ADELIC FOURIER SERIES

LEONARD TOMCZAK

Contents

1. Generalities

Let F be a number field. $\mathbb{A}$ denotes the additive group of adeles over F. Recall that F sits discretely and cocompactly in $\mathbb{A}$. Let ψ be a non-trivial character of $\mathbb{A}$, i.e. a continuous homomorphism $\psi : \mathbb{A} \to \mathbb{C}^\times$ with $|\psi(a)| = 1$ for every $a \in \mathbb{A}$. We assume that ψ is trivial on F. If $a \in \mathbb{A}$, we get another character ψ_a via $\psi_a(x) = \psi(ax)$.

Theorem 1. *The map $\mathbb{A} \to \widehat{\mathbb{A}}$ given by $a \mapsto \psi_a$ is an isomorphism of topological groups. Under this isomorphism $F^\perp$ corresponds to F.*

Proof. The first part essentially follows from the corresponding fact for local fields, see [Tat67, Theorem 4.1.1]. The second part is [Tat67, Theorem 4.1.4]. □

From this we deduce that $F \cong F^\perp = \widehat{\mathbb{A}/F}$ via $F \ni k \mapsto \psi_k \in F^\perp$. Take the Haar measure on $\mathbb{A}/F$ to be normalized so that $\mathbb{A}/F$ has volume 1. Then the dual measure on F is the counting measure. From general abstract harmonic analysis, see e.g. [Fol15, Chapter 4], we get the following definition and the subsequent statements.

Definition. *Let $f \in L^1(\mathbb{A}/F)$. The* Fourier transform *of f is the function $\widehat{f} : F \to \mathbb{C}$, given by*

$$\widehat{f}(k) = \int_{\mathbb{A}/F} f(x)\psi(-kx)\,\mathrm{d}x.$$

The Fourier series *of f is*

$$\sum_{k\in F} \widehat{f}(k)\psi(kx).$$

Theorem 2 (Plancherel theorem)**.** *The map*

$$L^2(\mathbb{A}/F) \longrightarrow \ell^2(F),$$
$$f \longmapsto \widehat{f}$$

is a unitary isomorphism.

Theorem 3 (Fourier inversion)**.** *If $f \in L^2(\mathbb{A}/F)$, then $\sum_{k\in F} \widehat{f}(k)\psi(kx)$ converges in $L^2(\mathbb{A}/F)$ to f. If $f \in C(\mathbb{A}/F)$ and $\widehat{f} \in \ell^1(F)$, then $\sum_{k\in F} \widehat{f}(k)\psi(kx)$ converges uniformly to f.*

Let $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{v|\infty} F_v$ be the product of the infinite places of F. Let $\mathbb{A}_f = \prod'_{v\nmid\infty}(F_v, \mathcal{O}_v)$ be the set of finite adeles. Then we have $\mathbb{A} = F_\infty \times \mathbb{A}_f$. If $a \in \mathbb{A}$ we write $a = a_\infty a_f$ with $a_\infty \in F_\infty, a_f \in \mathbb{A}_f$.

Recall that a function on $\mathbb{R}$ or $\mathbb{C}$ is smooth if it is infinitely often differentiable, and a function on a non-archimedean local field is smooth if it is locally constant. A function $\varphi : \mathbb{A} \to \mathbb{C}$ is called *smooth* if it is smooth at every place in a uniform way, more precisely: For every $x \in \mathbb{A}$ there is an open neighborhood $x \in U \subseteq \mathbb{A}$ and a smooth function $f : F_\infty \to \mathbb{C}$ such that $\varphi(a) = f(a_\infty)$ for all $a \in U$.

Proposition 4. *Let $\varphi : \mathbb{A} \to \mathbb{C}$ be smooth and periodic, i.e. $\varphi(a + x) = \varphi(a)$ for $a \in \mathbb{A}, x \in F$, so that φ defines a function on $\mathbb{A}/F$. Then $\widehat{\varphi} \in \ell^1(F)$, so the Fourier series of φ converges uniformly to φ.*

Proof. This is basically taken from an answer by Will Sawin on Math Overflow[1]. Using that $\mathbb{A}_f/F$ is compact and φ smooth and periodic, it is easy to show that there is an open subgroup $H \subseteq \mathbb{A}_f$ such that $\varphi(a+x) = \varphi(a)$ for $a \in \mathbb{A}, x \in x \in H$. Then φ descends to a function on $\mathbb{A}/(F+H)$. Consider the map $F_\infty \to \mathbb{A}/(F + H)$. Its kernel is a lattice Λ in F_∞ and by the strong approximation theorem it is surjective. Thus $F_\infty/\Lambda \cong \mathbb{A}/(F + H)$. Pulling back the function φ gives a smooth (in the usual sense) function on $F_\infty/\Lambda \cong (\mathbb{R}/\mathbb{Z})^{[F:\mathbb{Q}]}$. It is a basic fact that on a torus the Fourier coefficients of a smooth function are absolutely summable (by integration by parts). In fact one could even better decay bounds and explicitly relate the characters on $\mathbb{A}/F$ vanishing on H with the characters on F_∞/Λ. □

2. Application: Whittaker models of cuspidal automorphic representations

We continue to fix a non-trivial character ψ on A/F.

Theorem 5 ([Bum97, Theorem 3.5.5])**.** *Let $V \subseteq \mathcal{A}_0(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}), \omega)$ be a cuspidal automorphic representation where ω is a unitary character on $\mathbb{A}^\times/F^\times$. Then V has a Whittaker model. This means there is a space $\mathcal{W}$ of "nice"[2] functions W on $\mathrm{GL}_2(\mathbb{A})$ satisfying $W\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) = \psi(x)W(g)$ for $x \in F, g \in \mathrm{GL}_2(\mathbb{A})$, such that $\mathcal{W}$ is closed under the action of the global Hecke algebra $\mathcal{H}$ and is isomorphic to V as an $\mathcal{H}$-module.*

More precisely: For $\varphi \in V$ define the function $W_\varphi : \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$ by

$$W_\varphi(g) = \int_{A/F} \varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) \psi(-x)\,\mathrm{d}x.$$

[1]https://mathoverflow.net/questions/321284/sufficient-condition-for-the-absolute-convergence-of-fourier-series-of-a-functio

[2]This means: smooth and K-finite functions of moderate growth. Smoothness is defined similarly as for functions $\mathbb{A} \to \mathbb{C}$. K-finite means that the K-right translates generate a finite dimensional vector space, where $K = \prod_v K_v$ and $K_v = \mathcal{O}_v$ if $v \nmid \infty$ and $K_v = O(2)$ resp. $U(n)$ if v is real resp. complex. Moderate growth means that $W\left(\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix} g\right)$ is bounded by a polynomial in $|y|$. See [Bum97, p. 326].

Then the space $\mathcal{W}$ of functions W_φ with $\varphi \in V$ is a Whittaker model for V, and the map $V \to \mathcal{W}, \varphi \mapsto W_\varphi$ is an isomorphism of representations. Finally the following holds:

$$\varphi(g) = \sum_{k \in F^\times} W_\varphi\left(\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

Proof. Fix $g \in \mathrm{GL}_2(\mathbb{A})$ and define the function $f : \mathbb{A} \to \mathbb{C}$ by $f(x) = \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)$. f is smooth since φ is. Since φ is invariant by $\mathrm{GL}_2(F)$ on the left, f is periodic. Therefore we can apply the theory of Fourier series to f and get that

$$f(x) = \sum_{k \in F} \widehat{f}(k)\psi(kx)$$

uniformly where

$$\widehat{f}(k) = \int_{A/F} f(x)\psi(-kx)\,\mathrm{d}x = \int_{A/F} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)\psi(-kx)\,\mathrm{d}x.$$

Since the Fourier series converges pointwise, we may plug in $x = 0$ and obtain

$$\varphi(g) = f(0) = \sum_{k \in F} \widehat{f}(k).$$

We have $\widehat{f}(0) = 0$ since φ is cuspidal. For $k \neq 0$ we have

$$\begin{aligned}
\widehat{f}(k) &= \int_{A/F} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)\psi(-kx)\,\mathrm{d}x \\
&= \int_{A/F} \varphi\left(\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)\psi(-kx)\,\mathrm{d}x \\
&= \int_{A/F} \varphi\left(\begin{pmatrix} 1 & kx \\ 0 & 1 \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} g\right)\psi(-kx)\,\mathrm{d}x \\
&= \int_{A/F} \varphi\left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} g\right)\psi(-y)\,\mathrm{d}y \\
&= W_\varphi\left(\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} g\right).
\end{aligned}$$

In the last step we used that elements k in $F^\times$ have $|k| = 1$. Therefore

$$\varphi(g) = \sum_{k \in F^\times} W_\varphi\left(\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

It is relatively easy to see that the functions W_φ are "nice" in the desired sense, that $W_\varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \psi(x)W_\varphi(g)$ and that $\varphi \mapsto W_\varphi$ is a map of representations. It remains to show that it is injective. If $\varphi \not\equiv 0$, then $\varphi(g) \neq 0$ for some $g \in \mathrm{GL}_2(\mathbb{A})$. In the above notation we then must have $\widehat{f}(k) \neq 0$ for some $k \neq 0$, and therefore $W_\varphi\left(\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} g\right) \neq 0$, so $W_\varphi \not\equiv 0$. $\square$

References

[Bum97] D. Bump. *Automorphic Forms and Representations*. Vol. 55. Cambridge Studies in Advanced Mathematics. Cambridge, GBR: Cambridge University Press, 1997.

[Fol15] G. B. Folland. *A course in abstract harmonic analysis*. Second edition. Textbooks in mathematics ; 29. 2015.

[Tat67] J. Tate. "Fourier Analysis in Number Fields and Hecke's Zeta-Functions". In: *Algebraic Number Theory*. Ed. by J. W. S. Cassels and A. Fröhlich. 1967.

Department of Mathematics, Evans Hall, University of California, Berkeley, CA 94720, USA

Email address: `leonard.tomczak@berkeley.edu`