PART 1
The following problems are each worth 5 points.

1. Curtis p. 87 5.
2. Curtis p. 98 2.

The solutions to part 1 are all in the book this week (and the book solutions are fine). Notes: When proving, for example, that $T(0) = 0$ for every linear transformation $T$, you should explicitly state that $0 + T(0) = T(0 + 0) = T(0) + T(0)$ which then implies that $T(0) = 0$ by the cancellation law.

PART 2
Remember that the starred problem is non collaborative.

Problem 1 (10,5)

b) Consider again a system $Ax = b$ with solution $x = A^{-1}b$. We can explicitly solve for $x$ by computing $A^{-1}$. How does this compare (in terms of the number of steps that need to be executed) with the row reduction techniques of sections 8,9?

The methods of sections 8,9 are really just to compute the echelon form of the matrix and then back substitute. The inversion of the whole matrix, as described in section 12 proceeds by obtaining the echelon form, but then continuing to use row operations to make the upper right half of the matrix zero as well. Some thought should convince you that this will take the same as the back substitution technique ($(n(n+1)/2$ multiplications and additions).
Problem 2* (10)
If \( S, T \) are invertible linear transformations that commute, prove that their inverses also commute.

**Proof:** Let the inverses of \( S, T \) be \( S^{-1}, T^{-1} \) respectively.

\[
ST = TS \quad \Rightarrow \quad S^{-1}ST = S^{-1}TS
\]

\[
\Rightarrow \quad T^{-1} = T^{-1}S^{-1}TS
\]

\[
\Rightarrow \quad 1 = T^{-1}S^{-1}TS
\]

\[
\Rightarrow \quad S^{-1}T^{-1} = T^{-1}S^{-1}
\]

Problem 3 (15)
Let \( V \) be the vector space of all real convergent sequences \( \{x_n\} \). Define a transformation \( T : V \rightarrow V \) as follows: If \( x = \{x_n\} \) is a convergent sequence with limit \( a \), let \( T(x) = \{y_n\} \), where \( y_n = a - x_n \) for \( n \geq 1 \). Prove that \( T \) is linear and describe the null space and range of \( T \).

**Proof:** We need to show that \( T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \). Here \( x, y \) are sequences. That is, \( x = x_1, x_2, \ldots \) with \( \lim_{x \to \infty} = L_x \) and \( y = y_1, y_2, \ldots \) with \( \lim_{y \to \infty} = L_y \). Now \( T(\alpha x + \beta y) \) is the sequence \( z = z_1, z_2, \ldots \) where \( z_i = (\alpha L_x + \beta L_y) - \alpha x_i - \beta y_i \). This is clearly the same as \( \alpha T(x) + \beta T(y) \).

The null space of \( T \) consists of those sequences that get mapped to the zero sequence. These are all the sequences of the form \( x_i = c \) where \( c \) is a constant. The range of \( T \) consists of those sequences \( x \) with limit 0. This is because clearly any sequence \( x \) satisfies that \( T(x) \) is a sequence with limit 0, and furthermore, if \( y \) has limit 0, then \( y = T(x) \) where \( x_i = -y_i \).

Problem 4 (20)
For each of the following transformations \( T : R_2 \rightarrow R_2 \) determine if \( T \) is linear. If \( T \) is linear, describe its null space and range, and compute its nullity and rank.

Note: the solutions omit verification of linearity except for the last example.

\( a \) \( T \) rotates every point through the same angle \( \varphi \) about the origin. \( T \) maps 0 onto itself.

\( T \) is linear. The nullspace is 0 and the range is \( R_2 \).

\( b \) \( T \) maps each point onto its reflection with respect to a fixed line through the origin.
$T$ is linear. The nullspace is 0 and the range is $R_2$.

c) $T$ maps every point onto the point $(\pi, 2)$.

$T$ is not linear. For one thing, if $T$ is linear then $T(0)$ must be 0.

d) $T$ maps each point to its reflection across the origin.

$T$ is linear. The nullspace is 0 and the range is $R_2$.

e) $T$ maps the point $(x, y)$ to the point $(x, x)$.

$T$ is linear. This is because $T((ax_1, ay_1) + (bx_2, by_2)) = (ax_1 + bx_2, ax_1 + bx_2) = aT(x_1, y_1) + bT(x_2, y_2)$. The null space consists of the points $(0, y)$. The range consists of the space spanned by the vector $(1, 1)$ (just a line at 45 degrees through the origin).
PART 3 - Optional Problem

Recall that the rank-nullity theorem states that if $V$ (a vector space) is finite dimensional, and $T(V)$ ($T$ is a linear transformation) is also finite dimensional, then

$$\text{dim} N(T) + \text{dim} T(V) = \text{dim} V.$$ 

What can we say if $V$ is infinite dimensional? Prove that in this case at least one of $T(V)$ or $N(T)$ is infinite dimensional.

**Proof:** Assume to the contrary that $N(T)$ and $T(V)$ are both finite dimensional with dimensions $k$ and $r$ respectively. Suppose $e_1, e_2, \ldots, e_k$ is a basis for $N(T)$. Also, take $e_1, e_2, \ldots, e_{k+n}$ to be independent elements in $V$, where $n$ has been taken to be larger than $r$ (we can do this since $V$ is infinite dimensional). Notice that the elements $T(e_1), T(e_2), \ldots, T(e_{k+n})$ are dependent since $n > r$, the dimension of $T(V)$. We can follow the proof of the rank and nullity theorem from here on as follows: Suppose $\sum_{i=k+1}^{k+n} c_i T(e_i) = 0$. Then $T(\sum_{i=k+1}^{k+n} c_i e_i) = 0$. So $x = c_{k+1} e_{k+1} + \cdots + c_{k+n} e_{k+n}$ is in the nullspace of $T$. This means there are constants $c_1, c_2, \ldots, c_k$ such that $x = c_1 e_1 + \cdots + c_k e_k$. Now

$$x - x = \sum_{i=1}^{k} c_i e_i - \sum_{i=k+1}^{k+n} k + nc_i e_i = 0.$$ 

But since $e_1, \ldots, e_{k+n}$ are independent by assumption all the $c_i$s must be 0, which implies that $T(e_1), T(e_2), \ldots, T(e_{k+n})$s are independent. This is a contradiction, which means that $N(T)$ and $T(V)$ can’t both be finite dimensional.