IGUSA'S ZETA FUNCTIONS AND GENERALIZATIONS

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1. Igusa's Zeta Function

Main reference for this part: [Igu07].

Let F be a p-adic field, $\mathcal{O}_F$ its ring of integers and $\mathfrak{p}$ the maximal ideal. q denotes the cardinality of the residue field. Let $f \in \mathcal{O}_F[X_1, \ldots, X_m]$ be nonconstant. We are interested in the sequence

$$N_n = \#\{x \bmod \mathfrak{p}^n \mid f(x) \equiv 0 \bmod \mathfrak{p}^n\}$$

counting the number of solutions of $f \equiv 0 \bmod \mathfrak{p}^n$. Note that $N_0 = 1$. Let $P(t) = \sum_{n=0}^{\infty} N_n t^n$.

Theorem 1 (Igusa)**.** *$P(t)$ is a rational function in t.*

Remark. This is easy if $f = 0$ is non-singular mod $\mathfrak{p}$. Then Hensel's lemma gives that each $x \in \mathcal{O}_F/\mathfrak{p}$ has exactly $q^{(n-1)(k-1)}$ lifts to $\mathcal{O}_F/\mathfrak{p}^n$, hence $N_n = N_1 q^{(n-1)(m-1)}$ for $n \geq 1$ and we get

$$P(t) = 1 + N_1 \sum_{n=1}^{\infty} q^{(n-1)(k-1)} t^n = 1 + N_1 \frac{t}{1 - q^{m-1}t}.$$

How do we prove the general case? The strategy is as follows:

(1) Relate $P(t)$ to the integral $Z(s) = \int_{\mathcal{O}_F^k} |f(x)|^s \,\mathrm{d}x$ where $t = q^{-s}$.

(2) Use Hironaka's desingularization result to reduce to the case of $\int_{\mathcal{O}_F^m} |x_1|^{a_1 s + b_1} |x_2|^{a_2 s + b_2} \cdots |x_m|^{a_m s + b_m} \,\mathrm{d}x$.

(3) Compute this last integral.

In fact (2) and (3) generalize to the following [Igu07, Theorem 8.2.1]:

Theorem 2. *Let Φ be a Schwartz Bruhat function on F^k (i.e. locally constant and compactly supported), χ a character of $F^\times$, and $f \in F[X_1, \ldots, X_m] \setminus F$. For $s \in \mathbb{C}$ with* $\operatorname{Re} s > 0$ *define*

$$Z(\Phi, \chi, s) = \int_{F^m} |\chi(f(x))| \, |f(x)|^s \, \Phi(x) \mathrm{d}x.$$

Then Z is a rational function of $t = q^{-s}$ (for fixed Φ, χ).

Note that $Z(s)$ above is the special case where χ is trivial and $\Phi = \mathbb{1}_{\mathcal{O}_F^m}$.

We now go through steps (1), (2) and (3).

Step (1). Let $x_1, \ldots, x_{N_n}$ be representatives of the solutions of $f(x) \equiv 0 \bmod \mathfrak{p}^n$. Then we have $f^{-1}(\mathfrak{p}^n) = \coprod_{i=1}^{N_n} (x_i + (\mathfrak{p}^n)^{\times m})$, so $\operatorname{vol}(f^{-1}(\mathfrak{p}^n)) = N_n q^{-nm}$. Then

$$Z(s) = \int_{O_F^m} |f(x)|^s \,\mathrm{d}x = \sum_{n \geq 0} \int_{f^{-1}(\mathfrak{p}^n \setminus \mathfrak{p}^{n+1})} q^{-ns} \mathrm{d}x$$

$$= \sum_{n\geq 0} \Big(\mathrm{vol}(f^{-1}(\mathfrak{p}^n)) - \mathrm{vol}(f^{-1}(\mathfrak{p}^{n+1})) \Big) q^{-ns} = \sum_{n\geq 0} \Big(N_n q^{-nm} - N_{n+1} q^{-(n+1)m} \Big) t^n$$
$$= P(q^{-m}t)\big(1 - t^{-1}\big) + t^{-1}$$

Hence showing the rationality of $P(t)$ reduces to rationality of $Z(s)$ as a function of $t = q^{-s}$.

Step (2). We use the resolution of singularities in the following form:

Theorem. *Let F be a complete field of characteristic 0 and $f \in F[X_1, \ldots, X_m]$. There is an m-dimensional K-analytic manifold and a proper K-analytic map $h : Y \to K^m$, with the following property: For every $b \in Y$ there are local coordinates $y_1, \ldots, y_p, \ldots, y_m$ such that in a neighborhood of b we have*

$$h^* f = \varepsilon \prod_{i=1}^{p} y_i^{k_i}, \quad h^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m) = \eta \prod_{1\leq i\leq p} y_i^{l_i - 1} \bigwedge_{1\leq i\leq n} dy_i$$

for some integers k_i, l_i and functions ε, η, non-zero at b. Furthermore h is an isomorphism away from the singular locus of f.

The general statement is more precise, for example h is a composition of monoidal transformations, the $y_1, \ldots, y_p$ are local defining equations of submanifolds whose union is $f^{-1}(0)$.

Apply this to our situation. Let $Y_0 = h^{-1}(\mathcal{O}_F^m)$. Since h is an isomorphism up to measure 0 sets, we have

$$\int_{\mathcal{O}_K^m} |f(x)|^s \,\mathrm{d}x = \int_{Y_0} |f(h(x))|^s \,|h^*(\mathrm{d}x)|.$$

Now we can write Y_0 as a finite union of small enough open sets on which we have coordinates as in the theorem. By making them even smaller we may assume that the ε, η functions have constant absolute value on the coordinate neighborhoods, and that the coordinates $y_1, \ldots, y_m$ are parameterized by $(\mathfrak{p}^n)^{\times m}$ for some n. Hence the result is a sum of integrals of the form

$$\int_{(\mathfrak{p}^n)^{\times m}} |y_1^{k_1} \cdots y_p^{k_p}|^s |y_1^{l_1 - 1} \cdots y_p^{l_p - 1}| \mathrm{d}y_1 \cdots \mathrm{d}y_m$$

After rescaling we may replace $\mathfrak{p}^n$ by $\mathcal{O}_F$ and the reduction step (2) is done.

Step (3). We wish to compute $\int_{\mathcal{O}_F^k} |x_1|^{a_1 s + b_1} |x_2|^{a_2 s + b_2} \cdots |x_m|^{a_m s + b_m} \,\mathrm{d}x$. This is a straightforward computation using the geometric series:

$$\int_{\mathcal{O}_F^k} |x_1|^{a_1 s + b_1} |x_2|^{a_2 s + b_2} \cdots |x_m|^{a_m s + b_m} \,\mathrm{d}x = \prod_{i=1}^{m} \int_{\mathcal{O}_F} |x_i|^{s a_i + b_i} \,\mathrm{d}x = \prod_{i=1}^{m} \sum_{n=0}^{\infty} q^{-a_i n s} q^{-n b_i}$$
$$= \prod_{i=1}^{m} \frac{1}{1 - q^{-a_i s - b_i}} = \prod_{i=1}^{m} \frac{1}{1 - t^{a_i} q^{-b_i}}$$

Examples. [Igu07, pp. 168, 169, 171]

(1) Suppose $f \in \mathcal{O}_F[X_1, \ldots, X_m]$ is a homogeneous polynomial of degree d such that 0 is the only singular point of $f = 0 \bmod \mathfrak{p}$. Let $N = N_1$, then

$$Z(s) = \big[(1 - q^{-1})(1 - q^{-m})t + (1 - q^{-m}N)(1 - t)\big](1 - q^{-1}t)^{-1}(1 - q^{-m}t^d)^{-1}.$$

(2) If $f = X_1^2 - X_2^3$, then

$$Z(s) = (1-q^{-1})(1-q^{-1}t)^{-1}(1-q^{-5}t^6)^{-1}(1-q^{-2}t(1-t)-q^{-5}t^5).$$

Remark. Everything can be generalized to the case of multiple simultaneous equations, i.e. let $f_1, \ldots, f_k \in \mathcal{O}_F[X_1, \ldots, X_m]$ be given and denote by N_n the number of solutions of $f_1 = \cdots = f_k = 0$ in $\mathcal{O}_F/\mathfrak{p}^n$. Then $\sum_{n\geq 0} N_n t^n$ is a rational function. Again in the case where the variety defined is non-singular mod $\mathfrak{p}$, this is easy. In the general case one proceeds in a similar way as above. The problem reduces to computing the integral

$$\int_{\mathcal{O}_F^m} \max\{|f_1(x)|, |f_2(x)|, \ldots, |f_k(x)|\}^s \mathrm{d}x.$$

In this case the integrals to compute in Step (3) become more complicated. See [Meu81].

Remark. Archimedean analog also important! E.g. meromorphic continuation of $\int_{\mathbb{R}^n} |f(x)|^s \Phi(x)\mathrm{d}x$ for Schwartz functions $\Phi(x)$ gives the Malgrange-Ehrenpreis theorem [Ber73, Corollary 4.8].

Remarks. Some more things about $Z(f,s)$. Let K be a number field.

(1) If f is homogeneous and there exists a resolution of singularities such that Y is nonsingular mod $\mathfrak{p}$, then $\deg_t Z(f,s) = -\deg f$ [Den87, Theorem 4.1]. A counterexample that some kind of restriction is necessary is $f = x^2 + 2y^2$ over $\mathbb{Q}_2$. In particular if $f \in K[X_1, \ldots, X_m]\backslash K$, then $\deg_t Z_{\mathfrak{p}}(f,s) = -\deg f$ for almost all primes $\mathfrak{p}$ of K.

(2) Functional Equation: If f is homogeneous, there are $\alpha_1, \ldots, \alpha_k$ and a rational function $Z(x_1, \ldots, x_{k+1})$ such that $Z_L(s) = Z(\alpha_1^{-e}, \ldots, \alpha_k^{-e}, q_L^{-s}) =: Z(s,e)$ for all finite extensions L/F with residue field degree e (yes, this seems to be the standard notation here). Furthermore, if $f \in K[X_1, \ldots, X_m]\backslash K$ then for almost all places, Z satisfies the functional equation $Z(s,-e) = q^{-es\deg f} Z(s,e)$ [DM91].

(3) (Monodromy) Conjecture: For $f \in K[X_1, \ldots, X_m]\backslash K$, for almost all primes $\mathfrak{p}$, if s_0 is a pole of $Z_{\mathfrak{p}}(f,s)$, then $\operatorname{Re} s_0$ is a root of b_f, the Bernstein-Sato polynomial of f.

2. Generalization

Define a language $\mathcal{L}$ with ([DvdD88, 0.6])

- For each $m \geq 0$ and $f \in \mathbb{Z}_p\{X_1, \ldots, X_m\}$ (power series whose coefficients go to 0) an m-ary operation symbol f,
- a binary operation symbol D,
- for each $n > 0$ a unary relation P_n.

We are interested in its structure $\mathbb{Z}_p$ where the operation symbols f are interpreted in the obvious way by evaluation; D is defined by $D(x,y) = \frac{x}{y}$ if $|x| \leq |y|$ and $y \neq 0$, $D(x,y) = 0$ otherwise; and P_n is the set of n-th powers in $\mathbb{Z}_p$.

Theorem 3. *Suppose $S \subseteq \mathbb{Z}_p^m$ is a definable subset and $f, g : \mathbb{Z}_p^m \to \mathbb{Z}_p$ definable functions. Then*

$$Z(S,f,g,s) := \int_S |f(x)|^s |g(x)| \,\mathrm{d}x$$

is a rational function in $t = p^{-s}$.

Applications:

This was first used (in a slightly weaker form) to prove a conjecture of Serre: In the setup as in the beginning let $\widetilde{N}_n := \#\{x \bmod \mathfrak{p}^n \mid f(x) = 0\}$ be the number of solutions to $f = 0 \bmod \mathfrak{p}^n$ that can be lifted to solutions in $\mathcal{O}_F$. Let $\widetilde{P}(t) = \sum_{n=0}^{\infty} \widetilde{N}_n t^n$

Theorem 4 ([Den84])**.** *$\widetilde{P}(t)$ is a rational function.*

The point is that again we can express $\widetilde{P}(t)$ as a p-adic integral: If

$$I(s) = \int_D |w|^s \,\mathrm{d}x\,\mathrm{d}w$$

where $D = \{(x, w) \in \mathcal{O}_F^m \times \mathcal{O}_F \mid \exists y \in \mathcal{O}_F^m : x \equiv y \bmod w, f(y) = 0\}$, then $I(s) = \frac{q-1}{q}\widetilde{P}(q^{-m-1}t)$.

Theorem 5 ([dSau93, Theorem B])**.** *Let G be a compact p-adic analytic group. If a_n denotes the number of subgroups of index n. Then $\zeta_{G,p}(t) = \sum_{n=0}^{\infty} a_{p^n} t^n$ is a rational function.*

See [dSau93] for more of these results.

Idea of proof: First one considers the case where G is a *uniformly powerful* pro-p group. For a topological generating set $x_1, \ldots, x_d$ of G with d minimal consider the function $\mathbb{Z}_p^d \to G$, $(\lambda_1, \ldots, \lambda_d) \mapsto x_1^{\lambda_1} \cdots x_d^{\lambda_d}$. This is a homeomorphism. In this way we can associate to any finite-index subgroup $H \subseteq G$, a subset $\mathcal{M}(H) \subseteq M_{d\times d}(\mathbb{Z}_p)$ consisting of "good bases" for H. The measure of $\mathcal{M}(H)$ is related to the index of H and one can define functions $f, g : M_{r\times r}(\mathbb{Z}_p)$, independent of H, such that $[G : H]^{-s} = \int_{\mathcal{M}(H)} |f(x)|^s |g(x)| \,\mathrm{d}x$. Summing over all H gives $\zeta_{G,p}(t)$, and one only has to show that everything is definable. In the general case there is at least a uniformly powerful pro-p group G_1 of finite index in G. Then a finite index subgroup H can be split up into $H \cap G_1 = H_1$ and a set of coset representatives for H_1 in H. Then combining the set $\mathcal{M}(H_1)$ as in the first part together with these coset representatives one can represent $\zeta_{G,p}$ as a definable integral.

Here are two more rationality results that use Theorem 3.

Theorem 6 ([dSau00, Theorem 1.6])**.** *Fix integers c, d. Let a_n be the number of isomorphism classes of finite groups of order p^n, nilpotency class $\leq c$ and at most d generators. Then $\sum_{n\geq 1} a_n t^n$ is a rational function.*

Theorem 7 ([Jai06, Theorem 1.1])**.** *Let G be a compact p-adic analytic group with $p > 2$. Let r_n be the number of isomorphism classes of irreducible complex continuous representation of dimension n. Then there are integers $n_1, \ldots, n_k$ and rational functions $f_1, \ldots, f_k$ such that*

$$\sum_{n=1}^{\infty} r_n n^{-s} = \sum_{i=1}^{k} n_i^{-s} f_i(p^{-s}).$$

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