REPRESENTATIONS OF GL_2 OVER NONARCHIMEDEAN FIELDS

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Contents

Notes still in progress. Main references: [JL70], [BH06], [GH11] and Chapter 4 in [Bum97].

1. Preliminaries

1.1. Topological Stuff.

Definition. *A* locally profinite group *is a totally disconnected locally compact topological group.*

Here we assume topological groups to be Hausdorff.

Remark. This is the terminology from [BH06]. Sometimes, e.g. in [GH] or [Bor+79], these groups are called *td-groups.*

Theorem. *A topological group is locally profinite if and only the identity element has a neighborhood basis consisting of open compact subgroups.*

Proof. "If" is easy. The other direction is in [DE09, Proposition 4.1.5]. □

This justifies the terminology: A topological group is profinite if and only it is compact and locally profinite.

Definition. *Let G be a locally profinite group. A function $f : G \to \mathbb{C}$ is* smooth *if it is locally constant. The vector space of smooth (resp. smooth and compactly supported) functions on G is denoted $C^\infty(G)$ (resp. $C_c^\infty(G)$).*

1.2. **Representations of Locally Profinite Groups.** Let G be a locally profinite group.

Definition. *A representation (V, π) of G is a complex vector space V, together with a group homomorphism $\pi : G \to \mathrm{Aut}_\mathbb{C}(V)$. If $K \subseteq G$ is a subgroup, let V^K be the set of K-fixed vectors. (V, π) is called*

- smooth *if every vector is fixed by some open compact subgroup (such a vector is called* smooth*), i.e. if $V = \bigcup_K V^K$ where the union runs over the open compact subgroups,*
- admissible *if it is smooth and V^K is finite-dimensional for every open compact subgroup K,*
- irreducible *if V has no proper G-invariant non-zero subspace.*

Of course we also have the usual definitions of homomorphisms of representations, subrepresentations, quotients...

Theorem 1. *Let K be a* compact *locally profinite group. Let (V, π) be a smooth representation of K. Then π is semisimple, i.e. V is the direct sum of irreducible subrepresentations. Any irreducible smooth representation of K is finite-dimensional.*

Proposition 2 (Schur's Lemma)**.** *Let G be a locally profinite group and (V, π) a smooth irreducible representation. Assume that one of the following holds:*

(1) G/K is countable for some compact open subgroup K, or

(2) π is admissible.

Then $\mathrm{End}_G(V) = \mathbb{C}$, i.e. if $T : V \to V$ is an intertwining operator, there is $\lambda \in \mathbb{C}$ such that $T = \lambda\, \mathrm{id}_V$.

Corollary 3. *Assumptions as in the previous proposition. There is a quasi-character $\omega : Z(G) \to \mathbb{C}^\times$, called the* central quasi-character of π*, such that $\pi(z)v = \omega(z)v$ for all $z \in Z(G), v \in V$.*

Corollary 4. *Assumptions as in the previous proposition. If G is abelian, then $\dim V = 1$.*

Remark. Unlike in the case of finite groups (or more generally unitarizable representations), the converse of Schur's lemma does not hold, i.e. $\mathrm{End}_G(V) = \mathbb{C}$ does not imply that V is irreducible. For example if F is a local nonarchimedean field, χ a quasi-character of $F^\times$, then the principal series representation $(V, \pi) = \mathcal{B}(\chi, \chi\,|\cdot|)$ is reducible (Theorem 23), but $\dim_\mathbb{C} \mathrm{End}_G V = 1$ (Theorem 25).

Definition. *Let G be a locally profinite group, K a compact open subgroup. We denote by $\widehat{K}$ the set of equivalence classes of irreducible smooth representations of K. Let (V, π) be a smooth representation of G. If $\rho \in \widehat{K}$, denote by V^ρ the sum of all subspaces of V which are isomorphic to ρ as K-representations. We call it the* ρ-isotypic component *of (V, π).*

Theorem 5 ([Bum97, Proposition 4.2.2])**.** *In the setup as in the definition we have*

$$V = \bigoplus_{\rho \in \widehat{K}} V^\rho.$$

V is admissible if and only if each V^ρ is finite-dimensional.

Definition. *Let G be locally profinite and (V,π) a smooth representation. The contragredient representation $(\widehat{V},\widehat{\pi})$ is the representation of G where $\widehat{V}$ is the space of smooth vectors in the algebraic dual of (V,π), i.e.*

$$\widehat{V}=\{f:V\to\mathbb{C} \text{ linear}: \exists \text{compact open subgroup } K\subseteq G \text{ with } f(kv)=f(v) \text{ for all } k\in K, v\in V\},$$

and $\widehat{\pi}$ acts on this space by $(\widehat{\pi}(g)f)(v)=f(\pi(g^{-1})v)$.

If $f\in\widehat{V}, v\in V$ we also denote $f(v)$ by $\langle v,f\rangle$. Then $\langle \pi(g^{-1})v,f\rangle=\langle v,\pi(g)f\rangle$.

Definition. *Let (V,π) be a smooth representation of G. A* matrix coefficient *of π is a function $G\to\mathbb{C}$ of the form $g\mapsto\langle\pi(g)v,\widehat{v}\rangle$ with $v\in V,\widehat{v}\in\widehat{V}$.*

Proposition 6. *Let (V,π) be an admissible representation of G and K a compact open subgroup. Then the pairing between $V,\widehat{V}$ induces a non-degenerate pairing between V^K and $\widehat{V}^K$, so we can naturally identify $(V^K)^*=\widehat{V}^K$. $\widehat{V}$ is admissible and the natural map $V\to\widehat{\widehat{V}}$ is an isomorphism.*

Let G be a locally profinite group and $H\subseteq G$ a closed subgroup. Denote by δ_G,δ_H the modular quasi-characters of G,H respectively. Let (U,σ) be a smooth representation of H. This induces two representations of G in the following way: Let V the vector space of functions $f:G\to U$ satisfying

(i) $f(hg)=\delta_G(h)^{-1/2}\delta_H(h)^{1/2}f(g)$ for $h\in H, g\in G$.

(ii) There is a compact open subgroup $K\subseteq G$ such that $f(gk)=f(g)$ for $g\in G,k\in K$.

Let V_c be the subspace of V of functions f additionally satisfying

(iii) f has compact support modulo H, i.e. the image of the support of f is compact in G/H.

Letting G act on V (resp. V_c) gives us a representation, denoted $\operatorname{Ind}_H^G\sigma$ (resp. $c\text{-}\operatorname{Ind}_H^G\sigma$) and called the *induced representation* (resp. *induced representation with compact support*). Both $\operatorname{Ind}_H^G\sigma$ and $c\text{-}\operatorname{Ind}_H^G\sigma$ are smooth representations of G. Note that if G/H is compact, then they coincide.

Remark. In [BH06] the notation is slightly different, there this would be denoted $\boldsymbol{\iota}_H^G\sigma=\operatorname{Ind}_H^G(\delta_G^{-1/2}|_H\otimes\delta_H^{1/2}\otimes\sigma)$. The inclusion of the modular quasi-characters has the advantage that $c\text{-}\operatorname{Ind}_H^G\sigma$ will be unitarizable if σ is (see e.g. Theorem 27) and it behaves nicely under taking the contragredient, see Theorem 8.

Theorem 7 (Frobenius reciprocity, [BH06, 2.4, 2.5])**.** *Let $(V,\pi),(U,\sigma)$ be smooth representations of G,H respectively. Then there are canonical isomorphisms*

$$\operatorname{Hom}_G(\pi,\operatorname{Ind}_H^G\sigma)\cong\operatorname{Hom}_H(\pi|_H,\sigma\otimes\delta_G^{-1/2}\delta_H^{1/2}),$$
$$\operatorname{Hom}_G(c\text{-}\operatorname{Ind}_H^G\sigma,\pi)\cong\operatorname{Hom}_H(\sigma\otimes\delta_G^{-1/2}\delta_H^{1/2},\pi|_H).$$

Theorem 8 ([BH06, 3.5])**.** *Let (U,σ) be a smooth representation of H. Then there is an isomorphism*

$$\widehat{c\text{-}\operatorname{Ind}_H^G\sigma}\cong\operatorname{Ind}_H^G\widehat{\sigma}.$$

Let G be a locally profinite unimodular group and fix a Haar measure $\mathrm{d}g = \mu$. The *Hecke algebra* of G is $\mathcal{H} = \mathcal{H}(G) = C_c^\infty(G)$ the space of compactly supported locally constant functions on G. If $f_1, f_2 \in \mathcal{H}$, define the convolution $f_1 * f_2$ by

$$(f_1 * f_2)(g) = \int_G f_1(h) f_2(h^{-1}g)\,\mathrm{d}h.$$

Then $\mathcal{H}$ becomes an algebra with the convolution as multiplication. If (V, π) is a smooth representation of G, then V becomes a module over $\mathcal{H}$ via

$$\pi(f)v := \int_G f(g)\pi(g)v\,\mathrm{d}g$$

where $f \in \mathcal{H}(G), v \in V$. To make sense of the integral one can note that that it is really a finite sum, since f is locally constant of compact support and v is fixed by an open compact subgroup of G. If K is a compact open subgroup, let $\mathcal{H}_K$ be the subalgebra of $\mathcal{H}$ consisting of those functions that are biinvariant under K. Given $\rho \in \widehat{K}$, define a function $e_\rho \in \mathcal{H}$ by $e_\rho(k) = \frac{\dim \rho}{\mu(K)} \operatorname{Tr} \rho(k^{-1})$ when $k \in K$ and $e_\rho(k) = 0$ otherwise. If ρ is the trivial representation we also denote e_ρ by e_K. It is $\frac{1}{\mu(K)}$ times the characteristic function of K. We then have $\mathcal{H}_K = e_K * \mathcal{H} * e_K$ and $\mathcal{H}_K$ is a unital algebra with unit e_K.

Theorem 9 ([BH06, Proposition 4.4])**.** *Let (V, π) be a smooth representation of G. Then $\pi(e_\rho)$ is the projection $V = \bigoplus_{\rho' \in \widehat{K}} V^{\rho'} \to V^\rho$.*

For the next definition, note that if $T : V \to V$ is an endomorphism of a vector space V with finite-dimensional image W, then we may define the trace of T by $\operatorname{Tr} T := \operatorname{Tr} T|_{W' \to W'}$ where $W' \subseteq V$ is any finite-dimensional subspace containing W. A *distribution* on a locally compact totally disconnected space X is a linear functional $C_c^\infty(X) \to \mathbb{C}$.[1]

Definition. *Let (V, π) be an admissible representation of G. The* character *of π is the distribution $\operatorname{Tr} \pi : C_c^\infty(G) \to \mathbb{C}$, defined by*

$$\operatorname{Tr} \pi(f) = \operatorname{Tr}(\pi(f) : V \to V).$$

Note that if $f \in C_c^\infty(G)$, then $\pi(f)$ has finite rank, so the trace is well-defined by the remark before the definition.

Theorem 10 ([JL70, Lemma 7.1])**.** *Let $(V_1, \pi_1), \ldots, (V_n, \pi_n)$ be pairwise non-isomorphic irreducible admissible representation of G. Then their characters $\operatorname{Tr} \pi_1, \ldots, \operatorname{Tr} \pi_n$ are linearly independent.*

Note that if $0 \to \pi' \to \pi \to \pi'' \to 0$ is a short exact sequence of admissible representations, then $\operatorname{Tr} \pi = \operatorname{Tr} \pi' + \operatorname{Tr} \pi''$. Together with the theorem this easily implies

Corollary 11 ([Cas+08, Corollary 2.3.3])**.** *Let $(V, \pi), (V', \pi')$ be admissible representations of G of finite length. Then the irreducible composition factors and their multiplicities of π, π' coincide (i.e. π, π' have isomorphic semisimplifications) if and only if $\operatorname{Tr} \pi = \operatorname{Tr} \pi'$.*

[1] Note that unlike in the analytic case no continuity restriction is placed on such functionals.

1.3. **Matrix Decompositions.** Let F be any field and $G(F) = \mathrm{GL}_2(F)$. We introduce the following subgroups of G:

(Standard Borel subgroup) $$B(F) := \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$$

(Mirabolic group) $$M(F) := \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \right\}$$

(Standard unipotent group) $$N(F) := \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$$

(Standard split maximal torus) $$T(F) := \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}$$

$$Z(F) := Z(G(F)) = F^\times I_2.$$

We will usually drop the F in the notation. Note that $B = NT = TN = ZM$. We note the following matrices

$$w_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$w_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We have the *Bruhat decomposition*

$$G = B \dot\cup B w_0 B.$$

Now assume that F is a nonarchimedean local field. Let $\mathcal{O}_F$ be its ring of integers and ϖ a uniformizer. Let $K = \mathrm{GL}_2(\mathcal{O}_F)$. This is a compact open subgroup of G and every compact subgroup of G is conjugate to a subgroup of K. We have the following decompositions:

(Iwasawa decomposition) $$G = BK$$

(Cartan decomposition) $$G = \dot\bigcup_{n_1 \geq n_2} K \begin{pmatrix} \varpi^{n_1} & 0 \\ 0 & \varpi^{n_2} \end{pmatrix} K$$

1.4. **Haar Measures.** Let F be a local nonarchimedean field. We introduce the Haar measures on the various matrix groups from the previous section.

- G G is unimodular and the Haar measure is up to scalar given by $\frac{\mathrm{d}x}{|\det x|^2}$ where $\mathrm{d}x$ is the Haar measure on the additive group $M_{2\times 2}(F) \cong F^4$.
- B Write an element of B as $b = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}$. Then a right Haar measure on B is given by $\mathrm{d}_R b = \mathrm{d}x \mathrm{d}^\times y_1 \mathrm{d}^\times y_2$ and left Haar measure by $\mathrm{d}_L b = |y_2/y_1| \mathrm{d}_R b$. The modular function is $\delta(b) = \delta_B(b) = |y_1/y_2|$.

2. Representations of GL_2

In this section, let F be a local nonarchimedean field and $G = \mathrm{GL}_2(F)$. We will also use the matrix groups introduced in 1.3. The residue field cardinality is denoted by q.

2.1. Generalities.

Proposition 12. *Let (V, π) be a finite-dimensional irreducible smooth representation of G. Then V is one-dimensional and there is a quasi-character χ of $F^\times$ such that $\pi = \chi \circ \det$.*

Proof sketch. Since V is finite-dimensional $\ker \pi$ is an open normal subgroup. One can show that every open normal subgroup of G contains $\mathrm{SL}_2(F)$ from which the result follows. □

In the following given a quasi-character ϕ of $F^\times$ we identify it with a quasi-character of G by composition with det, but we will omit this from the notation and just write χ.

Theorem 13. *Let (V, π) be an irreducible admissible representation of G. The contragredient representation $\widehat{\pi}$ is isomorphic to the representation $g \mapsto \pi(g^{-T})$ on V. If ω is the central quasi-character, then it is also isomorphic to $\omega^{-1} \otimes \pi$.*

Theorem 14 ([JL70, Theorem 7.7])**.** *Let (V, π) be an irreducible admissible representation of G. Then its character $\mathrm{Tr}\,\pi$ is represented by a function in the following sense: There is a continuous function $\chi_\pi : G \to \mathbb{C}$ such that for every $f \in C_c^\infty(G)$ we have*

$$\mathrm{Tr}\,\pi(f) = \int_G f(g)\chi_\pi(g)\,\mathrm{d}g.$$

Let ψ be an additive character of F.

Definition. *Let (V, π) be a smooth representation of N. Let $V(\psi)$ denote the subspace of V spanned by elements of the form $\pi(n)v - \psi(n)v$ with $n \in N, v \in V$. The quotient is $V_\psi := V/V(\psi)$. If ψ is the trivial character we also write $V(N), V_N$ for $V(\psi), V_\psi$ and call $J(V) := V_N$ the* Jacquet module *of V.*

Note that V_N is the module of coinvariants, or the 0-th homology group $H_0(N, V)$.

Proposition 15. *Let (V, π) be a smooth representation of N and $v \in V$. Then $v \in V(\psi)$ if and only if*

$$\int_{\mathfrak{p}^{-n}} \psi(-x)\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) v\,\mathrm{d}x = 0$$

for sufficiently large n.

Note that if the integral is zero for some n_0, it will be zero for all $n \geq n_0$ as well (split it up into cosets $\mathfrak{p}^{-n}/\mathfrak{p}^{-n_0}$).

Corollary 16. *The functor $(V, \pi) \mapsto V_\psi$ is exact.*

Proof. It is clearly right exact and the proposition implies it is left exact. □

If (V, π) is a representation of B, note that $V(N)$ is an invariant subspace for T, so $J(V)$ is a T-module.

Theorem 17 ([Bum97, Theorem 4.4.4])**.** *Suppose (V, π) is an admissible representation of G. Then $J(V)$ is admissible as a representation of T.*

Theorem 18. *Suppose (V, π) is an irreducible admissible representation of G. Then $J(V)$ is at most two-dimensional.*

Proof. This requires some techniques from later sections. If $J(V) = 0$, there is nothing to show. Otherwise π is isomorphic to a subrepresentation of a principal series representation $\mathcal{B}(\chi_1, \chi_2)$ by Theorem 34 where χ_1, χ_2 are quasi-characters of $F^\times$. By exactness of the Jacquet functor it thus suffices to prove the statement for $\mathcal{B}(\chi_1, \chi_2)$. We make use of the following general fact:

Lemma 19 ([BH06, Restriction-Induction Lemma 9.3])**.** *Let (U, σ) be a smooth representation of T which we view as a representation of B via inflation. Let $(V, \pi) = \operatorname{Ind}_B^G \sigma$. Then we have a short exact sequence of representations of T:*

$$0 \to \sigma^w \otimes \delta_B|_T^{1/2} \to J(V) = \pi_N \to \sigma \otimes \delta_B|_T^{1/2} \to 0,$$ [2]

where $\sigma^w(t) = \sigma(wtw^{-1})$ with $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The right map is given by $f \mapsto f(1)$.

We apply this with σ the one-dimensional representation of T given by $\chi_1 \otimes \chi_2$ so that $(V, \pi) = \mathcal{B}(\chi_1, \chi_2)$. Then we get a short exact sequence of vector spaces

$$0 \to \mathbb{C} \to J(V) \to \mathbb{C} \to 0$$

and the claim follows. □

Now fix a *non-trivial* additive character ψ of F.

Definition. *Let (V, π) be a smooth representation of G. A* Whittaker functional *on V is a linear map $\Lambda : V \to \mathbb{C}$ satisfying $\Lambda(\pi(u)v) = \psi(u)\Lambda(v)$ for all $u \in N$.*

Equivalently a Whittaker functional is a linear functional on $V/V(\psi) = V_\psi$.

Theorem 20. *Let (V, π) be an irreducible admissible representation of G. Then the dimension of the space of Whittaker functionals on V is 1 if V is infinite-dimensional and 0 if V is one-dimensional.*

Let (V, π) be an infinite-dimensional irreducible admissible representation of G. Fix a non-zero Whittaker functional Λ on V. For $v \in V$ define the function $W_v : G \to \mathbb{C}$ by

$$W_v(g) = \Lambda(\pi(g)v).$$

Let $\mathcal{W} = \{W_v \mid v \in V\}$. This is a vector space of functions on G. Since $W_{\pi(g)v}(h) = W_v(hg)$, G acts on it by right translation and the map $V \to \mathcal{W}, v \mapsto W_v$ is an isomorphism of representations. $\mathcal{W}$ is the *Whittaker model* of π. Note that $\mathcal{W}$ is a subrepresentation of the induction $\operatorname{Ind}_N^G \psi$.

Similarly we can identify π with a space of functions on $F^\times$: For $v \in V$ define the function $\phi_v : F^\times \to \mathbb{C}$ by

$$\phi_v(a) = W_v\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = \Lambda\left(\pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)v\right).$$

Let $\mathcal{K}$ be the space of functions of this form. Then the map $V \to \mathcal{K}, v \mapsto \phi_v$ is bijective (surjective by definition, injective is not obvious) and thus identifies V with a representation of G on a space of

[2]Note that in [BH06] they use another convention for the modular quasi-character, our δ_B is their δ_B^{-1}.

functions on $F^\times$. This is called the *Kirillov model* of π. We can't easily describe the full action of G on $\mathcal{K}$, but the action of special subgroups is as follows: If $\phi \in \mathcal{K}$, then

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\phi(x) = \phi(ax), \tag{$*$}$$

$$\pi\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right)\phi(x) = \psi(bx)\phi(x), \tag{$**$}$$

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right)\phi(x) = \omega(a)\phi(x).$$

for $a \in F^\times, b \in F$ where ω is the central character of V. This tells us how B acts on $\mathcal{K}$. By the Bruhat decomposition, it would be be enough to know how w acts on $\mathcal{K}$ in order to understand the full action of G on $\mathcal{K}$.

Let $\phi \in \mathcal{K}$. $\phi(ax) = \pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\phi(x) = \phi(x)$ for $a \in F^\times$ close to 1 since the representation is smooth. Therefore $\mathcal{K} \subseteq C^\infty(G)$. Furthermore, if b is close to 0 we also have $\phi(x) = \psi(bx)\phi(x)$ for all $x \in F^\times$. If $|x|$ is large enough, we can find b close to 0 such that $\psi(bx) \neq 1$ (this is possible since ψ is non-trivial), so $\phi(x) = 0$ for $|x|$ large enough.

Theorem 21. *The Kirillov model contains $C_c^\infty(F^\times)$, in fact this is the kernel of the projection $V \to J(V)$ onto the Jacquet module.*

Proof. We may assume that $V = \mathcal{K}$ is equal to its Kirillov model. By definition the kernel of $V \to J(V)$ is $V(N)$, the subspace generated by elements of the form $\pi\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right)\phi - \phi$ with $b \in F$, $\phi \in V$. We have $\pi\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right)\phi(x) - \phi(x) = (\psi(bx) - 1)\phi(x)$. Since ψ is continuous, this is 0 if $|x|$ is small. Since we already saw above that all functions in V are locally constant and vanish for $|x|$ large, this shows that $V(N) \subseteq C_c^\infty(F^\times)$. By Theorem 18, $V(N) \neq 0$. Both $V(N)$ and $C_c^\infty(F^\times)$ are modules for $M = \left\{\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}\right\}$ via $(*)$ and $(**)$. It is not too difficult to show that $C_c^\infty(F^\times)$ is irreducible, hence $V(N) = C_c^\infty(F^\times)$. ∎

2.2. **Principal Series Representations.** Let χ_1, χ_2 be quasi-characters of $F^\times$ and view $\chi = \chi_1 \otimes \chi_2$ as a quasi-character of T and via the the quotient map $B \to B/N = T$ also as a quasi-character of B.

Definition. *A* principal series representation *of G is a representation of the form $\mathcal{B}(\chi_1, \chi_2) := \mathrm{Ind}_B^G(\chi)$.*

Explicitly the space $\mathcal{B}(\chi_1, \chi_2)$ consists of all locally constant functions $f : G \to \mathbb{C}$ satisfying

$$f\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g\right) = \delta\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}\right)\chi\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}\right) f(g) = \left|\frac{a}{b}\right|^{1/2} \chi_1(a)\chi_2(b)f(g)$$

for all $a, b \in F^\times, x \in F, g \in G$, and G acts on this space by right translation. Because of the Iwasawa decomposition $G = BK$, $f \in \mathcal{B}(\chi_1, \chi_2)$ is uniquely determined by its restriction to K. Since K is compact, this easily implies that $\mathcal{B}(\chi_1, \chi_2)$ is admissible.

Note that if χ_0 is another quasi-character of $F^\times$, we have $\mathcal{B}(\chi_0\chi_1, \chi_0\chi_2) = \chi_0 \otimes \mathcal{B}(\chi_1, \chi_2)$.

Remark. Write $\chi_j = |\cdot|^{s_j} \xi_j$ with ξ_j unitary characters of $F^\times$. In [GH11] this principal series representation is denoted $\mathcal{V}((s_1 + \frac{1}{2}, s_2 - \frac{1}{2}), (\xi_1, \xi_2))$.

Theorem 22. *Let χ_1, χ_2 be quasi-characters of $F^\times$. The pairing*

$$\mathcal{B}(\chi_1, \chi_2) \times \mathcal{B}(\chi_1^{-1}, \chi_2^{-1}) \longrightarrow \mathbb{C}$$
$$f, g \longmapsto \int_K f(k)g(k)\,\mathrm{d}k$$

is invariant and induces an isomorphism of $\mathcal{B}(\chi_1^{-1}, \chi_2^{-1})$ with the contragredient representation of $\mathcal{B}(\chi_1, \chi_2)$.

Theorem 23. *Let χ_1, χ_2 be quasi-characters of $F^\times$. The principal series representation $\mathcal{B}(\chi_1, \chi_2)$ is irreducible except when $\chi_1\chi_2^{-1} = |\cdot|^{\pm 1}$. If $\chi_1\chi_2^{-1} = |\cdot|^{-1}$, then $\mathcal{B}(\chi_1, \chi_2)$ has a one-dimensional subrepresentation and the quotient is irreducible, if $\chi_1\chi_2^{-1} = |\cdot|$, then $\mathcal{B}(\chi_1, \chi_2)$ has an irreducible codimension one subrepresentation.*

In the case $\chi_1\chi_2^{-1} = |\cdot|^{-1}$, the one-dimensional subrepresentation is spanned by

$$f(g) = \chi_1(\det g)\,|\det g|^{1/2}.$$

Conversely, every one-dimensional representation $\chi \circ \det$ of G occurs as a subspace of a principal series representation simply take $\chi_1 = \chi\,|\cdot|^{-1/2}$, $\chi_2 = \chi\,|\cdot|^{1/2}$.

If $\chi_1\chi_2^{-1} = |\cdot|^{\pm 1}$, the infinite-dimensional irreducible factor in the composition series of $\mathcal{B}(\chi_1, \chi_2)$ is called a *special representation* of G and is denoted $\sigma(\chi_1, \chi_2)$.

Theorem 24 (Jacquet module of a principal series representation)**.** *Let χ_1, χ_2 be quasi-characters of $F^\times$. Let $(V, \pi) = \mathcal{B}(\chi_1, \chi_2)$. Then $J(V)$ is two-dimensional and the T-module structure is given as follows. Let $\chi = \chi_1 \otimes \chi_2, \chi' = \chi_2 \otimes \chi_1$ the characters of T. If $\chi_1 \neq \chi_2$, then $J(V) \cong \delta^{1/2} \otimes (\chi \oplus \chi')$. If $\chi_1 = \chi_2$, then $J(V)$ is isomorphic to the representation given by*

$$t \mapsto (\delta^{1/2}\chi)(t) \begin{pmatrix} 1 & v(t_1/t_2) \\ 0 & 1 \end{pmatrix}$$

where $v : F^\times \to \mathbb{Z}$ is the valuation map.

Proof. In the notation of Lemma 19 we have a short exact sequence

$$0 \to \chi^w \otimes \delta^{1/2} \to J(V) \to \chi \otimes \delta^{1/2} \to 0.$$

Note that $\chi^w = \chi'$. If $\chi_1 \neq \chi_2$, we have $\chi^w \otimes \delta^{1/2} \neq \chi \otimes \delta^{1/2}$ and it is not difficult to see that this implies that $J(V) \cong (\chi^w \otimes \delta^{1/2}) \oplus (\chi \otimes \delta^{1/2})$.[3] For the other case see [Bum97, Theorem 4.5.4]. □

[3]Suppose G is an abelian group and (V, π) a two-dimensional representation that fits into a short exact sequence $0 \to \chi_1 \to V \to \chi_2 \to 0$ with χ_1, χ_2 *distinct* one-dimensional representations. Then $V \cong \chi_1 \oplus \chi_2$. Indeed, fix $g_0 \in G$ such that $\chi_1(g_0) \neq \chi_2(g_0)$. Then $\pi(g_0)$ has matrix $\begin{pmatrix} \chi_1(g_0) & * \\ 0 & \chi_2(g_0) \end{pmatrix}$ in some basis, so it has distinct eigenvalues $\chi_1(g_0), \chi_2(g_0)$ and is thus diagonalizable. Since all the operators $\pi(g)$, $g \in G$, commute with $\pi(g_0)$, they preserve the one-dimensional eigenspaces and that implies that $V \cong \chi_1 \oplus \chi_2$.

Theorem 25. *Let $\chi_1, \chi_2, \mu_1, \mu_2$ be quasi-characters of $F^\times$. Then*

$$\dim_{\mathbb{C}} \operatorname{Hom}_G(\mathcal{B}(\chi_1, \chi_2), \mathcal{B}(\mu_1, \mu_2)) = \begin{cases} 1 & \text{if } (\chi_1, \chi_2) = (\mu_1, \mu_2) \text{ or } (\chi_1, \chi_2) = (\mu_2, \mu_1), \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 26. *Assume $\mathcal{B}(\chi_1, \chi_2) \cong \mathcal{B}(\mu_1, \mu_2)$. Then either $(\chi_1, \chi_2) = (\mu_1, \mu_2)$ or $(\chi_1, \chi_2) = (\mu_2, \mu_1)$. If $\mathcal{B}(\chi_1, \chi_2)$ is irreducible, then $\mathcal{B}(\chi_1, \chi_2) \cong \mathcal{B}(\chi_2, \chi_1)$.*

Proof sketch of Theorem 25. Write $\mu = \mu_1 \otimes \mu_2, \chi = \chi_1 \otimes \chi_2$ for the quasi-characters on T and also for the quasi-characters on B obtained by inflation. By Frobenius reciprocity we have

$$\operatorname{Hom}_G(\mathcal{B}(\chi_1, \chi_2), \mathcal{B}(\mu_1, \mu_2)) = \operatorname{Hom}_B(\mathcal{B}(\chi_1, \chi_2)|_B, \mu \otimes \delta^{1/2}).$$

Since N acts trivially on μ, we get $\operatorname{Hom}_B(\mathcal{B}(\chi_1, \chi_2)|_B, \mu \otimes \delta^{1/2}) = \operatorname{Hom}_T(\mathcal{B}(\chi_1, \chi_2)_N, \mu \otimes \delta^{1/2})$ as the Jacquet module $\mathcal{B}(\chi_1, \chi_2)_N$ is the largest quotient on which N acts trivially. If $\chi_1 \neq \chi_2$, we have $\mathcal{B}(\chi_1, \chi_2)_N \cong \delta^{1/2} \otimes (\chi \oplus \chi')$ with $\chi' = \chi_2 \otimes \chi_1$. In this case the result immediately follows. Next assume $\chi_1 = \chi_2$. Then [BH06] says in Proposition 9.10 that "in this case, $\operatorname{Ind} \chi$ is irreducible and the result again follows", but it is not clear to me why that is the case. Note that if we knew the conclusion of the previous theorem on the structure of $\mathcal{B}(\chi_1, \chi_2)_N$ in the case $\chi_1 = \chi_2$, it would be easy. However, the proof in [Bum97] of that result makes use of the current theorem we are proving (or rather its corollary). □

[Bum97] approaches this differently. He proves the first part of the corollary using distributions and the second part as follows: We may assume $\chi_1 \neq \chi_2$. Write $\chi_j = |\cdot|^{s_j} \xi_j$ where ξ_j is a unitary character. Given $f \in \mathcal{B}(\chi_1, \chi_2)$, define

$$(Mf)(g) := \int_F f\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \mathrm{d}x.$$

If $\operatorname{Re}(s_1 - s_2) > 0$, this integral converges and then defines a non-zero intertwining operator $M : \mathcal{B}(\chi_1, \chi_2) \to \mathcal{B}(\chi_2, \chi_1)$. The integral can be analytically continued to all s_1, s_2 (except for $\chi_1 = \chi_2$, i.e. if $\xi_1 = \xi_2$, there will be a singularity at (s_1, s_2) with $q^{s_1 - s_2} = 1$), hence giving a non-zero intertwining operator in the remaining cases as well. To make sense of the analytic continuation we define *flat* sections: Fix the unitary characters ξ_1, ξ_2 and let $(V_{s_1,s_2}, \pi_{s_1,s_2}) = \mathcal{B}(|\cdot|^{s_1} \xi_1, |\cdot|^{s_2} \xi_2)$. A flat section is a mapping that associates to each $(s_1, s_2) \in \mathbb{C}^2$ a function $f_{s_1,s_2} \in V_{s_1,s_2}$ such that $f_{s_1,s_2}|_K$ is independent of s_1, s_2. For every function $f : K \to \mathbb{C}$ satisfying

$$f\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} k\right) = \xi_1(a)\xi_2(b)f(k),$$

for $a, b \in \mathcal{O}_F^\times, x \in \mathcal{O}_F, k \in K$, there is a unique flat section $(s_1, s_2) \mapsto f_{s_1,s_2}$ such that f is the restriction of the functions in the flat section to K. By analytic continuation of Mf we now mean that for every flat section the function $(s_1, s_2) \mapsto (Mf_{s_1,s_2})(g)$ admits an analytic continuation. Then M extends to a non-zero intertwining operator $V_{s_1,s_2} \to \mathcal{B}(|\cdot|^{s_2} \xi_2, |\cdot|^{s_1} \xi_1)$.

Useful matrix identity to keep in mind:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x^{-1} & -1 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}.$$

Theorem (Kirillov model of a principal series representation)**.** *Let χ_1, χ_2 be quasi-characters of $F^\times$ such that $\chi_1\chi_2 \neq |\cdot|^{\pm 1}$. Let V be the Kirillov model of $\mathcal{B}(\chi_1, \chi_2)$. Define $\phi_j : F^\times \to \mathbb{C}$ by $\phi_j(t) = |t|^{1/2}\chi_j(t)$ for $|t| < 1$ and $\phi_j(t) = 0$ otherwise, for $j = 1, 2$. If $\chi_1 \neq \chi_2$, then*

$$V = \mathbb{C}\phi_1 + \mathbb{C}\phi_2 + C_c^\infty(F^\times),$$

and if $\chi_1 = \chi_2$, then

$$V = \mathbb{C}\phi_1 + \mathbb{C}v\phi_1 + C_c^\infty(F^\times),$$

where $v : F^\times \to \mathbb{C}$ is the valuation map.

Proof sketch. We know from Theorem 24 that $C_c^\infty(F^\times)$ is of codimension 2 in V. So it suffices to prove that V is contained in the right hand side of the equation. For that also use Theorem 24. □

Theorem 27 ([Bum97, Theorem 4.6.7])**.** *Let χ_1, χ_2 be quasi-characters of $F^\times$. Then $\mathcal{B}(\chi_1, \chi_2)$ is unitarizable if and only if both χ_1 and χ_2 are unitary, or there is a unitary character ξ and a real number $-\frac{1}{2} < s < \frac{1}{2}$ such that $\chi_1 = |\cdot|^s \xi, \chi_2 = |\cdot|^{-s}\xi$.*

We can write down an explicit expression for the Whittaker functional on the principal series representation. So let ψ be a non-trivial additive character of F, χ_1, χ_2 be quasi-characters on $F^\times$ and write $\chi_j = |\cdot|^{s_j}\xi_j$ with ξ_j unitary. Define $\Lambda : \mathcal{B}(\chi_1, \chi_2) \to \mathbb{C}$ by

$$\Lambda(f) = \int_F f\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\psi(-x)\,\mathrm{d}x.$$

If $\mathrm{Re}(s_1 - s_2) > 0$ this converges absolutely and defines a Whittaker functional on $\mathcal{B}(\chi_1, \chi_2)$. This integral can be analytically continued to all s_1, s_2.

2.3. **Spherical Representations.** Let (V, π) be an irreducible admissible representation of G. Recall that $K = \mathrm{GL}_2(\mathcal{O}_F)$ is the standard maximal compact subgroup of G.

Definition. *π is called* spherical *if there is a non-zero K-fixed vector, i.e. if $V^K \neq 0$. Such a vector is called* spherical.

Proposition 28. *If π is spherical, so is $\widehat{\pi}$.*

Proof. This follows from Theorem 13. □

Recall that $\mathcal{H}_K$ is the space of locally constant compactly supported K-biinvariant functions $G \to \mathbb{C}$. It is an algebra under convolution. Matrix involution induces an involution on $\mathcal{H}_K$ and the (Cartan decomposition) implies that it must be the identity, so $\mathcal{H}_K$ is commutative. As a $\mathbb{C}$-algebra, $\mathcal{H}_K$ is generated by T, R, R^{-1} where T, R are the characteristic functions of $K\begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}K$ and $K\begin{pmatrix} \varpi & 0 \\ 0 & \varpi \end{pmatrix}K$ respectively.

Theorem 29. *If π is spherical,* $\dim V^K = 1$, *so a spherical vector is unique up to scalars.*

Proof. V^K is a finite dimensional simple module for the commutative ring $\mathcal{H}_K$. □

Denote by v_K any spherical vector in V (unique up to scalars by the theorem). Then there is a homomorphism $\xi : \mathcal{H}_K \to \mathbb{C}$ such that $\pi(\phi)v_K = \xi(\phi)v_K$ for $\phi \in \mathcal{H}_K$. This is the *character* of $\mathcal{H}_K$ associated to π.

Theorem 30. *Two irreducible admissible spherical representations are isomorphic if and only if the corresponding characters of $\mathcal{H}_K$ coincide.*

Example. A finite-dimensional irreducible admissible representation of G is one-dimensional and of the form $\chi \circ \det$ for a quasi-character χ of $F^\times$. It is spherical if and only if χ is unramified, i.e. trivial on $\mathcal{O}_F^\times$.

Example. Let χ_1, χ_2 be unitary unramified quasi-characters of $F^\times$ (hence of the form $|\cdot|^s$). Assume that $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}$, so that $(V,\pi) = \mathcal{B}(\chi_1,\chi_2)$ is irreducible. Write χ for the character of B. Consider the function $\phi_K : G \to \mathbb{C}$ defined by

$$\phi_K(g) = (\delta^{1/2}\chi)(b)$$

where we write $g = bk$ with $b \in B, k \in K$. This is independent of the choice of b, k. Then ϕ_K is a spherical vector in V.

Let $\alpha_1 = \chi_1(\varpi), \alpha_2 = \chi_2(\varpi)$. Since χ_1, χ_2 are unramified, these numbers determine the quasi-characters uniquely. To find the character ξ of $\mathcal{H}_K$ for this spherical representation, it suffices to know $\xi(T), \xi(R)$ which are given by:

Proposition 31. *Notation as above, we have $\pi(T)\phi_K = \lambda\phi_K$ and $\pi(R) = \mu\phi_K$ where*

$$\lambda = q^{1/2}(\alpha_1 + \alpha_2),$$
$$\mu = \alpha_1\alpha_2.$$

Proof. Evaluate both sides of $\pi(T)\phi_K = \lambda\phi_K$ at $I_2 \in G$ to get

$$\lambda = (\pi(T)\phi_K)(I_2) = \int_{K\begin{pmatrix}\varpi & 0\\ 0 & 1\end{pmatrix}K} \phi_K(g)\,\mathrm{d}g.$$

Then split it up into left cosets mod K and use explicit Hermite normal form coset representatives to compute this. Same for $\pi(R)\phi_K$. □

In fact the above two examples are exhaustive:

Theorem 32. *Let (V,π) be an irreducible admissible spherical representation of G. Then π is isomorphic to one of the two examples above.*

Proof. We make use of Theorem 30. Let ξ be the character of $\mathcal{H}_K$. Let $\lambda = \xi(T), \mu = \xi(R)$. Let α_1, α_2 be the roots of $X^2 - q^{1/2}\lambda X + \mu = 0$ and χ_1, χ_2 the unramified quasi-characters of $F^\times$ with $\chi_j(\varpi) = \alpha_j$. If $\mathcal{B}(\chi_1,\chi_2)$ is irreducible, it is spherical and the corresponding character of $\mathcal{H}_K$ is ξ by construction of α_1, α_2, the proposition and since $\mathcal{H}_K$ is generated by T, R, R^{-1}. Hence $(V,\pi) \cong \mathcal{B}(\chi_1,\chi_2)$. If $\mathcal{B}(\chi_1,\chi_2)$ is not irreducible, one argues similarly that π is isomorphic to the one-dimensional subrepresentation or quotient of $\mathcal{B}(\chi_1,\chi_2)$. □

2.4. **Supercuspidal Representations.**

Definition. *Let (V,π) be an admissible representation of G. π is called* supercuspidal *if $V_N = 0$.*

Explicitly π is supercuspidal if for every $v \in V$ there is some $n \in \mathbb{Z}$ such that $\int_{\mathfrak{p}^{-n}} \pi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) v\,\mathrm{d}x = 0$ (Proposition 15).

Theorem 33 ([Cas+08, Proposition 5.4.2])**.** *Any admissible supercuspidal representation of G is a countable direct sum of irreducible supercuspidal representations.*

Theorem 34. *Let (V,π) be an irreducible admissible representation of G. The following are equivalent:*

(i) π is supercuspidal.

(ii) π is not isomorphic to a subrepresentation of a principal series representation.

(iii) There is a matrix coefficient of π that is compactly supported modulo Z (i.e. whose support has compact image in G/Z).

(iv) The matrix coefficients of π are compactly supported modulo Z.

Proof. For "$(i) \Leftrightarrow (ii)$" use Frobenius reciprocity: $\mathrm{Hom}_G(\pi, \mathcal{B}(\chi_1,\chi_2)) = \mathrm{Hom}_B(\pi|_B, \chi\otimes\delta^{1/2}) = \mathrm{Hom}_T(\pi_N, \chi\otimes\delta^{1/2})$. This immediately gives "$(i)\Rightarrow(ii)$", for the other direction, one has to show that if $\pi_N \neq 0$, there is some quasi-character χ of T such that $\mathrm{Hom}_T(\pi_N,\chi\otimes\delta^{1/2})\neq 0$. This can be seen quickly as follows. π_N is admissible, hence so is its contragredient. Any admissible representation of $(F^\times)^k$ has a one-dimensional invariant subspace (see [Bum97, Proposition 4.2.9]), hence there exists $0\neq L\in\widehat{V_N}$ such that $L(\pi_N(t)v) = (\delta^{1/2}\chi)(t)v$ for some quasi-character χ of $T = F^\times\oplus F^\times$. Then $L\in\mathrm{Hom}_T(\pi_N,\chi\otimes\delta^{1/2})$. A different argument given in [BH06, Proposition 9.1] is to argue that V is finitely generated as a representation of G, hence V_N is finitely generated over T, and any finitely generated representation admits an irreducible quotient.

For "$(iii)\Leftrightarrow(iv)$" see [BH06, Theorem 10.2]. □

Corollary 35. *Let (V,π) be an irreducible admissible representation of G. Then π is isomorphic to exactly one of the following:*

- *A one-dimensional representation χ for some quasi-character χ of $F^\times$,*
- *A special representation $\sigma(\chi_1,\chi_2)$ for some quasi-characters χ_1,χ_2 of $F^\times$ with $\chi_1\chi_2 = |\cdot|^{\pm1}$,*
- *A principal series representation $\mathcal{B}(\chi_1,\chi_2)$ for some quasi-characters χ_1,χ_2 of $F^\times$ with $\chi_1\chi_2\neq|\cdot|^{\pm1}$,*
- *A supercuspidal representation.*

We can construct supercuspidal representations as follows. Let $k=\mathcal{O}_F/\mathfrak{m}$ be the residue field of F. Let (V_0,π_0) be a cuspidal representation[4] of $\mathrm{GL}_2(k)$. Via inflation along $\mathrm{GL}_2(\mathcal{O}_K)\to\mathrm{GL}_2(k)$ this gives a smooth representation of $K=\mathrm{GL}_2(\mathcal{O}_K)$. Let $\omega_0: k^\times\to\mathbb{C}^\times$ be the central character of π_0, we can lift it to a character of $\mathcal{O}_K^\times$ and extend it to a unitary character ω of $F^\times$. Then extend the representation π_0 on K to a representation π_1 on KZ by letting Z act via ω. Let $(V,\pi) = c\text{-}\mathrm{Ind}_{KZ}^G\,\pi_1$.

[4]An irreducible representation (V_0,π_0) of $\mathrm{GL}_2(k)$ is cuspidal if it has no $N(k)$-fixed vector (note given the decomposition $V_0 = V_0^N\oplus V_0(N)$ we see that this is analogous to the definition of supercuspidality for representation of $\mathrm{GL}_2(F)$). The cuspidal representations of $G(k)$ are obtained as follows. Let l/k be the quadratic field extension and θ a character of $l^\times$ with $\theta^q\neq\theta$. By viewing l as a two-dimensional k-vector space identify $l^\times$ with a subgroup E of $G(k)$. Let ψ a non-trivial character of $N(k)$. Define a character θ_ψ of $Z(k)N(k)$ by $\theta_\psi\left(\begin{pmatrix}a&0\\0&a\end{pmatrix}u\right)=\theta(a)\psi(u)$ for $a\in k^\times, u\in N(k)$. Then $\mathrm{Ind}_{Z(k)N(k)}^{G(k)}\theta_\psi - \mathrm{Ind}_E^{G(k)}\theta$ is an irreducible cuspidal representation. It is of dimension $q-1$ and every irreducible cuspidal representation of $G(k)$ is of this form. See [BH06, 6.4] for more. See also [Bum97, 4.1] for another method of constructing cuspidal representations of $G(k)$ via the Weil representation.

Theorem 36 ([Bum97, Theorem 4.8.1]). *The representation constructed this way is irreducible, admissible, supercuspidal and unitarizable.*

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