose is the weak-$*$ topology we obtain when viewing V as the dual of $\ell^1(\mathbb{Z}^2)$. By the Banach-Alaoglu theorem ([Con85, Theorem V.3.1]), the closed unit ball B of V is w^*-compact and clearly $C \subseteq B$, so it suffices to prove that C is a closed subset in the w^* topology. Given $(m,n) \in \mathbb{Z}^2$, let $\delta_{(m,n)} : V \to \mathbb{R}$ be defined by $\delta_{(m,n)}(f) = f(m,n)$. This is (by definition) continuous for the w^* topology. We can now write

$$C = B \cap \bigcap_{(m,n) \in \mathbb{Z}^2} \big(\delta_{(m,n)} - \frac{1}{4}(\delta_{(m,n+1)} + \delta_{(m,n-1)} + \delta_{(m+1,n)} + \delta_{(m-1,n)})\big)^{-1}(0),$$

showing that C is indeed closed. Thus, the conditions in the Krein-Milman theorem are satisfied and we obtain $C = \overline{\mathrm{conv}}^{w^*}\{-1, 1\}$. Note that $\mathrm{conv}\{-1, 1\}$ is the set of all constant functions on $\mathbb{Z}^2$ with values in $[-1, 1]$ which we can also write as

$$\mathrm{conv}\{0, 1\} = B \cap \bigcap_{(m,n) \in \mathbb{Z}^2} (\delta_{(m,n)} - \delta_{(0,0)})^{-1}(0).$$

From this we see that it is already w^*-closed. In particular, C consists of only constant functions.

References

[Con85] John B Conway. *A Course in Functional Analysis*. Graduate Texts in Mathematics 96. 1985.
[Rud91] Walter Rudin. *Functional analysis*. Second. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991.

Department of Mathematics, Evans Hall, University of California, Berkeley, CA 94720, USA

Email address: leonard.tomczak@berkeley.edu