Problem Set 8 (due Friday October 29)

MATH 110: Linear Algebra

Each problem is worth 10 points.

PART 1


PART 2

Problem 1 (20)
a) Prove that if $A$ is an $n \times n$ matrix with all its eigenvalues equal to $\lambda$, then

\[ e^{xA} = e^{\lambda x} \sum_{k=0}^{n-1} \frac{x^k}{k!} (A - \lambda I)^k. \]

b) A $3 \times 3$ matrix $A$ has all its eigenvalues equal to $\lambda$. Show that

\[ e^{xA} = \frac{1}{2} e^{\lambda x} \left( (\lambda^2 x^2 - 2\lambda x + 2)I + (-2\lambda x^2 + 2x)A + x^2 A^2 \right). \]

Problem 2 (15)
Curtis p. 226 2,3.

Problem 3 (10)
Let $A$ and $B$ be $n \times n$ matrices with $\det A = \det B$ and $\text{tr} A = \text{tr} B$. Prove that $A$ and $B$ have the same characteristic polynomial if $n = 2$, but that this need not be the case if $n > 2$.

Problem 4 (10)
Suppose that the minimal polynomial of a linear transformation $T : V \to V$ satisfies

\[ m(x) = (x - \alpha_1)^{r_1} \cdots (x - \alpha_s)^{r_s} \]

as in the Triangular Form Theorem. Find the rational canonical form of $T$.

PART 3 - Optional Problem
(Putzer’s Method)
Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of an $n \times n$ matrix $A$, and define a sequence of polynomials in $A$ as follows: $P_0(A) = I$, and for $k = 1, \ldots, n$:

\[ P_k(A) = \prod_{m=1}^{k} (A - \lambda_m I). \]
Show that
\[ e^{xA} = \sum_{k=0}^{n-1} r_{k+1}(x) P_k(A) \]

where the function \( r_1(x), \ldots, r_n(x) \) are defined recursively by the linear differential equations:
\[ r_1'(x) = \lambda_1 r_1(x), \quad r_1(0) = 1 \]
and for \( k = 1, \ldots, n - 1 \):
\[ r_{k+1}'(x) = \lambda_{k+1} r_{k+1}(x) + r_k(x), \quad r_{k+1}(0) = 0. \]