Problem Set 7 (due Friday October 22)
MATH 110: Linear Algebra

Each problem is worth 5 points.

PART 1

2. Curtis p. 192 5.
3. Curtis p. 215 1(a,b,c,d,e).

PART 2

Remember that the starred problem is non collaborative.

Problem 1 (10)

a) If \( T : V \to V \) has an eigenvalue \( \lambda \), prove that \( aT \) has the eigenvalue \( a\lambda \).

b) If \( x \) is an eigenvector for both \( T_1 \) and \( T_2 \), prove that \( x \) is an eigenvector for \( aT_1 + bT_2 \) and find the eigenvalues of \( aT_1 + bT_2 \) in terms of the eigenvalues of \( T_1 \) and \( T_2 \).

c) Suppose that \( x \) is an eigenvector of \( T \) with eigenvalue \( \lambda \). Show that \( x \) is an eigenvector of \( T^2 \) with eigenvalue \( \lambda^2 \).

d) Let \( P \) be a polynomial. Show that if \( x \) is an eigenvector of \( T \) with eigenvalue \( \lambda \) then \( x \) is an eigenvector of \( P(T) \) with eigenvalue \( P(\lambda) \).

Problem 2 (10)

If \( T : V \to V \) has the property that \( T^2 \) has a nonnegative eigenvalue \( \lambda^2 \), prove that at least one of \( \lambda \) or \( -\lambda \) is an eigenvalue for \( T \). (Hint: \( T^2 - \lambda^2 I = (T + \lambda I)(T - \lambda I) \)).

Problem 3 (10)

Let \( V \) be the linear space of all real convergent sequences \( \{x_n\} \). Define \( T : V \to V \) as follows: if \( x = \{x_n\} \) is a convergent sequence with limit \( a \), let \( T(x) = \{y_n\} \) where \( y_n = a - x_n \) for \( n \geq 1 \). Prove that \( T \) has only two eigenvalues \( \lambda = 0 \) and \( \lambda = -1 \) and determine the eigenvectors belonging to each such \( \lambda \).

more problems on page 2...
Problem 4 (20)

Let $V$ be the vector space of sequences $\{a_n\}$ over the real numbers. The shift operator $S : V \to V$ is defined by

\[ S(\{a_1, a_2, \ldots\}) = (a_2, a_3, a_4, \ldots). \]

Find the eigenvectors of $S$, and show that the subspace $W$ consisting of the sequences $\{x_n\}$ satisfying $x_{n+2} = x_{n+1} + x_n$ is a two dimensional, $S$-invariant subspace of $V$. Also, find an explicit basis for $W$.

Using these results, find an explicit formula for the $n$th Fibonacci number $f_n$ where $f_{n+2} = f_{n+1} + f_n$ and $f_1 = f_2 = 1$.

Problem 5* (10)

Prove that the eigenvalues of an upper triangular matrix are its diagonal entries.

PART 3 - Optional Problem

Recall that a graph is consists of two sets: a set of vertices, and a set of edges consisting of pairs of vertices. The adjacency matrix of a graph on $n$ vertices is an $n \times n$ graph with the $ij$th entry equal to 1 if vertex $i$ is adjacent to vertex $j$ and zero otherwise.

Let $f(x) = x^n + c_1x^{n-1} + c_2x^{n-2} + \ldots + c_n$ be the characteristic polynomial of the adjacency matrix of a graph. Show that $c_1 = 0$, $-c_2$ is the number of edges in the graph, and $-c_3$ is twice the number of triangles in the graph.