PART 1
The following problems are each worth 5 points.

1. Curtis p. 107 3

Answers for Part 1 are in the book.

PART 2
Problem 1 (5)
Determine the matrix of each of the following linear transformations of \(\mathbb{R}^n\) into \(\mathbb{R}^n\):

a) The identity transformation.
The \(n \times n\) identity matrix.

b) The zero transformation.
The \(n \times n\) zero matrix.

c) Multiplication by a fixed scalar \(c\).
The \(n \times n\) matrix with the constant \(c\) along the diagonal and zeros everywhere else.

Problem 2 (5)
Determine the matrix of each of the following projections:

a) \(T : \mathbb{R}^3 \rightarrow \mathbb{R}^2\) where \(T(x_1, x_2, x_3) = (x_1, x_2)\).
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

b) \(T : \mathbb{R}^3 \rightarrow \mathbb{R}^2\) where \(T(x_1, x_2, x_3) = (x_2, x_3)\).
\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

c) \(T : \mathbb{R}^5 \rightarrow \mathbb{R}^3\) where \(T(x_1, x_2, x_3, x_4, x_5) = (x_2, x_3, x_4)\).
\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

Problem 3 (15)
In the linear space \(P_n\) of all real polynomials of degree \(\leq n\) define
\[
(f, g) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)g\left(\frac{k}{n}\right).
\]
a) Prove that \((f, g)\) is an inner product for \(P_n\).

**Proof:** The function is clearly bilinear and commutative. Just make sure that \((f, f) > 0\) if \(f\) is nonzero. Now if \(f\) is nonzero, it can have at most \(n\) roots (since it has degree \(n\)), but we are summing its values at \(n + 1\) points, which means one of the evaluations must be nonzero (and hence positive when squared).

b) Compute \((f, g)\) when \(f(t) = t\) and \(g(t) = at + b\).

This is just calculation.

\[
(f, g) = \sum_{k=0}^{n} \frac{k}{n} \left( \frac{k}{n} + b \right).
\]

Now the RHS can be explicitly evaluated using parts of homework 1 and the result proved in class that \(\sum_{k=0}^{n} k = \frac{n(n+1)}{2}\). The answer you get is

\[
(f, g) = \frac{a}{6n} + \frac{a}{2} + \frac{b}{2} + \frac{na}{3} + \frac{nb}{2}.
\]

c) If \(f(t) = t\), find all linear polynomials \(g\) orthogonal to \(f\).

This corresponds to solving for \(a, b\) when \((f, g)\) in the formula above is equal to 0. In particular,

\[
a = b \cdot \frac{-3n^2 - 3n}{2n^2 + 3n + 1}.
\]

**Problem 4 (10)**

Curtis p. 130 9.

**Proof:** Lets call denote set of vectors perpendicular to \(W\) by \(W_p\). Since \(W\) is a subset of a finite dimensional subspace, it has an orthonormal basis \(e_1, e_2, \ldots, e_k\). Consider the transformation \(T : V \rightarrow V\) defined by

\[
T(x) = \sum_{i=1}^{k} (x, e_i)e_i.
\]

Notice that \(T\) it is clearly linear because the inner product is linear in the first coordinate. Now the null space of \(T\) consists precisely of the vectors \(x\) with \((x, e_i) = 0\) for every \(i\). These vectors are exactly the ones in \(W_p\). The range of \(T\) is exactly the space \(W\), since for any \(y \in W\), \(T(y) = y\), and the
image of any element is clearly in $W$. The theorem now follows from the “rank-nullity” theorem.

**PART 3 - Optional Problem**

Calculate the integrals

\[ \int e^{ax} \sin(bx) \, dx, \int e^{ax} \cos(bx) \, dx. \]

Let $V$ be the real vector space of all continuous functions defined on the real line, and let $a, b$ be real numbers such $a, b \neq 0$. (If either $a$ or $b$ are zero the solution of the integrals is easy). Follow these steps:

a) Let $u_1 = e^{ax} \sin(bx)$ and let $u_2 = e^{ax} \cos(bx)$ be two functions in $V$. Show that $u_1, u_2$ are independent. This is easy because if $\lambda_1 u_1 + \lambda_2 u_2 = 0$ we can plug in $x = 0$ and $x = \frac{\pi}{2}$ to conclude that both $\lambda_1$ and $\lambda_2$ are zero.

b) Let $S = S(u_1, u_2)$ be the subspace of $V$ of dimension 2 generated by $u_1$ and $u_2$. Consider the linear transformation $D : S \rightarrow S$ defined by $D(f) = f'$ (D is the differentiation operator). Compute $D(u_1)$ and $D(u_2)$. This is just calculus using the product and chain rule for derivatives. $D(u_1) = au_1 + bu_2$ and $D(u_2) = -bu_1 + au_2$.

c) Show that the rank of $D$ is 2. Notice that $D(u_1)$ and $D(u_2)$ are independent using the same trick as in part a. You will need that $a^2 + b^2 \neq 0$ but this is true because $a, b$ are both nonzero.

d) Find the matrix (call it $M$) of $D$ with respect to the basis $\{u_1, u_2\}$. You’ve actually already done this in part b. Just put the first column equal to $a$ $b$ and the second column $-b$ $a$.

e) Notice the transformation is invertible by c, and thus the matrix is invertible. It is easy to invert it, and apply it to the vector $1 0$ (corresponding to $u_1$) to get

\[ \int e^{ax} \sin(bx) \, dx = \frac{ae^{ax} \sin(bx) - be^{ax} \cos(bx)}{a^2 + b^2}. \]

The same can be done to calculate the integral of $u_2$. 

3