Log Geometry and the Moduli Space of Toric Varieties

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1 Introduction

1.1 Background

In [2], Alexeev constructed a proper moduli space of polarized toric varieties. However, in addition to the main component containing the toric varieties, there were additional irreducible components which were tricky to eliminate in a canonical way.

Later, Olsson [1] showed how adding a log structure to the toric varieties under consideration effectively restricted the problem enough to single out the primary irreducible component.

The goal of these notes is to give a broad overview of Olsson's work in [1]. We'll look at some examples which illustrate the fundamental ideas and go through the broad strokes of some of the proofs. In particular, we'll try to illustrate the importance of the log structure and the role they play in putting sufficiently strong conditions on our moduli spaces.

1.2 Outline

Generally, log structures are useful for remembering information about a scheme which is not encoded in the usual scheme structure. In our case, we're interested in toric varieties which can always be constructed out of certain polytopes contained in a vector space. Indeed, this is sometimes taken to be the definition.

Given this structure, it seems natural to construct a moduli space of toric varieties by parameterizing them by the polytopes used to generate them. However, as usual in these types of problems, this naive moduli space may not necessarily be compact. One issue is that a family of toric variety may not correspond to a nice family of the polytopes which generate them. So at the limit, we may get a variety which isn't generated by a polytope in a meaningful sense.

As we'll see, adding a log structure to a toric variety allows us to specify exactly which polytope was used to generate it. More specifically, giving a scheme a log structure is equivalent to specifying a submonoid of the sheaf of sections and in the case of a toric variety, this monoid will directly correspond to the polytope. The rigidity imposed by this additional structure can prevent the degenerate cases alluded to above.

For instance, suppose we are given a family of toric varieties together with their log structures. Since the log structures naturally correspond to the polytopes used to generate these varieties, this provides us with a nice family of polytopes as well. We can now see that the limit of the family of varieties is toric, and is generated by the limit of the corresponding family of polytopes.

It is also possible to think about the log structure in terms of the torus action. Since the polytope is what dictates the torus action on the corresponding toric variety, a toric variety with a log structure is a variety together with a specified torus action.

2 Toric Varieties

As a first order of business, we'll give a brief introduction to toric varieties and standardize notation. Since we're interested primarily in projective toric varieties with a specified (linearizable) line bundle, the construction given here will differ slightly from that of conventional sources such as Fulton's book on the subject.

Abstractly, a projective toric variety is a projective variety together with the action of a torus. The torus $\mathbb{T}^n$ is defined to be the product of n copies of the multiplicative group $\mathbb{G}_m$. I.e,

$$\mathbb{T}^n = \mathbb{G}_m \times \mathbb{G}_m \cdots \times \mathbb{G}_m$$

Let $\mathcal{P} = \text{Proj}(S_\bullet)$ be a projective scheme over k where $S_\bullet$ is a graded ring. Let

$$\mathcal{C}^0 = \text{Spec}(S_\bullet) \setminus V(\bigoplus_{n=1}^{\infty} S_n)$$

be the affine cone minus the origin. One way to construct $\mathcal{P}$ is to take the quotient of $\mathcal{C}^0$ under the natural $\mathbb{G}_m = \text{Spec}(k[t, t^{-1}])$ action corresponding to multiplication by a scalar.

In general, defining a $\mathbb{G}_m$ action on a ring $\text{Spec}(A)$ is equivalent to specifying a $\mathbb{Z}$ grading of $A = \bigoplus A_n$. Given the grading, the action $\mathbb{G}_m \times \text{Spec}(A) \to \text{Spec}(A)$ is induced by the ring map $A \to k[t, t^{-1}] \otimes A$ sending a homogeneous element $a_n \in A_n$ to $t^n \otimes a_n$. With a little work it's possible to show that every such action arises from a grading in this way.

For example, we can obtain $\mathbb{P}^1 = \text{Proj}(k[x, y])$ by taking the quotient of $\text{Spec}(k[x, y]_{(x,y)})$ by the action of $\mathbb{G}_m$ induced by the ring map sending x to $t \otimes x$ and y to $t \otimes y$. Or in coordinates, $\lambda \cdot (x, y) = (\lambda x, \lambda y)$.

In a similar fashion, the action of a torus $\mathbb{T}^n$ on an affine space $\text{Spec}(A)$ is determined by an X grading of A where X is a monoid. All together, this means that the action of $\mathbb{T}^n$ on $\mathcal{P}$ is given by a $\mathbb{Z} \times X$ grading of $S_\bullet$. We can use this interpretation to plot the action of $\mathbb{T}^n$ on $\mathcal{P}$ as a cone in $\mathbb{Z} \times X$. Since this presentation of the action is crucial to Olsson's construction, we'll spell out in detail how one goes back and forth between a projective scheme with an action of $\mathbb{T}^n$ and a cone in $\mathbb{Z} \times X$.

For starters, consider the projective scheme $\mathcal{P}$ with a line bundle $\mathcal{L}$ together with an action of $\mathbb{T}^n$. The monoid that will use is the monoid of characters $X = \text{Hom}(\mathbb{T}^n, \mathbb{G}_m)$ and the graded ring will be $R_\bullet = \bigoplus \Gamma(\mathcal{P}, \mathcal{L}^{\otimes n})$.

In this case, $\mathcal{P}$ is given by the quotient of $\text{Spec}((R_\bullet)_m)$ by the $\mathbb{G}_m$ action corresponding to the given grading.

Let $A = k[u_1, u_1^{-1}, \ldots, u_n, u_n^{-1}]$ and set $\mathbb{T}^n$ to be $\text{Spec}(A)$.

Since a character $\chi \in X$ is both a scheme map and a group homomorphism, it corresponds to a ring map sending t to an element $\chi(t) = u_1^{m_1} \ldots u_n^{m_n}$. In this way, we can identify X with $\mathbb{Z}^n$.

In addition, the action of $\mathbb{T}^n$ determines a map $R \xrightarrow{\phi} A \otimes R$. We can use this map to decompose R as

$$R = \bigoplus_{\chi \in X} R_\chi$$

where an element $r \in R_\chi$ gets mapped to $\phi(r) = \chi(t) \otimes r$.

Finally, since $\mathbb{T}^n$ is actually acting on $\mathcal{P}$, the X grading corresponding to the action of $\mathbb{T}^n$ on $\text{Spec}(R)$ is compatible with the $\mathbb{Z}$ grading coming from the $\mathbb{G}_m$ action. Therefore, we obtain the further decomposition

$$R = \bigoplus R_{n,\chi}$$

We use this grading to define a subset P of $\mathbb{Z} \times X$:

$$P = \{(n, \chi) | R_{n,\chi} \neq 0\}$$

Using the fact that the elements of X are multiplicative, it's easy to check that we in fact get a cone in the monoid $\mathbb{Z} \times X$. We use this cone to define a polytope $Q \subset X$ by $Q = P \cap (1 \times X)$.

We'll now show how to go in the other direction. Suppose we are given a polytope Q in $X = \mathbb{Z}^n$. We want to produce a projective scheme together with a line bundle and a torus action.

First of all, we define $P \subset \mathbb{Z} \times X$ to be the cone generated by $1 \times Q$. Working in reverse of the above construction, our graded ring will be $R = k[P]$ where $k[P]$ is the ring generated over k by elements of the form x^p with $p \in P$ and with relations $x^p x^q = x^{p+q}$. Naturally, we take $\mathcal{P}$ to be $\mathrm{Proj}(R)$ and $\mathcal{L}$ to be the associated $\mathcal{O}(1)$.

The action of $\mathbb{T}^n$ on $\mathcal{P}$ corresponds to an action of $\mathbb{G}_m \times \mathbb{T}^n$. But this action now comes for free since $k[P]$ naturally embeds into $k[t, t^{-1}] \otimes A$.

We'll denote the scheme and line bundle constructed from Q by $\mathcal{P}[Q]$ and $\mathcal{L}[Q]$. Note that we've obtained a scheme over $\mathbb{Z}$ so this construction can be done over any base.

Remark. We can subdivide the polytope Q into smaller ones, thus obtaining a *paving* S of Q. We can then perform the above construction to the cone over each of the polytopes in the subdivision. In addition, if the cone P generated by the intersection of the cones P_1 and P_2 generated by neighboring polytopes in the paving, then $k[P_1]$ and $k[P_2]$ naturally embed into $k[P]$ since more elements are invertible there. So we can glue the toric varieties generated by P_1 and P_2 along the one generated by P. By gluing everything together, a paving of a polytope gives us a scheme with possibly more than one component.

Let's see this correspondence in action with an example.

Example. Let $X = \mathbb{Z}$ and let $Q = [0, 1]$. In this case, $P = \{(n, k) | n \geq 0, 0 \leq k \leq n\}$. If we set $X = (1, 1)$ and $Y = (1, 0)$ then we get that $k[P] = k[X, Y]$. So $\mathcal{P} = \mathbb{P}^1$ and $\mathcal{L} = \mathcal{O}(1)$.

The action of the torus is given by the ring map $X \mapsto t \otimes X$ and $Y \mapsto 1 \otimes Y$ so in coordinates we get $\lambda \cdot [x : y] = [\lambda x : y]$.

3 First Attempt at a Moduli Space

By the previous discussion, it makes sense to try and classify polarized toric varieties by the polytope that they generate. It turns out that this isn't enough information and that in order to make the torus action work, Alexeev considers polarized toric varieties $(\mathcal{P}, \mathcal{L})$ together with a section θ of $\mathcal{L}$ whose zero locus does not contain any components of the stabilizer of the torus action. We also assume that over each geometric fiber, we get an object induced by a polytope Q as above. In other words, over a geometric fiber we get $\mathcal{P}[Q]$ and $\mathcal{L}[Q]$ for some polytope Q.

However, even this isn't quite enough. In order for our moduli space to be compact we'll have to consider possibly non smooth toric varieties with more than one component.

For example, let's focus on the moduli space of smooth polarized toric varieties with a degree 2 line bundle and a section as above. Given such an object $(\mathcal{P}, \mathcal{L}, \theta)$ with a $\mathbb{T}^1$ action, we can produce an isomorphism to $(\mathbb{P}^1, \mathcal{O}(2), ax^2 + bxy + cy^2)$ with the natural $\mathbb{G}_m$ action by removing on of the stabilizer points of $\mathcal{P}$, thus turning it into an affine with one stabilizer point, and then mapping that over to $\mathbb{P}^1 \setminus \infty$ with a ring map dictated by the line bundle $\mathcal{L}$ on $\mathcal{P}$ and $\mathcal{O}(2)$ on $\mathbb{P}^1$. The reason that such a map exists is that the cone given by a smooth toric variety via the above construction can be shown to span the entire monoid $\mathbb{Z} \times \mathbb{Z}^2$ so we can indeed map it's generators to x and y.

The upshot of this is that the smooth polarized toric varieties with a degree 2 line bundle and section can be parameterized by the coefficients a, b, c. Furthermore, since we can scale the map from $\mathcal{L}$ to $\mathcal{O}(1)$, these coefficients are only determined up to a scalar. And since $\mathbb{G}_m$ acts on $\mathbb{P}^1$ as $\lambda \cdot [x : y] = [\lambda x, y]$, scaling the identification of $\mathbb{T}^2$ with $\mathbb{G}_m$ corresponds to transforming (a, b, c) to $(\lambda^2 a, \lambda b, c)$. We also note that since the fixed points of the $\mathbb{G}_m$ action on $\mathbb{P}^1$ are $[1 : 0]$ and $[0 : 1]$, the condition on our section says that $a \neq 0$ and $c \neq 0$. So up to scaling, we can assume that our smooth polarized toric variety is of the form $(\mathbb{P}^1, \mathcal{O}(2), x^2 + bxy + y^2)$. This means that we can parameterize these objects as $\mathbb{A}^1_b$. We should note that there're still some automorphisms we haven't accounted for such as permuting variables and multiplication by $\lambda = -1$ which fixes a and c but sends b to $-b$. So technically we should be considering a stack. But for simplicity we'll continue to think of the moduli space as $\mathbb{A}^1_b$.

The first thing we notice is that this isn't a proper scheme. So in order to compactify it we need to add the polarized toric variety corresponding to infinity which will turn out to have two intersecting components.

We'll start by writing down a family of smooth polarized toric varieties with a degree 2 line bundle and a section parameterized by b. The family is given by

$$(\mathrm{Proj}(k[s,v,t]/(st-v^2)), \mathcal{O}(1), s+t+bv)$$

with the standard $\mathbb{T}^1$ action. Note that out map back to $\mathbb{P}^1$ sends s to x^2 and t to y^2 so $\mathcal{O}(1)$ will indeed get mapped to $\mathcal{O}(2)$.

We'll now produce a family in coordinates $k[1/b]$ and a gluing from this family to the previously family. This will give us a family parameterized by the proper scheme $\mathbb{P}^1$. We consider the family

$$(\mathrm{Proj}(k[(1/b)^2][x,y,z]/(xy-(1/b)^2z^2)), \mathcal{O}(1), x+y+z)$$

Note that we made an unramified base change to the square root of b in order to make this work. This has to do with the stackiness of our moduli space but we won't get into that here.

We glue the families together by the map $x \mapsto s$, $y \mapsto t$ and $z \mapsto bv$.

Under this parametrization, the toric variety corresponding to infinity is now the one corresponding to $(1/b)^2 = 0$ in the $k[1/b]$ coordinates. Plugging that into our family in those coordinates gives us $\mathrm{Proj}(k[x,y,z]/(xy))$ which is the intersection of two projective lines.

This example suggests that in order to get a complete moduli space, we need to account for degenerate cases with possibly more than one component. As we mentioned in the previous section, one natural way to obtain such varieties is by considering polytopes together with a paving.

4 Properness and Pavings

In order to see how exactly pavings will lead the way to properness we need to consider the problem a little more generally. Suppose $(\mathcal{P}, \mathcal{L}, \theta)$ is a toric variety over K and $A \subset K$ is a DVR. The valuative criteria for properness essentially says that we should be able to extend $(\mathcal{P}, \mathcal{L}, \theta)$ to an object $(\mathcal{P}_A, \mathcal{L}_A, \theta_A)$ over A. The scheme and line bundle don't pose too much of a problem since over a field (or it's algebraic extension) by assumption our scheme and line bundle are of the form $\mathcal{P}[Q]$ and $\mathcal{L}[Q]$ and as we mentioned above, this construction extends to A.

However, after reducing the section to R, possibly by multiplying by sufficient powers of the uniformizer, the base change to the geometric fiber could cause the section to have too many zeros to the effect that its zero locus could contain stabilizer points.

More precisely, let $(\mathcal{P}[Q]_K, \mathcal{L}[Q]_K, \theta_K)$ be given over K. This gives us the rings

$$R_A = \bigoplus \Gamma(\mathcal{P}_A, \mathcal{L}_A^{\otimes n}) \hookrightarrow \bigoplus \Gamma(\mathcal{P}[Q]_K, \mathcal{L}[Q]_K^{\otimes n}) = R_K$$

By considering the X grading as well, we get that $(R_A)_{n,\chi}$ is a sub A-module of $(R_K)_{n,\chi}$ which in turn implies that $(R_A)_{n,\chi} = \pi^{h(n,\chi)}\Gamma(\mathcal{P}[Q]_A, \mathcal{L}[Q]_A^{\otimes n})_\chi$ where π is the uniformizer of A and $h : \mathbb{Z} \times X \to \mathbb{Z}_{\geq 0}$ is some function.

Essentially, the function h is telling us how to change coordinates in order to extend our section to A in such a way that the section won't vanish over $A/(\pi)$.

Now, h is only defined pointwise but using the properties of χ we can check that it's possible to find a paving of Q such that on each component of the induced decomposition of the cone P, h is linear. This means that after possibly decomposing Q, we have a way to extend the toric variety given by each piece to A. So if we want our moduli space to be proper, we have to consider polytopes with a paving as opposed to just ordinary polytopes.

Example. As an example, suppose we start with $X = \mathbb{Z}$ and $Q = [0,2]$. Similarly to the example with $Q = [0,1]$, we now get $\mathcal{P}[Q] = \mathbb{P}^1$ and $\mathcal{L}[Q] = \mathcal{O}(2)$. In this case, the cone P is generated by $(1,0)$, $(1,1)$ and $(1,2)$ we will denote the corresponding elements in $k[P]$ by s, v and t. We can represent $\mathcal{P}[Q] = \mathrm{Proj}(k[P])$ as $\mathrm{Proj}(k[s,v,t]/(st-v^2))$. Let's consider the paving $[0,1] \cup [1,2]$ and define h by setting $h(1,0) = 1$, $h(1,1) = 1$ and $h(0,2) = 0$. We can extend this linearly to the cone over $[0,1]$ and over $[1,2]$ (but not to both at once).

We'll now look at what we get by changing coordinates via the h function. Since $(1,0)$ corresponds to s we get $s' = \pi^1 s = \pi s$. Similarly , $t' = \pi t$ and $v' = v$. Note that now $s't' = \pi^2 v^2 = \pi^2 v'^2$.

Therefore, reducing modulo π we get $\mathrm{Proj}(k[x,y,z]/(xy))$ which has two components.

5 Defining the Moduli Space

The previous section demonstrated that if we want to define a good moduli space for polarized toric varieties, we need to have some scheme theoretic way to talk about the toric varieties coming from specific polytopes and pavings. Interestingly, log schemes fit that bill precisely.

By definition, a log scheme is a scheme Y together with sheaf of monoids $\mathcal{M}$ and a map of sheaves $\mathcal{M} \to \mathcal{O}_Y$. This map of sheaves is called the *log structure* on Y. More details can be found in Olsson's book.

In our case, the schemes $\mathcal{P}[Q]$ have a natural log scheme structure given by the natural map $P \to k[P]$. It's slightly more complicated then that but the idea is to use the above map to get a log structure on affine cone $\mathrm{Spec}(k[P])$ and then take a quotient to get a log structure on $\mathcal{P}[Q]$. This turns $\mathcal{P}[Q]$ into a log scheme over the trivial log scheme $\mathrm{Spec}(\mathbb{Z})$.

The additional log scheme structure will give us enough control over our toric varieties to allow us to define a good moduli space as follows.

For a given polytope Q in a free abelian group X, we define $\mathcal{K}_Q$ to be the fibered category over $\mathrm{Spec}(\mathbb{Z})$ which associates to each $\mathrm{Spec}(\mathbb{Z})$-scheme B the groupoid

$$(M_B, f : (X, M_X) \to (B, M_B), \mathcal{L}, \theta, \rho)$$

Where

- M_B is a fine log structure on B
- f is log smooth and the underlying map of schemes $X \to B$ is proper.
- $\mathcal{L}$ is a relatively ample invertible sheaf on X/B.
- ρ is the torus action
- $\theta \in f_*\mathcal{L}$

  For all geometric sections $\bar{s} \to B$ we have:
- There exists a paving S of Q such that we have the following fibered diagram

$$\begin{array}{ccc} (X_{\bar{s}}, M_{X_{\bar{s}}}, \mathcal{L}_{X_{\bar{s}}}, \rho_{X_{\bar{s}}}) & \longrightarrow & (\mathcal{P}[S], M_{\mathcal{P}[S]}, \mathcal{O}(1), \rho_{\mathcal{P}[S]}) \\ \downarrow & & \downarrow \\ (\bar{s}, M_B|_{\bar{s}}) & \longrightarrow & \mathrm{Spec}(\mathbb{Z}) \end{array}$$

  where $\mathcal{P}[S]$ is the toric variety induced by the paved polytope S.
- The zero locus of $\theta_{\bar{s}}$ in $X_{\bar{S}}$ does not contain an orbit of the action of the torus.

Theorem 1 (Olsson)**.**
- *$\mathcal{K}_Q$ is a proper, irreducible algebraic stack with finite diagonal.*
- *$(\mathcal{K}_Q, M_Q) \to \mathrm{Spec}(\mathbb{Z})$ is log smooth.*

One of the key facts about $\mathcal{K}_Q$ is that etale locally, elements of $\mathcal{K}_Q$ look like base changes of $(\mathcal{P}[S], M_{\mathcal{P}[S]})$. Essentially, this is because it allows us to specify polytopes and pavings around geometric points which in turn tell us how the torus acts.

The proof of this fact contains some interesting ideas so we'll give a brief overview. As usual, Olsson's book contains the full details. The main idea is to try and apply general scheme theoretic machinery to take care of everything but the torus action, and then use the log structure to force the torus action to be the right one.

5.1 Deformation Theory

As we mentioned in the introduction, the addition of a log structure makes a toric variety much mor rigid under deformations.

Let k be an infinite field, $0 \to J \to A' \to A \to 0$ a surjection of local artinean rings with residue field k. Let $(\mathcal{P}_A, M_A, \mathcal{L}_A, \rho_A)$ be a polarized toric log variety over A and similarly $(\mathcal{P}_{A'}, M_{A'}, \mathcal{L}_{A'}, \rho_{A'})$ over A'. We call $(\mathcal{P}_{A'}, M_{A'}, \mathcal{L}_{A'}, \rho_{A'})$ a deformation of $(\mathcal{P}_A, M_A, \mathcal{L}_A, \rho_A)$ with respect to our surjection $A' \to A$ if the following diagram is fibered:

$$\begin{array}{ccc} (\mathcal{P}_A, M_A, \mathcal{L}_A, \rho_A) & \longrightarrow & (\mathcal{P}_{A'}, M_{A'}, \mathcal{L}_{A'}, \rho_{A'}) \\ \downarrow & & \downarrow \\ (\mathrm{Spec}(A), M_A) & \longrightarrow & (\mathrm{Spec}(A'), M_{A'}) \end{array}$$

Proposition 1. *Let S be a paving of Q. Let M_A, $M_{A'}$ and M_k be monoids and let $M_A \to A$, $M_{A'} \to A'$ and $M_k \to k$ be the trivial log structures sending all non zero elements to zero. Let $(\mathcal{P}_A, M_A, \mathcal{L}_A, \rho_A)$ be the polarized toric variety obtained by base changing from the standard toric variety generated by S. Furthermore, suppose that $(\mathcal{P}_{A'}, M_{A'}, \mathcal{L}_{A'}, \rho_{A'})$ is a deformation of $(\mathcal{P}_A, M_A, \mathcal{L}_A, \rho_A)$. Then $(\mathcal{P}_{A'}, M_{A'}, \mathcal{L}_{A'}, \rho_{A'})$ is also obtained via base change of the standard toric variety generated from S. I.e*

$$\begin{array}{ccccc} (\mathcal{P}_A, M_A, \mathcal{L}_A, \rho_A) & \longrightarrow & (\mathcal{P}_{A'}, M_{A'}, \mathcal{L}_{A'}, \rho_{A'}) & \longrightarrow & (\mathcal{P}[S], M_{\mathcal{P}[S]}, \mathcal{O}(1), \rho_{\mathcal{P}[S]}) \\ \downarrow & & \downarrow & & \downarrow \\ (\mathrm{Spec}(A), M_A) & \longrightarrow & (\mathrm{Spec}(A'), M_{A'}) & \longrightarrow & \mathrm{Spec}(\mathbb{Z}) \end{array}$$

The first step in the proof is to show that $(\mathcal{P}_{A'}, M_{A'}, \mathcal{L}_{A'})$ is isomorphic to the pull back of $(\mathcal{P}[S], M_{\mathcal{P}[S]}, \mathcal{O}(1))$ to $(\mathrm{Spec}(A'), M_{A'})$ as log schemes without worrying about the toric action.

We start by passing to the affine cone without the origin $\mathcal{C}^0_{A'}$. We also define $\mathcal{C}^0_k$ to be the base change of the cone to $(\mathrm{Spec}(k), M_k)$. The quotient of $\mathcal{C}^0_{A'}$ by $\mathbb{G}_m$ is $\mathcal{P}_{A'}$. By the standard deformation theory of log schemes, the group

$$H^1(\mathcal{C}^0_k, T_{(\mathcal{C}^0_k, M_{\mathcal{C}^0_k})/(\mathrm{Spec}(k), M_k)} \otimes J)$$

is a torsor on the set of deformations of $(\mathcal{P}_A, M_A)$. Since the log structure over k comes from the base change of the log structure $P \to \mathbb{Z}[P]$, by a general log geometry fact we have

$$\mathrm{dlog} : \mathcal{O}_{\mathcal{C}^0_k} \otimes_{\mathbb{Z}} P^{gp} \xrightarrow{\cong} \Omega^1_{(\mathcal{C}^0_k, M_{\mathcal{C}^0_k})/(\mathrm{Spec}(k), M_k)}$$

This implies that

$$H^1(\mathcal{C}^0_k, T_{(\mathcal{C}^0_k, M_{\mathcal{C}^0_k})/(\mathrm{Spec}(k), M_k)} \otimes J) \cong H^1(\mathcal{C}^0_k, \mathcal{O}_{\mathcal{C}^k_0}) \otimes \mathrm{Hom}(X, J))$$

So the difference between $(\mathcal{C}^0_{A'}, M_{\mathcal{C}^0_{A'}})$ and the base change of $(\mathcal{P}[S], M_{\mathcal{P}[S]})$ is an element of $H^1(\mathcal{C}^0_k, \mathcal{O}_{\mathcal{C}^k_0})$. Furthermore, this difference is clearly invariant under the $\mathbb{G}_m$ action so it is in fact an element of

$$H^1(\mathcal{C}^0_k, \mathcal{O}_{\mathcal{C}^k_0})^{\mathbb{G}_m} \cong H^1(\mathcal{P}[S]_k, \mathcal{O}_{\mathcal{P}[S]_k})$$

But as Alexeev [2, 2.52] shows, this last homology group is equal to 0. The idea is that since it comes from the standard toric construction, we can produce a Leray covering of $\mathcal{P}[S]_k$ in a way that is very similar to what we can do for a simplicial complex.

This shows that $(\mathcal{C}^0_{A'}, M_{\mathcal{C}^0_{A'}})$ and the base change of $(\mathcal{P}[S], M_{\mathcal{P}[S]})$ are equivalent except perhaps for the torus action. But we can now use the fact that they share the log structure to produce an isomorphism preserving the torus action as well.

5.2 Algebraization

Let $(A, \mathfrak{m})$ be a complete noetherian local ring. Let $k = A/\mathfrak{m}$ and $A_n = A/\mathfrak{m}^n$. Also, Let $(X_A, M_{X_A}, \mathcal{L}_A, \rho_A) \to$ (Spec(()A)$, M_A)$ be a log smooth proper morphism with polarization and a torus action such that the fiber $(X_k, M_{X_k}, \mathcal{L}_k, rho_k)$ is the standard toric variety coming from a paving S.

Proposition 2. *In the setting above,* $(X_A, M_{X_A}, \mathcal{L}_A, \rho_A) \to (\mathrm{Spec}(A), M_A)$ *is also the standard toric variety coming from a paving* S.

By the deformation theory step, the claim is true for each one of the $(X_{A_n}, M_{X_{A_n}}, \mathcal{L}_{A_n}, \rho_{A_n})$ We then apply the Grothendeik existence theorem.

5.3 Approximation

Let (B, M_B) be a fine log scheme, $\overline{b} \to B$ a geometric point and S a paving of Q. Also, let $(X_B, M_{X_B}, \mathcal{L}_B, \rho_B) \to (\mathrm{Spec}(B), M_B)$ be a proper log smooth morphism with polarization and torus action such that the base change $(X_{\overline{b}}, M_{X_{\overline{b}}}, \mathcal{L}_{\overline{b}}, \rho_{\overline{b}}) \to (\overline{b}, M_{\overline{b}})$ is isomorphic to the base change of standard toric variety coming from a paving S.

Proposition 3. *In the above setting, there exists an etale local neighborhood of* $\overline{b}$*,* $(X_B, M_{X_B}, \mathcal{L}_B, \rho_B) \to (\mathrm{Spec}(B), M_B)$ *is also the base change of the standard toric variety coming from a paving* S.

From the previous step, we know the statement is true at the completion of the local ring at $\overline{B}$. The proposition then follows from the artin approximation theorem.

References

[1] Martin Olsson, *Compactifying Moduli Spaces for Abelian Varieties*

[2] Valery Alexeev, *Complete Moduli in the Presence of Semiabelian Group Action* Annals of Mathematics, Second Series, Vol. 155, No. 3 (May, 2002), pp. 611-708