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Model Theoretic Topology

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1. Introduction

During the past few decades, the study of topology using model theoretic principles and vice versa has proven to be quite successful.

First of all, restricting our topology to definable sets provides a useful axiomatic framework for Grothendieck's *tame topology*. The idea behind tame topology is that when working with geometric or algebraic aspects of topology, we're primarily interested in sets that are intuitively "nice" and "orderly" as opposed to Cantor sets or arbitrarily complex Borel sets. As we'll soon see, by carefully choosing our model, the definable sets have many of the "nice" properties that we are looking for. However,there's more to be gained than just a convenient language. This formalization of nice sets allows us to use the full power of model theory in order to analyze tame topologies and obtain far reaching conclusions.

In addition, it turns out that structures with relatively elementary topological restrictions have powerful model theoretic properties and have much in common with stable theories. This implies a profound relationship between two different ways of looking at a structure.

To highlight these ideas we'll now look at o-minimal structures, which elegantly showcase the interplay of topology and logic. A structure M will be called *o-minimal* if it contains a dense linear order $<$, such that all of the subsets of M which are definable with parameters are finite unions of intervals and points. The natural topology to look at in this case is of course the order topology. Note that this topology has a basis which can be defined using a single parametrized formula:

$$\phi(x, y_1, y_2) = y_1 < x < y_2$$

.

The simplest example of an o-minimal structure is $(\mathbb{R}, <)$ which can be shown to have the above property by noting that it has quantifier elimination. Another basic example is any real closed field which can be shown to be o-minimal by Tarski's theorem. Using Wilkie's theorem, it has been proven that models of $\mathbb{R}_{exp}$, the reals with exponentiation, are o-minimal as well. As we'll see below, this model theoretic result can be used to give an elegant proof of Khovanski's theorem:

Theorem. *(Khovanski) If $f_1, \dots, f_m : \mathbb{R}^n \to \mathbb{R}$ are exponential polynomials then the set $\{x | f_1(x) = \cdots = f_m(x) = 0\}$ has finitely many connected components.*

In addition, using o-minimality results that we'll develop in section 2, it's possible to obtain information regarding the homeomorphism types:

Theorem. *For any given natural numbers m and n there are only finitely many homeomorphism types among the sets $\{x \in \mathbb{R}^n | f(x) = 0\}$ where $f(X_1, \dots, X_n) \in \mathbb{R}[X_1, \dots, X_n]$ has at most m monomials.*

This is one example of how model theoretic considerations can be used to prove purely topological results.

The first results related to o-minimal structures were given by Pillay and Steinhorn 1986 [2, 3]. In particular, they proved the monotonicity theorem which is the basis for many other results:

Theorem. *If R is an o-minimal structure, $a, b \in R \cup \{\pm\infty\}$ and $f : (a, b) \to R$ is definable then there're $a = a_0 < \cdots < a_n = b$ such that f is continuous on every subinterval (a_i, a_{i+1}) and on every such interval f is either strictly increasing, strictly decreasing or constant.*

They also proved the fundamental cell decomposition theorem which provides detailed information about the definable sets in R^n when R is o-minimal, and in particular shows that definable sets in such structures are indeed "nice". Using cell decomposition, it's possible to define good notions of dimension and Euler characteristic which are useful for analyzing the topology of these structures. Furthermore, they showed that every o-minimal theory has a unique prime model, which is reminiscent of what Shelach proved about ω-stable theories.

As a short but interesting example of the application of the Euler characteristic to model theory, consider the following proposition.

Proposition. *If R is an o-minimal structure and $A \subset R^m$ is definable, then there doesn't exist a definable map $f : A \to A$ such that $(A, f) \equiv (\mathbb{N}, s)$ where s is the successor function.*

After defining the Euler characteristic in section 2, we'll show how it's invariance can be used to easily prove the above result.

Another interesting application of the concepts of dimension and Euler characteristic to model theoretic questions about definability is the following corollary which will be proved in section 3.

Corollary. *Let $A \subset R^n$ and $B \subset R^m$ be definable subsets. Then there exists a definable bijection from A to B iff $dim(A) = dim(B)$ and $E(A) = E(B)$.*

An excellent source for the basics of o-minimal theory is Van den Dries's book [1]. In the first section we'll go over the main results that are necessary for the rest of the paper, though for a complete treatment it's better to refer to Van den Dries.

In section 3 we'll use a mix of model theoretic and topological techniques to prove even stronger results related to the structure of definable sets. Namely, the triangulation and trivialization theorems. These, together with the fact that $\mathbb{R}_{exp}$ is o-minimal, will allow us to prove Khovanski's theorem and the one stated after it with relative ease.

O-minimal theories provide us with yet another fascinating example of the interplay of logic, topology and algebra. Starting with a paper by Pillay in 1988 [9], much research has been done on the subject of definable groups in the o-minimal setting. An especially nice example of this is Strebonski's paper [7] about an extension of Sylow's theorems to arbitrary definable groups in o-minimal structures. Specifically, he showed that these groups behave like finite groups in the sense of Sylow theory when the cardinality of a group is replaced by its Euler characteristic. So again we see that despite not being finite or even stable, in many cases o-minimal structures behave in ways that are analogous to what one finds in those simpler structures.

In section 4 we'll generalize a curious result about definable functions in o-minimal expansions of the reals which was proved by Chris Miller [4]:

Theorem. *Let R be an o-minimal expansion of the reals and not polynomially bounded. Then the exponential function is definable.*

Miller later generalized this result to arbitrary o-minimal expansions of a real closed field where the idea of a polynomially bounded function is replaced with that of a power function [10]:

Theorem. *Let $R = (R, <, 0, 1, +, \cdot, \dots)$ be an o-minimal expansion of a real closed field. Then e^x is definable in R iff R isn't power bounded. If R is power bounded* *then for every definable function $f : R \to R$, if ultimately $f \neq 0$ then there exists an element $c \in R^*$ and a power function λ such that $f \sim c \cdot \lambda$.*

An alternative proof using an interesting growth dichotomy for o-minimal expansions of ordered groups was obtained by Chris Miller and Sergei Strachenko [8].

After going through the basics of the theory of power bounded functions, most of which can be found in [10], we'll obtain the previous theorem by proving that an additional property is equivalent to the definability of the exponent:

Theorem. *Let $R = (R, <, 0, 1, +, \cdot, \dots)$ be an o-minimal expansion of a real closed field. Then e^x is definable in R iff there exists a definable function $f : R \to R$ such that ultimately $f \neq 0$ and $lim_{x\to\infty} \frac{f'}{f} x = \infty$.*

We then obtain the original theorem by showing that the new property is equivalent to power-boundedness:

Proposition. *R is power bounded iff there doesn't exist a definable function $f : R \to R$ such that ultimately $f \neq 0$ and $lim_{x\to\infty} \frac{f'}{f} x = \infty$. In that case, for every definable function $f : R \to R$, if ultimately $f \neq 0$ then there exists an element $c \in R^*$ and a power function λ such that $f \sim c \cdot \lambda$.*

Towards the end of the section, we'll concentrate on what happens in the power bounded case and prove the following:

Theorem. *Let R power bounded, $A \subset R^m$ and let $f(x, \bar{y}) : R \times A \to R$ be a definable family of functions in R such that for all $\bar{a} \in A$, ultimately $f(x, \bar{a}) \neq 0$. Then there exists a finite set of power functions $\lambda_1, \dots, \lambda_n$ and a definable function $c : A \to R^*$ such that for all $\bar{a} \in A$, $f(x, \bar{a}) \sim c(\bar{a}) \cdot \lambda_i$ for some $1 \leq i \leq n$.*

This theorem was originally proved by Chris Miller [10], but we prove it in a model theoretic fashion using a finiteness result that we'll show beforehand.

After focusing on o-minimal structures, in section 5 we'll move to a more generalized setting. Instead of using the order topology we'll look at what happens when a structure has a topology whose basis is definable using one parametrized formula together with some assumptions on the generated topology. These will be called *first order topological structures* and they were first studied by Pillay [5]. As we noted above, o-minimal structures are a special case of first order topological structures. We'll be looking at a subset of these structures called *topologically totally transcendental* structures. Despite having a far weaker definition, the t.t.t structures share some of the nice structural and model theoretic properties described above.

Our primary focus will be on what types of dimension can be defined on definable subsets of t.t.t structures. In particular, we'll prove that two natural definitions of dimension coincide.

The first definition if derived from the following model theoretic construct.

Definition. (*rank)* Let M be a model and $A \subset M$ a definable subset.

(1) For any tuple $\bar{a} \in M^n$, $rk(\bar{a}/A)$ is the least cardinality of a subtuple $\bar{a}'$ of $\bar{a}$ such that $\bar{a} \in acl(\bar{a}'/A)$.
(2) for any type $p(\bar{x}) \in S_n(A)$, $rk(p/A) = rk(\bar{a}/A)$ for any $\bar{a} \in M^n$ realizing p.

The corresponding dimension in then defined as:

Definition. Let $X \subset M^n$ and $A \subset M$ be definable sets. Then we define:

$$rk(X) = max_{p\in S_n(A)}\{rk(p/A)|\text{p is realized in } X\}$$

The second definition is of a topological flavor.

Definition. Let M be a t.t.t model and $X \subset M^n$ be a definable subset. We define the *topological dimension* of X as:

$$dim(X) = max_{1\leq k\leq n}\{\exists 1 \leq i_1 < \cdots < i_k \leq n \,\text{s.t}\, int(\pi_{i_1,\ldots,i_k}(X)) \neq \emptyset\}$$

Our main result in section 5 will be:

Theorem. *Let M be an ω-saturated t.t.t structure an let $X \subset M^n$ be definable. Then $rk(X) = dim(X)$.*

This result has been proven by L. Mathews [11] under the assumption that the structure has the cell decomposition property and satisfies the exchange principle. We obtain it without assuming cell decomposition. The exchange principle is still implicitly assumed because as we'll see, it holds for any t.t.t structure

We'll also use some of the tools developed in that section in order to prove some statements about the definable sets in t.t.t structures. For example:

Proposition. *Suppose M is ω-saturated. Let $X \subset M^n$ be a dense definable set. Then $int(X) \subset M^n$ is dense as well.*

In addition, we'll prove a theorem of Pillay which states that every first order topological theory has a prime model. However, there is still much work to be on this topic as it's unclear which of the theorems we mentioned in the o-minimal case carry over to this generalized setting.

Finally, in the last section we'll take a closer look at the number of connected components of definable sets in t.t.t structures. First we show that the number of connected components is uniformally bounded over a definable family. This is used to prove that 1-dimensional ω-saturated t.t.t structures are preserved under elementary equivalence.

Our main result in this section is that for any 1-dimensional ω-saturated t.t.t structure, if removing any point divides the space into more that one connected component, then there exists a finite set $X \subset M_t$ such that each connected component of $M_t\backslash X$ is o-minimal:

Theorem. *Let M be a 1-dimensional ω-saturated t.t.t structure such that for all $x \in M_t$, $M\backslash\{x\}$ has more than one connected component. Then there exists a finite set $X \subset M_t$ such that all of the finite number of connected components of $M_t\backslash X$ are o-minimal.*

In order to prove this, we first obtain some intermediate results such as the fact that the equivalence relation specifying if y and z are in the same connected component of $M_t\backslash\{x\}$ is a definable relation in M_t^3. We also introduce a notion of "local flatness" which is used as a stepping stone between t.t.t structures and o-minimality.

2. O-minimal Preliminaries

In this section we'll give a quick overview of some of the most basic o-minimality definitions and theorems which will be used extensively in the next sections. The proofs of the theorems covered here can be found in Van den Dries's book [1] on the subject so I won't repeat them. Though in any case, I'll try to present a sketch of the proof when it seems like it could be illuminating.

Remark. From here on, definable will mean "definable with constants" unless stated otherwise.

Definition 1. Let M be a structure in a language $L = \{<, \dots\}$ where $<$ is interpreted as a dense linear order. M will be called an *o-minimal structure* if every definable set is a finite union of points and intervals of the form (a, b) where $a, b \in M \cup \{\pm\infty\}$.

Remark. When we talk about the topology of an o-minimal structure we're always referring to the order topology generated by $<$.

The following lemma is immediate but are worth noting as it'll be used extensively.

Lemma 2. *Let A be a definable subset of an o-minimal structure R. Then:*

(1) *Both $cl(A)$ and $int(A)$ are definable.*
(2) *The boundary of A is finite.*
(3) *If $|A| = \infty$ then A has an interior.*
(4) *If A is bounded then both $sup(A)$ and $inf(A)$ exist in R.*

We'll now explain what we mean when we talk about definable connected components in an o-minimal structure.

Definition 3. Let R be o-minimal and $A \subset R$ a definable subset. A will be called *definably connected* if there doesn't exist a pair of disjoint definable open subsets B_1 and B_2 such that $A = B_1 \cup B_2$.

2.1. **Monotonicity and Cell Decomposition.** Throughout the rest of this section, R will be an arbitrary o-minimal structure. The results appear in chapter 3 of [1].

One of the most basic theorems about o-minimal structures is the monotonicity theorem (1.2 in [1]):

Theorem. *(Monotonicity) If $a, b \in R \cup \{\pm\infty\}$ and $f : (a, b) \to R$ is definable then there're $a = a_0 < \cdots < a_n = b$ in R such that f is continuous on every subinterval (a_i, a_{i+1}) and on every such interval f is either strictly increasing, strictly decreasing or constant.*

Despite the fact that we won't prove the theorem here, the first step of the proof is worth stating since it's a very common technique. In general, when we want to prove that a *definable* property holds in an o-minimal structure up to a finite number of points, then it's enough to show that it holds on a subinterval of every interval. Because in that case, if we assume for contradiction that the set of points such that the property *doesn't* hold is infinite, then by lemma 2 that set contains an interval and we reach a contradiction.

So in this case, it's enough to prove that for every interval I, all of the following hold:

(1) There's a subinterval of I on which f is either constant or injective.
(2) If f is injective on I then f is strictly monotone on a subinterval of I.

(3) If f is strictly monotone on I then f is continuous on a subinterval of I.

We now present two easy corollaries (1.6 in [1]). The first one will be used extensively in section 3 when we discuss power functions.

Corollary 4. *Let $f : (a,b) \to R$ be definable. Then for each $c \in (a,b)$, the right and left limits of f at c exist in in $R \cup \{\infty\}$. Furthermore, the right limit of a and the left limit of b exist as well.*

The next corollary shows that closed intervals are similar to compact subsets in $\mathbb{R}$.

Corollary 5. *Let $f : [a,b] \to R$ be definable. Then f takes a minimum and maximum on $[a,b]$.*

The following lemma (1.7 in [1]) in an important ingredient of the proof of the cell decomposition theorem and is one of the first indications that definable subsets of an o-minimal structure can't be as complicated as we'd like.

Lemma 6. *(Finiteness lemma) Let $A \subset R^2$ be definable and assume that for each $x \in R$, the fiber $A_x = \{y \in R | (x,y) \in A\}$ is finite. Then there is an $N \in \mathbb{N}$ such that $|A_x| < N$ for all $x \in R$.*

Remark. The proof of this lemma is somewhat technical, but intuitively it's a consequence of the monotonicity theorem. Because if we define functions $f_1, f_2, \ldots$ such that $f_i(x)$ is equal to the i-th element in the fiber A_x, then by the monotonicity theorem, each of these functions behaves "nicely" and so A is the union of the graphs of continuous and either strictly monotonous or constant functions defined on subintervals of R. This means that locally, the set looks like a finite union of "nice" graphs. However, extending this to a global statement is tricky.

We'll now turn to the cell decomposition theorem which shows that any definable subset of R^n can be split up into a finite number of cells which will be defined shortly. Since the cells are of a relatively elementary form, this is an indispensable tool in analyzing o-minimal structures.

While working with cells, we use the following notion. Given a definable set $X \subset R^n$ and two definable and continuous functions $f, g : X \to R \cup \{\infty\}$ such that $f < g$, we'll define:

$$(f,g)_X = \{(x,r) \in X \times R | f(x) < r < g(x)\} \subset R^{n+1}$$

.

Definition 7. Let $(i_1, \ldots, i_m)$ be a sequence of zeros and ones. An $(i_1, \ldots, i_m)$-*cell* is a definable subset of R^m defined inductively as follows:

1. (1) A (0)-cell is a point $\{a\} \in R$ and a (1)-cell is an interval $(a,b) \in R$.
2. (2) Suppose we've defined $(i_1, \ldots, i_m)$-cells. Then a $(i_1, \ldots, i_m, 0)$ cell is the graph $\Gamma(f)$ of a definable continuous function $f : X \to R$ on an $(i_1, \ldots, i_m)$-cell X. In addition, a $(i_1, \ldots, i_m 1)$ cell is a set $(f,g)_X$ where $f, g : X \to R \cup \{\infty\}$ are definable and continuous functions such that $f < g$ and X is an $(i_1, \ldots, i_m)$-cell.

Remark. Note that the $(1, \ldots, 1)$-cells are exactly the cells which are open in their ambient space R^m.

Definition 8. A *cell* in R^m is some $(i_1, \ldots, i_m)$-cell.

Before continuing to the theorem, there're a few properties of cells which are worth mentioning.

First of all, it's easy to see that every $(i_1, \dots, i_m)$-cell in R^m is homeomorphic to an open subset of R^k for some $k \leq m$ under the projection to the coordinates $1 \leq j \leq m$ for which $i_j = 1$. Furthermore, by the definition of a cell, if $A \subset R^{m+1}$ is a cell and $\pi : R^{m+1} \to R^m$ is the projection then $\pi(A) \subset R^m$ is a cell as well. This allows us to prove many of the properties of cells by induction on m. For instance, one easy consequence is that every cell is definably connected.

Definition 9. A *cell decomposition* of R^m is a kind of partition of R^m into finitely many cells which is defined inductively on m:

1. A cell decomposition of R^1 is a finite union of points and intervals.
2. A cell decomposition of R^{m+1} is a partition of R^{m+1} into a finite set of cells A such that $\pi(A)$ is a cell decomposition of R^m.

It's sometimes helpful to look at this definition constructively. As stated in the definition, a cell decomposition of R is obtained by choosing a finite number of points and then taking the points together with the intervals between them. This is then turned into a cell decomposition of R^2 in two steps. In the first step, for each point a in our decomposition of R of we choose a finite number of points in R^2 which project onto a. We then take these points together with the intervals between them. In the second step, for each interval (a, b) we choose a finite number of definable continuous functions $f_1 < \cdots < f_n : (a, b) \to R$ and add to our partition the sets $\Gamma(f_1), \dots, \Gamma(f_n)$ and $(f_1, f_2), \dots, (f_{n-1}, f_n)$. This process is then iterated in order to obtain a cell decomposition of R^m.

A decomposition D of R^m is said to *partition* a set $S \subset R^m$ if each cell in D is either a subset of S or disjoint from S.

We now have the terminology enabling us to state the cell decomposition theorem (2.11 in [1]).

Theorem 10. *(Cell decomposition theorem)*

- *(I_m) Given any definable sets $A_1, \dots, A_k \subset R^m$ there is a cell decomposition of R^m partitioning each of $A_1, \dots, A_k$.*
- *(II_m) For each definable function $f : A \to R$, where $A \subset R^m$ is definable, there is a decomposition D of R^m partitioning A such that for each cell $B \in D$ contained in A, the restriction $f|_B : B \to R$ is continuous.*

Remark. The proof of the theorem is by induction on m and is quite long. The details can of course be found in [1] but I'll try to give a rough outline here.

First of all, note that I_1 follows by o-minimality and II_1 follows from the monotonicity theorem. The induction is carried out by first proving I_{m+1} and then using that together with the inductive hypothesis to prove II_{m+1}. A key part of the proof of I_{m+1} is to use the inductive hypothesis in order to generalize the finiteness lemma to definable subsets of R^{m+1}. After proving this, we can then look at the union of the boundaries of the definable sets:

$$Y = bd(A_1) \cup \cdots \cup bd(A_k)$$

. By o-minimality, the fibers of Y are finite subsets of R. This allows us to use the generalized finiteness lemma to achieve a uniform bound on the fiber size. We can then define a finite set of functions f_i on R^m where for each i, $f_i(\bar{a})$ is equal to the i-th point on the fiber $Y_{\bar{a}}$. These functions will be the ones used in the construction of the cell decomposition, similar to what was described after we defined decompositions.

Then, after proving I_{m+1}, we prove II_{m+1} by noting that it's now enough to assume that A is a cell. If A isn't an open cell then we can project A to an open cell in R^k for some $k \leq m$ and use the inductive hypothesis. So we can now assume that A is an open cell.

Using II_m and the monotonicity theorem, we can find a dense definable subset A^* of A such that for every $\bar{a} \in A$, f is continuous on a neighborhood of $\bar{a}$. This is obtained by applying the following topological lemma:

Lemma. *Let X be a topological space, $(R_1, <)$ and $(R_2, <)$ dense linear orderings without endpoints, and $f : X \times R_1 \to R_2$ a function such that for every $(x, r) \in X \times R_1$:*

(1) *$f(x, \cdot) : R_1 \to R_2$ is continuous and monotone on R_1.*
(2) *$f(\cdot, r) : X \to R_2$ is continuous at x.*

Then f is continuous.

Now, again by I_{m+1}, the existence of the set A^* is enough to prove II_{m+1} when A is an open cell.

The cell decomposition theorem has many applications. One immediate consequence is that every definable subset of R^m has a finite number of connected components. This follows from the fact that cells are definably connected.

2.2. **Dimension and Euler Characteristic.** Using the CDT, we'll now define two invariants which can be determined for any definable subset of R^m. We say that they're invariant because as we'll see, they're invariant under definable bijections.

As before, in this section R will be an o-minimal structure. The material from this section can be found in chapter 4 of [1].

Definition 11. Let $X \subset R^m$ be definable. Then the *dimension* of X is defined by:

$$dimX = max\{i_1 + \cdots + i_m | X \text{ contains an } (i_1, \ldots, i_m)\text{-cell}\}$$

If $X = \emptyset$ then we define $dimX = -\infty$.

Remark. Note that by this definition, $dim(X) \in \{-\infty, 0, 1, \ldots, m\}$. Furthermore, $dimX = m$ iff X contains an open cell. By the CDT we can use this to see that $dimX = m$ iff X has an interior.

The following proposition (1.3 in [1]) shows that the dimension we defined has the desired properties. The proof relies heavily on the CDT and isn't particularly difficult.

Proposition 12. *Let $X, Y \subset R^m$ be definable subsets. Then:*

(1) *If $X \subset Y$ then $dimX \leq dimY$.*
(2) *If $Z \subset R^n$ is definable and there's a definable bijection from X to Z then $dimX = dimZ$.*
(3) *$dim(X \cup Y) = max\{dim(X), dim(Y)\}$*

Remark. 1 and 3 are almost immediate. As for 2, the main idea is to use a projection to obtain a homeomorphism between any given cell and an open cell. After that, it's enough to show that if $A \subset R^m$ is an open cell and $f : A \to R^m$ is an injective map then $f(A)$ contains an open cell.

The dimension defined above has many other useful properties such as definable variation among a definable family which will be omitted here as they won't be used in any of the sections below. As usual, the details are contained in [1].

We now turn to our next invariant, the Euler characteristic.

Definition 13. Let $X \subset R^m$ be definable and let D be a cell decomposition of X. Then:

(1) For each cell $C \in D$ we define $E(C) = (-1)^{dim(C)}$.
(2) $E(X) = \sum_{C \in D} E(C)$

Remark. It's important to note that $E(X)$ doesn't depend of the choice of a partition. This can be shown by taking a common refinement of any two partitions.

The invariance of the Euler characteristic (2.4 in [1]) is much harder to prove than the invariance of dimension. The main difficulty is that a transposition of coordinates doesn't necessarily map a cell to a cell.

Proposition 14. *If $f : X \to R^m$ is a definable injection then $E(X) = E(f(X))$.*

As promised in the introduction, we'll now use the Euler characteristic to prove the following model theoretic claim:

Proposition 15. *Let $A \subset R^m$ be definable. Then there doesn't exist a definable map $f : A \to A$ such that $(A, f) \equiv (\mathbb{N}, s)$ where s is the successor function.*

Proof. Assume for contradiction that there exists a definable map $f : A \to A$ such that $(A, f) \equiv (\mathbb{N}, s)$. In $(\mathbb{N}, s)$, $s : \mathbb{N} \to \mathbb{N}$ is a bijection from $\mathbb{N}$ to $\mathbb{N}\backslash\{1\}$. Therefore, there exists an $a \in A$ such that f is a definable bijection from A to $A\backslash\{a\}$. By the invariance of the Euler characteristic, $E(A) = E(A\backslash\{a\})$. But since A has a cell partition where the point a is one of the cells, $E(A\backslash\{a\}) = E(A) - 1$ which is a contradiction. □

3. Triangulation and Trivialization in o-minimal Structures

3.1. **Introduction.** The primary goal of this section is to obtain a short proof of the trivialization theorem for o-minimal structures using a combination of topological an model theoretic techniques. First I'll introduce the triangulation theorem and use it to prove a few useful results, including some fiberwise properties about homeomorphism types. After that I'll prove some fiberwise properties that have to do with continuous functions. Then I'll use those results to prove the trivialization theorem. Finally, I'll discuss the applications of the theorem to the analysis of the set of zeros of a polynomial in a real closed field. Most of the material below comes from theorems and exercises in chapters 6,8 and 9 of [1].

3.2. **Triangulation.** In this section, $(R, <, 0, 1, +, -, \cdot)$ will be an o-minimal expansion of of an ordered, real closed field.

Definition 16. Let $a_0, \ldots, a_k \in R^n$ be an affine independent set of points. The following set:

$$(a_0, \ldots, a_k) = \{\sum t_i a_i | \forall i : t_i > 0, \sum t_i = 1\}$$

will be called a *k-simplex* in R^n. A *face* of $(a_0, \ldots, a_k)$ is a simplex spanned by a non-empty subset of $\{a_0, \ldots, a_k\}$.

Remark. Every $x \in (a_0, \dots, a_k)$ has a unique representation as $x = \sum t_i a_i$ where for all i, $t_i > 0$ and $\sum t_i = 1$.

Definition 17. A *complex* in R^n is a finite collection K of simplexes in R^n such that for all $\sigma_1, \sigma_2 \in K$, either $cl(\sigma_1) \cap cl(\sigma_2) = \emptyset$ or $cl(\sigma_1) \cap cl(\sigma_2) = cl(\tau)$ where τ is a common face of σ_1 and σ_2. In addition, we define $|K|$, the ***polyhedron*** spanned by K, as the union of all the complexes in K.

The following theorem is proved in [1]. It's proof is rather lengthy and technical so we omit it here.

Theorem 18. *(triangulation) Each definable set $S \subset R^m$ is definably homeomorphic to a polyhedron $|K|$ for some complex K in R^m.*

Let K be a complex. The *scheme* of the complex is the following pair:

$$(Vert(K),\ \{\{a_0, \dots, a_k\} \subset Vert(K) | (a_0, \dots, a_k) \in K\})$$

We say that the complexes L and K have isomorphic schemes if there's a bijective map: $V : K \to L$ such that $(a_0, \dots, a_k) \in K \iff (V(a_0), \dots, V(a_k)) \in L$. It's easy to see that V induces a definable homeomorphism from $|K|$ to $|L|$. This means that for any complex K, if $N = |Vert(K)|$ then there exists a definable homeomorphism from $|K|$ to $|L_N|$ where:

$$L_N \subset \{(e_{i_1}, \dots, e_{i_m}) | 1 \le m \le N,\ 1 \le i_1 < \dots < i_m \le N\}$$

and $e_1, \dots, e_N$ is the natural basis of R^N.

The fact that every definable set is definably homeomorphic to a polyhedron of a complex with a given scheme allows us to definably transform any definable set into a finite combinatorial object. This property will allow us to prove many finiteness results about the definable sets of an o-minimal structure and ultimately the trivialization theorem.

First of all, we'll use the theorem to show that the dimension and Euler characteristic uniquely characterize a definable set up to a definable bijection. The main idea is to translate this statement into a statement about complexes and use some basic results about simplexes which we'll prove below.

Lemma 19. *Let K be a k-simplex. Then there exist two k-simplexes K_1, K_2 and a $k-1$-simplex K_3 such that $K = K_1 \cup K_2 \cup K_3$ and the union is disjoint.*

Proof. We'll use induction on k. If $k = 1$ then $K = (a_0, a_1)$ and so we can write

$$K = (a_0, \frac{a_0 + a_1}{2}) \cup (\frac{a_0 + a_1}{2}) \cup (\frac{a_0 + a_1}{2}, a_1)$$

.

We now assume that the claim is true for k. Let $K = (a_0, \dots, a_k, a_{k+1})$ be the complex. According to the inductive hypothesis, there exist $K_1 = (b_0^1, \dots, b_k^1)$, $K_2 = (b_0^2, \dots, b_k^2)$, $K_3 = (b_0^3, \dots, b_{k-1}^3)$ such that $(a_0, \dots, a_k) = K_1 \cup K_2 \cup K_3$ is a disjoint union. We'll now show that:

$$K = (b_0^1, \dots, b_k^1, a_{k+1}) \cup (b_0^2, \dots, b_k^2, a_{k+1}) \cup (b_0^3, \dots, b_{k-1}^3, a_{k+1})$$

is a disjoint union. It's easy to see that

$$(b_0^1, \dots, b_k^1, a_{k+1}) \cup (b_0^2, \dots, b_k^2, a_{k+1}) \cup (b_0^3, \dots, b_{k-1}^3, a_{k+1}) \subset K$$

. In addition, if $x \in K$, then there exist $t_0, \ldots, t_{k+1}$ such that $t_i > 0$, $\sum t_i = 1$ and $x = \sum t_i a_i$. Therefore, if we define $t = \sum_{i=0}^{k} t_i$ then $x = t\sum_{i=0}^{k}(\frac{t_i}{t}a_i) + t_{k+1}a_{k+1}$. But $\sum_{i=0}^{k}(\frac{t_i}{t}a_i) \in (a_0, \ldots, a_k)$, $t > 0$, and $t + t_{k+1} = 1$ so

$$x \in (b_0^1, \ldots, b_k^1, a_{k+1}) \cup (b_0^2, \ldots, b_k^2, a_{k+1}) \cup (b_0^3, \ldots, b_{k-1}^3, a_{k+1})$$

.

The only thing that's left is to show that $(b_0^1, \ldots, b_k^1, a_{k+1})$, $(b_0^2, \ldots, b_k^2, a_{k+1})$, $(b_0^3, \ldots, b_{k-1}^3, a_{k+1})$ are pairwise disjoint. Let's assume for contradiction that there exists an $x \in K$ such that

$$x \in (b_0^1, \ldots, b_k^1, a_{k+1}) \cap (b_0^2, \ldots, b_k^2, a_{k+1})$$

. In a similar way to what we did above we can show that there exist $b^1 \in K_1$, $b^2 \in K_2$, $t^1 > 0$, $t^2 > 0$, $t_{k+1}^1 > 0$, $t_{k+1}^2 > 0$ such that $x = t^1 b^1 + t_{k+1}^1 a_{k+1} = t^2 b^2 + t_{k+1}^2 a_{k+1}$ and $1 = t^1 + t_{k+1}^1 = t^2 + t_{k+1}^2$. But x has a unique representation so $t_{k+1}^1 = t_{k+1}^2 \Rightarrow t^1 = t^2 \Rightarrow b^1 = b^2$ which is a contradiction to $K_1 \cap K_2 = \emptyset$. The proof that the other intersections are empty is similar. □

Lemma 20. *Let K, L be two k-simplexes. Then K and L are definably homeomorphic.*

Proof. Let $K = (a_0, \ldots, a_k)$ and $L = (b_0, \ldots, b_k)$. We define $f : K \to L$ by:

$$f(\sum_{i=0}^{k} t_i a_i) = \sum_{i=0}^{k} t_i b_i$$

. By the unique representation of elements in K and L, f is well defined and is an automorphism. It's easy to see that f and f^{-1} are continuous and that f is definable. □

Lemma 21. *If K is a k-simplex then $E(K) = (-1)^k$.*

Proof. By the previous lemma, it's enough to show this for k-simplexes of the form: $\Delta_k = (0, e_1, \ldots, e_k)$ where (e_i) is the natural basis of R^k. We'll prove by induction on k that Δ_k is an open cell in R^k.

For $k = 0$ there's nothing to show.

Let's assume that the claim is true for k. Now,

$$\Delta_{k+1} = \{(x_1, \ldots, x_{k+1}) | x_i > 0, \sum x_i = 1\} =$$

$$\{(x_1, \ldots, x_k, r) | (x_1, \ldots, x_k) \in \Delta_k,\ r > 0, \sum_{i=1}^{k} x_k + r = 1\} =$$

$$\{(x_1, \ldots, x_k, r) | (x_1, \ldots, x_k) \in \Delta_k, 0 < r < 1 - \sum_{i=1}^{k} x_k\}$$

By the inductive hypothesis, Δ_k is an open cell and this shows that Δ_{k+1} is an open cell as well. This finishes the proof because for every open cell C in R^k, $E(C) = (-1)^k$. □

We are now ready to prove the corollary mentioned above.

Corollary 22. *Let $A \subset R^n$ and $B \subset R^m$ be definable subsets. Then there exists a definable bijection from A to B iff $dim(A) = dim(B)$ and $E(A) = E(B)$.*

Proof. We already know that the conditions are necessary. We'll prove that they're sufficient. First of all, by the triangulation theorem we can assume that A and B are finite disjoint unions of simplexes in R^n and R^m respectively. We'll prove the theorem by induction on $k = dim(A) = dim(B)$.

If $k = 0$ then both A and B are a disjoint set of $E(A) = E(B)$ points and the theorem is trivial.

Let's assume that the theorem is true for all $l < k$. As we said above, we can assume that:

- $A = \sigma_1 \cup \cdots \cup \sigma_p \cup \cdots \cup \sigma_q$
- $B = \tau_1 \cup \cdots \cup \tau_r \cup \cdots \cup \tau_s$

where $dim(\sigma_i) = dim(\tau_j) = k$ for all $1 \leq i \leq p$ and $1 \leq j \leq r$ and $dim(\sigma_i), dim(\tau_j) < k$ for all $i > p$ and $j > r$. By lemma 19, we can assume that $p = r$ and that there exist $i > p$, $j > r$ such that $dim(\sigma_i) = dim(\tau_j) = k - 1$.

By lemma 20, there's a definable bijection from $\sigma_1 \cup \cdots \cup \sigma_p$ to $\tau_1 \cup \cdots \cup \tau_r$. In addition, by lemma 21, $E(\sigma_1 \cup \cdots \cup \sigma_p) = E(\tau_1 \cup \cdots \cup \tau_r)$ which means that $E(\sigma_{p+1} \cup \cdots \cup \sigma_q) = E(\tau_{r+1} \cup \cdots \cup \tau_s)$. In addition,

$$dim(\sigma_{p+1} \cup \cdots \cup \sigma_q) = dim(\tau_{r+1} \cup \cdots \cup \tau_s) = k - 1$$

so by the inductive hypothesis there also is a definable bijection from $\sigma_{p+1} \cup \cdots \cup \sigma_q$ to $\tau_{r+1} \cup \cdots \cup \tau_s$. This completes the proof. □

We'll now prove another corollary of the triangulation theorem which will play a key role in proving the trivialization theorem. Before proving it we'll need a useful lemma which will also be used in future sections.

Lemma 23. *(definable choice) Let (R,<,0,-,+) be an o-minimal expansion of an ordered abelian group and let $A \subset R^k$ be a non-empty definable subset. Then we can definably pick an element $e(A)$. In other words, for every formula $\psi(\overline{x}_k, \overline{y})$, there exists a formula $\phi(\overline{x}_k, \overline{y})$ such that $\forall \overline{y}(\forall \overline{x}_k(\phi(\overline{x}_k, \overline{y}) \rightarrow \psi(\overline{x}_k, \overline{y})) \wedge \exists! \overline{z}_k(\phi(\overline{z}_k, \overline{y}))$.*

Proof. We'll use induction on k. If $k = 1$, then we choose the smallest element. If no such element exists, then we take the leftmost interval $(a, b) \subset A$ and pick:

- $a = -\infty$, $b = +\infty$: 0
- $a = -\infty$, $b \in R$: $b - 1$
- $a \in R$, $b = +\infty$: $a + 1$
- $a \in R$, $b \in R$: $\frac{a+b}{2}$

Now let's assume that the claim is true for k. The projection $\pi : R^k \rightarrow R^{k-1}$ is definable and therefore $\pi(A) \subset R^{k-1}$ is definable and non-empty. So by the inductive hypothesis we can definably pick some $a \in \pi(A)$. In addition, the fiber $A_a \subset R$ is nonempty and definable so by the base case, we can definably pick some $b \in A_a$. Together, we can definably pick $(a, b) \in A$. □

We can now prove our next corollary to the triangulation theorem.

Corollary 24. *Let $S \subset R^{m+n}$ be a definable set. The the sets $\{S_a | a \in R^m\}$ fall into a finite number of homeomorphism types.*

More specifically, there exists some N and some definable function $f : S \rightarrow R^N$ such that for each $a \in R^m$, $f_a : S_a \rightarrow R^N$ is a homeomorphism from S_a to some union of faces of $(0, e_1, \ldots, e_N)$.

Proof. First we'll assume that R is ω-saturated. Let's assume that S is definable by $\phi(\overline{x}_m, \overline{y}_n)$. By the triangulation theorem, for any $\bar{a} \in R^m$, there exists an $N_{\bar{a}} \in \mathbb{N}$ and a definable homeomorphism $h_{\bar{a}}$ from $S_{\bar{a}}$ to a union of faces of $(0, e_1, \ldots, e_{N_{\bar{a}}})$. Let $\psi_{\bar{a}}(\overline{x}_n, \overline{y}_{N_{\bar{a}}}, \overline{c}_k)$ be the formula defining $h_{\bar{a}}$ where $\overline{c}_k$ are all the constants used in the formula. For any given $\bar{a} \in R^m$, the set

$$\{\overline{c}_k \in R^k | \psi_{\bar{a}}(\overline{x}_n, \overline{y}_{N_{\bar{a}}}, \overline{c}_k) \text{is a homeo' from} S_{\bar{a}} \text{to a subset of faces of} (0, e_1, \ldots, e_{N_{\bar{a}}})\}$$

is definable so by definable choice, we can definably pick a specific $\overline{c}_k$ from the set. This means that we can assume that $\psi_{\bar{a}}(\overline{x}_n, \overline{y}_{N_{\bar{a}}}, \overline{z}_m)$ is a formula without parameters such that $\psi_{\bar{a}}(\overline{x}_n, \overline{y}_{N_{\bar{a}}}, \overline{a})$ is the required homeomorphism. Let's assume for contradiction that there doesn't exist a finite subset H of $\{\psi_{\bar{a}} | \bar{a} \in R^m\}$ such that for every $\bar{b} \in R^m$ there exists a formula $\psi_{\bar{a}} \in H$ such that $\psi_{\bar{a}}(\overline{x}_n, \overline{y}_{N_{\bar{a}}}, \bar{b})$ is a homeomorphism of the required type. Since R is ω-saturated, there exists a $\bar{d} \in R^m$ such that for all $\bar{a} \in R^m$, $\psi_{\bar{a}}(\overline{x}_n, \overline{y}_{N_{\bar{a}}}, d)$ isn't a homeomorphism of the required type. This is clearly a contradiction. So there exists a $M \in \mathbb{N}$ such that if we define the sentence:

$$\alpha = \exists \bar{a}_1 \ldots \exists \bar{a}_M \forall \bar{b} \bigvee_{\bar{a} = \bar{a}_1, \ldots, \bar{a}_M} (\text{"}\psi_{\bar{a}}(\overline{x}_n, \overline{y}_{N_{\bar{a}}}, \bar{b}) \text{ is a homeomorphism}$$

$$\text{from } S_{\bar{b}} \text{ to a subset of faces of } (0, e_1, \ldots, e_{N_{\bar{a}}})\text{"})$$

then $R \models \alpha$.

Now, let R be general structure (i.e. not necessarily ω-saturated) and $S \subset R^{m+n}$ a subset definable with parameters $\overline{c} \in R^k$. Then we can find an ω-saturated elementary extension $R \prec \tilde{R}$. By our construction above, $\tilde{R} \models \alpha$. But the only parameters in α are $\overline{c}$ and therefore $R \models \alpha$.

Let $\bar{a}_1, \ldots, \bar{a}_M \in R^m$ be the constants whose existence is given by α. Let $N = max_{1 \le i \le M}\{N_{\bar{a}_i}\}$. We can now define a construct a function $f : S \to R^N$ in the following fashion. For every $\bar{b} \in R^m$, we choose the smallest $1 \le i \le M$ such that if $\bar{a} = \bar{a}_i$ then $\psi_{\bar{a}}(\overline{x}_n, \overline{y}_{N_{\bar{a}}}, \bar{b})$ defines a homeomorphism from $S_{\bar{b}}$ to some union of faces of $(0, e_1, \ldots, e_{N_{\bar{a}}})$. $f|_{S_{\bar{b}}}$ will be defined as the extension of that homeomorphism to R^N. Clearly f is definable and this completes the proof. □

3.3. **Trivialization.** In this section we use corollary 24 to prove the trivialization theorem. Before we prove the theorem itself, we'll prove a few lemmas about fiberwise properties of continuous functions.

Lemma 25. *Let $S \subset R^{m+n}$ be a definable set such that for each $x \in R^m$, S_x is open in R^n. Then there is a partition of R^m into cells $C_1, \ldots, C_k$ such that for all i, $S \cap (C_i \times R^n)$ is open in $C_i \times R^n$.*

Remark. Since the complement of a definable set is definable, the lemma is equivalent to one in which "open" is replaced with "closed".

Proof. We'll use induction on m. If $m = 0$ then there's nothing to show.

Let's assume the claim is true for all $k < m$. First we'll show that for every open cell $C \subset R^m$ there's an open cell $D \subset C$ such that $S \cap (D \times R^n)$ is open in $D \times R^n$.

Let C be an open cell in R^m. We define:

$$\tilde{C} = \{x \in C | (\{x\} \times S_x) \cap cl((C \times R^n) \backslash S) \neq \emptyset\}$$

Let's assume for contradiction that $\tilde{C}$ has a non-empty interior. Without loss of generality, $\tilde{C}$ is an open cell. By definable choice, for each $x \in \tilde{C}$ we can pick some $s(x) \in S_x$ such that $(x, s(x)) \in cl((\tilde{C} \times R^n) \backslash S)$. In addition, S_x is open in R^n so again by definable choice there's some $\epsilon(x) \in R$ such that for all $y \in R^n$, $|y - s(x)| < \epsilon(x) \Rightarrow y \in S_x$. By replacing $\tilde{C}$ with a suitable subcell we can assume that $s(x)$ and $\epsilon(x)$ are continuous. But then, the set:

$$\{(x, y) \in R^{m+n} : |y - s(x)| < \epsilon(x)\}$$

is an open subset of S in $\tilde{C} \times R^n$ which is a contradiction to the definition of $s(x)$.

Therefore, $\tilde{C}$ has an empty interior and so $C \backslash \tilde{C}$ has an open interior. Let $D \subset C \backslash \tilde{C}$ be an open cell. Then $(S \cap (D \times R^n)) \cap cl((S^c \cap (D \times R^n) = \emptyset$ and so $S \cap (D \times R^n)$ is open in $D \times R^n$.

We now define an additional set $A \subset R^m$ which contains all the points $x \in R^m$ such that there exists an open box $x \in D \subset R^m$ such that $(D \times R^n) \cap S$ is open in $D \times R^n$. Then A is definable and $S \cap (A \times R^n)$ is open in $A \times R^n$. In addition, by what we just showed above, $dim(R^m \backslash A) < m$. This means that we can write $R^m \backslash A$ as a disjoint union of cells $B_1, \ldots, B_r$ where for each $1 \leq i \leq r$, $dim(B_i) < m$.

Let $1 \leq i \leq r$ and let $dim(B_i) = d < m$. Then there's a projection $\pi : R^m \to R^d$ such that $\pi(B_i)$ is open in R^d and $\pi|B_i$ is a homeomorphism. We can now apply the inductive hypothesis to $\pi(B_i)$. This completes the induction. □

We now extend the previous lemma to a slightly more general case.

Lemma 26. *Let $\tilde{S} \subset S$ be definable sets in R^{m+n} and let $A \subset R^m$ be definable such that $\tilde{S}_a$ is open in S_a for all $a \in A$. Then there's a partition of A into definable subsets $A_1, \ldots, A_M$ such that for all $1 \leq i \leq M$, $\tilde{S} \cap (A_i \times R^n)$ is open in $S \cap (A_i \times R^n)$.*

Remark. As in the previous lemma, we can replace "open" with "closed".

Proof. For this lemma we'll prove the closed version. By replacing S with $S \cap (A \times R^n)$ we can assume that $\tilde{S}_x$ is closed in S_x for all $x \in R^m$. We now define an additional set:

$$S^* = \{(\overline{x}, \overline{y}) \in R^{m+n} | \overline{y} \in cl(\tilde{S}_x)\}$$

Then S^* is definable and $S^*_x \cap S_x = \tilde{S}_x$ for all $x \in R^m$ which means that $S^* \cap S = \tilde{S}$. By lemma 25 there exists a partition of R^m, $A_1, \ldots, A_M$ such that for each $1 \leq i \leq M$, $\tilde{S} \cap (A_i \times R^n)$ is closed in $A_i \times R^n$. Therefore, for each $1 \leq i \leq M$, $\tilde{S} \cap (A_i \times R^n) = S \cap (S^* \cap (A_i \times R^n)$ is closed in $S \cap (A_i \times R^n)$. □

Lemma 27. *Let $S \subset R^{m+n}$ be definable, $f : S \to R^k$ a definable map and $A \subset R^m$ be a definable subset such that for all $a \in A$, the map $f_a : S_a \to R^k$ is continuous. Then there's a partition $A_1, \ldots, A_M$ of A such that for each $1 \leq i \leq M$, $f|_{S \cap (A_i \times R^n)}$ is locally bounded.*

Proof. We'll use an inductive proof on m, similar to the proof of lemma 25. The case of $m = 0$ is trivial.

Let's assume the claim is true for $k < m$. We now define a set:

$$C = \{a \in R^m | \exists x \in \{a\} \times S_a \text{ s.t f isn't locally bounded at } x\}$$

. As in lemma 25, it's enough to show that C has an empty interior.

Assume for contradiction that C has an interior. With out loss of generality C is an open cell. We now use definable choice to define a definable function $g : C \to R^n$

such that for all $a \in C$, f isn't locally bounded at $(a, g(a))$. In addition, we define a function $h : C \to R^k$ such that for all $a \in C$, $h(a) = f((a, g(a))$. By cell decomposition we can assume that h and g are continuous on C.

Finally, we define an additional function $m : C \to R_{>0}$ such that for all $a \in C$ and $b \in \{a\} \times S_a$,

$$|b - g(a)| < m(a) \Rightarrow |f(a, b) - f(a, g(a))| < 1$$

. m exists by the assumption that f is continuous on every fiber S_a. As before, we can assume that m is continuous.

Now, the set:

$$B = \{(a, b) \in (C \times R^n) \cap S \ : \ |b - g(a)| < m(a)\}$$

is open in S. But by the definition of m and the fact that h is continuous, f is locally bounded on B. This is a contradiction to the fact that f is not locally bounded on $(a, g(a))$ for all $a \in C$. ☐

We're now ready to prove the first fiberwise property on continuous functions.

Lemma 28. *Let $S \subset R^{m+n}$ be definable, $f : S \to R^k$ a definable map and $A \subset R^m$ be a definable subset such that for all $a \in A$, the map $f_a : S_a \to R^k$ is continuous. Then there's a partition $A_1, \ldots, A_M$ of A such that for each $1 \le i \le M$, $f|_{S \cap (A_i \times R^n)}$ is continuous.*

Proof. We apply lemma 26 to the graph of f, $\Gamma(f) \subset S \times R^k$. For each $a \in A$, $\Gamma(f)_a = \Gamma(f_a)$ and therefore, since R is Hausdorff, $\Gamma(f)_a$ is closed in $S_a \times R^k$. This means that there's a partition $A_1, \ldots, A_M$ of A such that for each $1 \le i \le M$, $\Gamma(f|_{S \cap (A_i \times R^n)}) = \Gamma(f) \cap (A_i \times R^n)$ is closed in

$$(S \cap (A_i \times R^n)) \times R^k = (S \times R^k) \cap (A_i \times R^{n+k})$$

. By lemma 27 we can refine this partition such that for every A_i, $f|_{S \cap (A_i \times R^n)}$ is locally bounded. This proves that $f|_{S \cap (A_i \times R^n)}$ is continuous. (A straightforward consequence of lemma 1.7 from chapter 6 in [1] is that a definable locally bounded function with a closed graph is continuous.) ☐

Lemma 29. *Let $S \subset R^{m+n}$ be a definable subset, $f : S \to R^k$ a definable map and $A \subset R^m$ a definable subset such that $f|_{S \cap (A \times R^n)}$ is injective and for each $a \in A$, $f_a : S_a \to R^k$ is a homeomorphism from S_a to $f(S_a)$. Then there exists a partition $A_1, \ldots, A_M$ of A such that for each $1 \le i \le M$, $f|_{S \cap (A_i \times R^n)}$ is a homeomorphism from $S \cap (A_i \times R^n)$ to $f(S \cap (A_i \times R^n))$.*

Proof. By lemma 28, there's a partition $A_1, \ldots, A_M$ of A such that f is a continuous injection on each $S \cap (A_i \times R^n)$. Let $1 \le i \le M$.

We'll now show that there exists a partition $B_1, \ldots, B_K$ of A such that for all $1 \le i \le K$, $(f|_{S \cap (B_i \times R^n)})^{-1}$ is locally bounded.

We'll define the set $T = \{(a, x) \in A \times R^k | x \in f(S_a)\}$ and function $g : T \cap (A \times R^k) \to R^n$ by:

$$g(a, x) = y \iff f(a, y) = x$$

By lemma 27 and the fact that for each $a \in A$, $f_a : S_a \to R^k$ is a homeomorphism from S_a to $f(S_a)$, there exists a partition $B_1, \ldots, B_K$ of A such that $g|_{T \cap (B_i \times R^k)}$ is locally bounded for all $1 \le i \le K$.

Let $x_0 \in R^k$. Then there exist exactly one $1 \le i \le K$ and $b_i \in B_i$ such that $(b_i, x_0) \in (B_i \times R^k) \cap T$. By the way we defined $B_1, \dots, B_K$, there exist a bounded set $a_i \in U \subset B_i$, an open set $x_0 \in V \subset R^k$ and a bounded set $W \subset R^n$ such that for all $(a, x) \in T \cap (U \times V)$, $g(a, x) \in W$. But by the definition of g and the fact that f is injective, this means that for all $x \in V \cap f(B_i \times R^n)$, $(f|_{S \cap (B_i \times R^n)})^{-1}(x) \in U \times W$.

So without loss of generality, by taking a finer partition we can assume that for all $1 \le i \le M$, $(f|_{S \cap (A_i \times R^n)})^{-1}$ is locally bounded.

Finally, the graph of $(f|_{S \cap (A_i \times R^n)})^{-1}$ is closed because f is continuous on $S \cap (A_i \times R^n)$. Since $(f|_{S \cap (A_i \times R^n)})^{-1}$ is locally bounded, this means that $(f|_{S \cap (A_i \times R^n)})^{-1}$ is continuous which completes the proof. □

Lemma 30. *Let $S \subset R^{m+n}$ be a definable subset. Then there's a partition $A_1, \dots, A_K$ of R^m, an $N \in \mathbb{N}$, definable sets $\{F_i \subset R^N | 1 \le i \le K\}$ and definable homeomorphisms $\{h_i : S \cap (A_i \times R^n) \to A_i \times F_i \subset R^n | 1 \le i \le K\}$ such that for each $1 \le i \le K$ and $x \in S \cap (A_i \times R^n)$, $\pi_m^{m+n}(x) = \pi_m^{m+N}(h_i(x))$.*

Proof. By corollary 24, there exists a partition $A_1, \dots, A_K$ of R^m, definable sets $\{F_i \subset R^n | 1 \le i \le K\}$ and definable functions

$$\{f_i : S \cap (A_i \times R^n) \to F_i \subset R^n | 1 \le i \le K\}$$

such that for each $1 \le i \le K$ and $a \in A_i$, f_a is a homeomorphism from S_a to F_i. Let's define new functions $\tilde{f}_i : S \cap (A \times R^n) \to A_i \times F_i$ by $\tilde{f}_i((a, y)) = (a, f_i(a))$. By lemma 29, for each $1 \le i \le K$, there exists a partition $A_i^1, \dots, A_i^N$ of A_i such that for each A_i^j, $\tilde{f}_i|_{S \cap (A_i^j \times R^n)}$ is a homeomorphism. In addition, by its definition, for each $x \in S \cap (A_i^j \times R^n)$ we have $\pi_m^{m+n}(x) = \pi_m^{m+N}(\tilde{f}_i(x))$. □

Theorem 31. *(trivialization) Let $A \subset R^m$ and $B \subset R^n$ be definable sets and $f : B \to A$ a definable function. Then there exists a partition $A_1, \dots, A_K$ of A, an $N \in \mathbb{N}$, definable sets $\{F_i \subset R^N | 1 \le i \le K\}$ and definable homeomorphisms $\{h_i : B \cap f^{-1}(A_i) \to A_i \times F_i | 1 \le i \le K\}$ such that for each $1 \le i \le K$ and $x \in B \cap f^{-1}(A_i)$, $f(x) = \pi_m^{m+N}(h_i(x))$.*

Proof. Let's define the set:

$$S = \{(f(x), x) | x \in B\}$$

By lemma 30, there exists a partition $A_1, \dots, A_K$ of A, a $N \in \mathbb{N}$, definable sets $\{F_i \subset R^N | 1 \le i \le K\}$ and definable homeomorphisms

$$\{h_i : S \cap (A_i \times R^n) \to A_i \times F_i \subset R^{m+N} | 1 \le i \le K\}$$

such that for each $1 \le i \le K$ and $x \in B \cap f^{-1}(A_i)$, $f(x) = \pi_m^{m+n}(f(x), x) = \pi_m^{m+N}(h_i(f(x), x))$. For each $1 \le i \le K$, we now define

$$\tilde{h}_i : B \cap f^{-1}(A_i) \to A_i \times F_i$$

by $\tilde{h}_i(x) = h_i(f(x), x)$. By the previous line, the partition (A_i), subsets (F_i) and homeomorphisms $(\tilde{h}_i)$ have the required properties. □

3.4. **Applications of the Trivialization Theorem.** There're many interesting applications of the trivialization theorem. In this section we'll use it to prove the theorem mentioned in the introduction regarding the number of homeomorphism types among the sets

$$Z(f) = \{x \in \mathbb{R}^n | f(x) = 0\}$$

where $f \in \mathbb{R}[X_1, \ldots, X_n]$ has at most m monomials. Interestingly, despite the fact that the proof we'll present relies heavily on model theoretic machinery, the result is entirely classical.

First we note that the trivialization theorem can be used to obtain a short proof of corollary 24 which for convenience is stated below. Despite the fact that the corollary was used to prove the trivialization, there exists a topological proof of the trivialization theorem which can in turn be used to prove the corollary without the compactness theorem.

Corollary 32. *Let $S \subset R^{m+n}$ be a definable set. Then the definable sets S_x fall into a finite number of homeomorphism types as x ranges over R^m.*

Proof. Let $f : S \to R^m$ be the restriction of the projection map $\pi : R^{m+n} \to R^m$. By the trivialization theorem, there exists a partition $A_1, \ldots, A_M$ of R^m, an $N \in \mathbb{N}$, definable subsets $F_1, \ldots, F_M$ of R^N and homeomorphisms

$$\{h_i : S_{A_i} \to A_i \times F_i | 1 \leq i \leq M\}$$

such that for all $1 \leq i \leq M$ and $x \in S_{A_i}$, $f(x) = \pi^{m+N}_m(h_i(x))$. But since f is the projection, this means that for every $a \in A_i$, $h_i|_{S_a}$ is a homeomorphism from S_a to F_i. Therefore the sets S_x fall into at most M homeomorphism types.

We'll now use the lemma to prove the theorem about polynomials. □

Theorem 33. *For any given natural numbers m and n there are only finitely many homeomorphism types among the sets $Z(f) = \{x \in \mathbb{R}^n | f(x) = 0\}$ where $f(X_1, \ldots, X_n) \in \mathbb{R}[X_1, \ldots, X_n]$ has at most m monomials.*

Proof. The idea of the proof is taken from [1]. First of all, we'll use the fact stated in the introduction that $\mathbb{R}_{exp}$ is an o-minimal structure. After that, the general idea is that we can define x^y by:

$$x^y = exp(ylog(x))$$

which allows us to look at the sets $Z(f)$ as fibers of a definable set. We then use the previous lemma.

The proof is slightly more complicated due to the fact that $x^n = exp(nlog(x))$ is only well defined for $x > 0$. So in practice we'll have to split up into two cases, even and odd:

$$E(x, y) = \begin{cases} |x|^y & x \neq 0 \\ 1 & x = y = 0 \\ 0 & x = 0,\ y \neq 0 \end{cases}$$

$$O(x, y) = \begin{cases} x^y & x > 0 \\ -(-x)^y & x < 0 \\ 0 & x = 0 \end{cases}$$

Now, for an even n we have $x^n = E(x, n)$ and for an odd n we have $x^n = O(x, n)$.

Fix $m, n \in \mathbb{N}$ and let

$$f = \sum_{i=1}^{m} a_i \prod_{j=1}^{n} x_j^{\alpha_{ij}}$$

be an arbitrary polynomial in $\mathbb{R}[X_1, \ldots, X_n]$ where $a_i \in \mathbb{R}$ and $\alpha_{ij} \in \mathbb{N}$. In order to write f in the language of $\mathbb{R}_{exp}$, we have to know for each α_{ij} whether it's even or odd.

Let's define $Sgn(f) \subset \{0,1\}^{mn}$ by:

$$Sgn(f)_{ij} = \begin{cases} 0 & \alpha_{ij} \in 2\mathbb{N} \\ 1 & otherwise \end{cases}$$

. Using $Sgn(f)$, we can now define f in our language by:

$$f(x_1, \ldots, x_n) = \phi_{Sgn(f)}(x_1, \ldots x_n, a_1, \ldots, a_n, \alpha_{11}, \ldots \alpha_{mn}) =$$

$$\sum_{i=1}^{m} a_i \prod_{Sgn(f)_{ij}=0} E(x_i, \alpha_{ij}) \prod_{Sgn(f)_{ij}=1} O(x_i, \alpha_{ij})$$

. Since $Sgn(f)$ takes a finite number of values, it's enough to prove the theorem for polynomials g such that $Sgn(g) = Sgn(f)$ since f was arbitrary.

Now, if we define $S \subset \mathbb{R}^{mn+n+n}$ by:

$$\phi_{Sgn(f)}(x_1, \ldots x_n, a_1, \ldots, a_n, \alpha_{11}, \ldots \alpha_{mn}) = 0$$

, then the family of sets $Z(g)$ such that $Sgn(g) = Sgn(f)$ is the set of fibers $S_{\bar{a}}$ for $\bar{a} \in \mathbb{R}^{mn+n}$ where $\bar{a}$ represents the constants of the polynomial. The theorem then follows from the previous lemma.

□

4. Power Boundedness in O-minimal Structures

4.1. **Introduction and Preliminaries.** Chris Miller [4] proved that if an o-minimal expansion of the real numbers contains definable functions that are not polynomially bounded then the exponential function is definable. The central idea of his proof uses a result from the theory of the valuation of Hardy fields which was proven by Maxwell Rosenlicht [6]. Chris Miller later extended his result to o-minimal expansions of arbitrary real closed fields using similar methods [10]. In that setting, the idea of a polynomially bounded function is generalized to power bounded functions. An alternative proof using an interesting growth dichotomy for o-minimal expansions of ordered groups was obtained by Chris Miller and Sergei Strachenko [8].

In this section I'll present a proof using the same ideas as Miller's original proof which relies on only elementary properties of o-minimal structures together with some basic calculus. As an intermediate step, we'll obtain an additional property which is equivalent to the definability of the exponent. Afterwards, I'll use it to obtain a uniform bound on the growth of functions in o-minimal expansions of real closed fields. Finally, I'll use the uniform bound to prove some properties of power bounded structures.

For the rest of this section, $R = (R, <, 0, 1, +, \cdot, \dots)$ will be an o-minimal expansion of a real closed field.

We'll also use the following notion:

Definition 34. Let f and g be definable functions in R and $M \in R$ such that for all $x \geq M$, $g(x) \neq 0$. Then we'll write $f \sim g$ if $lim_{x\to\infty} f(x)/g(x) = 1$.

Remark. When we say *ultimately* we mean "for all sufficiently large x".

In addition, we don't distinguish between a function going to $+\infty$ and $-\infty$. In other words, when we say that $lim_{x\to\infty} f = \infty$ we mean $lim_{x\to\infty} f = \pm\infty$.

The following propositions can be found in [1] and will be used in the proofs below.

Proposition 35. *Let $a, b \in R \cup \{\pm\infty\}$ and $f : (a, b) \to R$ be a definable function. Then there're $a = a_0 < \cdots < a_n = b$ such that f is differentiable on every subinterval (a_i, a_{i+1}).*

Remark. By the previous proposition, any definable function $f : R \to R$ is ultimately differentiable.

Proposition 36. *Suppose $\cdot : R \times R \to R$ is a definable function such that $(R, <, \cdot)$ is an ordered group. Then any definable subgroup is either trivial or R.*

Furthermore, as shown in chapter 7 of [1] (2.2 and 2.12), the mean value theorem and L'hopital's rule hold in o-minimal expansions of a real closed field.

4.2. Power Functions and Exponential Functions.

Definition 37. We'll define four types of functions.

(1) A *power function* on R is a definable endomorphism of the multiplicative group $(R_{>0}, \cdot, 1)$.
(2) An *exponential function* on R is a definable ordered group isomorphism $E : (R, <, +) \to (R_{>0}, <, \cdot)$.
(3) The definable exponent E such that $E'(x) = E(x)$ will be called *the exponential function* of R and will also be written as e^x. As we'll see below, exponent exists whenever some exponential functions exists and it's unique.
(4) The inverse function of the exponential function from $R_{>0}$ to R will be called the *logarithmic function* of R.

Remark 38. We'll mention some immediate results regarding power functions and exponential functions which we'll use below.

(1) Since power functions are ultimately differentiable, they're differentiable everywhere. In addition, for every power function λ, $\lambda'(x) \cdot x = \lambda(1) \cdot \lambda(x)$. This follows by directly calculating the derivative:

$$\frac{\lambda'(x)}{\lambda(x)} x = lim_{h\to 0} \frac{\lambda(x+h)/\lambda(x) - 1}{h} x = lim_{h\to 0} \frac{\lambda(1 + \frac{h}{x})}{\frac{h}{x}} = \lambda'(1)$$

. From this we can see that if $\lambda'(1) > 0$ then λ is strictly increasing, if $\lambda'(1) < 0$ then λ is strictly decreasing, and if $\lambda'(1) = 0$ then $\lambda = 1$.

We'll now see that the set of power functions is closed under elementary operations.

- If f is a power function then $\frac{1}{f}$ is a power function and $(1/f)'(1) = -f'(1)$

- If f and g are power functions then $f \cdot g$ is a power function and $(f \cdot g)'(1) = f'(1) + g'(1)$.
- If f and g are power functions then $f \circ g$ is a power function with $(f \circ g)'(1) = f'(1) \cdot g'(1)$

We now define $K \subset R$ as the image of the map $\lambda \mapsto \lambda'(1)$ from the set of power functions to R. The map $\lambda \mapsto \lambda'(1)$ is injective because if f and g are power functions then

$$f'(1) = g'(1) \Rightarrow (f/g)'(1) = 0 \Rightarrow (f/g)' = 0 \Rightarrow (f/g) = 1 \Rightarrow f = g$$

. This, together with the remarks above, shows that K can be regarded as a field and $\lambda \mapsto \lambda'(1)$ is an embedding of K into R. K will be called the *field of exponents*.

(2) From the definition of an exponential function, since it's ultimately differentiable it must be differentiable everywhere. Furthermore, if E is an exponential function then $E'(x) = E(0)E(x)$ for all $x \in R$:

$$\frac{E'(x)}{E(x)} = lim_{h \to 0} \frac{E(x+h) - E(x)}{hE(x)} = lim_{h \to 0} \frac{E(h) - E(0)}{h} = E(0)$$

. Therefore, assuming that there exists some non-zero exponential function F, $E = F/F'(0)$ will be the exponential function. This function is unique because given two exponential functions E, F such that $E' = E$ and $F' = F$, we can show that $(E/F)(0) = 1$ and $(E/F)' = 0$ which means that $E/F = 1 \Rightarrow E = F$.

The following lemma provides a useful way to characterize power functions. It also shows that a power function can be defined using only it's derivative at 1.

Lemma 39. *Let λ be a power function. Then λ is the unique solution to the equation*

$$x \cdot y' = y'(1)y$$

.

Proof. By remark 38, it's the unique solution within the set of power functions. So it's enough to show that for every definable function f, if $x \cdot f' = f'(1)f(x)$ then f is a power function. Let f be a solution. We first note that $f(1) = 1$. Let a be an element in R. Then

$$\left(\frac{f(ax)}{f(x)}\right)' = \frac{af'(ax)f(x) - f'(x)f(ax)}{f^2(x)} = \frac{f(ax)\frac{f(x)}{x}f'(1) - f'(x)f(ax)}{f^2(x)} = 0$$

. Therefore, there exists a constant $c \in R$ such that $f(ax) = c \cdot f(x)$. But $f(a \cdot 1) = c \cdot 1$. So for all $x \in R$, $f(ax) = f(a) \cdot f(x)$. This proves the lemma. □

We'll now define the notions of power bounded and exponential fields.

Definition 40.

(1) We say that R is *power bounded* if for every definable function f, there exists a power function λ and an $M \in R$ such that $|f(x)| \leq \lambda(x)$ for any $x \geq M$.
(2) We say that R is *exponential* if it contains an exponential function.

Remark. Every polynomially bounded structure is power bounded. However, the converse isn't true.

For example, we can start with the structure $M = (\mathbb{R}, +, \cdot, 1, 0, <)$ and extend it to $M_1 = (\mathbb{R}, +, \cdot, 1, 0, <, N, e^x)$ where $N = \mathbb{N}$. Let M_2 be an elementary expansion of M_1 with a non-standard element c such that $c \in N^{M_2}$. Now let M_3 be the restriction of M_2 to the language $(+, \cdot, 1, 0, <, x^c)$. Clearly, M_3 isn't polynomially bounded. We claim that M_3 is power bounded.

By corollary 52, it's enough to show that e^x isn't definable in M_3. Assume for contradiction that e^x is definable in M_3.

We can write a sentence ψ in the language $(+, \cdot, 1, 0, <, N, e^x)$ saying that

"*there exists an element a such that $a \in N$ and $\phi(x, a)$ defines e^x*",

where $\phi(x, y)$ is a formula in the language $(+, \cdot, 1, 0, <, x^y)$. We note that by remark 38 we can say that a formula defines e^x using the language of M.

The function x^y is definable by using e^x. By our assumption, $M_2 \models \psi \Rightarrow M_1 \models \psi$. This means that there exists an $n \in \mathbb{N}$ such that e^x is definable in $(\mathbb{R}, +, \cdot, 1, 0, <, x^n)$. Since x^n is definable in M, we got that e^x is definable in M. But by Tarski's QE theorem, M is polynomially bounded which is a contradiction.

Lemma 41. *If R is exponential then any elementary equivalent structure is exponential as well.*

Proof. Let M be elementary equivalent to R. Then M is o-minimal. In addition, the exponential function E of R is unique and therefore 0-definable. This means that M defines E as well and in particular it's exponential. □

We now introduce a useful way of obtaining power functions which appears in Chris Miller's paper.

Lemma 42. *Let f be a definable function, ultimately non-zero, such that $lim_{x\to\infty} \frac{f(2x)}{f(x)} \in R_{>0}$. Then the function $P(f) : R_{>0} \to R_{>0}$ defined by*

$$P(f)(t) = lim_{x\to\infty} \frac{f(tx)}{f(x)}$$

is a power function.

Remark. In order to shorten the notion, we'll frequently write Pf instead of $P(f)$.

Proof. First we'll show that the function $P(f)$ is positive and defined for all $t \in R_{>0}$. The set:

$$A = \{t \in R_{>0} | lim_{x\to\infty} \frac{f(tx)}{f(x)} \in R_{>0}\}$$

is a definable subgroup of $(R_{>0}, \cdot, 1)$. Therefore, it's either trivial or $(R_{>0}, \cdot, 1)$. By the condition of the lemma, $2 \in A$ so $A = R_{>0}$. In addition,

$$Pf(st) = lim_{x\to\infty} \frac{f(xst)}{f(x)} = lim_{x\to\infty} \frac{f(xst)}{f(xs)} \cdot \frac{f(xs)}{f(x)} = Pf(t) \cdot Pf(s)$$

so Pf is a power function. □

We can think of $P(f)$ as the "power part" of f. We'll now prove one more lemma regarding this function.

Lemma 43. *Let f be a definable function, ultimately non-zero, such that $lim_{x\to\infty}\frac{f(2x)}{f(x)} \in R_{>0}$. Then $P(Pf/f) = 1$.*

Proof. For every $t \in R_{>0}$,

$$P(Pf/f)(t) = lim_{x\to\infty}\frac{Pf(xt)}{f(xt)}\cdot\frac{f(x)}{Pf(x)} = Pf(t)\cdot lim_{x\to\infty}\frac{f(x)}{f(xt)} = 1$$

□

4.3. **O-minimal Expansions of Real Closed Fields.** We'll prove some results related to o-minimal expansions of real closed fields. The motivation for these lemmas comes from the theory of valuations of Hardy fields. In addition, some of the proofs are simply generalizations of the proofs in Maxwell Rosenlicht's paper [6] to arbitrary o-minimal expansions of real closed fields. In addition, some of the proofs are similar to the ones in [10].

Lemma 44. *Let $a(x)$ and $b(x)$ be definable functions from R to R such that $lim_{x\to\infty}a(x) = lim_{x\to\infty}b(x) = \infty$ and $lim_{x\to\infty}\frac{a}{b} \in R$. Then $0 \le lim_{x\to\infty}\frac{a'/a}{b'/b} \le 1$.*

Proof. If $lim_{x\to\infty}\frac{a}{b} > 0$ then $lim_{x\to\infty}\frac{a'}{b'} = lim_{x\to\infty}\frac{a}{b}$ and so $lim_{x\to\infty}\frac{a'/a}{b'/b} = 1$.

Let's assume that $lim_{x\to\infty}\frac{a}{b} = 0$. This means that $lim_{x\to\infty}\frac{b}{a} = \infty$ which implies that ultimately, $\left(\frac{b}{a}\right)' > 0$. So ultimately,

$$\frac{b'a - a'b}{a^2} > 0 \Rightarrow b'a - a'b > 0 \Rightarrow 0 \le \frac{a'/a}{b'/b} \le 1$$

. Therefore, $0 \le lim_{x\to\infty}\frac{a'/a}{b'/b} \le 1$. □

Lemma 45. *Let $a(x)$ and $b(x)$ be definable functions from R to R such that $lim_{x\to\infty}a(x) \in R$ and $lim_{x\to\infty}b(x) = \infty$. Then $lim_{x\to\infty}a'\frac{b}{b'} = 0$.*

Proof. Without loss of generality we can assume that $lim_{x\to\infty}a \neq 0$. Because otherwise we can just look at $a + 1$. Therefore:

$$a = \frac{ab}{b} \sim \frac{a'b + b'a}{b'} = a + \frac{a'b}{b'}$$

Which means that $lim_{x\to\infty}a'\frac{b}{b'} = 0$. □

Lemma 46. *Let $g : R \to R$ be a definable function such that $lim_{x\to\infty}x\cdot g(x) = \infty$. Then there exists a definable function $u : R \to R$ such that $lim_{x\to\infty}\frac{u'}{g} \in R_{\ge 0}$ and $lim_{x\to\infty}u = \infty$.*

Proof. We define

$$u = \begin{cases} x & lim_{x\to\infty}g = \infty \\ g\cdot x & \text{otherwise} \end{cases}$$

. u is obviously definable. In addition, $lim_{x\to\infty}\frac{u}{x} \in R$ and $lim_{x\to\infty}u = \infty$. So by lemma 44, $lim_{x\to\infty}\frac{u'}{u}\cdot x \in R_{\ge 0}$. In addition, $lim_{x\to\infty}\frac{u}{g\cdot x} \in R_{\ge 0}$. So together:

$$lim_{x\to\infty}\frac{u'}{g} = lim_{x\to\infty}\frac{u'\cdot x}{u}\cdot\frac{u}{g\cdot x} \in R_{\ge 0}$$

. □

Lemma 47. *Let $f : R \to R$ be a definable function such that $lim_{x\to\infty} f = \infty$ and $lim_{x\to\infty} \frac{f'}{f} x = \infty$. Then there exists a definable function $h : R \to R$ such that $h' \sim \frac{1}{x}$.*

Proof. Let's define $g = \frac{f'}{f}$. By lemma 46, there exists a definable function $u : R \to R$ such that $lim_{x\to\infty} \frac{u'}{g} \in R$. We'll now show that:

$$\left(g \cdot \frac{(g \cdot u)/u'}{((g \cdot u)/u')'}\right)' \sim g$$

. First of all, $lim_{x\to\infty} u = \infty$ and since u is definable, it ultimately must be strictly monotonic and so ultimately, $u' \neq 0$. In addition, $lim_{x\to\infty} \frac{u'}{g} \in R_{\geq 0}$ and $lim_{x\to\infty} u = \infty$ so $lim_{x\to\infty} \frac{g \cdot u}{u'} = \infty$ which means that ultimately $\left(\frac{g \cdot u}{u'}\right)' \neq 0$. This means that the expression above is well defined.

In addition,

$$\left(g \cdot \frac{(g \cdot u)/u'}{((g \cdot u)/u')'}\right)' = \left(\frac{g}{((g \cdot u)/u')'} \cdot ((g \cdot u)/u')\right)' = g + ((g \cdot u)/u') \left(\frac{g}{((g \cdot u)/u')'}\right)'$$

so it's enough to show that:

$$lim_{x\to\infty} (u/u') \left(\frac{g}{((g \cdot u)/u')'}\right)' = 0$$

. Since $lim_{x\to\infty} u = \infty$, $lim_{x\to\infty} \frac{g \cdot u}{u'} = \infty$ and $lim_{x\to\infty} \frac{u'}{g} \in R_{\geq 0}$, by lemma 44:

$$lim_{x\to\infty} \frac{u'/u}{(gu/u')'/(gu/u')} \in R \Rightarrow lim_{x\to\infty} \frac{g}{(gu/u')'} \in R$$

. So by lemma 45,

$$lim_{x\to\infty} (u/u') \left(\frac{g}{((g \cdot u)/u')'}\right)' = 0$$

.

Now, let's look at the function:

$$v = g \cdot \frac{(g \cdot u)/u'}{((g \cdot u)/u')'}$$

. v is clearly definable and $v' \sim g = \frac{f'}{f}$. We now define: $h = v \circ f^{-1}$. We note that since f is definable, there's an $M \in R$ such that f is monotonic and hence invertible on $\{x \in R : x > M\}$ and we only need to define h on this subset of R since we're only concerned with it's asymptotic properties. Finally:

$$h' = \frac{v'(f^{-1})}{f'(f^{-1})} \sim \frac{1}{f'(f^{-1})} \cdot \frac{f'(f^{-1})}{f(f^{-1})} = \frac{1}{x}$$

□

4.4. **The Definability of the Exponent.** In this section we'll provide a necessary and sufficient condition for an o-minimal expansion of a real closed field to define e^x. Afterwards we'll show that this property is equivalent to the field not being power bounded.

Lemma 48. *Assume that there exists a definable function $f : R \to R$ such that ultimately $f \neq 0$ and $lim_{x\to\infty}\frac{f'}{f}x = \infty$. Then there exists a definable function $g : R \to R$ such that $lim_{x\to\infty}g = \infty$ and $lim_{x\to\infty}\frac{g'}{g}x = \infty$.*

Proof. If $lim_{x\to\infty}f = \infty$ then there's nothing to prove.

In addition, there doesn't exist a constant $0 \neq C \in R$ such that $lim_{x\to\infty}f = C$. Because otherwise, $lim_{x\to\infty}\frac{f}{x} \in R$ and so by lemma 44,

$$lim_{x\to\infty}\frac{f'/f}{1/x} = lim_{x\to\infty}\frac{f'}{f}x \in R$$

which is a contradiction.

Finally, if $lim_{x\to\infty}f = 0$, then we can take $h = 1/f$. Because then,

$$lim_{x\to\infty}\frac{h'}{h}x = lim_{x\to\infty} - \frac{f'}{f^2}\cdot f \cdot x = -lim_{x\to\infty}\frac{f'}{f}x = \infty$$

□

Theorem 49. *Let $R = (R, <, 0, 1, +, \cdot, \dots)$ be an o-minimal expansion of a real closed field. Then e^x is definable in R iff there exists a definable function $f : R \to R$ such that ultimately $f \neq 0$ and $lim_{x\to\infty}\frac{f'}{f}x = \infty$.*

Proof. First of all, if e^x is definable then

$$lim_{x\to\infty}\frac{(e^x)'}{e^x}x = lim_{x\to\infty}x = \infty$$

which proves the first direction.

Now, let's assume that there exists a definable function $f : R \to R$ such that $lim_{x\to\infty}\frac{f'}{f}x = \infty$. By lemma 48, with out loss of generality we can assume that $lim_{x\to\infty}f = \infty$.

According to lemma 47, there exists a definable function g such that $g' \sim \frac{1}{x}$.

Claim. For all $t \in R_{>0}\backslash\{1\}$, $lim_{x\to\infty}(g(x \cdot t) - g(x)) \in R\backslash\{0\}$.

Proof. We'll prove the claim under the assumption that $t > 1$. The case $0 < t < 1$ is identical.

Let $M \in R$ be such that g is differentiable for all $x > M$. According to the mean value theorem, for every $x > M$ there exists an $x < \epsilon(x) < xt$ such that $\frac{g(xt)-g(x)}{t-1} = x \cdot g'(\epsilon(x))$. In addition:

$$lim_{x\to\infty}x\cdot g'(x) = 1 \Rightarrow lim_{x\to\infty}\epsilon(x)\cdot g'(\epsilon(x)) = 1 \Rightarrow lim_{x\to\infty}\frac{\epsilon(x)}{x}\cdot(x\cdot g'(\epsilon(x))) = 1$$

. But

$$lim_{x\to\infty}(x \cdot g'(\epsilon(x))) \leq lim_{x\to\infty}\frac{\epsilon(x)}{x}\cdot(x\cdot g'(\epsilon(x))) \leq lim_{x\to\infty}t\cdot(x\cdot g'(\epsilon(x)))$$

so $lim_{x\to\infty}(x \cdot g'(\epsilon(x)))$ is positive and therefore

$$lim_{x\to\infty}(g(xt) - g(x)) = (t-1)lim_{x\to\infty}(x \cdot g'(\epsilon(x)))$$

is positive as well. □

Now let's define a function $G : R \to R$ by $G(t) = lim_{x\to\infty}(g(x \cdot t) - g(x))$. Since g is definable, G is definable as well. Furthermore, for all $s, t \in R_{>0}$:

$$G(t) + G(s) = lim_{x\to\infty}(g(xt) - g(x)) + lim_{x\to\infty}(g(xs) - g(x)) =$$

$$lim_{x\to\infty}(g(xst) - g(xs) + g(xs) - g(x)) = G(st)$$

So we've proved that for all $x, y \in R > 0$, $G(xy) = G(x) + G(y)$.

In addition, both the kernel and the image of G are definable subgroups of R so G is a bijection. We'll denote it's inverse by $F : R \to R_{>0}$. Clearly, F is a non-zero exponential function. By the remark 38, this implies that e^x is definable in R. $\square$

We'll know show that the condition in theorem 49 is equivalent to the condition that R isn't power bounded. The following lemma was proved by Chris Miller [10].

Lemma 50. *Assume that for every definable f in R, if ultimately $f \neq 0$ and $lim_{x\to\infty}\frac{f'}{f}x \notin R^*$ then $lim_{x\to\infty}f \in R^*$. Then R is power bounded. Furthermore, for every definable function $f : R \to R$, if ultimately $f \neq 0$ then there exists an element $c \in R^*$ and a power function λ such that $f \sim c \cdot \lambda$.*

Proof. Let $f : R \to R$. If $lim_{x\to\infty}f(x) \in R$ then f is bounded by the power function $\lambda(x) = x$. In addition, if $lim_{x\to\infty}f = c \in R^*$ then $f \sim c * 1$. And if $lim_{x\to\infty}f = 0$ then it's enough to find a power function λ and an element $c \in R^*$ such that $\frac{1}{f} \sim c \cdot \lambda$. Because if λ is a power function then so is $\frac{1}{\lambda}$.

So we can assume that $lim_{x\to\infty}f = \infty$.

First we'll prove that $lim_{x\to\infty}\frac{f(2x)}{f(x)} \in R^*$. Let $g = \frac{f(2x)}{f(x)}$. By the conditions of the lemma it's enough to show that $lim_{x\to\infty}\frac{g'}{g}x = 0$. But

$$\frac{g'}{g}x = x \cdot \frac{2f'(2x)f(x) - f'(x)f(2x)}{f^2(x)} \cdot \frac{f(x)}{f(2x)} = 2x \cdot \frac{f'(2x)}{2x} - x \cdot \frac{f'(x)}{f(x)}$$

and again by the conditions of the lemma, $lim_{x\to\infty}\frac{f'}{f}x \in R^*$. This means that $lim_{x\to\infty}\frac{g'}{g}x = 0$.

By lemma 42, Pf is a power function. Furthermore, by lemma 43, $P(Pf/f) = 1$. In order to show that f is power bounded, it's enough to show that $lim_{x\to\infty}\frac{Pf}{f} \in R^*$. Therefore, it's enough to show that for any definable function g, if $Pg = 1$ then $lim_{x\to\infty}g \in R^*$.

Claim. Let g be a definable function such that $Pg = 1$. Then, $lim_{x\to\infty}g \in R^*$.

Proof. First of all,

$$\frac{g(2x)}{g(x)} - 1 = x \cdot \frac{g'(\epsilon(x))}{g(x)}$$

for some $x < \epsilon(x) < 2x$. By definable choice, ϵ is a definable function from R to R. Since $Pg(2) = 1$,

$$lim_{x\to\infty}x \cdot \frac{g'(\epsilon(x))}{g(x)} = lim_{x\to\infty}\epsilon(x) \cdot \frac{g'(\epsilon(x))}{g(x)} = 0$$

. In addition,

$$lim_{x\to\infty}\frac{g(x)}{g(2x)} = lim_{x\to\infty}\frac{g(x)}{g(x)} = 1 \Rightarrow lim_{x\to\infty}\frac{g}{g \circ \epsilon} = 1$$

where we used the fact that g is ultimately monotonic.

Together, we get

$$lim_{x\to\infty}\frac{g'\circ\epsilon}{g\circ\epsilon}\epsilon = lim_{x\to\infty}\frac{g'\circ\epsilon}{g}x\cdot\frac{\epsilon}{x}\cdot\frac{g}{g\circ\epsilon}=0$$

which means that $lim_{x\to\infty}\frac{g'}{g}x = 0$. So by the assumption of the lemma, $lim_{x\to\infty}g \in R^*$. □

As we said above, from the claim it follows that that $lim_{x\to\infty}\frac{Pf}{f} \in R^*$. Therefore, there exists a $c \in R^*$ such that $f \sim c\cdot Pf$. □

Proposition 51. *R is power bounded iff there doesn't exist a definable function $f : R \to R$ such that ultimately $f \neq 0$ and $lim_{x\to\infty}\frac{f'}{f}x = \infty$. In that case, for every definable function $f : R \to R$, if ultimately $f \neq 0$ then there exists an element $c \in R^*$ and a power function λ such that $f \sim c\cdot\lambda$.*

Proof. For the first direction, let's assume that R is power bounded. Let $f : R \to R$ be a definable function such that ultimately $f \neq 0$ and $lim_{x\to\infty}\frac{f'}{f}x = \infty$. Furthermore, let λ be a power function bounding f. By remark 38, there exists some constant $C \in R^*$ such that $g'(x)x = Cg(x)$. In addition, $lim_{x\to\infty}\frac{f}{g} \in R$ and so by lemma 44, $lim_{x\to\infty}\frac{f'/f}{g'/g} \in R$. But,

$$lim_{x\to\infty}\frac{f'/f}{g'/g} = lim_{x\to\infty}\frac{(f'/f)x}{(g'x)/g} = \frac{1}{C}\cdot lim_{x\to\infty}\frac{f'}{f}x$$

which means that $lim_{x\to\infty}\frac{f'}{f}x \in R$. So there doesn't exist definable function $f : R \to R$ such that ultimately $f \neq 0$ and $lim_{x\to\infty}\frac{f'}{f}x = \infty$.

Now we'll prove the other direction. By lemma 50, we have to show that if f is a definable function and ultimately $f \neq 0$ then:

(1) $lim_{x\to\infty}\frac{f'}{f}x = 0 \Rightarrow lim_{x\to\infty}f \neq \infty$
(2) $lim_{x\to\infty}\frac{f'}{f}x = 0 \Rightarrow lim_{x\to\infty}f \neq 0$

Let's assume for contradiction that $lim_{x\to\infty}f = \infty$ and $lim_{x\to\infty}\frac{f'}{f}x = 0$. Since f is ultimately monotonic, we define $g = f^{-1}$ on $x > M$ for some $M \in R$. On that interval, ultimately $g(x) \neq 0$ and

$$lim_{x\to\infty}\frac{g'}{g}x = lim_{x\to\infty}\frac{x}{f'(f^{-1}(x))\cdot f^{-1}(x)} = lim_{x\to\infty}\frac{f(x)}{x\cdot f'(x)} = \infty$$

which contradicts the assumption of the proposition. This proves 1.

To prove 2 it's enough to note that if $lim_{x\to\infty}\frac{f'}{f}x = 0$ and $h = \frac{1}{f}$ then

$$lim_{x\to\infty}\frac{h'}{h}x = lim_{x\to\infty} - \frac{f'}{f^2}\cdot f\cdot x = -lim_{x\to\infty}\frac{f'}{f}x = 0$$

. Therefore, 2 follows from 1.

Finally, since we showed that the conditions of lemma 50 hold in this case, for every definable function $f : R \to R$, if ultimately $f \neq 0$ then there exists an element $c \in R^*$ and a power function λ such that $f \sim c\cdot\lambda$. □

By the previous proposition, the following corollary is immediate.

Corollary 52. *Let $R = (R, <, 0, 1, +, \cdot, \dots)$ be an o-minimal expansion of a real closed field. Then e^x is definable in R iff R isn't power bounded. If R is power bounded then for every definable function $f : R \to R$, if ultimately $f \neq 0$ then there exists an element $c \in R^*$ and a power function λ such that $f \sim c \cdot \lambda$.*

4.5. **Uniform Bounds in Power Bounded Structures.** By lemma 41, the definability of the exponent is preserved under elementary extensions. We can use this result to obtain uniform bounds on the property in theorem 49. We'll then use the uniform bounds to prove some interesting properties of power bounded structures.

Lemma 53. *Assume that R is power bounded. Then any structure elementarily equivalent to R is power bounded.*

Proof. Let R' be elementarily equivalent to R. First we note that R' is an o-minimal expansion of a real closed field. In addition, by corollary 52, e^x is not definable in R. So by lemma 41, e^x is not definable in R'. But again by corollary 52 this means that R' is power bounded. □

Lemma 54. *Let R be power bounded, $A \subset R^m$ and $f(x, \bar{y}) : R \times A \to R$ a definable family of functions in R such that for all $\bar{a} \in A$, ultimately $f(x, \bar{a}) \neq 0$. Then there exists some $M \in R$ such that for all $\bar{a} \in A$, $lim_{x\to\infty} \frac{f'(x,\bar{a})}{f(x,\bar{a})} x \leq M$.*

Proof. Let's assume for contradiction that for every $M \in R$ there exists a $\bar{a} \in A$ such that $lim_{x\to\infty} \frac{f'(x,\bar{a})}{f(x,\bar{a})} x > M$. Then by compactness, there exists an elementary extension $R \prec R'$ and an element $\bar{y} \in R'^m$ such that $lim_{x\to\infty} \frac{f'(x,\bar{y})}{f(x,\bar{y})} x = \infty$. By lemma 53, R' is also power bounded and this is a contradiction to proposition 51. □

The following proposition was also proved by Chris Miller [10]. We provide an alternative proof here using lemma 54.

Proposition 55. *Let R power bounded, $A \subset R^m$ and let $f(x, \bar{y}) : R \times A \to R$ be a definable family of functions in R such that for all $\bar{a} \in A$, ultimately $f(x, \bar{a}) \neq 0$. Then there exists a finite set of power functions $\lambda_1, \dots, \lambda_n$ and a definable function $c : A \to R^*$ such that for all $\bar{a} \in A$, $f(x, \bar{a}) \sim c(\bar{a}) \cdot \lambda_i$ for some $1 \leq i \leq n$.*

Proof. By lemma 51, there exists a function $c : A \to R^*$ and a set of power functions $\{\lambda_{\bar{a}} | \bar{a} \in A\}$ such that for all $\bar{a} \in A$, $f(x, \bar{a}) = c(\bar{a}) \cdot \lambda_{\bar{a}}(x)$. Since $lim_{x\to\infty} \frac{f'(x,\bar{a})}{f} x = \lambda'_{\bar{a}}(1)$, the function $\bar{a} \mapsto \lambda'_{\bar{a}}(1)$ and the set $B = \{\lambda'_{\bar{a}}(1) | \bar{a} \in A\}$ are definable. This means that the function c is definable as well because $f'(1, \bar{a}) = c(\bar{a}) \cdot \lambda'_{\bar{a}}(1)$.

In the argument below we'll use some of the properties mentioned in the first part of remark 38.

We have to prove that B is finite. This will imply that $\{\lambda_{\bar{a}} | \bar{a} \in A\}$ is finite as well. Let's assume for contradiction that B is infinite.

Since B is infinite, it contains an interval $I \subset R$. By dividing f by some $\lambda_{\bar{a}}$ we can assume that $0 \in B$. Therefore, there exists some $\epsilon \in R$ such that $(0, \epsilon) \subset B$. Let $\bar{a} \in A$ such that $\lambda'_{\bar{a}}(1) \in (0, \epsilon)$. Then $\lambda_{\bar{a}}$ is strictly increasing and $\lambda_{\bar{a}}^{-1}$ is a power function with

$$(\lambda_{\bar{a}}^{-1})'(1) = \frac{1}{(\lambda_{\bar{a}})'(1)}$$

. This means that be taking the inverse of all of the non-constant functions defined by f, we can assume that $(\frac{1}{\epsilon}, \infty) \subset B$. But this is a contradiction to lemma 54 because for all $\bar{a} \in A$. $lim_{x \to \infty} \frac{f'(x,\bar{a})}{f(x,\bar{a})} x = \lambda'_{\bar{a}}(1)$. □

We can use the previous proposition to obtain a nice result about the definability of power functions in a power bounded structure which was also proved by Chris Miller in [10].

Proposition 56. *Let R be power bounded. Then all of the power functions are 0-definable.*

Proof. Let λ be a power function.

By lemma 39, λ is the unique solution to the differential equation

$$x \cdot y' = y'(1)y$$

.

Therefore, it's enough to show that $\lambda'(1)$ is 0-definable. Let λ be defined by the formula $\phi(x, y, \bar{a})$ where $\bar{a}$ is a tuple of constants in R. We now look at the family of functions $f(x, \bar{z})$ defined by $\phi(x, y, \bar{z})$. By proposition 55, there exists a finite set $B \subset R$ such that $\lambda'(1) \in B$. But since R has order this means that $\lambda'(1)$ is zero definable. □

5. Dimension and t.t.t Structures

5.1. **Introduction.** In this section we'll generalize the notion of an o-minimal structure to that of a first order topological structure. How ever, in order to connect the topology to the structure we'll add some simple topological and model theoretic assumptions which will lead us to the definition of a topologically totally transcendental structure. Even this restricted setting is still a good generalization of the o-minimal one because as shown by Pillay [5], when the topology is the order topology, then connected one-dimensional t.t.t structures turn out to be o-minimal.

This leads us to the question of the exact relationship between one dimensional t.t.t structures and o-minimal structures. Very little is known to this extent, and currently there aren't examples of one dimensional t.t.t structures which aren't o-minimal. We'll approach the question here by analyzing two different notions of dimension on t.t.t structures and proving their equivalence. The tools developed throughout the proof will allow us to obtain some basic information about definable sets in such structures.

Definition 57. Let M be a two sorted L structure with sorts M_t and M_b and let $\phi(x, y_1, \ldots, y_k)$ be an L formula such that $\{\phi^{M_t}(x, \bar{a}) | \bar{a} \in M_b^k\}$ is a basis for a topology on M_t. Then the pair (M, ϕ) will be called a *first order topological structure*. When we talk about the topology of M_t we mean the one generated by the basis described above.

Remark. In Pillay's paper, first order topological structures were defined on a one-sorted structure where each element can be both a parameter for a basis set, and a point in the topological space. However, in practice this double meaning isn't needed, so we're using the two sorted definition both for clarity and in order to slightly strengthen some of the theorems.

In addition, we consider to following condition on a first order topological structure M:

(A) Every definable set $X \subset M_t$ is a boolean combination of definable open subsets.

In this section we assume that M_t is Hausdorff and (M, ϕ) is a first order topological structure satisfying (A).

The following topological result is also helpful in this context.

Lemma 58. *Let V be a topological space, and $W \subset V$ a non-empty subset. Let $A \subset V$ be a boolean combination of open subsets of V and let $B = V \setminus A$. Then either $W \cap A$ or $W \cap B$ has an interior with respect to the induced topology on W.*

The proof of the lemma is purely topological so I won't provide it here. It can be done relatively easily by showing that the sets $A \subset V$ with the desired property contain the open sets and are closed under boolean operations.

Definition 59. Let M be a first order topological structure satisfying (A) and $X \subset M_t$ be a closed definable subset of M_t. The ordinal valued $D_M(X)$ is defined by:

(1) If $X \neq \emptyset$ then $D_M(X) \geq 0$.
(2) If δ is a limit ordinal and $D_M(X) \geq \alpha$ for all $\alpha < \delta$ then $D_M(X) \geq \delta$.
(3) If there's a closed definable $Y \subset M_t$ such that $Y \subset X$, Y has no interior in X and $D_M(Y) \geq \alpha$ then $D_M(X) \geq \alpha + 1$.

Remark. We'll write $D_M(X) = \alpha$ if $D_M(X) \geq \alpha$ and $D_M(X) \ngeq \alpha + 1$. We'll write $D_M(X) = \infty$ if $D_M(X) \geq \alpha$ for all α.

Definition 60. We say that M *has dimension* if $D_M(X) \neq \infty$ for all closed definable subsets $X \subset M_t$.

In addition, we define the number of connected components for definable subsets of our topology:

Definition 61. Let $X \subset M_t$ be definable. Then $d_M(X)$ is the maximum $d < \omega$ such that there are disjoint definable clopen $X_1, \ldots, X_d \subset X$ with $X = \cup_{i=1}^{d} X_i$, and ∞ if no such d exists.

And now for the main definition:

Definition 62. We say that M is *topologically totally transcendental (t.t.t)* if M is a first order topological structure satisfying (A) with dimension such that for every definable $X \subset M_t$, $d_M(X) < \infty$. We say that a theory T is t.t.t is every model of T is t.t.t.

The following lemma was proved by Pillay [5] and plays a key role in most of the proofs in this paper.

Lemma 63. *Let M be a 1-dimensional t.t.t structure. Then:*

(1) *For any closed and definable $X \subset M_t$, $D(X) = 0$ iff X is finite.*
(2) *The set of isolated points of M_t is finite.*
(3) *For any definable $X \subset M_t$ there are pairwise disjoint definably connected definable open subsets $X_1, \ldots, X_m \subset M_t$ and a finite set $Y \subset M_t$ such that $X = (\cup_{i=1}^{m} X_i) \cup Y$.*
(4) *For any definable $X \subset M_t$, the set of boundary points of X is finite.*

Remark. One consequence of part 3 of lemma 63 which will be used many times below is that if $A \subset M_t$ is definable then the statement *"A is infinite"* is expressible in first order logic as it's equivalent to the statement *"A has no interior"*.

5.2. **Stability and Prime Models.** Before defining the dimension in t.t.t structures, I'd like to present two model theoretic properties of these structures which were also proved by Pillay [5].

Proposition 64. *Let T be a t.t.t theory. Then over any set A there's a prime model.*

Proof. The idea of the proof is that given a definable set S, we can use the finiteness of the dimension and the number of connected components to find a definable subset $X \subset S$ such that there are no definable proper subsets of X. This will show that the isolated 1-types over A are dense.

Let $\phi(x)$ be a formula with parameters from A. We choose another formula $\psi(x)$ with parameters in A such that $\vDash \psi(x) \to \phi(x)$, first minimizing the dimension of the closure of ψ^{M_t}, and then minimizing the number of connected components of ψ^{M_t}. We note that both the dimension and the number of connected components are the same for all models of T.

In order to show that ψ isolates a complete type, we'll show that for any model $M \vDash T$ containing A, if $X = \psi^{M_t}$ then there doesn't exist a definable set $Y \subset M_t$ such that $Y \subsetneq X$.

Let $Y \subset X$ be a definable subset. By the choice of ψ, $D(\bar{Y}) = D(\bar{X})$ and Y isn't clopen. This means that Y has a non-empty boundary. Let Z be the boundary of Y. By lemma 58, Z has an empty interior. Because if $intZ \neq \emptyset$, then both $intZ \cap Y$ and $intZ \backslash Y$ have an empty interior which is a contradiction to lemma 58.

This means that $D(Z) < D(X)$ which is a contradiction to the choice of ψ. This proves that ψ isolates a 1-type. □

The next proposition is true about topological models in general.

Proposition 65. *Let (M, ϕ) be a first order topological structure whose topology is T_1 and not discrete. Then $Th(M)$ has the strict order property and in particular it isn't stable.*

Proof. We use the fact that the topology is T_1 and not discrete in order to construct an infinitely decreasing sequence of definable open sets. This will prove the strict order property.

Let $a \in M_t$ be a non-isolated point. Let $\bar{b} \in M_b$ be some point such that $M \vDash \phi(a, \bar{b})$. Since a isn't isolated, there exists a point $a_1 \in M_t$ such that $M \vDash \phi(a_1, \bar{b})$. By the T_1 property and that fact that the sets definable by ϕ form a basis, there exists some $\bar{b}_1 \in M_b$ such that $M \vDash \phi(a, \bar{b})$ and $\phi^M(x, \bar{b}_1) \subset \phi^M(x, \bar{b})$ but $M \vDash \neg\phi(a_1, \bar{b}_1)$. Continuing in this fashion, we can find a sequence $(\bar{b}_i)_{i=1}^{\infty}$ in M_b such that $M \vDash \exists x(\neg\phi(x, \bar{b}_i) \wedge \phi(x, \bar{b}_j)) \iff i < j$. □

5.3. **Defining Dimension in t.t.t Structures.**

Definition 66. (*exchange*) Let M be a first order structure. We say that M has the *exchange property* if for every $a, b \in M$ and a set $A \subset M$, if $b \in acl(A \cup a)$ and $b \notin acl(A)$ then $a \in acl(A \cup b)$.

The following theorem was proved by Pillay[5]:

Theorem 67. *Let M be a t.t.t structure. Then M_t has the exchange property.*

In the next section we'll obtain this theorem as a corollary of a more general topological result.

We'll now introduce some constructs used in our first definition of dimension and prove their elementary properties. For this part we'll leave the t.t.t setting and will only need to assume that our structure has the exchange property.

Definition 68. (*rank*) Let M be a model and $A \subset M$.

(1) For any tuple $\bar{a} \in M^n$, $rk(\bar{a}/A)$ is the least cardinality of a subtuple $\bar{a}'$ of $\bar{a}$ such that $\bar{a} \in acl(\bar{a}'/A)$.
(2) for any type $p(\bar{x}) \in S_n(A)$, $rk(p/A) = rk(\bar{a}/A)$ for any $\bar{a} \in M^n$ realizing p.

Remark. It's easy to see that the second part of the definition doesn't depend on the choice of the element which realizes p.

Lemma 69. *Let M be a model with the exchange property, $A \subset M$, and $\{a_1, \ldots, a_n\} \subset M$ an algebraically independent set over A. In addition, let $b \in M$ have the property that $b \notin acl(\{a_1, \ldots, a_n\}/A)$. Then $\{a_1, \ldots, a_n, b\}$ is an algebraically independent set over A.*

Proof. Suppose for contradiction that $\{a_1, \ldots, a_n, b\}$ is not algebraically independent over A. Then there's some $1 \le i \le n$ such that

$$a_i \in acl(\{a_1 \ldots, \hat{a_i}, \ldots, a_n, b\}/A)$$

. By the assumption, $a_i \notin acl(\{a_1 \ldots, \hat{a_i}, \ldots, a_n\}/A)$. But since M has the exchange property, this means that $b \in acl(\{a_1 \ldots, a_i, \ldots, a_n\}/A)$ which is a contradiction. ☐

Lemma 70. *Let M be a model with the exchange property. Then for any $\bar{a} = (a_1, \ldots, a_n) \in M^n$ and $A \subset M$, $rk(\bar{a}/A)$ is the cardinality of any maximal algebraically independent over A subtuple of $\bar{a}$.*

Proof. Let $\{b_1, \ldots, b_k\} \subset \{a_1, \ldots, a_n\}$ be a subset of least cardinality such that $\bar{a} \in acl(\{b_1, \ldots, b_k\}/A)$. By definition 2, $rk(\bar{a}/A) = k$. $\{b_1, \ldots, b_k\}$ is obviously a maximal algebraically independent subset of $\{a_1, \ldots, a_n\}$ over A.

Let $\{c_1, \ldots, c_m\} \subset \{a_1, \ldots, a_n\}$ be a maximal algebraically independent subset of $\{a_1, \ldots, a_n\}$ over A. We'll now show that $k = m$.

Let's assume for contradiction that $m > k$. We'll use induction to show that for every $0 \le i \le k$, there exists a subset $\{d_1 \ldots d_k\}$ of $\{a_1, \ldots, a_n\}$ such that:

(1) $\bar{a} \in acl(\{d_1, \ldots, d_k\}/A)$
(2) $\{d_1, \ldots, d_i\} \subset \{c_1, \ldots, c_m\}$
(3) $\{d_{i+1}, \ldots, d_k\} \subset \{b_1, \ldots, b_k\}$
(4) $\{d_1, \ldots, d_i\}$ is algebraically independent over A

For $i = 0$ there's nothing to show.

Let's assume that the claim is true for i. Let $\{d_1 \ldots d_k\}$ be the set guaranteed by the inductive hypothesis. Since $\{c_1, \ldots, c_m\}$ is algebraically independent over A, there exists some $1 \le j \le m$ be such that $c_j \notin \{d_1 \ldots d_i\}$. Since $\bar{a} \in acl(\{d_1, \ldots, d_k\}/A)$, there exists some $1 \le l \le k$ such that $c_j \in acl(d_1, \ldots, d_l/A)$. With out loss of generality l is the smallest number with this property so $c_j \notin acl(d_1, \ldots, d_{l-1}/A)$.

By the exchange property,

$$d_l \in acl(\{d_1, \ldots, d_{l-1}, c_j\}/A)$$

. In addition, $\{c_1, \ldots, c_m\}$ is algebraically independent over A so $l > i$. By lemma 69, $\{d_1, \ldots, d_i, c_j\}$ is algebraically independent over A. Let $\{d'_1, \ldots, d'_k\}$ be the set obtained by taking $\{d_1, \ldots, d_k\}$ and replacing d_l with c_j and switching between c_j and d_{i+1}. It's easy to see that $\{d'_1, \ldots, d'_k\}$ has the desired properties. This finishes the induction.

Now, by the k-th step of the induction, there exists a subset $\{c'_1, \ldots, c'_k\} \subset \{c_1, \ldots c_m\}$ such that $\bar{a} \in acl(\{c'_1, \ldots, c'_k\}/A)$. This is a contradiction to the fact that $\{c_1, \ldots, c_m\}$ is algebraically independent over A. □

The following is an easy consequence of the previous lemma but is easier to use in practice.

Lemma 71. *Let M be a model with the exchange property. Let $\bar{a} = (a_1, \ldots, a_n) \in M^n$, $\bar{a}'$ a subtuple of $\bar{a}$ and $A \subset M$. If $\bar{a}'$ is algebraically independent over A and $\bar{a} \in acl(\bar{a}'/A)$ then the cardinality of $\bar{a}'$ is $rk(\bar{a}/A)$.*

Proof. Since $\bar{a} \in acl(\bar{a}'/A)$, $\bar{a}'$ is maximal algebraically independent over A and so by lemma 70, the cardinality of $\bar{a}'$ is $rk(\bar{a}/A)$. □

From now on we will assume that M is a model with the exchange property. In addition, we will assume that M is sufficiently saturated so that the dimension of a type doesn't depend on the specific model we're using.

Lemma 72. *Let $A, B \subset M$ and $\bar{a} \in M^s, \bar{b} \in M^t$. Then:*

(1) *If $A \subset B$ then $rk(\bar{a}/A) \geq rk(\bar{a}/B)$.*
(2) *$rk(\bar{a}\hat{\ }\bar{b}/A) = rk(\bar{a}/A \cup \bar{b}) + rk(\bar{b}/A)$.*
(3) *$rk(\bar{a}/A \cup \bar{b}) = rk(\bar{a}/A) \iff rk(\bar{b}/A \cup \bar{a}) = rk(\bar{b}/A)$.*
(4) *If $p \in S_n(A)$ and $A \subset B$ then there exists a type $q \in S_n(B)$ such that $p \subset q$ and $rk(q/B) = rk(p/A)$.*

Proof. We'll prove each part separately.

(1) This follows directly from the definition of rk.
(2) We'll prove this by induction on t. If $t = 0$ then there's nothing to show. Let's assume that the claim is true for t. In order to prove the inductive step we'll first show the following claim:

Claim. Let $\bar{a}$ and A be as above and $b \in M$. If $b \notin acl(A)$ then $rk(\bar{a}\hat{\ }b/A) = rk(\bar{a}/A \cup b) + 1$.

Proof. Let's assume that $rk(\bar{a}\hat{\ }b) = k$. According to lemma 71, without loss of generality we can assume that there exist $1 \leq i_1, \ldots, i_{k-1} \leq s$ such that $(a_{i_1}, \ldots, a_{i_{k-1}}, b)$ is a subtuple of $\bar{a}\hat{\ }b$ which is algebraically independent over A such that $\bar{a}\hat{\ }b \in acl((a_{i_1}, \ldots, a_{i_{k-1}}, b)/A)$. This means that $(a_{i_1}, \ldots, a_{i_{k-1}})$ is algebraically independent over A and

$$\bar{a} \in acl((a_{i_1}, \ldots, a_{i_{k-1}})/A \cup b)$$

. So by lemma 71, $rk(\bar{a}/A \cup b) = k - 1$. □

Now, assuming the claim:

$$rk(\bar{a}\hat{\ }\bar{b}_t/A) = rk(\bar{a}\hat{\ }\bar{b}_{t-1}\hat{\ }b_t/A) = rk(\bar{a}\hat{\ }\bar{b}_{t-1}/A \cup b_t) + 1 =$$

$$rk(\bar{a}/A \cup b_t \cup \bar{b}_{t-1}) + rk(\bar{b}_{t-1}/A \cup b_t) + 1 = rk(\bar{a}/A \cup \bar{b}) + rk(\bar{b}/A)$$

where we used the claim in the second and forth equalities and the inductive hypothesis for the third.

(3) This follows immediately from 2.

(4) By 1, it's enough to find some $q \in S_n(B)$ such that $p \subset q$ and $rk(q/B) \geq rk(p/A)$. Let's assume that $rk(p/A) = k$. Then by lemma 70, there exists a tuple $\bar{c} = (c_1, \ldots, c_n) \in M^n$ satisfying p and a subtuple $\bar{a} = (a_1, \ldots, a_k)$ that's independent over A. It's enough to find a tuple $(b_1, \ldots, b_k)$ that is independent over B such that

$$tp((a_1, \ldots, a_k)/A) = tp((b_1, \ldots, b_k)/A)$$

.

By induction on i, we prove that for every $0 \leq i \leq k$ there exists a tuple $(b_1, \ldots, b_i)$ that is independent over B such that $tp((a_1, \ldots, a_i)/A) = tp((b_1, \ldots, b_i)/A)$.

If $i = 0$ then there's nothing to show.

Let's assume the claim is true for i and that $(b_1, \ldots, b_i)$ is the tuple with the required properties. We now define a set of formulas $\tilde{\Gamma}_i \subset L_{B\cup\{b_1,\ldots,b_l\}}$ as the set of formulas in $L_{B\cup\{b_1,\ldots,b_l\}}$ with one free variable that define a finite subset of M. In addition, we define:

$$\Gamma_i = \{\neg\phi(x) | \phi(x) \in \tilde{\Gamma}_i\}$$

and:

$$\Psi_i = \{\psi(b_1, \ldots, b_i, x) | \psi(y_1, \ldots, y_i, x) \in tp((a_1, \ldots, a_{i+1})/A)\}$$

.

We'll now show that $\Gamma_i \cup \Psi_i$ is consistent. Let

$$\psi_1(y_1, \ldots, y_i, x), \ldots, \psi_l(y_1, \ldots, y_i, x) \in tp((a_1, \ldots, a_{i+1})/A)$$

and $\gamma_1(x), \ldots, \gamma_m(x) \in \Gamma_i$. First of all,

$$\psi_1(y_1, \ldots, y_i, x) \wedge \cdots \wedge \psi_l(y_1, \ldots, y_i, x) \in tp((a_1, \ldots, a_{i+1})/A)$$

. In addition, since $a_{i+1} \notin acl(\{a_1, \ldots, a_i\}/A)$, there exist an infinite number of elements $c \in M$ such that $M \models \psi_1(a_1, \ldots, a_i, c) \wedge \cdots \wedge \psi_l(a_1, \ldots, a_i, c)$. By the inductive hypothesis, this means that there exist an infinite number of elements $c \in M$ such that

$$M \models \psi_1(b_1, \ldots, b_i, c) \wedge \cdots \wedge \psi_l(b_1, \ldots, b_i, c)$$

. By the definition of Γ_i, there are only a finite number of elements that *don't* satisfy $\gamma_1(x) \vee \ldots, \vee\gamma(x)$. This means that there exists an element satisfying

$$\{\psi_1(b_1, \ldots, b_i, x), \ldots, \psi_l(b_1, \ldots, b_i, x), \gamma_1(x), \ldots, \gamma_m(x)\}$$

. This shows that $\Gamma_i \cup \Psi_i$ is consistent.

Let b_{i+1} be an element satisfying $\Gamma_i \cup \Psi_i$. By the definition of Γ_i, $b_{i+1} \notin acl((b_1, \ldots, b_i)/B)$. So by lemma 69, $(b_1, \ldots, b_{i+1})$ is algebraically independent over B. In addition, by the definition of Ψ_i,

$$tp((a_1, \ldots, a_{i+1})/A) = tp((b_1, \ldots, b_{i+1})/A)$$

. This completes the induction.

□

We are now ready to define our first concept of dimension for a structure with the exchange property.

Definition 73. Let M be a structure with the exchange property, $X \subset M^n$ a definable subset and $A \subset M$. Then we define:

$$rk(X) = max_{p \in S_n(A)}\{rk(p/A) | \text{p is realized in } X\}$$

Remark. Note that under our assumption that M is sufficiently saturated, by part 4 of lemma 72, $rk(X)$ doesn't depend on the choice of A.

Let $X \subset M^n$ be definable, $A \subset M$ and $\bar{a} \in X$. Then $\bar{a}$ will be called a *generic point of* X *over* A if $rk(\bar{a}/A) = rk(X)$.

We'll now give our second definition of dimension. In this definition we need to assume that M has some definable topology. For the purposes of this paper we'll assume that M is a t.t.t structure.

Furthermore, given a set $X \subset M^n$ and indices $1 \leq i_1 < \cdots < i_k \leq n$, $\pi_{i_1,\ldots,i_k}(X)$ is the projection of X onto the coordinates $i_1, \ldots, i_k$.

Definition 74. Let M be a t.t.t structure and $X \subset M_t^n$ be a definable subset. We define the *topological dimension* of X as:

$$dim(X) = max_{1 \leq k \leq n}\{\exists 1 \leq i_1 < \cdots < i_k \leq n \text{ s.t } int(\pi_{i_1,\ldots,i_k}(X)) \neq \emptyset\}$$

5.4. **The Equivalence of the Dimensions.** In this section we'll prove that for any ω-saturated t.t.t structure, the two definitions of dimension we gave above agree on all definable sets. In this section, M will denote a t.t.t structure.

Lemma 75. *Let $\phi(x, y)$ be a formula in M and $X = \phi^{M_t}$. In addition, let $U \subset M_t$ be a definable open set such that for all $u \in U$, $|\{y \in M_t : (u, y) \in X\}| \geq \aleph_0$. Then, for every $k \in \mathbb{N}$ there exists a $y \in M_t$ such that $|\{x \in U : (x, y) \in X\}| > k$.*

Proof. We'll generalize some of the ideas in Pillay's proof of the exchange property in [5]. Let's assume for contradiction that there exists a $k \in \mathbb{N}$ such that for all $y \in M_t$, $|\{x \in U : (x, y) \in X\}| \leq k$. For all $u \in U$ we'll define $X_u = \{y \in M : (u, y) \in M_t\}$. U and X are definable and therefore X_u is definable as well. In addition, we know that for all $u \in U$, $|X_u| \geq \aleph_0$. So according to lemma 63, X_u contains an open set. We now define another set:

$$X_0 = \{c \in M_t : c \in \overline{X}_u \backslash int(X_u) \text{ for some } u \in U\}$$

.

First we'll assume that X_0 is finite and reach a contradiction. Since $|\{u \in U : (u, y) \in X\}| \leq k$ for all $y \in M_t$, we have the following:

(*) for only a finite number of $u \in U$ there exists a $c \in X_0$ such that $(u, c) \in X$.

Let's define $N = (\cup_{u \in U} X_u) \backslash X_0$ and for all $u \in U$, $Z_u = X_u \cap N = X_u \backslash X_0$. By (*), there're an infinite number of $u \in U$ such that $Z_u \neq \emptyset$. We'll now show that for each $u \in U$, Z_u is clopen in N. First of all, Z_u is open in M_t and therefore it's also open in N. In addition, N is open so if c is a boundary point of Z_u in N then it's a boundary point of Z_u and therefore also a boundary point of X_u. But that means that $c \in X_0$ which is a contradiction to the definition of N.

Now, by our assumption for contradiction, for any distinct $u_1, \ldots, u_{k+1} \in U$,

(**) $\cap_{i=1}^{k+1} Z_{u_i} = \emptyset$

We now show that for any $n \in \mathbb{N}$, we can find n clopen definable disjoint sets $V_1, \ldots, V_n$ where each V_i is of the form $Z_{u_1} \cap \ldots, \cap Z_{u_m}$ for some $u_1, \ldots, u_m \in U$. Let $\tilde{U} = \{u \in U : Z_u \neq \emptyset\}$. As we mentioned above, $\tilde{U}$ is infinite. For $n = 1$, We can define $V_1 = Z_u$ for any $u \in \tilde{U}$. Let's assume that we've already found sets $V_1, \ldots, V_n$ with the properties mentioned above. We choose some $u_1 \in \tilde{U}$ that isn't used in the definition of any of the V_i. We define $V_{n+1}^1 = Z_{u_1}$. We now construct a sequence V_{n+1}^i , $1 \leq i \leq k$, inductively. We already have V_{n+1}^1. Let's say that we've defined V_{n+1}^i. If there exists some $u_{i+1} \in \tilde{U}$ such that $V_{n+1}^i \cap Z_{u_{i+1}}$ then we define $V_{n+1}^{i+1} = V_{n+1}^i \cap Z_{u_{i+1}}$. Otherwise, we define $V_{n+1}^{i+1} = V_{n+1}^i$. We now define $V_{n+1} = V_{n+1}^k$. According to (**), the sequence $V_1, \ldots, V_{n+1}$ now has the required properties. But this is a contradiction to the fact that $d(N) \in \mathbb{N}$.

Now we assume that X_0 is infinite. Let W_0 be the interior of X_0. For each $u \in U$, let $W_u = int X_u$. We'll now inductively find a sequence $u_1, u_2, \dots \in U$ such that for all $n \in \mathbb{N}$,

$$W_0 \cap W_{u_1} \cap \cdots \cap W_{u_n} \neq \emptyset$$

. For $n = 0$ there's nothing to show. Let's assume that we've found some $u_1, \ldots, u_n \in U$ with the desired property. We choose some $c \in W_0 \cap \cdots \cap W_{u_n}$. Since $c \in X_0$, there exists some $u \in U \backslash \{u_1, \ldots, u_n\}$ such that c is a boundary point of X_u. But X_u has a finite number of boundary points and so by the Hausdorffness of M_t, every neighborhood of c contains points in the interior of X_u. Specifically, $(W_0 \cap \cdots \cap W_{u_n}) \cap W_u \neq \emptyset$ so we can set $u_{n+1} = u$. Now we choose some $y \in W_0 \cap \cdots \cap W_{u_{k+1}}$. This means that $(y, u_i) \in X$ for all $1 \leq i \leq k+1$ which is a contradiction to our assumption on X. □

Proposition 76. *Suppose that M is ω-saturated. Let $\phi(x, y)$ be a formula in M, $X = \phi^{M_t}$, and $U \subset M_t$ an open definable subset such that $|\{y \in M_t : (u, y) \in X\}| \geq \aleph_0$ for all $u \in U$. Then $X \cap (U \times M_t)$ has a non-empty interior.*

Proof. Let $\alpha(x)$ be the formula in M defining U.

Claim. There exists a $y \in M_t$ such that $|(M_t \times \{y\}) \cap X \cap (U \times M_t)| \geq \aleph_0$.

Proof. Let $n \leq \omega$. According to lemma 75 there exists some $c \in M_t$ such that $|(M_t \times \{y\}) \cap X| \geq n$. So if we define

$$\psi_n(y) = \exists x_1 \ldots \exists x_n ((\bigwedge_{i \neq j} x_i \neq x_j) \wedge (\bigwedge_i (\alpha(x_i) \wedge \phi(x_i, y)))$$

then $M \models \psi_n[c]$. Since M is ω-saturated, there exists some $d \in M_t$ such that $M \models \psi_n[d]$ for all $n < \omega$. Therefore, $|(M_t \times \{d\}) \cap X \cap (U \times M_t)| \geq \aleph_0$ which completes the claim. □

Claim. There exists an open definable set $V \subset U$ and an infinite number of elements $y \in M_t$ such that for all $v \in V$, $(v, y) \in X$.

Proof. By the definition of a t.t.t, there exists some formula $\beta(x, y_1, \ldots, y_k)$ such that $\{\beta^{M_t}(x, \bar{a}) | \bar{a} \in M_b^k\}$ is a basis for the topology on M. We first show that for every $n < \omega$:

(***) there exists a $\bar{b}_n \in M_b^k$ and $c_1, \ldots, c_n \in M_t$ such that if $B_n = \beta^{M_t}[\bar{b}_n]$ then $B_n \subset U$ and $(u, c_i) \in X$ for all $u \in B_n$ and all $1 \leq i \leq n$.

According to the first claim there exists a $c_1 \in M_t$ such that the definable set $(M_t \times \{c_1\}) \cap X \cap (U \times M_t)$ is infinite. Therefore, its projection onto U is infinite so there's some $\bar{b}_1 \in M_b^k$ such that $B_1 = \beta^{M_t}[\bar{b}_1] \subset U$ is contained in the projection. This means that $(u, c_1) \in X$ for all $u \in B_1$. This shows that (***) is true for $n = 1$.

Now let's assume that (***) is true for $n \in \mathbb{N}$. The set

$$\tilde{X} = \{(x, y) \in X : \forall 1 \leq i \leq n, y \neq c_i\}$$

and the open definable set B_n fulfill the conditions of the prior claim (where $\tilde{X}$ is instead of X and B_n is instead of U). This means that we can find an element $c_{n+1} \in M_t$ such that

$$|(M_t \times \{c_{n+1}\}) \cap \tilde{X} \cap (B_n \times M_t)| \geq \aleph_0$$

. So exactly like in the case of $n = 1$, there exists some $\bar{b}_{n+1} \in M_b^k$ such that $B_{n+1} = \beta^M[\bar{b}_{n+1}] \subset B_n$ and $(u, c_{n+1}) \in \tilde{X} \subset X$ for all $u \in B_{n+1}$. Also, by the definition of $\tilde{X}$, $c_{n+1} \neq c_i$ for all $1 \leq i \leq n$. Finally, since $B_{n+1} \subset B_n$, $(u, c_i) \in X$ for all $u \in B_{n+1}$ and all $1 \leq i \leq n+1$. So we showed that (***) holds for all $n < \omega$.

Therefore, if we define the formula:

$$\gamma_n(\bar{x}) = \exists c_1 \ldots \exists c_n ((\bigwedge_{i \neq j} c_i \neq c_j) \wedge (\forall u (\beta(u, \bar{x}) \rightarrow ((\bigwedge_i \phi(u, c_i)) \wedge \alpha(u)))))$$

then for each $n < \omega$ there exists a $\bar{b} \in M_b^k$ such that $M \vDash \gamma_n[\bar{b}]$. But M is ω-saturated so there's some $\bar{b} \in M_b^k$ such that $M \vDash \gamma_n[\bar{b}]$ for all $n < \omega$, i.e, if $B = \beta^{M_t}[\bar{b}]$ then $B \subset U$ and the set $C = \{y \in M_t : \forall u \in B, (u, y) \in X\}$ is infinite. This finishes the proof of the claim. □

Now, let B and C be the sets defined in the end of the proof of the second claim. Let C_0 be the (non empty) interior of C. Then by the definition of C, $B \times C_0 \subset X \cap (U \times M_t)$ is open and which completes the proof of the proposition. □

Lemma 77. *Suppose M is ω-saturated. Let $X \subset M_t^{n+1}$ be a definable subset such that $U = \pi_{1,\ldots,n}(X)$ is a basis set in the product topology on M_t^n. If for all $\bar{x} \in U$, $|(\{\bar{x}\} \times M_t) \cap X| = \infty$, then X has a non-empty interior.*

Proof. We'll use induction on n.

For $n = 1$, this lemma follows proposition 76.

Let's assume that the claim is true for $n - 1$. We define $A = \pi_{2,\ldots,n}(X)$ and $B = \pi_{2,\ldots,n+1}(X)$.

By proposition 76, for every ,$\bar{a} \in A$ the set $(M_t \times \{\bar{a}\} \times M_t) \cap X$ has a non-empty interior. In particular, this means that $|(\{\bar{a}\} \times M_t) \cap B| = \infty$. Now, we define the set:

$$C = \{(\bar{y}, z) \in B : \bar{y} \in A,\ z \in M_t\ |(M_t \times \{(\bar{y}, z)\}) \cap X| = \infty\}$$

Claim. There exists a basis set $U \subset \pi_{1,\dots,n}(X)$ and basis set $V \subset A$ such that

$$|\{x \in M_t : U \times \{x\} \subset (M_t \times V \times \{x\}) \cap X\}| = \infty$$

.

Proof. By what we just showed, it follows that for every $\bar{a} \in A$,

$$|(\{\bar{a}\} \times M_t) \cap C| = \infty$$

. By the inductive hypothesis, there exists a basis set $\tilde{V_1} \subset A$ and a $x_1 \in M_t$ such that $\tilde{V_1} \times \{x_1\} \subset C$. By the definition of C and the inductive hypothesis, $(M_t \times \tilde{V_1} \times \{z\}) \cap X$ has a non-empty interior. Therefore, there exists a basis set $U_1 \subset \pi_{1,\dots,n}(X)$ such that $U_1 \times \{x_1\} \subset (M_t \times \tilde{V_1} \times \{x_1\}) \cap X$.

In addition, let's define $V_1 = \pi_{2,\dots,n}(U_1) \subset A$. Together, we found a pair of basis sets $U_1 \subset \pi_{1,\dots,n}(X)$ and $V_1 \subset \pi_{2,\dots,n}(U_1)$ such that

$$U_1 \times \{x_1\} \subset (M_t \times V_1 \times \{x_1\}) \cap X$$

.

Now, lets look at the sets

$$X_2 = [(U_1 \times M_t) \cap X] \backslash [M_t \times V_1 \times \{x_1\}]$$

and $A_2 = \pi_{2,\dots,n}(X_2)$. Since we only removed a finite number of elements from each fiber of X, X_2 has the properties required by the proposition. This means that we can repeat the above process again and obtain a basis set $U_2 \subset \pi_{1,\dots,n}(X_2) = U_1$, a basis set $V_2 \subset \pi_{2,\dots,n}(U_2) \subset A_2 \subset V_1$ and an element $x_2 \in M_t$ such that

$$U_2 \times \{x_2\} \subset (M_t \times V_2 \times \{x_2\}) \cap X$$

. Furthermore, since $U_2 \subset U_1$ and $V_2 \subset V_1$, we also have $U_2 \times \{x_1\} \subset (M_t \times V_2 \times \{x_1\}) \cap X$. Therefore:

$$|\{x \in M_t : U_2 \times \{x\} \subset (M_t \times V_2 \times \{x\}) \cap X\}| \geq 2$$

By continuing this process n times, we can find a pair of basis sets $U_n \subset \pi_{1,\dots n}(X)$ and $V_n \subset \pi_{2,\dots,n}(U_n) \subset A$ such that:

$$|\{x \in M_t : U_n \times \{x\} \subset (M_t \times V_n \times \{x\}) \cap X\}| \geq n$$

Since M is ω-saturated and basis sets are definable with a tuple of constants from M_b, there exists a pair of basis sets $U \subset \pi_{1,\dots n}(X)$ and $V \subset \pi_{1,\dots,n}(U) \subset A$ such that:

$$|\{x \in M_t : U \times \{x\} \subset (M_t \times V \times \{x\}) \cap X\}| = \infty$$

□

Let U and V be the basis sets given by the claim. Since M is t.t.t, there exists a basis set $W \subset M$ such that for all $w \in W$, $U \times \{w\} \subset (M_t \times V \times \{w\}) \cap X\}$. Therefore, $U \times W \subset X$ which finishes the induction and proves the lemma. □

Before proceeding to prove the the theorem about the equivalence of the dimensions, we use lemma 77 to obtain an interesting corollary.

Corollary 78. *Suppose M is ω-saturated. Let $X \subset M_t^n$ and $Y \subset X$ be definable sets. If X has an interior in M_t^n and Y does not, then $X \backslash Y$ has an interior in M_t^n.*

Proof. We use induction on n.

For $n = 1$, the lemma follows directly from the fact that M is a t.t.t structure.

Let's assume the claim is true for n. Let $X \subset M_t^{n+1}$ be a definable set with an interior and $Y \subset X$ be a definable set with no interior. In addition, we define $\tilde{X} = \pi_{1,\ldots,n}(X)$, $\tilde{Y} = \pi_{1,\ldots,n}(Y) \subset \tilde{X}$ and a set $\tilde{Z} \subset \tilde{Y}$:

$$\tilde{Z} = \{\bar{y} \in \tilde{Y} : |(\{\bar{y}\} \times M_t) \cap Y| = \infty\}$$

. Since X has an interior, without loss of generality we can assume that for every $\bar{x} \in \tilde{X}$, $|(\{\bar{x}\} \times M_t) \cap X| = \infty$. Furthermore, by lemma 77, $\tilde{Z}$ has no interior. So by the inductive hypothesis, $\tilde{U} = \tilde{X} \backslash \tilde{Z}$ has an interior.

Let $\bar{u} \in \tilde{U}$. Since $|(\{\bar{u}\} \times M_t) \cap X| = \infty$ and $|(\{\bar{u}\} \times M_t) \cap Y| < \infty$,

$$|(\{\bar{u}\} \times M_t) \cap (X \backslash Y)| = \infty$$

. So by lemma 77, $X \backslash Y$ has an interior in M_t^{n+1}.

This completes the induction and the corollary. □

We can use the corollary to prove a proposition about dense definable sets in M_t^n.

Proposition 79. *Suppose M is ω-saturated. Let $X \subset M_t^n$ be a dense definable set. Then $int(X) \subset M_t^n$ is dense as well.*

Proof. Let $a \in M_t^n$ be a point and $a \in U \subset M_t$ a basis set including a. Since X is dense, $U \backslash X$ has an empty interior and so by corollary 78, $U \cap X$ has an interior. Therefore, there exists an element $b \in int(X)$ such that $b \in U$. This finishes the proof. □

Proposition 80. *Suppose M is ω-saturated. Let $X \subset M_t^n$ be definable over A, $0 \leq k \leq n$, $1 \leq i_1 < \cdots < i_k \leq n$ and $\bar{a} \in X$ such that $(a_{i_1}, \ldots, a_{i_k})$ is algebraically independent over A. Then $\pi_{i_1,\ldots,i_k}(X)$ has an interior.*

Proof. We use induction on n.

If $n = 1$, then since $a_{i_1} \notin acl(A)$, X is infinite and thus has an interior.

Let's assume the claim holds for n.

First we assume that $i_k < n+1$. In this case, the claim follows directly from the inductive hypothesis.

Now let's assume that $i_k = n + 1$. We define $Y = \pi_{i_1,\ldots,i_{k-1},n+1}(X)$, $Z = \pi_{i_1,\ldots,i_{k-1}}(X) = \pi_{i_1,\ldots,i_{k-1}}(Y)$ and:

$$C = \{\bar{z} \in Z : |(\{\bar{z}\} \times M_t) \cap Y| = \infty\}$$

. Since $(a_{i_1}, \ldots, a_{i_k})$ is algebraically independent over A, $(a_{i_1}, \ldots, a_{i_{k-1}}) \in C$. So by the inductive hypothesis, C has a non-empty interior and by lemma 77, Y has an interior.

This completes the induction and the proposition. □

Proposition 81. *Suppose M is ω-saturated. Let $X \subset M_t^n$ be definable over A and $0 \leq k \leq n$ and $1 \leq i_1 < \cdots < i_k \leq n$ such that $\pi_{i_1,\ldots,i_k}(X)$ has an interior. Then there exists a tuple $\bar{a} \in X$ such that $(a_{i_1}, \ldots, a_{i_k})$ is algebraically independent over A.*

Proof. We use induction on n.

Let $n = 1$ and $X \subset M_t$ a subset definable over A. For $k = 0$ there's nothing to show.

If $k = 1$ then X is infinite so by compactness there exists an element $x \in X$ such that $x \notin acl(A)$. Therefore, we can take $a_{i_1} = x$.

Let's assume the claim is true for n. Let $X \subset M_t^{n+1}$ be definable over A and $0 \leq k \leq n+1$ such that $dim(X) = k$ and $\pi_{i_1,\ldots,i_k}(X)$ has an interior.

First we assume that $i_k < n+1$.

Let's define $Y = \pi_{1,\ldots n}(X)$. According to the assumption, $\pi_{i_1,\ldots,i_k}(Y)$ has an interior. So by the inductive hypothesis, there exists a tuple $\bar{y} \in Y$ such that $(y_{i_1}, \ldots, y_{i_k})$ is algebraically independent over A. Since $\bar{y} \in Y$, there exists a $x \in M_t$ such that $\bar{a}\hat{\ }x \in X$.

Now let's assume that $i_k = n+1$.

Let's define $Y = \pi_{i_1,\ldots,i_{k-1},n+1}(X)$. According to the assumption, Y has a non-empty interior. This means that there exist basis sets $U \subset \pi_{i_1,\ldots,i_{k-1}}(X)$ and $V \subset M_t$ such that $U \times V \subset Y$. According to the inductive hypothesis, there exists a tuple $\bar{u} = (u_1, \ldots, u_k) \in U$ such that $(u_1, \ldots, u_k)$ is algebraically independent over A. In addition, since V is infinite, we can find an element $v \in V$ such that $v \notin acl(\bar{u}/A)$. By lemma 69, $\bar{u}\hat{\ }v \in A$ is algebraically independent over A.

This finishes the induction and proves the proposition. □

Theorem 82. *Suppose M is ω-saturated. Let $X \subset M_t^n$ be definable. Then $rk(X) = dim(X)$.*

Proof. Let's assume that X is definable over A.

We first prove that $rk(X) \leq dim(X)$.

Let $0 \leq k \leq n$ such that $rk(X) = k$. By the definition of $rk(X)$, there exists a tuple $\bar{a} \in X$ and indices $1 \leq i_1 < \cdots < i_k \leq n$ such that $(a_{i_1}, \ldots, a_{i_k})$ is algebraically independent over A. Then by proposition 80, $\pi_{i_1,\ldots,i_k}(X)$ has an interior and by the definition of $dim(X)$, $dim(X) \geq k$.

We now prove that $rk(X) \geq dim(X)$.

Let $0 \leq k \leq n$ such that $dim(X) = k$. By the definition of $dim(X)$, there exist indices $1 \leq i_1 < \cdots < i_k \leq n$ such that $\pi_{i_1,\ldots,i_k}(X)$ has an interior. Therefore, by proposition 81, there exists a tuple $\bar{a} \in X$ such that $(a_{i_1}, \ldots, a_{i_k})$ is algebraically independent over A. So by the definition of $rk(X)$, $rk(X) \geq k$.

So together, we proved that $rk(X) = dim(X)$. □

5.5. **Connected Components in ω-Saturated 1-Dimensional t.t.t Structures.** In this section we'll focus on the connected components of a 1-dimensional ω-saturated t.t.t structure. First we show that the number of connected components is uniformly bounded over a definable family. This is used to prove that 1-dimensional ω-saturated t.t.t structures are preserved under elementary equivalence.

Our main result in this section is that for any 1-dimensional ω-saturated t.t.t structure, if removing any point divides the space into more that one connected component, then there exists a finite set $X \subset M_t$ such that each connected component of $M_t \backslash X$ is o-minimal. In order to prove this, we first obtain some intermediate results such as the fact that the equivalence relation specifying if y and z are in the same connected component of $M_t \backslash \{x\}$ is a definable relation in M_t^3. We also introduce a notion of "local flatness" which is used as a stepping stone between t.t.t structures and o-minimality.

In this section we're assuming that M_t has no isolated points. This doesn't pose a problem because M_t has at most a finite number of isolated points so we can remove them without affecting any of our results.

Lemma 83. *Let (M, ϕ) be a 1-dimensional ω-saturated t.t.t structure. Then there exists a $K \in \mathbb{N}$ such that for all $b \in M_b$, $|bd(\phi^{M_t}(b)| \leq K$.*

Proof. By lemma 63, for each $b \in M_b$, $|bd(\phi^{M_t}(b)|$ is finite. The lemma then follows from the fact that M is ω-saturated. □

Lemma 84. *Let (M, ϕ) be a definably connected 1-dimensional ω-saturated t.t.t structure, K be the constant from the previous lemma, and $X \subset M_t$ a definable subset such that $bd(X) = n$. Then $d_M(X) \leq n \cdot K$.*

Proof. Let $N = d_M(X)$ and let $\{Y_1, \ldots, Y_N\}$ be pairwise disjoint clopen (in X) subsets of X such that $X = \cup_{i=1}^N Y_i$. In addition, we denote the elements of $bd(X)$ by $bd(X) = \{a_1, \ldots, a_n\}$.

By the Hausdorffness of M_t, we can find basis sets $\{U_1, \ldots, U_n\}$ such that for all $1 \leq i \leq n$:

(1) $a_i \in U_i$
(2) For all $1 \leq j \leq N$, if $Y_j \neq \{a_i\}$ then $Y_j \backslash U_i \neq \emptyset$.

Claim. For all $1 \leq j \leq N$, if Y_j isn't a point then there exists an $1 \leq i \leq n$ such that $a_i \in \bar{Y}_i$ and $bd(U_i) \cap Y_j \neq \emptyset$.

Proof. Let $1 \leq j \leq N$ be chosen such that Y_j isn't a point. Without loss of generality, $Y_j \neq X$ because otherwise X would be connected and the lemma would be trivial. Since M_t is definably connected, $bd(Y_j) \neq \emptyset$. But Y_j is clopen in X so $bd(Y_j) \subset bd(X)$. Therefore, there exists some $1 \leq i \leq n$ such that $a_i \in \bar{Y}_i$.

We'll now see that $bd(U_i) \cap Y_j \neq \emptyset$.

Assume for contradiction that $bd(U_i) \cap Y_j = \emptyset$. Then both $U_i \cap Y_j$ and $U_i^c \cap Y_j$ are non-empty clopen subsets of X, which is a contradiction to the fact that Y_j is a connected component.

This completes the claim. □

Without loss of generality, let's assume that $1 \leq L \leq N$ such that $\{Y_1, \ldots, Y_L\}$ are points and $\{Y_{L+1}, \ldots, Y_N\}$ are not points. According to the claim, for each $L+1 \leq j \leq N$ there exists an integer $1 \leq i \leq n$ and a point y_j such that $y_j \in bd(U_i)$ and $a_i \in \bar{Y}_i$. We note that from the fact that $a_i \in \bar{Y}_i$, it follows that a_i is not an isolated point. Since $y_k \neq y_l$ for $L+1 \leq k < l \leq N$ and $bd(X)$ contains at least L isolated points, by the choice of $\{U_1, \ldots, U_n\}$ we get that $N - L \leq (n - L) \cdot K$. But $K \geq 1$ so $N \leq n \cdot K$. □

Proposition 85. *Let (M, ϕ) be a 1-dimensional ω-saturated t.t.t structure and $\alpha(x, y_1, \ldots, y_l) \in L$. Then there exists a constant $C \in \mathbb{N}$ such that for all $c_1, \ldots, c_l \in M$, $d_M(\alpha^{M_t}(c_1, \ldots, c_l)) < C$.*

Proof. First of all, let K be the constant from the previous lemmas. By lemma 63, for each $\bar{c} \in M^l$ there exists an $n_{\bar{c}} \in \mathbb{N}$ such that $bd(\alpha^{M_t}(\bar{c})) < n_c$. Therefore, since M is ω-saturated, there exists some $n \in \mathbb{N}$ such that for all $\bar{c} \in M^l$, $bd(\alpha^{M_t}(\bar{c})) < n$.

We'll show that we can choose C to be $d_M(M_t) \cdot K \cdot n$. Let $m = d_M(M_t)$ and let $\{Y_1, \ldots, Y_m\}$ be pairwise disjoint definably connected subsets such that $M_t = \cup_{i=1}^m Y_i$. By lemma 84, for each $1 \leq i \leq m$ and $\bar{c} \in M^l$, $d_M(\alpha^{M_t}(\bar{c}) \cap Y_i) < n \cdot K$. The proposition then follows immediately. □

The following equivalence relation is useful when analyzing what happens when a point is removed from a structure.

Definition 86. Let M be a 1-dimensional t.t.t structure. Let $x, a, b \in M_t$. Then $a \sim_x b$ will be a relation which is true iff a and b are in the same definable connected component of $M_t \backslash \{x\}$.

Remark. Note that by proposition 85, there exists an $N \in \mathbb{N}$ such that for all $x \in M_t$, $\sim_x$ has less that N equivalence classes.

Our first goal is to show that if for every $x \in M_t$ we have $d_M(M_t \backslash \{x\})$, then $\sim_x \subset M_t^3$ is definable.

We start by show that for any x such that $d_M(M_t \backslash \{x\}) > 2$, $x \in acl(\emptyset)$.

Lemma 87. *Let M be a 1-dimensional ω-saturated t.t.t structure, $C \subset M_t$ an open connected definable subset, $a \neq b \in C$ and $2 \leq k, l \in \mathbb{N}$ such that $d_M(C \backslash \{a\}) = k$ and $d_M(C \backslash \{b\}) = l$. Let $A_1, \dots, A_k$ and $B_1, \dots, B_l$ be the connected components of $C \backslash \{a\}$ and $C \backslash \{b\}$ respectively such that $a \in B_1$ and $b \in A_1$. Then for all $1 < i \leq k$ and $1 < j \leq l$, $A_i \cap B_j = \emptyset$.*

Proof. According to the assumptions, $a \notin cl(\cup_{j=2}^{l} B_j)$, $bd(\cup_{j=2}^{l} B_j) = \{b\}$ and $b \in A_1$. Furthermore, since C is connected, $\{b\} \cup (\cup_{j=2}^{l} B_j)$ is connected as well. Therefore, since $b \in A_1$, $A_1 \cup (\cup_{j=2}^{l} B_j)$ is a connected component of $C \backslash \{a\}$. In particular, this means that $(\cup_{j=2}^{l} B_j) \subset A_1$. Similarly, $(\cup_{i=2}^{k} A_i) \subset B_1$. This proves the lemma. $\square$

Lemma 88. *Let M be a 1-dimensional ω-saturated connected t.t.t structure. Let $D \subset M_t$ be an open definable subset, $E(x, a, b) \subset M_t^3$ a definable relation and $N \in \mathbb{N}$ such that:*

1. *$N \geq 2$.*
2. *For every $x \in D$ and $a, b \in M_t$, $a \sim_x b \Rightarrow E(x, a, b)$.*
3. *For every $x \in D$, $E(x, a, b)$ is an equivalence relation with N classes.*

Then for all $a \in D$, there exists a $b \in D$ such that the definable set $X = \{x \in D | \neg E(x, a, b)\}$ is infinite.

Proof. Let $a \in D$. Without loss of generality, D is connected. Otherwise, we'll look at the connected component containing a. We'll now show that there exists a $b \in D$ such that for an infinite number of points $x \in D$, $\neg E(x, a, b)$. In order to do this, we'll inductively construct a sequence of points $(b_1, b_2, \dots)$ in D such that for each $n \in \mathbb{N}$ and each $1 \leq j < n$, $a \sim_{b_n} b_j$ and $\neg E(b_j, a, b_n)$.

For $n = 1$, we can choose any $b_1 \in D \backslash \{a\}$.

Let's assume that we constructed the sequence up to b_n. Let $X_1, \dots, X_{c(b_n)}$ be the connected components of $M_t \backslash \{b_n\}$ such that $a \in X_2$. We choose $b_{n+1} \in D$ to be some point such that $\neg E(b_n, a, b_{n+1})$. By our assumptions on $E(x, a, b)$, $b_{n+1} \notin X_2$. So without loss of generality, in $b_{n+1} \in X_1$. Let $Y_1, \dots, Y_{c(b_{n+1})}$ be the connected components of $M \backslash \{b_{n+1}\}$ such that $b_n \in Y_1$. By lemma 87, for all $1 < j \leq c(b_{n+1})$, $Y_j \cap X_2 = \emptyset$. By the inductive hypothesis, $b_j \in X_2$ for all $1 \leq j < n$. This means that for all $1 \leq j < n$, $b_j \in Y_1$. Similarly, $a \in Y_1$ and we already know that $b_n \in Y_1$. Together we've shown that $a \sim_{b_{n+1}} b_j$ for all $1 \leq j < n + 1$.

We'll now show that for all $1 \leq j < n$, $\neg E(b_j, a, b_{n+1})$. This will be enough because we already know that $\neg E(b_n, a, b_{n+1})$.

Let $1 \leq j < n$. Let $X_1, \ldots, X_{c(b_j)}$ be the connected components of $M_t \backslash \{b_j\}$ such that $a \in X_2$ and $b_n \in X_1$. In addition, let $Y_1, \ldots, Y_{c(b_n)}$ be the connected components of $M_t \backslash \{b_n\}$ such that $b_j, a \in Y_1$ and $b_{n+1} \in Y_2$. By lemma 87, $Y_2 \subset X_1$ so $b_n \sim_{b_j} b_{n+1}$. Therefore, $E(b_j, b_n, b_{n+1}) \Rightarrow \neg E(b_j, a, b_{n+1})$.

Now by the ω-saturation, there exists a $b \in D$ such that $|\{x \in D : \neg E(x, a, b)\}| = \infty$. $\square$

Lemma 89. *Let M be a 1-dimensional ω-saturated connected t.t.t structure. Let $D \subset M_t$ be an open definable subset, $E(x, a, b) \subset M_t^3$ a definable relation and $N \in \mathbb{N}$ such that:*

1. *For every $x \in D$ and $a, b \in M_t$, $a \sim_x b \Rightarrow E(x, a, b)$.*
2. *For every $x \in D$, $E(x, a, b)$ is an equivalence relation with N classes.*

Then $N \leq 2$.

Proof. Assume for contradiction that $N > 2$. For ease of notion, we define $c(x) = d_M(M_t \backslash \{x\})$ for each $x \in M_t$. We note that for all $x \in D$, $c(x) > 2$.

By lemma 88, there exist $a, b \in M_t$ such that for an infinite number of points $x \in D$, $\neg E(x, a, b)$.

We denote the infinite set $\{x \in D : \neg E(x, a, b)\}$ by X.

Let $x, y \in X$, let $X_1, \ldots, X_{c(x)}$ be the connected components of $M_t \backslash \{x\}$ such that $a \in X_1$ and $b \in X_2$ and let $Y_1, \ldots, Y_{c(y)}$ be the connected components of $M_t \backslash \{y\}$ such that $a \in Y_1$ and $b \in Y_2$.

First we note that, for all $3 \leq j \leq c(x)$, $y \notin X_j$. Because let $3 \leq j \leq n(x)$ and assume for contradiction that $y \in X_j$. Then, by using lemma 87 in the same way as in the proof of the claim, we get that if $x \in Y_j$ for some $1 \leq j \leq c(y)$, then $a, b \in Y_j$ which is a contradiction. In an analogous fashion, $x \notin Y_j$ for all $3 \leq j \leq c(y)$. Therefore, $x \in Y_1 \cup Y_2$ and $y \in X_1 \cup X_2$.

By lemma 87, this means that:

- for all $3 \leq i \leq c(x)$ and $3 \leq j \leq c(y)$, $X_i \cap Y_j = \emptyset$
- for all $3 \leq i \leq c(x)$, $X_i \cap X = \emptyset$.

From these two results it's easy to see that $M_t \backslash X$ has an infinite number of definable connected components. Because for each $x \in X$, the classes of $E(x, a, b)$ not containing a and b are definable sets which are contained and clopen in $M_t \backslash X$. Furthermore, all the sets obtained this way are disjoint. But since X is both infinite and definable, this is a contradiction to the fact that M is t.t.t. $\square$

Now, let p be some type in $S(\emptyset)$. We now show that there exist a $\emptyset$-definable relation $R_p(x, a, b) \in M_t^3$ and an infinite $\emptyset$-definable set $D_p \subset M_t$ such that:

1. For all $x \models p$ and $a, b \in M_t$, $R_p(x, a, b) \iff a \sim_x b$.
2. For all $x \in D_p$ and $a, b \in M_t$, $a \sim_x b \Rightarrow R_p(x, a, b)$.
3. For all $x \in D_p$, $R_p(x, a, b) \subset M_t^2$ is an equivalence relation with $d_M(M_t \backslash \{y\})$ equivalence classes where y is some element realizing p.
4. For every x that realizes p, $x \in D_p$.

We construct R_p and D_p in the following way. First, let x be some realization of p and let $N = d_M(M_t \backslash \{x\})$. Then there exist $\phi_1(x, \bar{y}), \ldots, \phi_N(x, \bar{y})$ such that:

(*) for some $\bar{y}$, $\phi_1^{M_t}[\bar{y}], \ldots, \phi_N^{M_t}[\bar{y}]$ partition $M_t \backslash \{x\}$ into N disjoint clopen sets. Furthermore, for any other $\bar{z}$, if $(\phi_1^{M_t}[\bar{z}], \ldots, \phi_N^{M_t}[\bar{z}])$ is a partition of $M_t \backslash \{x\}$ into disjoint clopen sets then it's the same partition as $(\phi_1^{M_t}[\bar{y}], \ldots, \phi_N^{M_t}[\bar{y}])$.

Since this is a first order statement, (*) holds for all $x \models p$.

Now, we define D_p as the set of all $x \in M_t$ such that (*) holds for x with the formulas $\phi_1(x, \bar{y}), \dots, \phi_N(x, \bar{y})$. We then define $R_p(x, a, b)$ as a relation which is true iff for some $\bar{y}$ is guaranteed by (*) for x, $\phi_1^{M_t}[\bar{y}], \dots, \phi_N^{M_t}[\bar{y}]$ partition $M_t \backslash \{x\}$ into N disjoint clopen sets such that a and b are in the same section of the partition.

Proposition 90. *Let M be a 1-dimensional ω-saturated connected t.t.t structure and let $x \in M_t$ such that $d_M(M_t \backslash \{x\}) > 2$. Then D_p is finite and in particular, $x \in acl(\emptyset)$.*

Proof. Let $N = d_M(M_t \backslash \{x\})$ and $p = tp(x/\emptyset)$. Since $N > 2$, by applying lemma 89 with $D = int(D_p)$ and $E = R_p$, $int(D_p)$ is finite. Therefore, D_p is finite. □

We now look at what happens if $d_M(M_t \backslash \{x\}) = 2$.

As before, let $p \in S(\emptyset)$ be a type with an element x that realizes p such that $d_M(M_t \backslash \{x\}) = 2$. We define $\tilde{D}_p \subset D_p$ as the set of points in D_p such that there exist $a, b \in M_t$ and a basis set $U \subset M_t$ containing x such that for all $u \in U$, $\neg R_p(u, a, b)$.

Proposition 91. *Let M be a 1-dimensional ω-saturated connected t.t.t structure and let $p \in S_1(\emptyset)$ be a complete type in M_t. In addition, assume that for some (all) $x \models p$, $\sim_x$ has 2 equivalence classes. Then, for all x realizing p, exactly one of the two hold:*

1. *There exists a finite $\emptyset$-definable subset of D_p containing x and in particular, $x \in acl(\emptyset)$.*
2. *$int(\tilde{D}_p)$ is a set containing x such that for all $y \in int(\tilde{D}_p)$, $d_M(M_t \backslash \{y\}) = 2$.*

Proof. First of all, if D_p is finite then the first case holds for all $x \models p$.

Let's assume that D_p is infinite. Now, suppose that for all x realizing p:

(*) for all $a, b \in M_t$ and for every basis set U containing x, there exists a $u \in U$ such that $R(u, a, b)$.

We define $C \subset D_p$ as the set of points in D_p with the property (*). C is clearly $\emptyset$-definable. Furthermore, for all $x \models p$, $x \in C$. Assume for contradiction that C is infinite. Therefore, by lemma 88, there exist points $a, b \in M_t$ and a basis set $U \subset C$ such that for all $u \in U$, $\neg R(u, a, b)$. This is clearly a contradiction to (*). This means that C is finite so again we're in the first case for all $x \models p$.

Therefore, we can assume that for all elements x realizing p, $x \in \tilde{D}_p$. If $\tilde{D}_p$ is finite then again we're in the first case for all $x \models p$.

We'll now see that if $\tilde{D}_p$ is infinite then for all $y \in int(\tilde{D}_p)$, $d_M(M_t \backslash \{y\}) \leq 2$. This will finish the proposition because we already know that for all $x \in \tilde{D}_p$, $d_M(M_t \backslash \{x\}) \geq 2$. We also note that if $x \in \tilde{D}_p \backslash int(\tilde{D}_p)$ then clearly we're in the first case as $|\tilde{D}_p \backslash int(\tilde{D}_p)| < \infty$.

Let's assume for contradiction that $y \in int(\tilde{D}_p)$ and $d_M(M_t \backslash \{y\}) > 2$. Since $y \in \tilde{D}_p$, there exist $a, b \in M_t$ and a basis set $U \subset \tilde{D}_p$ containing y such that for all $u \in U$, $\neg R_p(u, a, b)$. Let Y_1, Y_2, Y_3 be three definable disjoint clopen sets partitioning $M_t \backslash \{y\}$ such that $a \in Y_1$ and $b \in Y_2$. Since M_t is connected, there exists a z such that $z \in Y_3 \cap U$. Since $z \in D_p$, $k = d_M(M_t \backslash \{z\}) \geq 2$. Let $Z_1, \dots, Z_k$ be definable pairwise disjoint clopen sets partitioning $M_t \backslash \{z\}$ such that $y \in Z_1$. By lemma 87, $a, b \in Z_1$ which is a contradiction to the fact that $\neg R_p(z, a, b)$. □

We now use the previous two propositions to show that if for all $x \in M_t$ we have $d_M(M_t \backslash \{x\}) > 1$, then the relation $a \sim_x b \subset M_t^3$ is $\emptyset$-definable.

Proposition 92. *Let M be a 1-dimensional ω-saturated t.t.t structure such that for all $x \in M_t$, $d_M(M_t \backslash \{x\}) > 1$. Then the relation $a \sim_x b \subset M_t^3$ is $\emptyset$-definable.*

Proof. First of all, $_t$ has a finite number of definable connected components. Therefore, it's enough to prove the proposition under the assumption that M_t is connected.

We'll show that both $a \sim_x b$ and $a \nsim_x b$ are $\bigvee$-definable by formulas without parameters.

$a \nsim_x b$ is clearly $\bigvee$-definable by formulas without parameters because $a \nsim_x b$ iff there exist two open sets whose boundary is $\{x\}$ such that one contains a and the other contains b.

We'll now prove that $a \sim_x b$ is $\bigvee$-definable by formulas without parameters. This is done by showing that for each $x \in M_t$, there exists a set $C_x \subset M_t^3$ which is definable without parameters such that:

(1) For all $y, a, b \in M_t$, $(y, a, b) \in C_x \Rightarrow a \sim_y b$
(2) $(x, a, b) \in C_x \iff a \sim_x b$

Let $x \in M_t$ and $p = tp(x/\emptyset)$.

If $d_M(M_t \backslash \{x\}) = N > 2$, then by proposition 90, D_p is a finite $\emptyset$-definable set containing x. Furthermore, for every $y \in D_p$, $d_M(M_t \backslash \{x\}) \geq N$. Let $D_p = \{y_1, \dots, y_k\}$. Without loss of generality, there exists some $0 \leq l < k$ such that for all $1 \leq i \leq l$, $d_M(M_t \backslash \{y_i\}) > N$ and for all $l+1 \leq i \leq k$, $d_M(M_t \backslash \{y_i\}) = N$. It's easy to see that for each $1 \leq i \leq l$,$x \notin D_{tp(y_i/\emptyset)}$. Therefore, we can define:

$$C_x = (D_p \backslash (\bigcup_{i=1}^{l} D_{tp(y_i/\emptyset)}) \times M_t^2) \cap R_p$$

. Now let's assume that $d_M(M_t \backslash \{x\}) = 2$.

If D_p contains a finite $\emptyset$-definable set containing x, then we can define C_x in the same way as in the previous case. Otherwise, by proposition 91, $int(\tilde{D}_p) \subset D_p$ is a set containing x such that for all $y \in int(\tilde{D}_p)$, $d_M(M_t \backslash \{y\}) = 2$. Therefore, we can define:

$$C_x = (int(\tilde{D}_p) \times M_t^2) \cap R_p$$

. This finishes the proof of the proposition. ☐

We'll now prove that, under the condition that removing any point creates at least two connected components, there exist a finite number of points such that after removing them, the remaining finite number of connected components are o-minimal. This is done by first showing that up to a finite number of points the structure is "locally o-minimal", and then showing that local o-minimality implies global o-minimality. In addition, the definability of the relation $a \sim_x b \subset M_t^3$ will play a crucial role.

We start by defining a notion of "local flatness" and then showing that locally flat points have a neighborhood which behaves similarly to an o-minimal one.

Definition 93. Let M t.t.t structure. We say that the point $x \in M_t$ is *locally flat* if there exist points $a, b \in M_t$ and a basis set U such that for all $u \in U$, $a \nsim_u b$. We say that a set D is locally flat if all of it's points are locally flat.

We first show that in the type of structures that we're currently interested in, all but a finite number of points are locally flat.

Proposition 94. *Let M be a 1-dimensional ω-saturated connected t.t.t structure such that for all $x \in M_t$, $d_M(M_t \backslash \{x\}) > 1$. Then for all but a finite number of $x \in M_t$, x is locally flat.*

Proof. Let $X \subset M_t$ be the set of points $x \in M_t$ such that for all $a, b \in M_t$ and for every basis set U, there exists a point $y \in U$ such that $a \sim_y b$. By proposition 92, X is definable.

Assume for contradiction that X is infinite. Then, by lemma 88, there exist $a, b \in M_t$ such that the set $\tilde{X} = \{x \in X | a \nsim_x b\}$ is infinite. In addition, $\tilde{X}$ is definable so there exists a basis set U which is contained in $\tilde{X} \subset X$. This is clearly a contradiction to the definition of X. □

The next few propositions will show that points which are locally flat have a neighborhood on which we can define a linear order. This motivates the "flatness" in the definition.

Lemma 95. *Let M be a 1-dimensional ω-saturated connected t.t.t structure such that for all $x \in M_t$, $d_M(M_t \backslash \{x\}) > 1$. Let $x \in M_t$ be locally flat and let $U \subset M_t$ be an open connected definable set containing x. Then $U \backslash \{x\}$ has two connected components.*

Proof. Since $d_M(M_t \backslash \{x\}) > 1$ and M_t is connected, it's enough to show that $U \backslash \{x\}$ has no more than two connected components.

Assume for contradiction that $U_1, \ldots, U_k$ are the connected components of $U \backslash \{x\}$ with $k > 2$. In addition, let $a, b \in M_t$ be points and let $V \subset U$ be a basis set containing x such that for all $v \in V$, $a \nsim_v b$.

Let $X_1, X_2 \subset M_t \backslash \{x\}$ be a clopen partition of $M_t \backslash \{x\}$ such that $a \in X_1$ and $b \in X_2$. By the connectedness of M_t, we can assume without loss of generality that $U_1 \subset X_1$ and $U_2 \subset X_2$.

Now, let y be a point in $U_3 \cap V$. By lemma 87, U_1 and U_2 are in the same connected component of $U \backslash \{y\}$. So since $U_1 \subset X_1$ and $U_2 \subset X_2$, both X_1 and X_2 are in the same connected component of $M_t \backslash \{y\}$. This means that $a \sim_y b$ which is a contradiction to the fact that $y \in V$. □

One consequence of lemma 95 which will be used later is that if we remove all of the finite number of points which aren't locally flat, for each of the remaining connected components C, and for each $x \in C$, $C \backslash \{x\}$ will have exactly two connected components. This will be used to show that C is o-minimal.

In addition, it follows from this that if D is a definable open and connected locally flat set with $a, b, x \in D$ and $D \backslash \{x\} = X_1 \cup X_2$ is a clopen partition of D, then $a \nsim_x b$ iff a and b are in different sections of the partition.

Our next goal is to define an order on some neighborhood of each locally flat point. Let $x_0 \in U$ be locally flat and let $U \subset M_t$ be a definable neighborhood of x_0. We define an order $<_{x_0,U}$ on U in the following way.

By lemma 95, $U \backslash \{x_0\}$ has two connected components which we'll denote by V_+ and V_-. Let x and y be points in U. We'll say that $x <_{x_0,U} y$ if one of the following hold:

- $x, y \in V_+$ and $x_0 \nsim_x y$

- $x, y \in V_-$ and $x_0 \nsim_y x$
- $y \in V_+$ and $x \in V_-$.
- $y = x_0$ and $x \in V_-$.
- $x = x_0$ and $y \in V_+$.

Note that for any given $x, y \in U$, at most one of the above conditions can hold.

We now show that if x_0 is locally flat then there exists a neighborhood $x_0 \in U$ such that $<_{x_0,U}$ defines a dense linear order on U.

Proposition 96. *Let M be a 1-dimensional ω-saturated t.t.t structure such that for all $x \in M_t$, $d_M(M_t \backslash \{x\}) > 1$. Let $D \subset M_t$ be a connected open definable subset such that for all $x \in D$, $d_M(D \backslash \{x\}) = 2$. Let $x_0 \in D$ be locally flat. Then there exists a connected open neighborhood $U \subset D$ of x_0 such that $<_{x_0,U}$ defines a dense linear order on U.*

Proof. Let $a, b \in M_t$ and $V \subset D$ be the elements guaranteed by the fact that x_0 is locally flat. We can assume that V is connected because otherwise we can take the connected component containing x. In addition, let V_+ and V_- be the two connected components of $V \backslash \{x_0\}$ (by lemma 95 there are exactly two) and let C_a and C_b be the connected components of $M_t \backslash \{x\}$ such that $a \in C_a$ and $b \in C_b$. In addition, without loss of generality $V_+ \subset C_a$ and $V_- \in C_b$. This follows from the fact that $C_a \cap V$ and $C_b \cap V$divide V into two clopen sets, and by lemma 95, one must equal V_+ and the other must equal V_-.

We'll show that $<_{x_0,V}$ is a dense linear order on V. Let $x, y, z \in V$. In addition, let X_a and X_b be the connected components of $M_t \backslash \{x\}$ such that $a \in X_a$ and $b \in X_b$. Y_a, Y_b, Z_a and Z_b are defined analogously for y and z.

(1) $x \not<_{x_0,V} y \Rightarrow y <_{x_0,V} x$:

We assume that $x, y \in V_+$. The other possibilities are either identical or trivial. Since $x, y \in V_+ \subset C_a$, $x_0 \in X_b$ and $x_0 \in Y_b$. Because if we assume for contradiction that $x_0 \in X_a$, then by lemma 87 $C_b \cap X_b = \emptyset$ which is a contradiction to the fact that $b \in C_b \cap X_b$.

In addition, according to the assumption, $x_0 \sim_x y$ which means that $y \in X_b$ as well. Now let's assume for contradiction that $x \in Y_b$. Since $y \in X_b$ we get from lemma 87 that $Y_a \cap X_a = \emptyset$ which is a contradiction since $a \in X_a \cap Y_a$. Therefore, $x \in Y_a$ which together with $x_0 \in Y_b$ gives $y < x_{o,V} x$.

(2) $x <_{x_0,V} y \Rightarrow y \not<_{x_0,V} x$:

Also in this case we'll assume that $x, y \in V_+$. Since $x <_{x_0,V} y$ and $x_0 \in X_b$, we get that $y \in X_a$. Now, we claim that $x \in Y_b$. For otherwise, if $x \in Y_a$ then we'd get from lemma 87 that $Y_b \cap X_b = \emptyset$ which is a contradiction.

Therefore, since $x_0 \in Y_b$ as well, $y \not<_{x_0,V} x$.

(3) $x <_{x_0,V} y \wedge y <_{x_0,V} z \Rightarrow x <_{x_0,V} z$:

According to the assumptions and the inclusions we saw in the previous two steps, $x_0, x \in Y_b$, $z \in Y_a$, $x_0 \in X_b$ and $y \in X_a$. We have to prove that $z \in X_a$ as well. But again by lemma 87, $y \in X_a \wedge x \in Y_b \Rightarrow Y_a \subset X_a$.

These three claims show that $<_{x_0,V}$ is indeed a linear order. We'll now show that if $x <_{x_0,V} y$ then there exists a $z \in V$ such that $x <_{x_0,V} z <_{x_0,V} y$. Again we'll assume that $x, y \in V_+$. Therefore, by what we proved in the claims and by lemma

87,

$$V = (X_b \cap V) \cup \{x\} \cup (X_a \cap Y_b \cap V) \cup \{y\} \cup (Y_a \cap V)$$

where all of the sets in the union are disjoint. Therefore, since V is connected, $X_a \cap Y_b \cap V \neq \emptyset$. Finally, by the definition of $<_{x_0,V}$ and the fact that it's linear, if $z \in X_a \cap Y_b \cap V$ then $x <_{x_0,V} z <_{x_0,V} y$. □

Before extending the order defined to connected components, we introduce the notion of an interval in a t.t.t structure and prove some useful properties.

Definition 97. Let M be a 1-dimensional t.t.t structure and $x, y \in M_t$ such that $d_M(M_t \backslash \{x\}) = d_M(M_t \backslash \{y\}) = 2$. In addition, let X_1 and X_2 be clopen in $M_t \backslash \{x\}$ and let Y_1 and Y_2 be clopen in $M_t \backslash \{y\}$ such that $x \in Y_1$ and $y \in X_1$. Then *the interval between x and y* will be defined as $I(x, y) = X_1 \cap Y_1$. If $x = y$ then $I(x, y) = \emptyset$.

Remark. By lemma 87, under the assumptions of the lemma, if $V \subset M_t$ is an open definable subset and $x, y \in V$ then:

$$V = (X_2 \cap V) \cup \{x\} \cup (I(x, y) \cap V) \cup \{y\} \cup (Y_2 \cap V)$$

and the union is disjoint.

Lemma 98. *Let M be a 1-dimensional ω-saturated t.t.t structure such that for all $x \in M_t$, $d_M(M_t \backslash \{x\}) > 1$. Let $D \subset M_t$ be a connected open definable subset which is locally flat. Let $x \neq y$ be points in D. Then:*

1. *$I(x, y) \cap D$ is a non-empty definable open connected set.*
2. *$x, y \in bd(I(x, y))$.*
3. *If $a, b \in D \backslash \{x, y\}$ such that $a \sim_x b$ and $a, b \notin I(a, b)$, then $a \sim_y b$.*
4. *If $a, b \in D \backslash \{x, y\}$ such that $a \nsim_x b$ and $a, b \notin I(a, b)$, then $a \nsim_y b$.*
5. *If $a, b \in D$ such that $a, b \in I(x, y)$ then $I(a, b) \subset I(x, y)$.*

Proof. Let X_1 and X_2 be the connected components of $M_t \backslash \{x\}$ and Y_1 and Y_2 be the connected components of $M_t \backslash \{y\}$. Note that by proposition 95, both $D \backslash \{x\}$ and $D \backslash \{y\}$ have exactly two connected components which are given by $X_1 \cap D$, $X_2 \cap D$ and $Y_1 \cap D$, $Y_2 \cap D$ respectively. Without loss of generality, $x \in Y_1$ and $y \in X_1$ so $I(x, y) = X_1 \cap Y_1$, $X_2 \subset Y_1$ and $Y_2 \subset X_1$. This means that

(*) $D = (X_2 \cap D) \cup \{x\} \cup (I(x, y) \cap D) \cup \{y\} \cup (Y_2 \cap D)$.

We'll now prove the four parts of the lemma. □

1. First of all, since X_1 and Y_1 are definable, $I(x, y)$ is definable as well. If $I(x, y) \cap D$ were empty, then by (*) we'd get that D isn't connected. $I(x, y) \cap D$ is open as the intersection of open sets.

   Now, assume for contradiction that $I(x, y) \cap D$ has more than one connected component. Since D is connected, it follows that either x or y has a connected open neighborhood $U \subset D$ such that removing x or y creates more than two connected components in U. This is a contradiction to lemma 95.
2. If either $x \notin bd(I(x, y))$ or $y \notin bd(I(x, y))$ then by (*) we'd get a contradiction to the fact that D is connected.
3. If $a, b \in X_2$, then $a, b \in Y_2$ so $a \sim_y b$. So we can assume that $a, b \in X_1$. By the assumption, $a, b \notin Y_1 \Rightarrow a, b \in Y_2 \Rightarrow a \sim_y b$.

(4) Without loss of generality, $a \in X_1$ and $b \in X_2$. Therefore, $b \in Y_1$. In addition, $Y_2 \subset X_1$ and $a \notin Y_1 \cap X_1$ so $a \in Y_2$. This means that $a \nsim_y b$.
(5) Let A_1 and A_2 be the connected components of $M_t\backslash\{a\}$ and let B_1 and B_2 be the connected components of $M_t\backslash\{b\}$. By the assumption, $a, b \in X_1 \cap Y_1$. Without loss of generality, $x \in A_1 \cap B_1$, $y \in A_2 \cap B_2$ and $b \in A_2$. By a standard application of lemma 87, this means that $a \in B_1$. Therefore, $I(a,b) = A_2 \cap B_2$. But again by lemma 87, $A_2 \subset X_1$ and $B_1 \subset Y_1$. Therefore, $I(a,b) \subset I(x,y)$.

Lemma 99. *Let M be a 1-dimensional ω-saturated t.t.t structure such that for all $x \in M_t$, $d_M(M_t\backslash\{x\}) > 1$. Let $D \subset M_t$ be a connected open definable subset which is locally flat. Let $x_0 \in D$. Then there exist points $a, b \in D$ such that $x_0 \in I(a,b)$ and $<_{x_0,I(a,b)}$ defines a dense linear order on $I(a,b)$.*

Proof. By proposition 96, there exists a definable connected open neighborhood $U \subset D$ of x_0 such that $<_{x_0,U}$ defines a dense linear order on U. Since M_t is Hausdorff and $bd(U)$ is finite, we can assume that $<_{x_0,U}$ defines a dense linear order on $\overline{U}$. Let a be an the point on $bd(U)$ such that $x_0 <_{x_0,U} a$ and for all $y \in bd(U)$ with $x_0 <_{x_0,U} y$, $a \leq_{x_0,U} y$. Similarly, Let b be the point on $bd(U)$ such that $b <_{x_0,U} x_0$ and for all $y \in bd(U)$ with $y <_{x_0,U} x_0$, $y \leq_{x_0,U} b$.

We'll now see that $I(a,b) \subset U$. By lemma 98, $I(x,y) \cap D$ is a non-empty open connected set. Furthermore, by the definition of $<_{x_0,U}$ and the fact that it's a linear order together with the choice of a and b, $x_0 \in I(a,b)$. Assume for contradiction that $I(a,b)\backslash U \neq \emptyset$. Then by the connectedness of $I(a,b) \cap D$, $I(a,b) \cap bd(U) \neq \emptyset$. Let y be a point in $I(a,b) \cap D$. Then by the definition of $<_{x_0,U}$ and the fact that it's a linear order, $b <_{x_0,U} y <_{x_0,U} a$ which is clearly a contradiction to the choice of a and b. □

Now, for all x such that there exists an open neighborhood U of x which is locally flat, we define $V_x = I(a,b)$ where a and b are any two points given by lemma 99. Even though V_x depends on U, a and b, for ease of notion we omit the parameters as U will always be clear from the context (we'll write $V_x \subset U$) and the exact choice of a and b doesn't change the properties of V_x. We note that by lemma 98, V_x is an open connected neighborhood of x. In addition, for all $c, d \in V_x$, it follows from part 5 of lemma 98 that $I(c,d) \subset I(a,b) = V_x$ which means that:

$$\{y \in V_x | c <_{x,V_x} y <_{x,V_x} d\} = I(c,d)$$

. In other words, in V_x, the notion of an interval we defined above coincides with the interval induced by the order $<_{x,V_x}$.

We'll now prove three lemmas about locally flat points which together will show that the order we defined above can be extended from locally flat points to connected locally flat sets.

Lemma 100. *Let M be a 1-dimensional ω-saturated t.t.t structure such that for all $x \in M_t$, $d_M(M_t\backslash\{x\}) > 1$. Let $D \subset M_t$ be a connected open definable subset which is locally flat. Let $x \neq y \in D$. In addition, let $a, b \in D$ and $V_x = I(a,b)$ such that $<_{x_0,V_x}$ defines a dense linear order on V_x. Then there exists an $a \in V_x$ such that $x \nsim_a y$.*

Proof. Let X_1 and X_2 be a clopen partition of $D\backslash\{x\}$. Without loss of generality, $y \in X_1$.

First let's assume that $y \in V_x$. Let $a \in I(x,y)$. Then either $y <_{x,V_x} a <_{x,V_x} x$ or $x <_{x,V_x} a <_{x,V_x} y$. In either case, $x \nsim_a y$.

Now let's assume that $y \notin V_x$. We choose a to be some point in $I(x,y) \cap V_x$. Then $a \in X_1$. Let A_1 and A_2 be a clopen partition of $D \backslash \{a\}$ such that $x \in A_1$. Assume for contradiction that $y \in A_1$. Then $y \in I(a,x) \subset V_x$ which is a contradiction. Therefore, $y \in A_2$ and $x \nsim_a y$. □

Lemma 101. *Let M be a 1-dimensional ω-saturated t.t.t structure such that for all $x \in M_t$, $d_M(M_t \backslash \{x\}) > 1$. Let $D \subset M_t$ be a connected open definable subset which is locally flat. Let $x, a, b \in D$ such that for every open set $x \in U \subset D$ there exists some $u \in U$ such that $a \nsim_u b$. Then $a \nsim_x b$.*

Proof. Since M_t is Hausdorff, there exists an open definable connected set $U \subset D \backslash \{a,b\}$ containing x. Let $V_x \subset U$ be the set described above.

By the assumptions of the lemma, there exists some $y \in V_x$ such that $a \nsim_y b$. Since $a, b \notin V_x$, $a, b \notin I(x,y) \subset V_x$. So by lemma 98, $a \nsim_x b$. □

Lemma 102. *Let M be a 1-dimensional ω-saturated t.t.t structure such that for all $x \in M_t$, $d_M(M_t \backslash \{x\}) > 1$. Let $D \subset M_t$ be a connected open definable subset which is locally flat. Let $x, a, b \in D$ such that $a \nsim_x b$. The there exists a definable open set $U \subset D$ containing x such that for every $u \in U$, $a \nsim_u b$.*

Proof. As in the previous lemma, there exists an open definable connected set $U \subset D \backslash \{a,b\}$ containing x. Let $V_x \subset U$ be the set described above. Let $y \in V_x$. Since $a, b \notin V_x$ it follows that $a, b \notin I(x,y) \subset V_x$. Therefore, by lemma 98, $a \nsim_y b$. □

We now use the previous three lemmas to show that in some well defined sense, locally flat sets look like a line.

Lemma 103. *Let M be a 1-dimensional ω-saturated t.t.t structure such that for all $x \in M_t$, $d_M(M_t \backslash \{x\}) > 1$. Let $D \subset M_t$ be a connected open definable subset which is locally flat. Then, there doesn't exist a definable connected closed subset $U \subset D$ such that $bd(U) > 2$.*

Proof. Assume for contradiction that $U \subset D$ is a definable closed connected subset and that $a, b, c \in bd(U)$.

Let U_{ab} denote the set of points $x \in U$ such that $a \nsim_x b$. By lemmas 100 and 87, $U_{ab} \neq \emptyset$. By lemmas 102 and 101, U_{ab} is clopen. Therefore, $U = U_{ab}$.

Similarly, if U_{ac} and U_{bc} are defined in the analogous fashion, $U = U_{ac} = U_{bc} = U_{ab}$. We'll now show that this is a contradiction.

Let $x \in int(U)$. Since $d_M(D \backslash \{a\}) = 2$, let X_1 and X_2 be the two connected components of $D \backslash \{x\}$. Either X_1 or X_2 will contain two out of a, b, and c. Without loss of generality, $a, b \in X_1$. However, this is a contradiction to the fact that $x \in U_{ab} \Rightarrow a \nsim_x b$. □

We're now ready to show that every locally flat set is o-minimal. In order to do this, we'll extend our previous notion of order from neighborhoods of locally flat points to locally flat sets.

Let D be a definable open connected locally flat set. Let $a \in D$ be some arbitrary point which we'll think of as the center. In addition, let D_+ and D_- be the two connected components of $D \backslash \{a\}$ which we'll think of as the "positive side" and the "negative side". Finally let $x, y \in D$. We say that $x <_D y$ if one of the following holds:

- $x, y \in D_+$ and $a \not\sim_x y$
- $x, y \in D_-$ and $a \not\sim_y x$
- $y \in D_+$ and $x \in D_-$
- $y = a$ and $x \in D_-$
- $x = a$ and $y \in D_+$

Obviously, only one of the possibilities can hold for a given x and y. It is also clear that $<_D$ is definable.

The next proposition shows that $<_D$ defines a dense linear order on D such that the induced interval topology is equivalent to the topology induced by M_t.

Proposition 104. *Let M be a 1-dimensional ω-saturated t.t.t structure such that for all $x \in M_t$, $d_M(M_t \backslash \{x\}) > 1$. Let $D \subset M_t$ be a connected open definable subset which is locally flat. Then $<_D$ defines a dense linear order on D such that the induced interval topology is equivalent to the topology induced by M_t.*

Proof. Let D_+, D_- and $a \in D$ be the elements used in the definition of $<_D$ above.

Let $x, y, z \in D$. In addition, let X_1 and X_2 be the connected components of $D \backslash \{x\}$ such that $a \in X_1$. Y_1, Y_2, Z_1 and Z_2 are defined analogously for y and z.

(1) $x \not<_{x_0,V} y \Rightarrow y <_{x_0,V} x$:

We assume that $x, y \in D_+$. The other possibilities are either identical or trivial. By the assumption, $y \in X_1$. Assume for contradiction that $x \in Y_1$. Since $a \in X_1 \cap Y_1$ and $x, y \in D_+$, $U = X_1 \cap Y_1 \cap D_+$ is an open set such that:

$$D = D_- \cup X_2 \cup Y_2 \cup U \cup \{a\} \cup \{x\} \cup \{y\}$$

We'll now show that U is connected and that $bd(U) = \{a, x, y\}$. Let C be some connected component of U. Then since D is connected, $bd(C) = \{a, x, y\}$. In addition, by lemma 95, each one of a, x, and y is the boundary point of only one connected component of U. Therefore, U is connected and $bd(U) = \{a, x, y\}$.

However, this is a contradiction to lemma 103.

(2) $x <_D y \Rightarrow y \not<_D x$:

Also in this case we'll assume that $x, y \in V_+$. Since $x <_D y$ and $a \in X_1$, we get that $y \in X_2$. Now, we claim that $x \in Y_1$. For otherwise, if $x \in Y_2$ then we'd get from lemma 87 that $Y_1 \cap X_1 = \emptyset$ which is a contradiction to the fact that $a \in X_1 \cap Y_1$.

(3) $x <_{x_0,V} y \wedge y <_{x_0,V} z \Rightarrow x <_{x_0,V} z$:

According to the assumptions and the previous result, $a, x \in Y_1$, $z \in Y_2$, $a \in X_1$ and $y \in X_2$. We have to prove that $z \in X_2$ as well. But again by lemma 87, $y \in X_2 \wedge x \in Y_1 \Rightarrow Y_2 \subset X_2$.

This shows that $<_D$ is a linear order. In addition, by lemma 100 $<_D$ is dense. Because let $x, y \in D$ such that $x <_D y$. Again let's assume that $x, y \in D_+$. Since M_t is Hausdorff, we can find a neighborhood $U \subset D$ containing x but not y, a neighborhood $x \in V_x \subset U$, and a point $b \in V_x$ such that $x \not\sim_b y$. If $b <_D x$ or $y <_D b$ then we'd get a contradiction to $x \not\sim_b y$. Therefore, $x <_D b <_D y$.

We'll now see that the order topology induced on D by $<_D$ is equivalent to the topology on D induced by M_t.

As a first step, we note that if $x, y \in D$ and $x <_D y$, then

$$I(x, y) \cap D = \{z \in D | x <_D z <_D y\}$$

. This is immediate from the definitions of $I(x,y)$ and $<_D$.

Let $U \subset D$ be an open set in D with $x \in U$. Let $V_x \subset U$ be the set guaranteed by lemma 99. Let V_1 and V_2 be the two connected components of $V_x \backslash \{x\}$ and let v_1 and v_2 be elements in V_1 and V_2 respectively. As we mentioned after lemma 99, $I(v_1, v_2) \subset V_x$. Without loss of generality, $v_1 <_D v_2$. Therefore,

$$\{z \in D | v_1 <_D z <_D v_2\} = I(v_1, v_2) \subset V_x \subset U$$

. The other direction is trivial as $I(x,y)$ is open in D for every $x, y \in D$. □

Proposition 105. *Let M be a 1-dimensional ω-saturated t.t.t structure such that for all $x \in M_t$, $d_M(M_t \backslash \{x\}) > 1$. Let $D \subset M_t$ be a connected open definable subset which is locally flat. Then every definable subset of D is a finite union of points and intervals generated by the order $<_D$.*

Proof. First of all, we'll denote interval in $<_D$ by $(a,b)_D$ where $a, b \in D$.

Let $X \subset D$ be a definable set. Since X is a finite union of points and definable connected open sets, it's enough to show that every definable connected open subset of D is given by an interval of the form $(a,b)_D$.

Let $U \subset D$ be a definable connected open subset. By lemma 103, $bd(U)$ has either zero, one or two points. Before continuing, we note that by lemma 98, all intervals of the form $(a,b)_D$ are definable, open and connected. Because as we stated previously, if $x, y \in D$ and $x <_D y$, then $I(x,y) \cap D = (x,y)_D$.

- If $|bd(U)| = 0$ then since D is connected, $U = D$ so $U = (-\infty, \infty)_D$.
- If $bd(U) = \{a\}$, then we claim that $U = (a, \infty)_D$ or $U = (-\infty, a)_D$.
  First of all, since U is open and connected, either $U \subset (a,\infty)_D$ or $U \subset (-\infty, a)_D$. Without loss of generality, $U \subset (a,\infty)_D$. Assume for contradiction that there exists some $x \notin U$ such that $a <_D x$. Then, $U \cap (a,x)_D$ and $U \cap (x, \infty)_D$ is a clopen partition of U which is a contradiction to the fact that U is connected.
- If $bd(U) = \{a, b\}$ with $a <_D b$ then we claim that $U = (a,b)_D$.
  First of all $U \subset (a,b)_D$ because otherwise we'd easily get a clopen partition as above.
  Similarly, assume for contradiction that there exists some $x \in (a,b)_D \backslash U$. Then $U \cap (a,x)$ and $U \cap (x,b)$ is a clopen partition of U which is a contradiction to the fact that U is connected.

This proves the proposition. □

We now obtain our primary result as an immediate consequence of propositions 94, 105 and 104.

Theorem 106. *Let M be a 1-dimensional ω-saturated t.t.t structure such that for all $x \in M_t$, $d_M(M_t \backslash \{x\}) > 1$. Then there exists a finite set $X \subset M_t$ such that all of the finite number of connected components of $M_t \backslash X$ are o-minimal.*

Proof. Let X be the definable set of points in M_t which aren't locally flat. By proposition 94, X is finite. Let D be a connected component if $M_t \backslash X$. Since there're only a finite number of connected components, D is a connected open definable subset which is locally flat. By propositions 104 and 105, D is o-minimal. □

We'll now use the uniform bound from proposition 85 to prove that a certain set of first order properties are necessary and sufficient for an ω-saturated first order topological structure to be t.t.t.

Theorem 107. *Let (M,ϕ) be an ω-saturated first order topological structure such that M_t is Hausdorff. Then, M is a t.t.t structure iff M has the following properties:*

1. *For every formula $\alpha(x,y_1,\ldots,y_l)\in L$, there exists some $C\in\mathbb{N}$ such that for every $\overline{c}\in M^l$, there exist C points $x_1,\ldots,x_C$ in $\alpha^{M_t}(\overline{c})$ such that $\alpha^{M_t}(\overline{c})\backslash\{x_1,\ldots,x_C\}$ is open.*
2. *For every formula $\alpha(x,y_1,\ldots,y_l)\in L$, there exists a constant $C\in\mathbb{N}$ such that for all $c_1,\ldots,c_l\in M$, $d_M(\alpha^{M_t}(c_1,\ldots,c_l))<C$.*
3. *For any pair of formulas $\alpha(x,y_1,\ldots,y_s)$ and $\beta(x,y_1,\ldots,y_t)$ in L, and for all $\overline{a}\in M^s$ and $\overline{b}\in M^t$, if $B=\beta^{M_t}(\overline{b})\subset\alpha^{M_t}(\overline{a})=A$ is closed and non empty and doesn't have an interior in A, then A has an interior in M_t.*

Proof. First we'll see that the three properties are sufficient. Assume that (M,ϕ) is an ω-saturated first order topological structure such that M_t is Hausdorff and has the three properties in the theorem.

By property 1, every definable set X is a boolean combination of open sets so M has property (A). By property 2, every definable set has a finite number of definably connected components. Finally, by property 3, $D(M)=1$.

Now we'll see that the properties are necessary. Let (M,ϕ) be an ω-saturated t.t.t structure. By the definition of t.t.t, M_t is Hausdorff. We'll now prove that M has each one of the required properties.

1. Let $\alpha(x,y_1,\ldots,y_l)\in L$. Since M is t.t.t, for every $\overline{c}\in M^l$, there exist C points $x_1,\ldots,x_C$ in $\alpha^{M_t}(\overline{c})$ such that $\alpha^{M_t}(\overline{c})\backslash\{x_1,\ldots,x_C\}$ is open. Since M is ω-saturated, we can choose C uniformly for all $\overline{c}\in M^l$.
2. This property is essentially proposition 85.
3. This follows from the fact that $D(M)=1$.

□

Corollary 108. *Let $\phi(x,y_1,\ldots,y_k)$ be a formula and let (M,ϕ) be a 1-dimensional t.t.t structure which is ω-saturated. In addition, let N be a model such that $N\equiv M$. Then (N,ϕ) is a 1-dimensional t.t.t structure.*

Proof. This is immediate from the fact that all of the properties in theorem 107 can be expressed in first order logic. □

References

[1] van den Dries, Lou (1998). "Tame Topology and o-minimal Structures". Cambridge University Press.

[2] Pillay, Anand; Steinhorn, Charles (1986). "Definable Sets in Ordered Structures I". Transactions of the American Mathematical Society 295 (2): 565–592.

[3] Knight, Julia; Pillay, Anand; Steinhorn, Charles (1986). "Definable Sets in Ordered Structures II". Transactions of the American Mathematical Society 295 (2): 593–605

[4] Miller, Chris (1994). "Exponentiation is Hard to Avoid". Proceedings of the American Mathematical Society , Vol. 122, No. 1 (Sep., 1994), pp. 257-259

[5] Pillay, Anand (1987). "First Order Topological Structures and Theories". The Journal of Symbolic Logic , Vol. 52, No. 3 (Sep., 1987), pp. 763-778

[6] Rosenlicht, Maxwell (1983). "The Rank of a Hardy Field". Transactions of the American Mathematical Society , Vol. 280, No. 2 (Dec., 1983), pp. 659-671

[7] A.Strzebonski (1994). "Euler characteristic in semialgebraic and other o-minimal structures", J.Pure Appl. Algebra 96 (1994) 173-201.

[8] Miller, Chris; Starchenko, Sergei(1998). "A Growth Dichotomy for O-Minimal Expansions of Ordered Groups", Transactions of the American Mathematical Society , Vol. 350, No. 9 (Sep., 1998), pp. 3505-3521

[9] Pillay, Anand (1988). “On groups and fields definable in o-minimal structures”, J. Pure and Appl. Algebra, 53 (1988), pp. 239–255.

[10] Miller, Chris (1996). “"A growth dichotomy for o-minimal expansions of ordered fields, Logic: from foundations to applications, Oxford University Press, (1996), pp.385–399.

[11] Mathews, L., "Cell decomposition and dimension functions in first-order topological structures", Proceedings of the London Mathematical Society. Third Series, vol. 70 (1995), pp. 1–32.