Problem Set 5 (due Friday October 8)
MATH 110: Linear Algebra

Each part of each problem is worth 5 points.

PART 1

2. Curtis p. 130 11.

Remember that the starred problem is non collaborative.

Problem 1 (10)
An inner product on a vector space $V$ is a function which assigns to each pair of vectors $u, v$ in $V$ a real number such that the following conditions are satisfied ($c$ is any real number, $w$ is any vector in $V$):
1. $(u, v) = (v, u)$.
2. $(u, v + w) = (u, v) + (u, w)$.
3. $c(u, v) = (cu, v)$.
4. $(u, u) > 0$ if $u \neq 0$.

Show that this definition of an inner product is equivalent to the axiomatic definition given in Curtis, page 119.

Problem 2 (10)
Let $C(R)$ be the vector space of all polynomials with inner product

$$(x, y) = \int_{-1}^{1} x(t)y(t)dt.$$ 

Consider the subspace of $C(R)$ generated by the polynomials $\{1, x, x^2, x^3, x^4\}$. Use the Gram-Schmidt process to find polynomials $y_0, y_1, y_2, y_3, y_4$ that are orthonormal and span the same subspace.

Problem 3* (10)
Let $U$ be a subspace of a vector space $V$. Show that if 0 is the only vector orthogonal to all the vectors in $U$ then $U = V$.

PART 3 - Optional Problem
a) Let $S$ be a finite dimensional subspace of a Euclidean space $V$ and let $e_1, e_2, \ldots, e_n$ be an orthonormal basis for $S$. For any $x \in V$, let

$$s = \sum_{i=1}^{n} (x, e_i)e_i$$
be defined as the **projection of** $x$ **onto** $S$. Prove that for any $t \in S$, 

$$\| x - s \| \leq \| x - t \|$$

with equality iff $t = s$.

b) Find the linear polynomial nearest to $\sin \pi t$ on the interval $[-1, 1]$ (Here nearest is based on defining norm using the inner product defined in problem 3).