Hodge Decomposition

Daniel Lowengrub

April 27, 2014

1 Introduction

All of the content in these notes in contained in the book *Differential Analysis on Complex Manifolds* by Raymond Wells. The primary objective here is to highlight the steps needed to prove the Hodge decomposition theorems for real and complex manifolds, in addition to providing intuition as to how everything fits together.

1.1 The Decomposition Theorem

On a given complex manifold X, there are two natural cohomologies to consider. One is the *de Rham Cohomology* which can be defined on a general, possibly non complex, manifold. The second one is the *Dolbeault cohomology* which uses the complex structure.

We'll quickly go over the definitions of these cohomologies in order to set notation but for a precise discussion I recommend Huybrechts or Wells.

If X is a manifold, we can define the *de Rahm Complex* to be the chain complex

$$0 \to \mathcal{E}(\Omega^0)(X) \xrightarrow{d} \mathcal{E}(\Omega^1)(X) \xrightarrow{d} \dots$$

Where $\Omega = T^*$ denotes the cotangent vector bundle and Ω^k is the alternating product $\Omega^k = \Lambda^k \Omega$. In general, we'll use the notation $\mathcal{E}(E)$ to denote the sheaf of sections associated to a vector bundle E.

The boundary operator is the usual differentiation operator. The de Rham cohomology of the manifold X is defined to be the cohomology of the de Rham chain complex:

$$H^n_{dR}(X, \mathbb{R}) = \frac{\text{Ker}(\mathcal{E}(\Omega^n)(X) \xrightarrow{d} \mathcal{E}(\Omega^{n+1})(X))}{\text{Im}(\mathcal{E}(\Omega^{n-1})(X) \xrightarrow{d} \mathcal{E}(\Omega^n)(X))}$$

We can also consider the de Rham cohomology with complex coefficients by tensoring the de Rham complex with $\mathbb{C}$ in order to obtain the de Rham complex with complex coefficients

$$0 \to \mathcal{E}(\Omega^0_{\mathbb{C}})(X) \xrightarrow{d} \mathcal{E}(\Omega^1_{\mathbb{C}})(X) \xrightarrow{d} \dots$$

where $\Omega_{\mathbb{C}} = \Omega \otimes \mathbb{C}$ and the differential d is linearly extended. By taking the cohomology of this complexified chain complex we obtain the de Rham cohomology with complex coefficients $H^n_{dR}(X, \mathbb{C})$.

The other natural cohomology comes from decomposing the complex tangent bundle $T_{\mathbb{C}} = T \otimes \mathbb{C}$ into the two eigenspaces of the natural almost complex structure on X

$$T_{\mathbb{C}} = T^{1,0} \oplus T^{0,1}$$

By dualizing this and extending it to alternating products we obtain the decomposition

$$\Omega^k_{\mathbb{C}} = \bigoplus_{p+q=k} \Omega^{p,q}$$

where

$$\Omega^{p,q} = \Lambda^p (T^{1,0})^* \otimes \Lambda^q (T^{0,1})^*$$

It isn't hard to show that this partition of Ω respects the differential d. So by restricting d to each of these factors we obtain two additional differential operators

$$\begin{aligned} \partial &: \mathcal{E}(\Omega^{p,q}) \to \mathcal{E}(\Omega^{p+1,q}) \\ \bar{\partial} &: \mathcal{E}(\Omega^{p,q}) \to \mathcal{E}(\Omega^{p,q+1}) \end{aligned}$$

We thus obtain the *Dolbeault double complex*

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle \bar{\partial}} & & \uparrow{\scriptstyle \bar{\partial}} & & \uparrow{\scriptstyle \bar{\partial}} & & \\
\mathcal{E}(\Omega^{0,2}) & \xrightarrow{\partial} & \mathcal{E}(\Omega^{1,2}) & \xrightarrow{\partial} & \mathcal{E}(\Omega^{2,2}) & \xrightarrow{\partial} & \dots \\
\uparrow{\scriptstyle \bar{\partial}} & & \uparrow{\scriptstyle \bar{\partial}} & & \uparrow{\scriptstyle \bar{\partial}} & & \\
\mathcal{E}(\Omega^{0,1}) & \xrightarrow{\partial} & \mathcal{E}(\Omega^{1,1}) & \xrightarrow{\partial} & \mathcal{E}(\Omega^{2,1}) & \xrightarrow{\partial} & \dots \\
\uparrow{\scriptstyle \bar{\partial}} & & \uparrow{\scriptstyle \bar{\partial}} & & \uparrow{\scriptstyle \bar{\partial}} & & \\
\mathcal{E}(\Omega^{0,0}) & \xrightarrow{\partial} & \mathcal{E}(\Omega^{1,0}) & \xrightarrow{\partial} & \mathcal{E}(\Omega^{2,0}) & \xrightarrow{\partial} & \dots
\end{array}$$

Using this double complex we can compute the Dolbeault cohomology

$$H^{p,q}_{dB}(X) = H^q((\mathcal{E}(\Omega^{p,\bullet})(X), \bar{\partial}))$$

We can now wonder about the relationship between $H^{\bullet}_{dR}(X, \mathbb{C})$ and $H^{\bullet,\bullet}_{dB}(X)$. In general we can't say much, but if X is compact and Kahler then we have the following theorem.

Theorem 1. *(Hodge Decomposition) Let X be a compact Kahler manifold. Then we have the direct sum decomposition*

$$H^k_{dR}(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_{dB}(X)$$

In fact, the proof of this theorem gives us an even more explicit relation. To see what this is, we'll first introduce the notion of a harmonic form on a complex manifold.

1.2 Harmonic Forms

Let X be a compact Riemannian manifold with a metric $\langle , \rangle$. As usual, we can extend the inner product to the exterior powers $\Omega^k(X)$ for every k. Let $d\mu$ be the volume form associated to the metric. In other words, $d\mu$ is a positive differential form of degree n. In this case, we can use $d\mu$ in order to define an inner product on $\mathcal{E}(\Omega^k)(X)$ by the formula

$$(\alpha, \beta) = \int_X \langle \alpha, \beta \rangle d\mu$$

Now that we have an inner product on $\mathcal{E}(\Omega^k)(X)$, we can define

$$d^* : \mathcal{E}(\Omega^{k+1})(X) \to \mathcal{E}(\Omega^k)(X)$$

to be the adjoint operator to d^*. I.e, for $\alpha \in \mathcal{E}(\Omega^k)(X)$ and $\beta \in \mathcal{E}(\Omega^k)(X)$ we have

$$(d\alpha, \beta) = (\alpha, d^*\beta)$$

The existence of d^* can be shown using the Hodge star operator but we'll obtain a more general statement below. We can now define the *laplacian* to be

$$\Delta = d^*d + dd^*$$

A quick calculation shows that $\Delta : \mathcal{E}(\Omega^k)(X) \to \mathcal{E}(\Omega^k)(X)$ is self adjoint.

Definition 1. *We will call a form* $\alpha \in \mathcal{E}(\Omega^k)(X)$ harmonic *if* $\Delta(\alpha) = 0$. *We'll denote by* $\mathcal{H}^k(X)$ *the space of harmonic forms in* $\mathcal{E}(\Omega^k)(X)$ *the space of harmonic forms.*

One of the useful properties of harmonic forms is that every cohomology class of $H^n_{dR}(X)$ has a unique representative in $\mathcal{H}^n(X)$. This follows immediately from the following decomposition theorem.

Theorem 2. *Let* X *be a compact oriented Riemannian manifold. Then, with respect to the given metric, we have the following orthogonal decomposition*

$$\mathcal{E}(\Omega^k)(X) = d(\mathcal{E}(\Omega^{k-1})(X) \oplus \mathcal{H}^k(X) \oplus d^*\mathcal{E}(\Omega^{k+1})(X)$$

Furthermore, $\mathcal{H}^k(X)$ *is finite dimensional for all* k.

If our manifold X is hermitian complex manifold, we can linearly extend our inner product from $\mathcal{E}(\Omega^k)(X)$ to $\mathcal{E}(\Omega^k_{\mathbb{C}})(X)$ and use this to construct adjoints

$$\partial^* : \mathcal{E}(\Omega^{p+1,q}) \to \mathcal{E}(\Omega^{p,q})$$
$$\bar{\partial}^* : \mathcal{E}(\Omega^{p,q+1}) \to \mathcal{E}(\Omega^{p,q})$$

As before, we define $\Delta_{\partial} = \partial^*\partial + \partial\partial^*$ and $\Delta_{\bar{\partial}} = \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*$

We can use these to obtain the following harmonic forms.

Definition 2. *Let* X *be a complex hermitian manifold. We call a form* $\alpha \in \mathcal{E}(\Omega^k_{\mathbb{C}})$ $\bar{\partial}$-harmonic *if* $\Delta_{\bar{\partial}}(\alpha) = 0$. *Furthermore,*

$$\mathcal{H}^k_{\bar{\partial}}(X) = \{\alpha \in \mathcal{E}(\Omega^k_{\mathbb{C}}) : \Delta_{\bar{\partial}}(\alpha) = 0\}$$
$$\mathcal{H}^{p,q}_{\bar{\partial}}(X) = \{\alpha \in \mathcal{E}(\Omega^{p,q}_{\mathbb{C}}) : \Delta_{\bar{\partial}}(\alpha) = 0\}$$

We similarly define the ∂-harmonic *forms together with* $\mathcal{H}^k_{\partial}(X)$ *and* $\mathcal{H}^{p,q}_{\partial}(X)$.

Finally, we also have the analogous decomposition theorem.

Theorem 3. *Let* X *be a compact hermitian manifold. Then there exist two orthogonal decompositions*

$$\mathcal{E}(\Omega^{p,q})(X) = \partial(\Omega^{p-1,q})(X) \oplus \mathcal{H}^{p,q}_{\partial}(X) \oplus \partial^*\mathcal{E}(\Omega^{p+1,q})(X)$$

and

$$\mathcal{E}(\Omega^{p,q})(X) = \bar{\partial}\mathcal{E}(\Omega^{p,q-1})(X) \oplus \mathcal{H}^{p,q}_{\bar{\partial}}(X) \oplus \bar{\partial}^*\mathcal{E}(\Omega^{p,q+1})(X)$$

Furthermore, $\mathcal{H}^{p,q}_{\partial}(X)$ *and* $\mathcal{H}^{p,q}_{\bar{\partial}}(X)$ *are finite dimensional for all* p *and* q.

If X is Kahler, then we can use the Kahler identities to show that $\Delta_{\bar{\partial}} = \Delta_{\partial}$ so in particular we get that $\mathcal{H}^{p,q}_{\bar{\partial}} = \mathcal{H}^{p,q}_{\partial}$. In this case, we'll denote this common space by $\mathcal{H}^{p,q}$.

These two decomposition results show us that for a compact manifold $H^n_{dR}(X, \mathbb{C}) \cong \mathcal{H}^n(X)_{\mathbb{C}}$, and if X is Kahler then we have in addition that $H^{p,q}_{dB}(X) \cong \mathcal{H}^{p,q}(X)$. Together, we get the desired Hodge decomposition result by the following identifications

$$H^n_{dR}(X, \mathbb{C}) \cong \mathcal{H}^n(X)_{\mathbb{C}} = \bigoplus_{p+q=n} \mathcal{H}^{p,q}(X) \cong \bigoplus_{p+q=k} H^{p,q}_{dB}(X)$$

where the middle equality follows from the fact that a section $\alpha \in \mathcal{E}(\Omega^n_{\mathbb{C}})(X)$ is harmonic if and only if the projection of α to each one of the components in the direct sum $\mathcal{E}(\Omega^n_{\mathbb{C}})(X) = \bigoplus_{p+q=k} \mathcal{E}(\Omega^{p,q})(X)$ is harmonic.

1.3 Outline For These Notes

Our ultimate goal in these notes is to prove theorems 2 and 3. These two results look familiar and indeed, they are both special cases of a more general theorem regarding elliptic complexes.

We'll define elliptic complexes precisely below, but in general, it's a complex of the form

$$0 \to \mathcal{E}(E_0) \xrightarrow{L} \mathcal{E}(E_1) \xrightarrow{L} \dots$$

where the E_k-s are differentiable vector bundles. As we'll see, we can define an adjoint L^* and a laplacian $\Delta = L^*L + LL^*$ as above, and obtain a decomposition theorem reminiscent of theorems 2 and 3.

Suppose we wanted to prove such a decomposition theorem. In order to motivate the rest of these notes, let's suppose for a moment that the spaces $\mathcal{E}(E^n)$ were Hilbert spaces. This is almost never the case, but a large part of the machinery developed in these notes will essetially be used to transfer the proof in the Hilbert space case over to the general case.

Now, since we're assuming that $\mathcal{E}(E^n)$ is a Hilbert space and the space of harmonic forms is the kernel of Δ, we have a natural projection map H from $\mathcal{E}(E^n)$ to $\text{Ker}(\Delta)$. In addition, by a general theorem about Hilbert spaces,

$$\overline{\text{Im}(\Delta)} = \text{Ker}(\Delta^*)^\perp$$

In our case, $\Delta = \Delta^*$ and we'll see that the image is closed. So we get that $\text{Im}(\Delta) = \text{Ker}(\Delta)^\perp$.

Therefore, by the Banach open mapping theorem, we can find an inverse operator

$$G : \text{Ker}(\Delta)^\perp \to \text{Ker}(\Delta)^\perp$$

such that on $\text{Ker}(\Delta)^\perp$, $G \circ \Delta = \text{Id}$. By extending G by zero to $\text{Ker}(\Delta)$, we get a map $G : \mathcal{E}(E^n) \to \mathcal{E}(E^n)$ such that

$$G \circ \Delta = \Delta \circ G = \text{Id} + H$$

After replacing Δ with $L^*L + LL^*$,we can then conclude that we have an orthogonal decomposition

$$\mathcal{E}(E^n) = L(L^*G(\mathcal{E}(E^n)) \oplus \text{Ker}(\Delta) \oplus L^*(LG(\mathcal{E}(E^n))$$

which as we'll see, gives us the Hodge decomposition theorems as a special case.

As we said, the main problem with this is that the space of sections is not in general even a Banach space. Our general strategy will be to slowly enlarge the our space of sections to an increasing sequence of Banach spaces, which we'll call Sobolev spaces, in which we'll allow greater and greater deviations from smoothness. We will also extend our class of differential operators to the class of pseudodifferential operators. In addition, we'll show that we can naturally extend our Δ operator to this more general context. We can then apply the above argument to get the decomposition theorem for the Sobolev spaces.

The next part of the puzzle will be to show why the G and H functions we obtained for these larger spaces restrict to the space of smooth sections. The key ingredient here will be the existence of a partial inverse to Δ which we'll call the parametrix. More precisely, we'll prove the existance of an operator P which sends smooth sections to smooth sections, and a pseudodifferential smoothing operator S such that $\Delta \circ P = \text{Id} - S$.

We'll define smoothing operators more precisely below, but the important property is that they make functions "more smooth". This implies that if α is a possibly non smooth section in one of the Sobolev spaces, and $\Delta\alpha = \beta$ is smooth, then $\alpha = P\beta - S\alpha$ must have been smooth as well.

2 Sobolev Spaces

Let X be a compact differentiable manifold and let E be a hermitian differential vector bundle on X with inner product $\langle , \rangle_E$.

In this situation, we can also construct a positive smooth measure $d\mu$ on X. One way to construct $d\mu$ is to start by constructing a Riemannian metric g which locally looks like $g_{ij}(x_1, \dots, x_n)$. We can then define $d\mu$ locally by $d\mu = |\det g_{i,j}(\bar{x})| d\bar{x}$. This extends to a well defined positive smooth measure on X by virtue of g being a Riemannian metric.

We'll denote by $\mathcal{E}_k(X, E)$ the k-th order differentiable functions on E over X. We'll also define $\mathcal{E}(X, E)$ to be the differentiable functions on E over X and $\mathcal{E}$ to be the sheaf of differentiable functions on X.

Using $d\mu$ we can define an inner product on $\mathcal{E}(X, E)$ by defining

$$(\alpha, \beta) = \int_X \langle \alpha, \beta \rangle_E d\mu$$

Let $lVert\alpha\|_0 = (\alpha, \alpha)^{1/2}$ be the subsequent L^2 norm and let $W^0(X, E)$ be the completion of $\mathcal{E}(X, E)$ with respect to this norm. This gives us a Banach space containing $\mathcal{E}(X, E)$ which is what we desired, but it turns out that we've added too much in the process. For this reason, we'll now define a sequence of smaller and smaller Banach spaces containing $\mathcal{E}(X, E)$ which will ultimately give us enough control to prove our theorems.

Let $f : \mathbb{R}^n \to \mathbb{C}$ be a compactly supported function and $s \in \mathbb{Z}$. We define the *Sobolev norm* to be

$$\|f\|^2_{s,\mathbb{R}^n} = \int_{\mathbb{R}^n} |\hat{f}(\bar{y})|^2 (1 + |\bar{y}|^2)^s d\bar{y}$$

Where

$$\hat{f}(\bar{y}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\langle \bar{x}, \bar{y} \rangle} f(\bar{x}) d\bar{x}$$

is the Fourier transform of f. For a function $f : \mathbb{R}^n \to \mathbb{C}^m$, we define $\|f\|_{s,\mathbb{R}^n}$ to be the Sobolev norm of $\|f\|$.

We extend this to a norm $\langle \cdot \rangle_{s,E}$ on E via the usual procedure of taking a sum over a partition of unity.

For a tuple $\alpha = (\alpha_1, \dots, \alpha_n) \subset \mathbb{N}^n$, we define $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\bar{y}^\alpha = {y_1}^{\alpha_1} \cdots {y_n}^{\alpha_n}$ and $D^\alpha(f) = -i^{|\alpha|} \frac{\partial^n}{\partial^{\alpha_1} y_1 \dots \partial^{\alpha_n} y_n} f$.

The key property of Fourier transforms which motivates the above definition is the following.

Lemma 1. *Let $f : \mathbb{R}^n \to \mathbb{C}$ be a compactly supported function and $\alpha = (\alpha_1, \dots, \alpha_n) \subset \mathbb{Z}^n$. Then*

$$\widehat{D^\alpha f}(\bar{y}) = \bar{y}^\alpha \hat{f}(\bar{y})$$

Proof. The statement essentially follows from integration by parts. First suppose that $f : \mathbb{R} \to \mathbb{C}$. Then

$$\widehat{D^1 f}(y) = \frac{1}{2\pi} \int e^{-ixy} (\text{-i}) f'(x) dx = -\frac{i}{2\pi} e^{-ixy} f(x)|_{x=-\infty}^{\infty} - \frac{1}{2\pi} \int y e^{-ixy} f(x) dx = y\hat{f}(y)$$

The lemma follows from applying this basic case over and over for each one of the partial derivatives. ☐

This means that if the norm $\|f\|_{s,\mathbb{R}^n}$ is bounded, then we can expect f to be differentiable about s times. Because by taking the inverse Fourier transform, we get for some constant c

$$D^\alpha f(\bar{x}) = c \int_{\mathbb{R}^n} e^{-i\langle \bar{x}, \bar{y} \rangle} \bar{y}^\alpha \hat{f}(\bar{y}) d\bar{y}$$

assuming the integral is defined. But if $\|f\|_{s,\mathbb{R}^n} < \infty$, we can reasonably expect the integral to converge.

With this in mind, we define $W^s(X, E)$ to be the completion of $\mathcal{E}(X, E)$ with respect to $\|\cdot\|_{s,E}$. We can think of this as a Banach space approximating the space of k-differentiable functions on E over X. Note that $W^s(X, E)$ inherits the inner product on $\mathcal{E}(X, E)$ and we thus obtain a sequence of Hilbert spaces

$$\cdots \supset W^s(X, E) \subset W^{s+1}(X, E) \supset \cdots \supset W^{s+j}(X, E) \supset \ldots$$

Note that the space $W^0(X, E)$ according to this definition coincides with the previous one.

The properties alluded to above are formalized in the following theorems.

Theorem 4. *(Sobolev) Let* $n = \dim_{\mathbb{R}} X$ *and suppose that* $s > [n/2] + k + 1$. *Then*

$$W^s(X, E) \subset \mathcal{E}_k(X, E)$$

For the second theorem, recall that a *compact operator* between Banach spaces is one which send a closed ball to a compact set.

Theorem 5. *(Rellich) The natural inclusion*

$$j : W^t(X, E) \subset W^s(X, E)$$

where $s < t$ *is a compact operator.*

The main idea of Rellich's theorem is that given a collection of functions $f_n \in W^s(\mathbb{R}^n)$ such that $\|f_n\|_{s,\mathbb{R}^n} \leq 1$, by taking a convolution with a function with compact support we can find a subsequence f_{n_k} such that $|\widehat{f_{n_k}}|$ is uniformally bounded on compact sets. Similarly, we can find a subsequence f_{n_k} such that all the derivatives of $\widehat{f_{n_k}}$ are uniformally bounded on compact sets. By Ascoli's theorem this means that there exists a subsequence where $\widehat{f_{n_k}}$ converges to a smooth function.

Also, since $s < t$, there is some compact ball $B \subset \mathbb{R}^n$ such that outside this ball,

$$\frac{1}{(1 + |\bar{x}|)^{t-s}} < \epsilon$$

The theorem then follows because

$$\|f_{n_k} - f_{n_l}\|^2_{t,\mathbb{R}^n} = \int_{\mathbb{R}^n} \frac{|(\widehat{f_{n_k}} - \widehat{f_{n_l}})(\bar{x})|^2}{(1 + |\bar{x}|^2)^{t-s}} (1 + |\bar{x}|^2)^t d\bar{x}$$

$$\leq \int_B |(\widehat{f_{n_k}} - \widehat{f_{n_l}})(\bar{x})|^2 (1 + |\bar{x}|^2)^s d\bar{x} + \epsilon \int_B |(\widehat{f_{n_k}} - \widehat{f_{n_l}})(\bar{x})|^2 (1 + |\bar{x}|^2)^t d\bar{x} \leq \int_B |(\widehat{f_{n_k}} - \widehat{f_{n_l}})(\bar{x})|^2 (1 + |\bar{x}|^2)^s d\bar{x} + 2\epsilon$$

The since $\widehat{f_{n_k}}$ converges, we can make this last integral as small as we want. Full proofs of these theorems can be found in Wells.

3 Symbols of Differential Operators

In the previous section we defined a sequence of spaces extending the space of smooth sections $\mathcal{E}(X, E)$. With a great extension of your space comes a great enlargement of the set of functions between such spaces that we'd like to consider. In this section, we'll revisit the definition of a differentiable operator of order k and isolate the key component in this definition which we'll then be able to generalize to a broader context.

Let $U \subset X$ be an open subset. We call a function

$$\phi : \mathcal{E}(U)^p \to \mathcal{E}(U)^q$$

a *linear partial differential operator of order* k if there exists a collection of matrices of smooth functions $A_\alpha \in M_{p,q}(\mathcal{E}(U))$ for $|\alpha| \leq k$ such that for $f = (f_1, \ldots, f_p) \in \mathcal{E}(U)^p$,

$$\phi(f) = \sum_\alpha A_\alpha \cdot D^\alpha f$$

Let E and F be differentiable $\mathbb{C}$ vector bundles over a manifold X. Let

$$L : \mathcal{E}(X, E) \to \mathcal{E}(X, F)$$

be a $\mathbb{C}$-linear map. We'll call L a *differential operator* if for any choice of local coordinates and trivializations there exists a linear partial differential operator $\tilde{L}$ such that the following diagram commutes

$$\begin{array}{ccc}
\mathcal{E}(U)^p & \xrightarrow{\tilde{L}} & \mathcal{E}(U)^q \\
\cong \uparrow & & \cong \uparrow \\
\mathcal{E}(U, U \times \mathbb{C}^p) & \longrightarrow & \mathcal{E}(U, U \times \mathbb{C}^q) \\
\uparrow & & \uparrow \\
\mathcal{E}(X, E)|_U & \xrightarrow{L} & \mathcal{E}(X, F)|_U
\end{array}$$

We'll say that L is of *order* k if there exists a cover such that for each open U, the corresponding $\tilde{L}$ is of order at most k. We'll denote by $\mathrm{Diff}_k(E, F)$ the space of differential operators of order k. For most of the remaining part of this section we'll find a more intrinsic way to describe the matrices A_α and the tuples α appearing in the above definition and use it to obtain a more general definition.

We start by defining a class of operators extending $\mathrm{Diff}_k(E, F)$. Define $\mathrm{OP}_k(E, F)$ to be the vector space of $\mathbb{C}$-linear mappings

$$T : \mathcal{E}(X, E) \to \mathcal{E}(X, F)$$

such that for all $s \in \mathbb{Z}$ there is a continuous extension of T

$$\tilde{T}_s : W^s(X, E) \to W^{s-k}(X, F)$$

We call the elements of $\mathrm{OP}_k(E, F)$ *operators of order* k from E to F. The reason we're interested in this more general notion is that it captures the effect that the order of a differential operator has on the Sobolev spaces $W^K(X, E)$. This is formalized in the following lemma.

Lemma 2. $\mathrm{Diff}_k(E, F) \subset \mathrm{OP}_k(E, F)$

This follows immediately from lemma 1.

We're now ready to define the *symbol* of a differential operator. As we mentioned above, the idea is to obtain an intrinsic way to describe the matrices and D^α-s involved in the definition of the order of a linear partial differential operator. Once we define the symbol, we'll be able to extend the notion of order to a broader class of operators.

Let $T^*(X)$ denote the real cotangent bundle of X and let $T'(X)$ denote the real cotangent bundle with the zero section deleted. Let $T'(X) \xrightarrow{\pi} X$ denote the projection map. Furthermore, let $\pi^* E$ and $\pi^* F$ denote the pullbacks of E and F to $T'(X)$. Then for every point $(x, v) \in T'(X)$, the fiber of (x, v) in $\pi^* E$ is E_x and similarly for $\pi^* F$.

Definition 3. *Let* E *and* F *be as above. For every* $k \in \mathbb{Z}$ *we define the space of* k-symbols *from* E *to* F *to be*

$$\mathrm{Smbl}_k(E, F) = \{\sigma \in \mathrm{Hom}(\pi^* E, \pi^* F) \mid \sigma(x, \rho v) = \rho^k(x, v),\ (x, v) \in T'(X),\ \rho > 0\}$$

The idea is that the symbol is a linear map between the vector bundles E and F which records information pertaining to the k-th order directional derivatives. Slightly more precisely,

In order to motivate this definition we'll now show how to produce a k-symbol $\sigma_k(L)$ from each element $L \in \mathrm{Diff}_k(E, F)$ and see that it indeed allows to recover the highest order differentials used in the definition of L. Intuitively, starting with the operator L, for each point $x \in X$ with a vector v, We obtain a linear map from E_x to F_x corresponding to the k-th derivative of L in the direction v.

Let (x, v) be a point in $T'(X)$. By the definition, $\sigma_k(L)(x, v)$ should be a linear map from E_x to F_x. Let $e \in E_x$ be the vector that we want to send to F_x. Since L only knows how to send functions on E to functions on F, we'll proceed by associating to (x, v) and e a pair of functions $f, g \in \mathcal{E}(X, E)$. The natural way to do this is to pick a section f such that $f(x) = e$ and a section g such that $dg_x = v$. We then define

$$\sigma_k(L)(x, v)e = L(\frac{i^k}{k!}(g - g(x))^k f)(x) \in F_x$$

To justify this, we'll now see that $\sigma_k(L)$ allows us to recover the matrices and D^α-s used to locally define L. In a sense, $\sigma_k(L)$ is the k-th differential of the operator L. It's a linear map recording local differential information.

Locally on an open U, L is a partial differential linear map

$$L : \mathcal{E}(U)^p \to \mathcal{E}(U)^q$$

of the form

$$L = \sum_{|\alpha| \leq k} A_\alpha D^\alpha$$

for some collection of matrices $A_\alpha \in M_{p,q}(\mathcal{E}(U))$. We'll now calculate $\sigma_k(L)(x, v) : \mathbb{C}^p \to \mathbb{C}^q$ for some point $x \in X$ and tangent vector $v = \xi d\bar{x} = \xi_1 dx_1 + \cdots + \xi_n dx_n$.

Let $e \in \mathbb{C}^p$, $f \in \mathcal{E}(U)$ with $f(x) = e$ and $g \in \mathcal{E}(U)$ with $dg_x = v$. Then

$$\sigma_k(L)(x, v)(e) = \sum_{|\alpha| \leq k} A_\alpha D^\alpha \left(\frac{i^k}{k!}(g - g(x))^k e \right)(x)$$

The terms with $|\alpha| < k$ will vanish since we'll be left with a factor of $(g(x) - g(x))$. So by taking the derivatives in the expression above we obtain

$$\sigma_k(L)(x, v)(e) = \sum_{|\alpha| = k} A_\alpha \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} = \sum_{|\alpha| = k} A_\alpha \xi^\alpha$$

And indeed, we see the symbol of a linear operator allows us to recover the essential properties of the operator, the important one for us being it's order.

The calculation above leads us to the following proposition.

Proposition 1. *The symbol map σ_k gives an exact sequence*

$$0 \to \mathrm{Diff}_{k-1}(E, F) \hookrightarrow \mathrm{Diff}_k(E, F) \xrightarrow{\sigma_k} \mathrm{Smbl}_k(E, F)$$

The above calculation also leads us to an additional justification for thinking about the symbol $\sigma_k(L)$ as the differential of L.

Proposition 2. *Let $L_1 \in \mathrm{Diff}_k(E, F)$ and $L_2 \in \mathrm{Diff}_m(F, G)$ be differential operators. Then $L_2 \circ L_1 \in \mathrm{Diff}_{k+m}(E, G)$ and*

$$\sigma_{k+m}(L_2 \circ L_1) = \sigma_m(L_2) \cdot \sigma_k(L_1)$$

Before moving on to the next section, we'll calculate the symbols for two of our favorite operators. Namely, the boundary maps of the de Rham and Dolbeault complexes.

Example 1. Consider the de Rham complex

$$0 \to \mathcal{E}(X, \Omega^0_{\mathbb{C}}) \xrightarrow{d} \mathcal{E}(X, \Omega^1_{\mathbb{C}}) \xrightarrow{d} \dots$$

For $(x, v) \in T'(X)$, we'll calculate the 1-symbols

$$0 \to (\Omega^0_{\mathbb{C}})_x \xrightarrow{\sigma_1(d)(x,v)} (\Omega^1_{\mathbb{C}})_x \xrightarrow{\sigma_1(d)(x,v)} \dots$$

Since the vector space $(\Omega^k_{\mathbb{C}})_x$ is locally spanned by the elements $dx_\alpha = dx_{\alpha_1} \wedge \dots \wedge dx_{\alpha_k}$ for an ordered tuple $\alpha \subset \{1, \dots, n\}$ of length k, it will be enough to calculate the image of dx_α under the map $\sigma_1(d)(x, v)$.

Suppose that locally v is of the form $\xi_1 dx_1 + \dots \xi_n dx_n$. In order to calculate $\sigma_1(d)(x, v)(dx_\alpha)$, we first choose some section $f \in \mathcal{E}(E, \Omega^k_{\mathbb{C}})$ that is locally of the form dx_α and a section $g \in \mathcal{E}(X)$ such that $dg_x = \xi_1 dx_1 + \dots \xi_n dx_n$. Then by plugging this into the definition we get

$$\sigma_1(d)(x, v)(dx_\alpha) = d(\frac{i}{1}(g - g(x))dx_\alpha)(x) = i \cdot d(g dx_\alpha) = i \cdot v \wedge dx_\alpha$$

This intuitively makes sense since on the local level, d operates in the "direction" dx_i by sending dx_α to $dx_i \wedge dx_\alpha$.

Example 2. In this example we'll look at the p-th column of the Dolbeault complex

$$0 \to \mathcal{E}(X, \Omega^{p,0}) \xrightarrow{\bar{\partial}} \mathcal{E}(X, \Omega^{p,1}) \xrightarrow{\bar{\partial}} \dots$$

and for each $(x, v) \in T'(X)$ calculate the symbols

$$0 \to (\Omega^{p,0})_x \xrightarrow{\sigma_1(\bar{\partial})(x,v)} (\Omega^{p,1})_x \xrightarrow{\sigma_1(\bar{\partial})(x,v)} \dots$$

Suppose that v is of the form $v = v^{1,0} + v^{0,1} \in T_x(X)$. Similarly to before, the vector space $\Omega^{p,q}$ is spanned by vectors of the form $dz_\alpha \wedge d\bar{z}_{\bar{\alpha}}$ where α and $\bar{\alpha}$ are tuples of length p and q respectively.

By essentially the same calculation as in the previous example we obtain

$$\sigma_1(\bar{\partial})(x, v)(dz_\alpha \wedge d\bar{z}_{\bar{\alpha}}) = i \cdot v^{0,1} \wedge dz_\alpha \wedge d\bar{z}_{\bar{\alpha}}$$

Later on, we'll define the notion of an *elliptic complex* which will generalize these example.

An important property of the differential operators is the existence of an adjoint.

Proposition 3. *Let* $L \in \mathrm{Diff}_k(E, F)$ *be a differential operator of order* k*. Then* L *has an adjoint* L^* *with respect to the inner products on* $\mathcal{E}(X, E)$ *and* $\mathcal{E}(X, F)$ *and* $L^* \in \mathrm{Diff}_k(F, E)$*.*

Furthermore, $\sigma_k(L^*) = \sigma_k(L)^*$ *where* $\sigma_k(L)^*$ *is the usual dual of linear maps.*

The general gist of the proof is to locally write down the integral defining $\langle L\alpha, \beta \rangle$ and plug in the definition of L in terms of the elements $A_\alpha D^\alpha$. By applying the chain rule to the D^α terms one can obtain an expression of L^* together with it's symbol.

4 Pseudo-differential Operators

As we discussed in the introduction, we would like to expand the set of operators between our Sobolev spaces $W^k(X, E)$ in order to try and obtain inverses for functions such as $d * d + dd*$. In order to enlarge our space of sections of differential vector bundles, we noticed that we could capture the differentiability of a section by the convergence of certain integrals involving the Fourier transform of the section. We then defined our new spaces based on this convergence alone.

In order to generalize the notion of a differentiable function we'll follow a similar route.

Let $U \subset \mathbb{R}^n$ be an open set and $p(\bar{x}, \bar{y})$ a polynomial of degree n in $\bar{x}$ and degree m in $\bar{y}$. Then we can associate to p the differential operator $P = p(x, D)$ by replacing monomials $\bar{y}^\alpha = y_1^{\alpha_1} \cdots y_m^{\alpha_m}$ with the operators D^α.

By using the Fourier transform $\hat{f}$ we can write, for a section $f \in \mathcal{E}(U)$ with compact support,

$$Pf(\bar{x}) = p(\bar{x}, D)f(\bar{x}) = \int p(\bar{x}, \bar{y})\hat{f}(\bar{y})e^{i\langle \bar{x}, \bar{y}\rangle} d\bar{y} \tag{1}$$

The second equality follows from lemma 1.

With this in mind, we'll try to generalize the notion of a m-th order differentiable operator by replacing the polynomial p with a function with suitable asymptotic properties. We formalize these properties in the next definition.

Definition 4. *Let $U \subset \mathbb{R}^n$ be an open subset and let m me an integer.*

- *We define $\tilde{S}^m(U)$ to be the class of functions $p(\bar{x}, \bar{y})$ defined on $U \times \mathbb{R}^m$ with the property that for any compact set $K \subset U$ and tuples of indices α and β, there exists a constant $C_{\alpha,\beta,K}$ depending only on α, β, K and p such that*

$$|D_{\bar{x}}^\beta D_{\bar{y}}^\alpha p(\bar{x}, \bar{y})| \leq C_{\alpha,\beta,K}(1 + |\bar{y}|)^{m-|\alpha|}$$

  for all $\bar{x} \in K$ and $\bar{y} \in \mathbb{R}^n$.

- *Let $S^m(U)$ denote the set of $p \in \tilde{S}^m(U)$ such that the limit*

$$\sigma_m(p)(\bar{x}, \bar{y}) \stackrel{\text{def}}{=} \lim_{\lambda\to\infty} \frac{p(\bar{x}, \lambda\bar{y})}{\lambda^m}$$

  exists for all $\bar{y} \neq 0$ and in addition, for any cutoff function $\psi \in C^\infty(\mathbb{R}^n)$ with $\psi(\bar{y}) = 0$ near $\bar{y} = 0$ and $\psi(\bar{y}) = 1$ outside the unit ball, we have

$$p(\bar{x}, \bar{y}) - \psi(\bar{y})\sigma_m(p)(\bar{x}, \bar{y}) \in \tilde{S}^{m-1}(U)$$

- *Let $\tilde{S}_0^m(U)$ denote the class of functions $p \in \tilde{S}^m(U)$ such that there exists a compact $K \subset U$ so that for any $\bar{y} \in \mathbb{R}^n$, the function $p(\bar{x}, \bar{y})$ considered as a function of $\bar{x}$ has compact support in K. We furthermore denote $S_0^m(U) = S^m(U) \cap \tilde{S}_0^m(U)$.*

One thing to note is that if $p(\bar{x}, \bar{y})$ is a polynomial of degree m in $\bar{y}$ then the first two properties in the definition are satisfied. Furthermore, if the coefficient have compact support of functions of $\bar{x}$ then $p \in S_0^m(U)$.

Also, note that $\sigma_m(p)$ picks out the m-th homogeneous part of p in the variable $\bar{y}$. This is generalized in the next lemma.

Lemma 3. *Let $p \in S^m(U)$. Then $\sigma_m(p)(\bar{x}, \bar{y})$ is a smooth function on $U \times (\mathbb{R}^n \setminus 0$ and homogeneous of degree m in $\bar{y}$.*

The proof of the theorem uses the Arzela-Ascoli theorem. Specifically, in order to show that $\lim_{\lambda\to\infty} \frac{p(\bar{x},\lambda\bar{y})}{\lambda^m}$ is smooth, we show that on any compact subset $K \subset U$, the limit converges uniformally and that all of the derivatives of the expression inside the limit are uniformally bounded. But the boundedness of the derivatives follows easily from the fact that $p \in \tilde{S}^m(U)$. The homogeneity is a straightforward calculation.

As promised, we now use the class of functions $\tilde{S}^m(U)$ to replace the polynomial in equation 1 in order to obtain a broader class of operators.

For any $p \in \tilde{S}^m(U)$ and $f \in \mathcal{E}(U)$ with compact support, we define

$$L(p)f(\bar{x}) \stackrel{\text{def}}{=} \int p(\bar{x}, \bar{y})\hat{f}(\bar{y})e^{i\langle \bar{x}, \bar{y}\rangle} f\bar{y}$$

and call $L(p)$ a *canonical pseudo-differential operator of order* m. The next lemma guarantees that we don't get garbage after applying $L(p)$ to a compactly supported function.

Lemma 4. *With* p *and* f *as above,* $L(p)$ *is an element of* $\mathcal{E}(U)$*.*

To prove the lemma, we use the fact that f has compact support to show that the integrand in the definition of $L(p)f$ has nice enough convergence properties to pull the derivative under the integral sign.

Recall that in the previous section we defined a linear operator $T : \mathcal{E}(X, E) \to \mathcal{E}(X, F)$ to be *of order* k if for all $s \in \mathbb{Z}$ there is a continuous extension $\tilde{T}_s : W^s(X, E) \to W^{s-k}(X, F)$. The space of such operators was denoted by $OP_m(E, F)$. The next theorem shows why this notion of order is a good way to generalize the space of k-th order differentiable functions.

In order to stop repeating the words "compact support", let's define $\mathcal{E}_c(X) \subset \mathcal{E}(X)$ to be the space of compactly supported functions.

Theorem 6. *Let* $p \in \tilde{S}^m_0(U)$*. Then* $L(p) : \mathcal{E}(U)_c \to \mathcal{E}(U)$ *is an operator of order* m*.*

The proof of this theorem isn't difficult, but it's somewhat technical and involves a couple of unenlightening estimations that you can find in Wells.

We're now ready to define pseudo-differential operators globally. We first consider the one dimensional case.

Definition 5. *Let* L *be a linear map* $\mathcal{E}_c(X) \to \mathcal{E}(X)$*. We call* L *a* pseudo-differential operator *on* X *if for every coordinate chart* $U \subset X$ *and any open set* $U' \subset \overline{U'} \subset U$ *there exists a* $p \in S^m_0(U)$ *such that for all* $f \in \mathcal{E}_c(U')$*, after extending* f *by zero to* U *we get that*

$$Lf = L(p)f$$

This definition generalizes naturally to the case of vector bundles.

Definition 6. *Let* E *and* F *be differentiable vector bundles over* X *of rank* p *and* r *respectively. Let* L *be a linear mapping* $L : \mathcal{E}_c(X, E) \to \mathcal{E}(X, F)$*. We call* L *a* pseudo-differential operator *on* X *if for any coordinate chart* $U \subset X$ *and any open* $U' \subset \overline{U'} \subset U$*, there exists a matrix* $p_{ij} \in \mathrm{Mat}_{r\times p}(S^m_0(U))$ *such that the induced map*

$$L : \mathcal{E}_c(U')^p \to \mathcal{E}(U)^r$$

,obtained by extending an element of $\mathcal{E}_c(U')^p$ *by zero to* $\mathcal{E}(U)^p$*, is a matrix of canonical pseudo-differentiable operators of the form* $L(p_{ij})$*.*

In a continuation of our analogy with differential operators, we'd like to define a symbol corresponding to a pseudo-differential operator. In the differentiable case, we saw that the m-th symbol of an element of $\mathrm{Diff}_m(E, F)$ turned out to locally look like the matrices A_α corresponding to the D^α-s with $|\alpha| = m$. By lemma 3, it makes sense that in the pseudo-differentiable case, the functions $\sigma_m(p)(\bar{x}, \bar{y})$ should play the role of the elements in the matrices A_α. Inspired by this realization, we make the following definition.

Definition 7. *Let* local m-symbol *of a pseudo-differential operator* $L : \mathcal{E}_c(X, E) \to \mathcal{E}(X, F)$ *is, with respect to a coordinate chart* U *and trivializations of* E *and* F *over* U*, the matrix*

$$\sigma_m(L)_U(\bar{x}, \bar{y}) = [\sigma_m(p_{ij})(\bar{x}, \bar{y})]$$

where $i = 1, \ldots, r$ *and* $j = 1, \ldots, p$*.*

Note that the m could depend on the open set U. Because in our definition of a pseudo-differentiable operator, we required that $p_{ij} \in \mathrm{Mat}_{r\times p}(S^m_0(U))$ without requiring the m to be the same for all charts U.

At this stage, we would like to extend this local definition of a symbol to a global one. The problem is that it's not clear that our local definition is preserved under a change of coordinates. In order to see exactly what coordinate transformation properties we're looking for let's step back to the differentiable case for a moment.

Suppose a differentiable operator is locally expressed by

$$L = \sum_{|\alpha| \le m} A_\alpha(\bar{x}) D_{\bar{x}}^\alpha$$

and we make a change of coordinates $\bar{y} = F(\bar{x})$. Then using the chain rule we can find a corresponding collection of matrices $\tilde{A}_\alpha(\bar{y})$ such that in the new coordinates, the same operator is expressed by

$$\tilde{L} = \sum_{|\alpha| \le m} \tilde{A}_\alpha(\bar{y}) D_{\bar{y}}^\alpha$$

Furthermore, these two representation are related by

$$L(f \circ F)(\bar{x}) = \sum_{|\alpha| \le m} \tilde{A}_\alpha(F(\bar{x})) D_{\bar{y}}^\alpha f(F(\bar{x})) = \tilde{L}(f)(F(\bar{x}))$$

We thus obtain an operator $\tilde{L}$ of the same order. Moreover, the m-th homogeneous part of L is transformed to the m-th homogeneous part of $\tilde{L}$ in a precise manner, i.e, multiplication by the Jacobian of F.

The following theorem tells us the the same sort of transformation holds for pseudo-differential operators as well. In particular, it says that a pseudo-differential operator remains pseudo-differential after a change of coordinates, and that the symbol is changed in an explicit fashion.

Theorem 7. *Let U be an open relatively compact subset of $\mathbb{R}^n$ and let $p \in S_0^m(U)$. Suppose $F : U \to U$ is a diffeomorphism. Also, let $U' \subset \overline{U'} \subset U$ be an open set and define the linear map*

$$\tilde{L} : \mathcal{E}_c(U') \to \mathcal{E}(U)$$

by

$$\tilde{L}(f)(\bar{x}) = L(p)(f \circ F^{-1})(F(\bar{x}))$$

Then, there is a function $q \in S_0^m(U)$ such that $\tilde{L} = L(q)$. Furthermore, we have the following identity for the symbol

$$\sigma_m(q)(\bar{x}, \bar{y}) = \sigma_m(p)\left(F(\bar{x}), \left(\frac{\partial F}{\partial \bar{x}}\right)^{-t} \bar{y}\right)$$

The proof of the theorem is somewhat technical, and again I'll refer you to theorem 3.9 in Wells. There isn't anything terribly deep going on - the main idea is to substitute the definition of the Fourier transform $\hat{F}$ into the definition of $L(p)f$, change the coordinates of the integral with the standard Jacobian rule, and then obtain an explicit expression of q. Then some more work has to be done to show that this q indeed has the required properties.

Now that we know that our local definition of order is preserved under a change of coordinates, we're able to give a global definition of the order of a pseudo-differential operator. Furthermore, the identity we obtained for the symbol under a change of coordinates will allow us to globally assign a symbol to a pseudo-differential operator.

Definition 8. *Let X be a differentiable manifold and let $\mathcal{E}_c(X) \to \mathcal{E}(X)$ be a pseudo-differential operator. We say that L is a* pseudo-differential operator of order m *on X if, for any local coordinate chart $U \subset X$, the corresponding canonical pseudo-differential operator L_U on U is of order m. Or in other words, there exists a $p \in S_0^m(U)$ such that $L_U = L(p)$.*

We'll denote the class of pseudo-differential operators on X of order m by $\mathrm{PDiff}_m(X)$.

As we've already seen in theorem 6, for $p \in S_0^m(U)$, $L(p) \in \mathrm{OP}_m(U)$. In other words, $L(p)$ is an operator of order m. Therefore, the following proposition is immediate.

Proposition 4. *Let X be a compact differentiable manifold. Then $\mathrm{PDiff}_m(X) \subset \mathrm{OP}_m(X)$.*

In particular, this means that the definition of order in terms of the form of the symbol corresponds with the original definition of the order of a linear operator.

Let $\mathrm{Smbl}_m(X)$ denote $\mathrm{Smbl}_m(X \times \mathbb{C}, X \times \mathbb{C})$.

Proposition 5. *There exists a canonical linear map*

$$\sigma_m : \mathrm{PDiff}_m(X) \to \mathrm{Smbl}_m(X)$$

which is locally defined on the chart $U \subset X$ by

$$\sigma_m(L_U)(\bar{x}, \nu) = \sigma_m(p)(\bar{x}, \bar{y})$$

where $(\bar{x}, \nu) \in T'(X)$.

This proposition is a corollary of theorem 7 since the explicit formula for the change of basis tells us that the symbol is well defined.

This is a good generalization of our previous association of a symbol to a differential operator. To see this, notice that in the case that $p(\bar{x}, \bar{y})$ is a polynomial, it locally corresponds to an operator of the form

$$L = p(\bar{x}, D) = \sum_{\alpha} A_\alpha(\bar{x}) D^\alpha$$

As we saw in the previous section, the symbol corresponding to the differential operator L is $\sum_\alpha A_\alpha(\bar{x})\bar{y}^\alpha$. But this is exactly the symbol that the previous proposition associated to $L(p)$.

It's straightforward to extend the above definition to the higher dimensional case and obtain the following.

Proposition 6. *Let E and F be vector bundles on a differentiable manifold X. Then there exists a canonical linear map*

$$\sigma_m : \mathrm{PDiff}_m(E, F) \to \mathrm{Smbl}_m(E, F)$$

defined locally on a chart $U \subset X$ by

$$\sigma_m(L_U)(\bar{x}, \nu) = [\sigma_m(p_{ij})(\bar{x}, \nu)]$$

where $L_U = [L(p_i j)]$ is a matrix of canonical pseudo-differential operators and $(\bar{x}, \nu) \in T'(X)$.

One of our key techniques for analyzing pseudo-differential operators will be to move over to their symbols which are simpler linear operators, prove results about them, and then transfer the results back to the pseudo-differential operator we started with. This explain the key importance of the next theorem which strengthens proposition 1.

Theorem 8. *Let E and F be vector bundles over a differentiable manifold X. Then the following sequence is exact:*

$$0 \to K_m(E, F) \hookrightarrow \mathrm{PDiff}_m(E, F) \xrightarrow{\sigma_m} \mathrm{Smbl}_m(E, F) \to 0$$

Where $K_m(E, F)$ is the kernel of σ_m. Furthermore, $K_m(E, F) \subset \mathrm{OP}_{m-1}(E, F)$ if X is compact.

The proof of this theorem involves two steps. The first and easier one is to show that if $L \in \mathrm{PDiff}_m(E, F)$ and $\sigma_m(L) = 0$ then $L \in \mathrm{PDiff}_{m-1}(E, F)$. For this, it suffices to note that locally, $L_U = [L(p_{ij})]$ for $p_{ij} \in S^m(U)$ such that $\sigma_m(p_{ij}) = 0$. But by the definition of $S^m(U)$, it follows that for any cutoff function ψ, $p_i j = p_i j - \psi \cdot 0$ is in $\tilde{S}^{m-1}(U)$. This shows that L is in $\tilde{S}^m(E, F)$ which by theorem 6, together with the implies that it's of order $m - 1$.

The second part of the theorem is to show that for every symbol $s \in \mathrm{Smbl}_m(E, F)$ there exists an operator $L \in \mathrm{PDiff}_m(E, F)$ such that $\sigma_m(L) = s$. The idea of the proof is fairly straightforward. Let U_μ be a coordinate chart of X. We want to use s to produce a family of elements $p^\mu_{ij} \in S^m_0(U_\mu)$ and locally define L by $[L(p^\mu_{ij})]$.

More precisely, let ϕ_μ by a partition of unity subordinate to U_μ. Define $s^\mu = \phi_\mu s$. In addition, let χ be a smooth function on $\mathbb{R}^n$ that equals 0 around the origin and 1 outside the unit ball.

Since s^μ is a symbol, with respect to a trivialization of E and F it looks like a matrix $s^\mu = [s^\mu_{ij}]$ of homogeneous functions $s^\mu_{ij} : U_\mu \times (\mathbb{R}^n \setminus 0) \to \mathbb{C}$ satisfying

$$s^\mu_{ij}(\bar{x}, \rho\bar{y}) = \rho^m s^\mu_{ij}(\bar{x}, \bar{y})$$

is order to extend s^μ_{ij} to a function on all of $U_\mu \times \mathbb{R}^n$, we multiply it by the function χ and thus obtain

$$p^\mu_{ij}(\bar{x}, \bar{y}) = \chi(\bar{y}) s^\mu_{ij}(\bar{x}, \bar{y})$$

By homogeneity, together with the fact that χ is equal to 1 outside the unit ball, the symbol of $p^\mu_{ij}(\bar{x}, \bar{y})$ is s^μ_{ij}. Furthermore, it's support is contained in ϕ_μ so it's compactly supported. Therefore, $p^\mu_{ij} \in S^m_0(U_\mu)$. This means that we can locally define

$$L_\mu : \mathcal{E}_c(U_\mu)^p \to \mathcal{E}(U_\mu)^r$$

by

$$L_\mu(f) = [L(p^\mu_{ij})]f$$

and by the definition of a symbol we get that $\sigma_m(L_\mu) = [\sigma_m(p^\mu_{ij})] = [(s^\mu_{ij})]$

The final step is to glue the L_μ together to get a global pseudo-differential operator. We do this by defining a family of smooth functions $\psi_\mu \in \mathcal{E}_c(U_\mu)$ that equal the identity on the support of the partition of unity element ϕ_μ. We then define:

$$Lf = \sum_\mu \psi_\mu(L_\mu f)$$

The functions ψ_μ are necessary in order to extend the functions $L_\mu f$ to global sections in $\mathcal{E}(X, F)$. As before, it's easy to check that since $\psi_\mu = 1$ on $\operatorname{supp} s_\mu$, the local m-symbol of L is s_μ so $\sigma_m(L) = s$.

As we alluded to before the above theorem, we're interested in how operations on pseudo-differential operators effect the symbols and vice versa. The next theorem states the basic properties to this effect and generalizes propositions 2 and 3.

Theorem 9. *Let E, F and G be differentiable vector bundles over a compact differentiable manifold X. Then*

- *If* $Q \in \text{PDiff}_r(E, F)$ *and* $P \in \text{PDiff}_s(F, G)$ *then* $P \circ Q \in \text{PDiff}_{r+s}(E, G)$ *and*

$$\sigma_{r+s}(P \circ Q) = \sigma_s(P) \cdot \sigma_r(Q)$$

- *If* $P \in \text{PDiff}_m(E, F)$ *then* P^**, the adjoint of* P*, exists and*

$$\sigma_m(P^*) = \sigma_m(P)^*$$

The proof of this theorem is fairly technical. For the second part, the idea is to locally write out the full integral expression for $(f, P^*g) = (P_U f, g)$ and isolate the part of the integrand, q, which would put the integral in the form of $(f, L(q)g)$. The last step is to verify that $q \in S^k_0(U)$. The proof of the first part is similar. As usual, the details can be found in theorem 3.17 in Wells.

The upshot of these two theorems is that just like differential operators, pseudo-differential operators are locally dictated by linear functions.

5 A Parametrix for Elliptic Differential Operators

As we've discussed before, one of the reasons that we defined the space $\text{PDiff}_m(E, F)$ was to obtain a class of functions large enough to include an inverse for functions such as the Lagrangian $d^*d + dd^*$ from the de Rham complex. In this section we'll show how to produce such an inverse called the *parametrix* for an operator $L \in \text{PDiff}_m(E, F)$ assuming that the symbol of L is nice enough.

Definition 9. *Let $s \in \text{Smbl}_k(E, F)$. Then s is called* elliptic *if for any $(\bar{x}, \nu) \in T'(X)$, the linear map*

$$s(\bar{x}, \nu) : E_x \to F_x$$

is an isomorphism.

We'll mostly be interested in this property in the case that $E = F$.

Definition 10. *Let $L \in \text{PDiff}_k(E, F)$. We call L* elliptic *if $\sigma_k(L)$ is elliptic.*

For reasons that will become clear later on, we'll call any operator $L\text{OP}_{\text{-}1}(E, F)$ a *smoothing operator*. Intuitively, functions in $\text{OP}_{\text{-}1}(E, F)$ are nice because for every $s \in \mathbb{Z}$ we can continuously extend them to a function $T : W^s(E, F) \to W^{s+1}(E, F)$ so they sections to "even more differentiable" sections.

We'll now formalize what we mean by an inverse to a pseudo-differential operator.

Definition 11. *Let $L \in \text{PDiff}(E, F)$. Then $\tilde{L} \in \text{PDiff}(E, F)$ is called a* parametrix *for L if*

$$L \circ \tilde{L} - I_F \in \text{OP}_{\text{-}1}(F)$$
$$\tilde{L} \circ L - I_E \in \text{OP}_{\text{-}1}(E)$$

The next theorem shows us that for elliptic operators, our quest for an inverse has finally come to an end.

Theorem 10. *Let k be an integer and let $L \in \text{PDiff}_k(E, F)$ be elliptic. Then there exists a parametrix for L.*

Proof. In this proof we'll witness the power of the combinations of the exact sequence from theorem 8 and the symbol operations from theorem 9.

Let $s = \sigma_k(L)$. Since L is elliptic, $s^{\text{-}1}$ exists as a linear transformation and clearly $s^{\text{-}1} \in \text{Smbl}_{\text{-}k}(F, E)$. By the short exact sequence, there exists an operator $\tilde{L} \in \text{PDiff}_{\text{-}k}(F, E)$ such that $\sigma_{\text{-}k}(\tilde{L}) = s^{\text{-}1}$. Therefore,

$$\sigma_0(L \circ \tilde{L} - I_F) = \sigma_0(L \circ \tilde{L}) - \sigma_0(I_F) = \sigma_k(L) \cdot \sigma_{\text{-}1}(\tilde{L}) - 1_F = 1_F - 1_F = 0$$

where $1_F \in \text{Smbl}_0(F, F)$ is the identity symbol. Again by the short exact sequence, it follows that $L \circ \tilde{L} - I_F \in \text{OP}_{\text{-}1}(F, F)$. Similarly, $\tilde{L} \circ L - I_E \in \text{OP}_{\text{-}1}(E, E)$. □

By this theorem, we see that elliptic operators have inverses up to a smoothing operator. The following proposition is one of the reasons that smoothing operators are so well behaved. But first we need to introduce a definition.

Definition 12. *Let X be a compact manifold and let $L \in \text{OP}_m(E, F)$. We say that L is* compact *if for every $s \in \mathbb{Z}$, the extension $L_s : W^s(E) \to W^{s-m}(F)$ is a compact operator as a mapping of Banach spaces.*

Proposition 7. *Let X be a compact manifold and let $S \in \text{OP}_{\text{-}1}(E, E)$. Then S is a compact operator of order 0.*

This proposition follows immediately from Rellich's theorem (theorem 5) since the map $\tilde{S}_s : W^s(E) \to W^s(E)$ obtained by considering S as an element of $OP_0(E, E)$ via the inclusion $OP_{-1}(E, E) \supset OP_0(E, E)$ factors as

$$W^s(E) \xrightarrow{S_s} W^{s+1}(E) \hookrightarrow W^s(E)$$

Now, recall that one of the reasons that we're interested in elliptic operators is that we want to study the kernel and cokernel of our laplacian $d^*d + dd^*$. For instance, we'd like to show that they're finite dimensional. In order to bring the power of Hilbert to the table, we'll consider these spaces as subspaces of $W^0(E) = W^0(X, E)$ and $W^0(F) = W^0(X, F)$. Since this subspace of $W^0(E)$ isn't necessarily the kernel of a linear operator on Banach spaces, it will also be helpful to look at the kernels of the induced maps L_s between $W^s(E)$ and $W^s(F)$. We'll show later on that these kernels $W^s(E)$ actually consists entirely of smooth operators so we can have our kernel of operator cake and eat it too.

For the rest of this section, E and F will be Hermitian vector bundles over a compact manifold X. Recall that in this case, the inner products on E and F allow us to define an inner product between smooth sections of E and F by integration over X. We denoted the completions of $\mathcal{E}(X, E)$ and $\mathcal{E}(X, F)$ under these inner products by $W^0(E)$ and $W^0(F)$ respectively.

Also, we'll consider an operator $L \in \mathrm{Diff}_m(E, F)$ and it's adjoint L^* with respect to the inner products on sectioned mentioned above as given by proposition 3.

Furthermore, we define the space

$$\mathcal{H}_L = \{\alpha \in \mathcal{E}(X, E) | L\alpha = 0\}$$

and set $\mathcal{H}_L^{\perp}$ to be the orthogonal complement of $\mathcal{H}_L$ in the Hilbert space $W^0(E)$. Since the vanishing of an inner product is a closed condition, $\mathcal{H}_L^{\perp}$ is a closed subspace of $W^0(E)$. We'll see soon that if L is elliptic, $\mathcal{H}_L$ is finite dimensional and hence closed as well.

Before we continue, we'll need the following theorem from analysis.

Proposition 8. *Let B be a Banach space and let $S : B \to B$ be a compact operator. Let $T = I - S$. Then*

1. $\mathrm{Ker}\, T$ *is finite dimensional.*
2. $T(B)$ *is closed in* B *and* $\mathrm{Coker}\, T$ *is finite dimensional.*

For the first part of the theorem, note that is C is the unit ball in $\mathrm{Ker}\, T$, then $S(C) = C$ so by the definition of a compact operator, C is compact. Since any locally compact topological vector space is finite dimensional, $\mathrm{Ker}\, T$ must be finite dimensional.

For the second part, we use a similar argument to show that $\mathrm{Ker}\, T*$ is finite dimensional, and use this to prove the finite dimensionality of $\mathrm{Coker}\, T$.

We call an operator on a Banach space *Fredholm* if both it's kernel and cokernel are finite dimensional.

By combining theorem 10, proposition 7 and proposition 8, we immediately obtain the following theorem.

Theorem 11. *Let $Ł \in \mathrm{PDiff}_m(E, F)$ be an elliptic pseudo-differential operator. Then there exists a parametrix P for L such that $L \circ P$ and $P \circ L$ have continuous extensions of* Fredholm *operators $L \circ P : W^s(F) \to W^s(F)$ and $P \circ L : W^s(E) \to W^s(E)$ for every integer s.*

The important thing to notice about this theorem is that we a priori don't know much about $P \circ L$ as an operator from $\mathcal{E}(X, E)$ to itself, but after extending it to a map on the Banach spaces $W^s(E)$ we can use the compactness on $P \circ L - I$ on that space to prove that we get a Fredholm operator.

This shows us that even though we don't have an actual inverse of L, after composing it with a suitable P we get an operator only "finitely far" from being finite.

The next theorem will show that after extending our operator L to the Banach space $W^s(E)$, the inverse image of the smooth sections are smooth. In particular, this means that we don't pick up any non-smooth functions in the kernel.

Theorem 12. *Let* $L \in \mathrm{Diff}_m(E,F)$ *be an elliptic differential operator and suppose that* $\alpha \in W^s(E)$ *has the property that* $L_s\alpha = \beta \in \mathcal{E}(X,F)$*. Then* $\alpha \in \mathcal{E}(X,E)$*.*

Proof. As expected, the main ingredient in the proof of this theorem is the parametrix P of L. By the definition of P, $P \circ L - I = S$ for some $S \in OP_{-1}(E)$. Since $L\alpha \in \mathcal{E}(X,F)$ and $P : \mathcal{E}(X,F) \to \mathcal{E}(X,E)$, we get that $(P \circ L)\alpha \in \mathcal{E}(X,E)$. Furthermore, α is equal to $(P \circ L)\alpha$ up to the error of $S\alpha$.

$$\alpha = (P \circ L)\alpha - S\alpha$$

Our goal is to show that even after adding in $S\alpha$ the resulting operator is smooth. But since $\alpha \in W^s(E)$ and $S \in OP_{-1}(E)$, it follows that $S\alpha \in W^{s+1}(E)$. Together this implies that $\alpha = (P \circ L)\alpha + S\alpha \in W^{s+1}(E)$

Continuing in this manner we find that $\alpha \in W^{s+k}(E)$ for all $k > 0$. But by Sobolev's theorem (theorem 4), it follows that $\alpha \in \mathcal{E}(X,E)$. □

Now it is clear why we call elements of $OP_{-1}(E)$ smoothing operators. Since as we saw in the above proof, if we have a section $\alpha \in W^s(E)$ such that α is equal to a smooth section plus $S\alpha$ where S is a smoothing operator, then α must be smooth.

The previous two theorems easily imply the following useful finiteness theorem for elliptic operators.

Theorem 13. *Let* $L \in \mathrm{Diff}_k(E,F)$ *be an elliptic differential operator. Define*

$$\mathcal{H}_{L_s} = \mathrm{Ker}\, L_s : W^s(E) \to W^{s-k}(F)$$

Then

- $\mathcal{H}_{L_s} \subset \mathcal{E}(X,E)$ *and hence* $\mathcal{H}_{L_s} = \mathcal{H}_L$ *for all* s*.*
- $\dim \mathcal{H}_{L_s} = \dim \mathcal{H}_L < \infty$ *and* $\dim W^{s-k}/L_s(W^s(E)) < \infty$

The first part follows immediately from theorem 12.

For the second part, note that by theorem 11 there exists a parametrix P for L such that for every s,

$$P \circ L : W^s(E) \to W^s(F)$$

has a finite kernel and cokernel.

Before we continue to the last part of the puzzle, we record the following fact which follows immediately from the definitions.

Proposition 9. *Let* $L \in \mathrm{Diff}_m(E,F)$*. Then* L *is elliptic if and only if* L^* *is elliptic.*

We've already seen that if $L \in \mathrm{Diff}_m(E,F)$ and $L\alpha = \beta$ is smooth then alpha is smooth. We now prove existence and uniqueness.

Theorem 14. *Let* $L \in \mathrm{Diff}_m(E,F)$ *be elliptic and suppose that* $\beta \in \mathcal{H}_{L^*}^{\perp} \cap \mathcal{E}(X,F)$*. Then there exists a unique section* $\alpha \in \mathcal{E}(X,E)$ *such that* $L\alpha = \beta$ *and such that* α *is orthogonal to* $\mathcal{H}_L$ *in* $W_0(E)$*.*

For the existence, we first prove an elementary result from functional analysis.

Lemma 5. *Let* $T : A \to B$ *be a bounded linear operator from a Banach space to a Hilbert space. Then* $\overline{T(A)} = (\mathrm{Ker}\, T^*)^{\perp}$ *where we are identifying* B *and* B^* *via the inner product in the usual fashion. I.e, for* $x \in A$ *and* $y \in B$ *we have*

$$\langle Tx, y \rangle = (T^*y)(x)$$

Proof. For the first direction, suppose that $y = T(x)$ and that $T^*(z) = 0$. Then $\langle y, z \rangle = \langle x, T^*z \rangle = 9$ so $T(A) \subset (\mathrm{Ker}\, T^*)^{\perp}$. Since $(\mathrm{Ker}\, T^*)^{\perp}$ is closed, $\overline{T(A)} \subset (\mathrm{Ker}\, T^*)^{\perp}$.

For the second direction, we'll show that $\overline{T(A)}^{\perp} \subset (\mathrm{Ker}\, T^*)$. Let $z \in B$ such that for every $x \in A$, $\langle z, T(x) \rangle = 0$. This implies by duality that for every $x \in A$, $(T^*z)(x) = 0$ which means that $T^*z = 0$. □

By applying the lemma to $L_m : W^m(E) \to W^0(F)$, we see that $L_m(W^m(E))$ is dense in $\mathcal{H}^{\perp}_{L^*_m}$. (Notice that the dual of L_m as used in the lemma coincides with the dual of L_m with respect to the inner products on $W^m(E)$ and $W^m(F)$ by uniqueness.)

Furthermore, since we've already seen that L_m has a finite dimensional cokernel, $L_m(W^m(E))$ is closed and hence $L_m(W^m(E)) = \mathcal{H}^{\perp}_{L^*_m}$.

This shows that there exists an $\alpha \in W^m(E)$ such that $L_m(\alpha) = \beta$. But by theorem 12, α is a smooth section. By taking the orthogonal projection of α along $\operatorname{Ker} L_m = \mathcal{H}_L$ we obtain our unique solution.

We're now ready to prove the theorem about elliptic operators which we'll use in the next section to prove the general case of the Hodge decomposition theorem.

We call an operator $L \in \operatorname{Diff}_m(E) = \operatorname{Diff}_m(E, E)$ *self-adjoint* if $L = L^*$.

Theorem 15. *Let* $L \in \operatorname{Diff}_m(E)$ *be a self-adjoint elliptic operator. Then there exist linear mappings* H_L *and* G_L

$$H_L : \mathcal{E}(X, E) \to \mathcal{E}(X, E)$$
$$G_L : \mathcal{E}(X, E) \to \mathcal{E}(X, E)$$

such that

1. $H_L(\mathcal{E}(X, E)) = \mathcal{H}_L$ *and* $\dim_c \mathcal{H}_L < \infty$
2. $L \circ G_L + H_L = G_L \circ L + H_L = I_E$ *where* I_E *is the identity on* $\mathcal{E}(X, E)$.
3. H_L *and* G_L *are in* $\mathrm{OP}_0(E)$ *and in particular, extend to bounded operators on* $W^o(E)$.
4. $\mathcal{E}(X, E) = \mathcal{H}_L \oplus G_L \circ L(\mathcal{E}(X, E)) = \mathcal{H}_L \oplus L \circ G_L(\mathcal{E}(X, E))$ *and this decomposition is orthogonal with respect to the inner product on* $W^0(E)$.

Proof. First of all, we choose H_L to be the orthogonal projection in $W^0(E)$ onto the closed subspace $\mathcal{H}_L$ which by theorem 13 is finite dimensional.

As we saw in theorem 14, we have a continuous bijection

$$L_m : W^m(E) \cap \mathcal{H}^{\perp}_L \to W^0(E) \cap \mathcal{H}^{\perp}_{L^*}$$

so by the Banach open mapping theorem, L_m has a continuous inverse

$$G_0 : W^0(E) \cap \mathcal{H}^{\perp}_L \to W^m(E) \cap \mathcal{H}^{\perp}_{L^*}$$

We can extend G_0 to all of $W^0(E)$ by defining that $G_0(\alpha) = 0$ for $\alpha \in \mathcal{H}_L$. In addition, since $W^m(E) \subset W^0(E)$, we obtain a map

$$G_0 : W^0(E) \to W^0(E)$$

Furthermore, on both $\mathcal{H}_L$ and $\mathcal{H}^{\perp}_L$ we have that

$$L_m \circ G_0 + H_L = I_E$$

so this must hold on all of $W^0(E)$. Similarly,

$$G_0 \circ L_m + H_L = I_E$$

Finally, for any $\alpha \in \mathcal{E}(X, E)$ we have $L_m(G_0(\alpha)) = \alpha \in \mathcal{E}(X, E)$. By theorem 12 this implies that $G_0(\alpha) \in \mathcal{E}(X, E)$. Therefore, $G_0(\mathcal{E}(X, E) \subset \mathcal{E}(X, E)$ so we can restrict G_0 to $\mathcal{E}(X, E)$ and obtain G_L.

G_L and H_L clearly satisfy the conditions in the theorem. □

6 Elliptic Complexes

We're now finally ready to prove the general form of the Hodge decomposition theorem. In the previous section we saw a version of the theorem for elliptic operators. However, elliptic hypothesis is stronger than what we have in the case of the de Rham and Dolbeault complexes. This leads us to the following definition.

Definition 13. *Let $E_0, E_1, \dots E_N$ be a sequence of Hermitian differential vector bundles on a compact differentiable manifold X. Let π be the map $\pi : T'(X) \to X$. We also adopt the notation of $\mathcal{E}(E_i) = \mathcal{E}(X, E_i)$*

Suppose in addition that there is a sequence of differential operators $L_0, L_1, \dots L_{N-1}$ of fixed order k

$$\mathcal{E}(E_0) \xrightarrow{L_)} \mathcal{E}(E_1) \xrightarrow{L_1} \dots \xrightarrow{L_{N-1}} \mathcal{E}(E_N)$$

We call such a sequence of operators a complex *if $L_i \circ L_{i-1} = 0$. We call it an* elliptic complex *if the associated symbol sequence*

$$0 \to \pi^* E_0 \xrightarrow{\sigma(L_0)} \pi^*(E_1) \xrightarrow{\sigma(L_1)} \dots \xrightarrow{\sigma(L_{N-1})} \pi^* E_N \to 0$$

is exact.

Notice that if $L \in \mathrm{Diff}(E_0, E_1)$ is elliptic then $\mathcal{E}(E_0) \xrightarrow{L} \mathcal{E}(E_1)$ is an elliptic complex.

Let E denote an elliptic complex. Then as we saw in previous sections, each operator $L_i : \mathcal{E}(E_i) \to \mathcal{E}(E_{i+1})$ comes with an adjoint $L_j^* : \mathcal{E}(E_{j+1}) \to \mathcal{E}(E_j)$ with respect to the inner products on the $\mathcal{E}(E_i)$-s induced by the inner products on the vector bundles.

We define the *Laplacian operators* of the complex E to be

$$\Delta_j = L_j^* L_j + L_{j-1} L_{j-1}^* : \mathcal{E}(E_j) \to \mathcal{E}(E_j)$$

It is a simple matter of linear algebra to check that the exactness of the symbol sequence of E implies that for each j,

$$\sigma(\Delta_j) = \sigma(L_j)^* \sigma(L_j) + \sigma(L_{j-1}) \sigma(L_{j-1})^*$$

is an isomorphism as a linear map. It's also clear that Δ_j is self adjoint. We further denote by

$$\mathcal{H}(E_j) = \mathcal{H}_{\Delta_j}(E_j) = \mathrm{Ker}\, \Delta_j \subset \mathcal{E}(E_j)$$

the Δ_j-harmonic sections.

Since Δ_i is self-adjoint and elliptic, by theorem 15, we have a projection

$$H_j : \mathcal{E}(E_j) \to \mathcal{E}(E_j)$$

onto the closed finite dimensional subspace $\mathcal{H}(E_j)$ together with a map

$$G_j = G_{\Delta_j} : \mathcal{E}(E_j) \to \mathcal{E}(E_j)$$

satisfying the conditions in theorem 15. We thus obtain our much sought after theorem.

Theorem 16. *Let $E = (\mathcal{E}(E_i), L_i)$ be an elliptic complex. Then*

- *There is an orthogonal decomposition*

$$\mathcal{E}(E_j) = L_j(L_j^* G_j \mathcal{E}(E_j)) \oplus \mathcal{H}(E_j) \oplus L_j^*(L_j G_j \mathcal{E}(E_j))$$

- *The following relations hold*

  1. $G_j \Delta_j + H_j = \Delta_j G_j + H_j = I_{\mathcal{E}(E_j)}$
  2. $H_j G_j = G_j H_j = H_j \Delta_j = \Delta_j H_j = 0$

3. $L_j\Delta_j = \Delta_j L_j$, $L_j^*\Delta_j = \Delta_j L_j^*$

- $\dim_{\mathbb{C}}\mathcal{H}(E_j) < \infty$ *and there is a canonical isomorphism* $\mathcal{H}(E_j) \cong H^j(E)$

For the first part of the theorem, note that from theorem 15 we know that

$$\mathcal{E}(E_j) = (L_jL_j^* + L_j^*L_j)G_j\mathcal{E}(E_j)) \oplus \mathcal{H}(E_j)$$

The orthogonality follows from

$$(L_jL_j^*G_j\alpha, L_j^*L_jG_j\beta) = L_j^2L_j^*G_j\alpha, L_jG_j\beta) = 0$$

as $L_j^2 = 0$.

The other parts follow easily from the orthogonal decomposition together with theorem 15.

To conclude, we'll mention how this theorem applies in the special cases of the de Rham and Dolbeault complexes.

For the de Rham complex, since X is compact, we can put a Hermitian structure on the cotangent bundle $\Omega^p_{\mathbb{C}}$. By our calculation in example 1 $\sigma(d)$ is an isomorphism for every d so the de Rham complex is indeed an elliptic complex. The Laplacian now becomes the usual Laplacian $dd^8 + d^*d$ and the decomposition theorem for the de Rham complex is now a consequence of theorem 16.

The result for the Dolbeault complex follows similarly from example 2.