

MATH 55 COMMON MISTAKES

Mistake 1: Proving the converse instead of the original statement.

Example:

Problem: Prove that if $n^2 + 2$ is even, then n is even.

Wrong solution: Suppose that n is even. Then $n = 2k$ for some integer k . Hence

$$n^2 + 2 = 4k^2 + 2 = 2(2k^2 + 1),$$

so $n^2 + 2$ is even.

Explanation: This proves that if n is even, then $n^2 + 2$ is even. This is the converse of the statement that was asked for, not the original statement.

Corrected solution: We prove the contrapositive. Suppose that n is odd. Then $n = 2k + 1$ for some integer k . Therefore

$$n^2 + 2 = (2k + 1)^2 + 2 = 4k^2 + 4k + 3 = 2(2k^2 + 2k + 1) + 1,$$

which is odd. Hence, by contraposition, if $n^2 + 2$ is even, then n is even.

Mistake 2: Mixing up the order of quantifiers.

Example:

Problem: What does it mean for a function $f : A \rightarrow B$ to be surjective?

Wrong solution: There is an $x \in A$ such that for every $y \in B$, $f(x) = y$.

Explanation: The order of the quantifiers matters. The wrong statement says that there is one fixed $x \in A$ that maps to every $y \in B$. If this were true and A is nonempty, this would force B to have only a single element since by definition of a function every $x \in A$ maps to only one $y \in B$.

Corrected solution: For every $y \in B$ there is an $x \in A$ such that $f(x) = y$.

Mistake 3: Using variables without defining/introducing them.

Example:

Problem: Assume $f : B \rightarrow C$ is injective and $g, h : A \rightarrow B$ are functions. Prove that if $f \circ g = f \circ h$, then $g = h$.

Wrong solution: $f(g(a)) = f(h(a))$, $f(b) = f(c)$. Then $b = c$ since f is injective. So $g(a) = h(a)$ for every $a \in A$.

Explanation: What are a, b, c ? How are they related? What are the quantifiers?

Corrected solution: Let $a \in A$. Then $f(g(a)) = f(h(a))$. Let $b = g(a)$, $c = h(a)$. Then $f(b) = f(c)$, so $b = c$ since f is injective. So $g(a) = h(a)$ for every $a \in A$.

Introducing b, c , may make it clearer, but we can do without using them at all (but a still needs to be introduced): Let $a \in A$. Then $f(g(a)) = f(h(a))$. Then $g(a) = h(a)$ since f is injective. So $g(a) = h(a)$ for every $a \in A$.

Mistake 4: Introducing variables incorrectly/not how they are supposed to be used

Example:

Problem: Assume $f : A \rightarrow B$ is surjective and $g, h : B \rightarrow C$ are functions. Prove that if $g \circ f = h \circ f$, then $g = h$.

Wrong solution: Let $a \in A$. Define $b = f(a)$. Then $g(b) = g(f(a)) = h(f(a)) = h(b)$, hence $g(b) = h(b)$ for every $b \in B$.

Explanation: It is concluded that $g(b) = h(b)$ holds for every $b \in B$. However, b was not introduced as an arbitrary element of B : it was defined to be $f(a)$. At that point, the argument only shows the equality for an element of B that arises as the image of the chosen a .

Because f is surjective, every $b \in B$ is of the form $f(a)$ for some $a \in A$, so the idea can be made correct. However, this use of surjectivity needs to be stated explicitly. It is cleaner to begin with an arbitrary $b \in B$ and then use surjectivity to choose $a \in A$ such that $f(a) = b$.

Corrected solution: Let $b \in B$. Since f is surjective, there is $a \in A$ with $f(a) = b$. Then $g(b) = g(f(a)) = h(f(a)) = h(b)$. Hence, $g(b) = h(b)$ for every $b \in B$.

Mistake 5: Using only words and not introducing variables

Example:

Problem: What does it mean for a function $f : A \rightarrow B$ to be bijective?

Imprecise solution: For every output there is a unique input.

Explanation: This is unclear, how exactly are input/output related? Also what does the word “output” mean: an element of the range or an element of the codomain? If it means an element of the range this only expresses injectivity. If it means an element of the codomain, then it expresses bijectivity.

Introducing variables and stating $f(a) = b$ removes this ambiguity and makes the definition easier to use in proofs.

Corrected solution: For every $b \in B$ there is a unique $a \in A$ such that $f(a) = b$.

Using only words can also introduce mistakes, for example I have seen this on multiple occasions as the proposed meaning of injectivity of a function $f : A \rightarrow B$:

Wrong solution: For every input there is a unique output.

Explanation: The word “unique” is especially dangerous here because it is often used informally to mean “distinct.” For example, someone might say that the elements of a set are “unique” to mean that no two of them are the same. In this informal sense, a student saying that “every input has a unique output” may be trying to say that different inputs have different outputs, which *would* express injectivity.

However, the statement “for every input there is a unique output” is normally read mathematically as

$$\forall a \in A \exists! b \in B (f(a) = b).$$

Here “unique” refers to the output associated to a *fixed* input. This is true for every function and does not express injectivity.

Corrected solution: If $a_1, a_2 \in A$ and $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$. Alternative that is closer to the wording in the wrong solution: For every $b \in B$ there is at most one $a \in A$ such that $f(a) = b$.

Mistake 6: Question not read/answered completely.

Example:

Problem: Consider the statement $\forall x \exists y (x + y = 0)$ where the domain of x, y are the integers. For an arbitrary integer x , give an integer y (in terms of x) for which $x + y = 0$. Hence, state whether the statement is true or false.

Incomplete solution: For any integer x take $y = -x$. Then $x + y = 0$ is true.

Explanation: It was not stated whether the statement is true or false.

Corrected solution: For any integer x , take $y = -x$. Then $x + y = 0$ is true. Therefore, the statement is true.

Mistake 7: Negating an inequality by only flipping the inequality sign

Example:

Problem: Let x, y be integers. Negate the statement $x < y$.

Wrong solution: The negation is $x > y$.

Explanation: The statement $x < y$ is also false when $x = y$.

Corrected solution: The negation is $x \geq y$ (or $y \leq x$).

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