Problem Set 7 Solutions
MATH 110: Linear Algebra

Each problem is worth 5 points.

PART 1

2. Curtis p. 192 5.
3. Curtis p. 215 1(a,b,c,d,e).

Solutions in book.

PART 2

Problem 1 (10)

a) If $T : V \rightarrow V$ has an eigenvalue $\lambda$, prove that $aT$ has the eigenvalue $a\lambda$.

Proof: Let $x$ be an eigenvector for the eigenvalue $\lambda$. $(aT)(x) = a(T(x)) = a(\lambda x) = \lambda(ax)$ so $ax$ is an eigenvector of $aT$ with eigenvalue $\lambda$.

b) If $x$ is an eigenvector for both $T_1$ and $T_2$, prove that $x$ is an eigenvector for $aT_1 + bT_2$ and find the eigenvalues of $aT_1 + bT_2$ in terms of the eigenvalues of $T_1$ and $T_2$.

Proof: Let $\lambda$ be the eigenvalue of $x$ for $T_1$ and let $\mu$ be the eigenvalue of $x$ for $T_2$. $(aT_1 + bT_2)(x) = (aT_1)(x) + (bT_2)(x) = (\lambda a + \mu b)x$.

c) Suppose that $x$ is an eigenvector of $T$ with eigenvalue $\lambda$. Show that $x$ is an eigenvector of $T^2$ with eigenvalue $\lambda^2$.

Proof: $T^2(x) = T(T(x)) = T(\lambda x) = \lambda(T(x)) = \lambda(\lambda x) = \lambda^2 x$.

d) Let $P$ be a polynomial. Show that if $x$ is an eigenvector of $T$ with eigenvalue $\lambda$ then $x$ is an eigenvector of $P(T)$ with eigenvalue $P(\lambda)$.

Proof: By induction on the degree of $P$ (we will denote the degree with the variable $n$). Suppose $n = 0$: the polynomial is a constant, and certainly $x$ is an eigenvector of $P(T)$ (every vector actually is) with eigenvalue $P(\lambda)$. Now suppose the theorem is true for all polynomials of degree less than or equal to $n$. Consider $P$ of degree $n + 1$. If $P$ has no constant term, then $P(T) = TQ(T)$ where $Q$ has degree $n$. By induction $x$ is an eigenvector of $Q$ with eigenvalue $Q(\lambda)$. Therefore, $TQ(T)(x) = T(Q(\lambda)x) = Q(\lambda)T(x) = Q(\lambda)(\lambda x) = \lambda Q(\lambda)x$. Thus $x$ is an eigenvector of $P$ with eigenvalue $\lambda Q(\lambda) = P(\lambda)$. Now, if $P$ has a constant term $c$, then $P - c$ is a polynomial of degree
n + 1 with no constant term. By the above argument x is an eigenvector of \( P - c \) with eigenvalue \( (P - c)(\lambda) \). Notice that x is also an eigenvector of c with eigenvalue c (proved for the base case). By part (b) x is an eigenvector of the polynomial \( P - c + c = P \) with eigenvalue \( (P - c)(\lambda) + c = P(\lambda) \).

**Problem 2** (10)

If \( T : V \to V \) has the property that \( T^2 \) has a nonnegative eigenvalue \( \lambda^2 \), prove that at least one of \( \lambda \) or \( -\lambda \) is an eigenvalue for \( T \). (Hint: \( T^2 - \lambda^2 I = (T + \lambda I)(T - \lambda I) \)).

**Proof:** Let \( x \) be an eigenvector of \( T^2 \) with the eigenvalue \( \lambda^2 \). \( (T^2 - \lambda^2 I)(x) = 0 \). Therefore, \( (T + \lambda I)(T - \lambda I)(x) = 0 \). If \( (T - \lambda I)(x) = 0 \) then \( x \) is an eigenvector of \( T \) with eigenvalue \( \lambda \). Otherwise, \( (T - \lambda I)(x) = y \) where \( y \) is some nonzero vector. In this case, \( (T + \lambda I)(y) = 0 \), which means that \( y \) is an eigenvector of \( T \) with eigenvalue \( -\lambda \). Thus, either \( \lambda \) or \( -\lambda \) is an eigenvalue.

**Problem 3** (10)

Let \( V \) be the linear space of all real convergent sequences \( \{x_n\} \). Define \( T : V \to V \) as follows: if \( x = \{x_n\} \) is a convergent sequence with limit \( a \), let \( T(x) = \{y_n\} \) where \( y_n = a - x_n \) for \( n \geq 1 \). Prove that \( T \) has only two eigenvalues \( \lambda = 0 \) and \( \lambda = -1 \) and determine the eigenvectors belonging to each such \( \lambda \).

**Solution:** Let \( \lambda \) be an eigenvalue of \( T \). Then \( a - x_n = \lambda x_n \) for each \( n \), which means that \( x_n = \frac{a}{\lambda^{n+1}} \) (assuming that \( \lambda \neq -1 \)). Taking the limit, as \( n \to \infty \), we see that \( \frac{a}{\lambda^{n+1}} = a \) which implies that \( \lambda = 0 \). Thus \( \lambda \) is either 0 or \(-1\). If \( \lambda = 0 \) this means that \( x_n \) is a constant sequence, thus the eigenvectors corresponding to eigenvalue 0 are the constant sequences. If \( \lambda = -1 \) then we see that there is no restriction on \( x_n \), other than that \( a = 0 \). Thus, the sequences with limit 0 are the eigenvectors with eigenvalue \(-1\).

**Problem 4** (20)

Let \( V \) be the vector space of sequences \( \{a_n\} \) over the real numbers. The shift operator \( S : V \to V \) is defined by

\[ S((a_1, a_2, \ldots)) = (a_2, a_3, a_4, \ldots). \]

Find the eigenvectors of \( S \), and show that the subspace \( W \) consisting of the sequences \( \{x_n\} \) satisfying \( x_{n+2} = x_{n+1} + x_n \) is a two dimensional, \( S \)-invariant subspace of \( V \). Also, find an explicit basis for \( W \).

Using these results, find an explicit formula for the \( n \)th Fibonacci number \( f_n \) where \( f_{n+2} = f_{n+1} + f_n \) and \( f_1 = f_2 = 1 \).
Solution: Suppose that \((a_1, a_2, \ldots)\) is an eigenvector with eigenvalue \(\lambda\). Then \(S((a_1, \ldots)) = \lambda(a_2, a_3, \ldots)\) which means that \(a_2 = \lambda a_1, a_3 = \lambda a_2, \ldots\). Thus, the eigenvectors are of the form \(a_1(1, \lambda, \lambda^2, \ldots)\).

Now suppose that \(x = (x_1, x_2, \ldots) \in W\). \(x\) is completely determined by its two initial values \(x_1, x_2\). Therefore, the dimension of \(W\) is at most 2. If an element of \(W\) is an eigenvector, it must be associated with an eigenvalue \(\lambda^2 = \lambda + 1\), which gives the two possible eigenvalues
\[
\lambda_1 = \frac{1 + \sqrt{5}}{2}, \lambda_2 = \frac{1 - \sqrt{5}}{2}.
\]
A basis for \(W\) is therefore \(b_1 = (\lambda, \lambda^2, \lambda^3, \ldots)\) and \(b_2 = (\lambda^2, \lambda^4, \ldots)\) which is clearly invariant under \(S\). In order to express the Fibonacci sequence in terms of this basis, we need to find constants \(k_1, k_2\) such that
\[
1 = k_1 \lambda_1 + k_2 \lambda_2
\]
\[
1 = k_1 \lambda_1^2 + k_2 \lambda_2^2.
\]
Solving for \(k_1, k_2\), we find that
\[
f_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right).
\]

Problem 5∗ (10)
Prove that the eigenvalues of an upper triangular matrix are its diagonal entries.

Proof: Let \(A\) be the upper triangular matrix. Consider \(\text{det}(\lambda I - A)\). This is just
\[
\prod_{i=1}^{n} (\lambda - a_{ii})
\]
where \(a_{ii}\) is the \(i\)th diagonal entry. This expression equals zero if, and only if, \(\lambda = a_{ii}\) for some \(i\). Thus, the diagonal entries of the matrix are exactly the eigenvalues.

PART 3 - Optional Problem
Recall that a graph is consists of two sets: a set of vertices, and a set of edges consisting of pairs of vertices. The adjacency matrix of a graph on \(n\) vertices is an \(n \times n\) graph with the \(ij\)th entry equal to 1 if vertex \(i\) is adjacent to vertex \(j\) and zero otherwise.
Let \( f(x) = x^n + c_1x^{n-1} + c_2x^{n-2} + \ldots + c_n \) be the characteristic polynomial of the adjacency matrix of a graph. Show that \( c_1 = 0 \), \( -c_2 \) is the number of edges in the graph, and \( -c_3 \) is twice the number of triangles in the graph.