DIMENSION IN TTT STRUCTURES

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Abstract. In this paper we consider two types of dimension that can be defined for products of one-dimensional topologically totally transcendental (t.t.t) structures. The first is topological and considers the interior of projections of the set onto lower dimensional products. The second one is based on algebraic dependence. We show that these definitions are equivalent for ω-saturated one-dimensional t.t.t structures. We also prove that sets which are dense in products of these structures are comeager.

1. Introduction

There are a several of different ways to think about dimension in finite products of o-minimal structures. One of the first ways to do this was described in [Knight et al.(1986)Knight, Pillay, and Steinhorn] using the cell decomposition theorem. The idea is to first define dimension on relatively simple sets called cells, and then to generalize this to arbitrary sets by relying on the fact that any set can be decomposed into a finite number of cells.

An alternative approach was introduced by Pillay in [A.Pillay(1988)]. This definition is of a more algebraic flavor and it is based on the notion of algebraic dependence. Assuming that this dependence is well behaved, we can obtain a concept of dimension in a way that is quite similar to what is done in the case of linear independence in linear algebra.

A third possibility is to take a topological route and define the dimension of a set X as the largest integer $k \in \mathbb{N}$ such that some projection onto M^k has an interior.

It is natural to ask whether the various definition coincide. We can also ask whether it is possible to extend these types of dimension beyond the o-minimal setting. And assuming we can, how how does this affect their relationship with one another?

In [A.Pillay(1988), 1.4] Pillay showed that for o-minimal structures, the second and third definitions are equivalent. As he mentions in [A.Pillay(1986), 1.5], they are also equivalent to the first definition.

A natural generalization of o-minimal structures is that of a *first order topological structure* which was introduced by Pillay [Pillay(1987)]. In particular, a subset of these structures called *one dimensional topologically totally transcendental* (1-t.t.t) structures share several important characteristics with the o-minimal ones. For example, they have the exchange property which allows us to define dimension using algebraic dependence.

Mathews proved in [Mathews(1995), 8.8] that the equivalence mentioned above holds in the generalized setting of first order topological structures which have both the exchange property and what he defined as the cell decomposition property.

In this paper we prove the equivalence of the second two kinds of dimension for ω-saturated 1-t.t.t structures. We note that for ω-saturated connected first order topological structures, Mathew's cell decomposition implies 1-t.t.t.

In addition, we'll use some of the machinery developed during the proof in order to obtain additional information about dense sets in this type of structure. Specifically, we'll show that dense sets must be comeager.

Proposition 1. *Suppose M is an ω-saturated 1-t.t.t structure. Let $X \subset M_t^n$ be a dense definable set. Then $int(X) \subset M_t^n$ is dense as well.*

2. Preliminaries

We start by presenting the definitions from [Pillay(1987)] necessary for introducing t.t.t structures.

Definition 2. Let M be a two sorted L structure with sorts M_t and M_b and let $\phi(x, y_1, \dots, y_k)$ be an L formula such that $\{\phi^{M_t}(x, \bar{a}) | \bar{a} \in M_b^k\}$ is a basis for a topology on M_t. Then the pair (M, ϕ) will be called a *first order topological structure*. When we talk about the topology of M_t we mean the one generated by the basis described above.

We'll also be using the following property:

(A) Every definable set $X \subset M_t$ is a boolean combination of definable open subsets.

Definition 3. Let M be a first order topological structure satisfying (A) such that M_t is Hausdorff and let $X \subset M_t$ be a closed definable subset of M_t. The ordinal valued $D_M(X)$ is defined by:

1. If $X \neq \emptyset$ then $D_M(X) \geq 0$.
2. If δ is a limit ordinal and $D_M(X) \geq \alpha$ for all $\alpha < \delta$ then $D_M(X) \geq \delta$.
3. If there's a closed definable $Y \subset M_t$ such that $Y \subset X$, Y has no interior in X and $D_M(Y) \geq \alpha$ then $D_M(X) \geq \alpha + 1$.

Furthermore, we'll write $D_M(X) = \alpha$ if $D_M(X) \geq \alpha$ and $D_M(X) \not\geq \alpha + 1$. We'll write $D_M(X) = \infty$ if $D_M(X) \geq \alpha$ for all α.

Definition 4. Let M be a first order topological structure satisfying (A) such that M_t is Hausdorff. We say that M *has dimension* if $D_M(X) \neq \infty$ for all closed definable subsets $X \subset M_t$.

Definition 5. Let M be a first order topological structure satisfying (A) such that M_t is Hausdorff. Let $X \subset M_t$ be a definable subset. Then $d_M(X)$ is the maximum $d < \omega$ such that there are disjoint definable clopen $X_1, \dots, X_d \subset X$ with $X = \cup_{i=1}^d X_i$, and ∞ if no such d exists.

We're now ready to introduce t.t.t structures.

Definition 6. We say that M is *topologically totally transcendental (t.t.t)* if M is a first order topological structure satisfying (A) with dimension such that M_t is Hausdorff and for every definable set $X \subset M_t$, $d_M(X) < \infty$. We say that a theory T is t.t.t is every model of T is t.t.t.

As mentioned in the introduction, 1-t.t.t will stand for *one-dimensional t.t.t*.

The following lemma was proved by Pillay [Pillay(1987), 6.6] and will be used extensively.

Lemma 7. *Let M be a 1-t.t.t structure. Then:*

(1) *For any closed and definable $X \subset M_t$, $D(X) = 0$ iff X is finite.*
(2) *The set of isolated points of M_t is finite.*
(3) *For any definable $X \subset M_t$ there are pairwise disjoint definably connected definable open subsets $X_1, \ldots, X_m \subset M_t$ and a finite set $Y \subset M_t$ such that $X = (\cup_{i=1}^{m} X_i) \cup Y$.*
(4) *For any definable $X \subset M_t$, the set of boundary points of X is finite.*

3. Defining Dimension in t.t.t Structures

Definition 8. *(exchange)* Let M be a first order structure. We say that M has the *exchange property* if for every $a, b \in M$ and a set $A \subset M$, if $b \in acl(A \cup a)$ and $b \notin acl(A)$ then $a \in acl(A \cup b)$.

The following theorem was proved by Pillay [Pillay(1987), 6.7]:

Theorem 9. *Let M be a 1-t.t.t structure. Then M_t has the exchange property.*

The first type of dimension that we'll look at was introduced by Pillay [A.Pillay(1988)]. For this part we'll leave the t.t.t setting and will only need to assume that our structure has the exchange property.

Definition 10. *(rank)* Let M be a structure with the exchange property and $A \subset M$.

(1) For any tuple $\bar{a} \in M^n$, $rk(\bar{a}/A)$ is the least cardinality of a subtuple $\bar{a}'$ of $\bar{a}$ such that $\bar{a} \in acl(\bar{a}'/A)$.
(2) for any type $p(\bar{x}) \in S_n(A)$, $rk(p/A) = rk(\bar{a}/A)$ for any $\bar{a} \in M^n$ realizing p.

Remark. It's easy to see that the second part of the definition doesn't depend on the choice of the element which realizes p.

The following lemma is immediate but will be used in the next section.

Lemma 11. *Let M be a structure with the exchange property, $A \subset M$, and $\{a_1, \ldots, a_n\} \subset M$ an algebraically independent set over A. In addition, let $b \in M$ have the property that $b \notin acl(\{a_1, \ldots, a_n\}/A)$. Then $\{a_1, \ldots, a_n, b\}$ is an algebraically independent set over A.*

Proof. Suppose for contradiction that $\{a_1, \ldots, a_n, b\}$ is not algebraically independent over A. Then there's some $1 \leq i \leq n$ such that

$$a_i \in acl(\{a_1 \ldots, \hat{a_i}, \ldots, a_n, b\}/A)$$

. By the assumption, $a_i \notin acl(\{a_1 \ldots, \hat{a_i}, \ldots, a_n\}/A)$. But since M has the exchange property, this means that $b \in acl(\{a_1 \ldots, a_i, \ldots, a_n\}/A)$ which is a contradiction. □

For now we will assume that M is a structure with the exchange property which is sufficiently saturated so that the dimension of a type doesn't depend on the specific model we're using.

The following lemma was proved by Pillay [A.Pillay(1988), 1.2]

Lemma 12. *Let M be a structure with the exchange property, $A, B \subset M$, and $\bar{a} \in M^s, \bar{b} \in M^t$. Then:*

(1) *If $A \subset B$ then $rk(\bar{a}/A) \geq rk(\bar{a}/B)$.*

(2) $rk(\bar{a}\,\hat{}\,\bar{b}/A) = rk(\bar{a}/A \cup \bar{b}) + rk(\bar{b}/A)$.
(3) $rk(\bar{a}/A \cup \bar{b}) = rk(\bar{a}/A) \iff rk(\bar{b}/A \cup \bar{a}) = rk(\bar{b}/A)$.
(4) *If* $p \in S_n(A)$ *and* $A \subset B$ *then there exists a type* $q \in S_n(B)$ *such that* $p \subset q$ *and* $rk(q/B) = rk(p/A)$.

We are now ready to define our first concept of dimension for a structure with the exchange property.

Definition 13. Let M be a structure with the exchange property, $X \subset M^n$ a definable subset and $A \subset M$. Then we define:

$$rk(X) = max_{p \in S_n(A)}\{rk(p/A)|\text{p is realized in } X\}$$

Remark 1. Note that under our assumption that M is sufficiently saturated, by part 4 of lemma 12, $rk(X)$ doesn't depend on the choice of A.

We can make this more explicit by changing the definition and only requiring p to be realized $X(N)$ where N is some elementary extension of M.

We'll now give our second definition of dimension. In this definition M has to have some definable topology. Therefore, we'll assume that M is a t.t.t structure.

Furthermore, given a set $X \subset M^n$ and indices $1 \leq i_1 < \cdots < i_k \leq n$, let $\pi_{i_1,\ldots,i_k}(X)$ be the projection of X onto the coordinates $i_1, \ldots, i_k$.

Definition 14. Let M be a t.t.t structure and $X \subset M_t^n$ be a definable subset. We define the *topological dimension* of X as:

$$dim(X) = max_{1 \leq k \leq n}\{\exists 1 \leq i_1 < \cdots < i_k \leq n \,\text{s.t}\, int(\pi_{i_1,\ldots,i_k}(X)) \neq \emptyset\}$$

4. The Equivalence of the Dimensions

In this section we'll prove that for any ω-saturated 1-t.t.t structure, the two definitions of dimension we gave above agree on all definable sets.

Lemma 15. *Let* M *be a 1-t.t.t structure. Let* $\phi(x, y)$ *be a formula and* $X = \phi^{M_t}$. *In addition, let* $U \subset M_t$ *be a definable open set such that for all* $u \in U$, $|\{y \in M_t : (u, y) \in X\}| \geq \aleph_0$. *Then, for every* $k \in \mathbb{N}$ *there exists a* $y \in M_t$ *such that* $|\{x \in U : (x, y) \in X\}| > k$.

The proof of this lemma is nearly identical to Pillay's proof of the exchange property in [Pillay(1987), 6.7] but is modified for our purposes. For convenience we give the complete proof here.

Proof. Let's assume for contradiction that there exists a $k \in \mathbb{N}$ such that for all $y \in M_t$, $|\{x \in U : (x, y) \in X\}| \leq k$. For all $u \in U$ we'll define $X_u = \{y \in M : (u, y) \in X\}$. U and X are definable and therefore X_u is definable as well. In addition, we know that for all $u \in U$, $|X_u| \geq \aleph_0$. So according to lemma 7, X_u contains an open set. We now define another set:

$$X_0 = \{c \in M_t : c \in \overline{X}_u \backslash int(X_u) \,\text{for some}\, u \in U\}$$

First we'll assume that X_0 is finite and reach a contradiction. Since $|\{u \in U : (u, y) \in X\}| \leq k$ for all $y \in X_0$, we have the following:

(*) for only a finite number of $u \in U$ there exists a $c \in X_0$ such that $(u, c) \in X$.

Let's define $N = (\cup_{u \in U} X_u) \backslash X_0$ and for all $u \in U$, $Z_u = X_u \cap N = X_u \backslash X_0$. By (*), there're an infinite number of $u \in U$ such that $Z_u \neq \emptyset$. We'll now show that for each $u \in U$, Z_u is clopen in N. First of all, Z_u is open in M_t and therefore it's also open in N. In addition, if c is a boundary point of Z_u in N then it's a boundary point of Z_u and therefore also a boundary point of X_u. But that means that $c \in X_0$ which is a contradiction to the definition of N.

Now, by our assumption for contradiction, for any distinct $u_1, \dots, u_{k+1} \in U$,

(**) $\cap_{i=1}^{k+1} Z_{u_i} = \emptyset$

We now show that for any $n \in \mathbb{N}$, we can find n clopen definable disjoint sets $V_1, \dots, V_n$ where each V_i is of the form $Z_{u_1} \cap \dots, \cap Z_{u_m}$ for some $u_1, \dots, u_m \in U$. Let $\tilde{U} = \{u \in U : Z_u \neq \emptyset\}$. As we mentioned above, $\tilde{U}$ is infinite. For $n = 1$, We can define $V_1 = Z_u$ for any $u \in \tilde{U}$. Let's assume that we've already found sets $V_1, \dots, V_n$ with the properties mentioned above. We choose some $u_1 \in \tilde{U}$ that isn't used in the definition of any of the V_i. We define $V_{n+1}^1 = Z_{u_1}$. We now construct a sequence V_{n+1}^i , $1 \leq i \leq k$, inductively. We already have V_{n+1}^1. Let's say that we've defined V_{n+1}^i. If there exists some $u_{i+1} \in \tilde{U}$ such that $V_{n+1}^i \cap Z_{u_{i+1}} \neq \emptyset$ then we define $V_{n+1}^{i+1} = V_{n+1}^i \cap Z_{u_{i+1}}$. Otherwise, we define $V_{n+1}^{i+1} = V_{n+1}^i$. We now define $V_{n+1} = V_{n+1}^k$. According to (**), the sequence $V_1, \dots, V_{n+1}$ now has the required properties. But this is a contradiction to the fact that $d(N) \in \mathbb{N}$.

Now we assume that X_0 is infinite. Let W_0 be the interior of X_0. For each $u \in U$, let $W_u = intX_u$. We'll now inductively find a sequence $u_1, u_2, \dots \in U$ such that for all $n \in \mathbb{N}$,

$$W_0 \cap W_{u_1} \cap \dots \cap W_{u_n} \neq \emptyset$$

For $n = 0$ there's nothing to show. Let's assume that we've found some sequence $u_1, \dots, u_n \in U$ with the desired property. We choose an element $c \in W_0 \cap \dots \cap W_{u_n}$. Since $c \in X_0$, there exists some point $u \in U \setminus \{u_1, \dots, u_n\}$ such that c is a boundary point of X_u. But X_u has a finite number of boundary points and so by the Hausdorffness of M_t, every neighborhood of c contains points in the interior of X_u. Specifically, $(W_0 \cap \dots \cap W_{u_n}) \cap W_u \neq \emptyset$ so we can set $u_{n+1} = u$. Now we choose some $y \in W_0 \cap \dots \cap W_{u_{k+1}}$. This means that $(y, u_i) \in X$ for all $1 \leq i \leq k+1$ which is a contradiction to our assumption on X. $\square$

Proposition 16. *Suppose that M is an ω-saturated 1-t.t.t structure. Let $\phi(x, y)$ be a formula, $X = \phi^{M_t}$, and $U \subset M_t$ an open definable subset such that*

$$|\{y \in M_t : (u, y) \in X\}| \geq \aleph_0$$

for all $u \in U$. Then $X \cap (U \times M_t)$ has a non-empty interior.

Proof. Let $\alpha(x)$ be the formula in M defining U.

Claim. There exists a $y \in M_t$ such that $|(M_t \times \{y\}) \cap X \cap (U \times M_t)| \geq \aleph_0$.

Proof. Let $n \leq \omega$. According to lemma 15 there exists some $c \in M_t$ such that $|(M_t \times \{y\}) \cap X| \geq n$. So if we define

$$\psi_n(y) = \exists x_1 \dots \exists x_n ((\bigwedge_{i \neq j} x_i \neq x_j) \wedge (\bigwedge_i (\alpha(x_i) \wedge \phi(x_i, y)))$$

then $M \vDash \psi_n[c]$. Since M is ω-saturated, there exists some $d \in M_t$ such that $M \vDash \psi_n[d]$ for all $n < \omega$. Therefore, $|(M_t \times \{d\}) \cap X \cap (U \times M_t)| \geq \aleph_0$ which completes the claim. $\square$

Claim. There exists an open definable set $V \subset U$ and an infinite number of elements $y \in M_t$ such that for all $v \in V$, $(v, y) \in X$.

Proof. By the definition of a t.t.t, there exists some formula $\beta(x, y_1, \ldots, y_k)$ such that $\{\beta^{M_t}(x, \bar{a}) | \bar{a} \in M_b^k\}$ is a basis for the topology on M_t. We first show that for every $n < \omega$:

(***) there exists a tuple $\overline{b}_n \in M_b^k$ and distinct elements $c_1, \ldots, c_n \in M_t$ such that if $B_n = \beta^{M_t}[\overline{b}_n]$ then $B_n \subset U$ and $(u, c_i) \in X$ for all $u \in B_n$ and all $1 \leq i \leq n$.

According to the first claim there exists a $c_1 \in M_t$ such that the definable set $(M_t \times \{c_1\}) \cap X \cap (U \times M_t)$ is infinite. Therefore, its projection onto U is infinite so there's some $\overline{b}_1 \in M_b^k$ such that $B_1 = \beta^{M_t}[\overline{b}_1] \subset U$ is contained in the projection. This means that $(u, c_1) \in X$ for all $u \in B_1$. This shows that (***) is true for $n = 1$.

Now let's assume that (***) is true for $n \in \mathbb{N}$. The set

$$\tilde{X} = \{(x, y) \in X : \forall 1 \leq i \leq n, y \neq c_i\}$$

and the open definable set B_n fulfill the conditions of the prior claim (where $\tilde{X}$ is instead of X and B_n is instead of U). This means that we can find an element $c_{n+1} \in M_t$ such that

$$|(M_t \times \{c_{n+1}\}) \cap \tilde{X} \cap (B_n \times M_t)| \geq \aleph_0$$

So exactly like in the case of $n = 1$, there exists some $\overline{b}_{n+1} \in M_b^k$ such that $B_{n+1} = \beta^M[\overline{b}_{n+1}] \subset B_n$ and $(u, c_{n+1}) \in \tilde{X} \subset X$ for all $u \in B_{n+1}$. Also, by the definition of $\tilde{X}$, $c_{n+1} \neq c_i$ for all $1 \leq i \leq n$. Finally, since $B_{n+1} \subset B_n$, $(u, c_i) \in X$ for all $u \in B_{n+1}$ and all $1 \leq i \leq n+1$. So we showed that (***) holds for all $n < \omega$.

Therefore, if we define the formula:

$$\gamma_n(\overline{x}) = \exists c_1 \ldots \exists c_n((\bigwedge_{i \neq j} c_i \neq c_j) \wedge (\forall u(\beta(u, \overline{x}) \to ((\bigwedge_i \phi(u, c_i)) \wedge \alpha(u)))))$$

then for each $n < \omega$ there exists a tuple $\overline{b} \in M_b^k$ such that $M \vDash \gamma_n[\overline{b}]$. But M is ω-saturated so there is some $\overline{b} \in M_b^k$ such that $M \vDash \gamma_n[\overline{b}]$ for all $n < \omega$, i.e, if $B = \beta^{M_t}[\overline{b}]$ then $B \subset U$ and the set $C = \{y \in M_t : \forall u \in B, (u, y) \in X\}$ is infinite. This finishes the proof of the claim. □

Now, let B and C be the sets defined in the end of the proof of the second claim. Let C_0 be the (non empty) interior of C. Then by the definition of C, $B \times C_0 \subset X \cap (U \times M_t)$ is open and which completes the proof of the proposition. □

Lemma 17. *Suppose M is an ω-saturated 1-t.t.t structure. Let $X \subset M_t^{n+1}$ be a definable subset such that $\pi_{1,\ldots,n}(X)$ is a basis set in the product topology on M_t^n. If for all $\bar{x} \in \pi_{1,\ldots,n}(X)$, $|(\{\bar{x}\} \times M_t) \cap X| = \infty$, then X has a non-empty interior.*

Proof. We'll use induction on n.

For $n = 1$, this lemma follows proposition 16.

Let's assume that the claim is true for $n - 1$. We define $A = \pi_{2,\ldots,n}(X)$ and $B = \pi_{2,\ldots,n+1}(X)$.

In addition, we define the set:

$$C = \{\bar{b} \in B : |(M_t \times \{\bar{b}\}) \cap X| = \infty\}$$

By proposition 16, for every tuple $\bar{a} \in A$ the set $(M_t \times \{\bar{a}\} \times M_t) \cap X$ has a non-empty interior. In particular, this means that

$$|(\{\bar{a}\} \times M_t) \cap C| = \infty$$

Claim. There exists a basis set $U \subset \pi_{1,\ldots,n}(X)$ such that

$$|\{x \in M_t : U \times \{x\} \subset X\}| = \infty$$

Proof. As we showed above, for every tuple $\bar{a} \in A$,

$$|(\{\bar{a}\} \times M_t) \cap C| = \infty$$

By the inductive hypothesis, there exists a basis set $V \subset A$ and a point $x_1 \in M_t$ such that $V \times \{x_1\} \subset C$. By the definition of C and another application of the inductive hypothesis, $(M_t \times V \times \{x_1\}) \cap X$ has a non-empty interior. Therefore, there exists a basis set $U_1 \subset \pi_{1,\ldots,n}(X)$ such that $U_1 \times \{x_1\} \subset X$.

Now, lets define the set

$$X_2 = [(U_1 \times M_t) \cap X] \backslash [U_1 \times \{x_1\}]$$

Since we only removed a finite number of elements from each fiber of U_1, X_2 has the properties required by the proposition. This means that we can repeat the above process again and obtain a basis set $U_2 \subset U_1 = \pi_{1,\ldots,n}(X_2)$ and an element $x_2 \in M_t$ such that $x_1 \neq x_2$ and

$$U_2 \times \{x_2\} \subset X_2 \subset X$$

Furthermore, since $U_2 \subset U_1$, we also have

$$U_2 \times \{x_1\} \subset X$$

Therefore:

$$|\{x \in M_t : U_2 \times \{x\} \subset X\}| \geq 2$$

By continuing this process n times, we can find a basis set $U_n \subset \pi_{1,\ldots n}(X)$ such that:

$$|\{x \in M_t : U_n \times \{x\} \subset X\}| \geq n$$

Since M is ω-saturated and basis sets are definable with a tuple of constants from M_b, there exists a basis set $U \subset \pi_{1,\ldots n}(X)$ such that:

$$|\{x \in M_t : U \times \{x\} \subset X\}| = \infty$$

□

Let U be the basis set given by the claim. Since M is 1-t.t.t, there exists a basis set $W \subset M$ such that for all $w \in W$, $U \times \{w\} \subset X$. Therefore, $U \times W \subset X$ which finishes the induction and proves the lemma. □

Before proceeding to prove the the theorem about the equivalence of the dimensions, we use lemma 17 to obtain an interesting corollary.

Corollary 18. *Suppose M is an ω-saturated 1-t.t.t structure. Let $X \subset M_t^n$ and $Y \subset X$ be definable sets. If X has an interior in M_t^n and Y does not, then $X \backslash Y$ has an interior in M_t^n.*

Proof. We use induction on n.

For $n = 1$, the lemma follows directly from the fact that M is a 1-t.t.t structure.

Let's assume the claim is true for n. Let $X \subset M_t^{n+1}$ be a definable set with an interior and $Y \subset X$ be a definable set with no interior. In addition, we define $\tilde{X} = \pi_{1,\ldots,n}(X)$, $\tilde{Y} = \pi_{1,\ldots,n}(Y) \subset \tilde{X}$, and a set $\tilde{Z} \subset \tilde{Y}$:

$$\tilde{Z} = \{\bar{y} \in \tilde{Y} : \ |(\{\bar{y}\} \times M_t) \cap Y| = \infty\}$$

Since X has an interior, without loss of generality we can assume that for every $\bar{x} \in \tilde{X}$, $|(\{\bar{x}\} \times M_t) \cap X| = \infty$. Furthermore, by lemma 17, $\tilde{Z}$ has no interior. So by the inductive hypothesis, $\tilde{U} = \tilde{X} \backslash \tilde{Z}$ has an interior.

Let $\bar{u}$ be an element in $\tilde{U}$. Since $|(\{\bar{u}\} \times M_t) \cap X| = \infty$ and $|(\{\bar{u}\} \times M_t) \cap Y| < \infty$,

$$|(\{\bar{u}\} \times M_t) \cap (X \backslash Y)| = \infty$$

. So by lemma 17, $X \backslash Y$ has an interior in M_t^{n+1}.

This completes the induction and the corollary. $\square$

We can use the corollary to prove a proposition about dense definable sets in M_t^n.

Proposition 19. *Suppose M is an ω-saturated 1-t.t.t structure. Let $X \subset M_t^n$ be a dense definable set. Then $int(X) \subset M_t^n$ is dense as well.*

Proof. Let $a \in M_t^n$ be a point and $U \subset M_t$ a basis set containing a. Since X is dense, $U \backslash X$ has an empty interior and so by corollary 18, $U \cap X$ has an interior. Therefore, there exists an element $b \in int(X)$ such that $b \in U$. This finishes the proof. $\square$

Each of following two propositions will be used to show one of the inequalities which together will prove the equivalence of the dimensions.

Proposition 20. *Suppose M is an ω-saturated 1-t.t.t structure. Let $X \subset M_t^n$ be definable over A, $0 \leq k \leq n$, $1 \leq i_1 < \cdots < i_k \leq n$ and $\bar{a} \in X$ such that $(a_{i_1}, \ldots, a_{i_k})$ is algebraically independent over A. Then $\pi_{i_1,\ldots,i_k}(X)$ has an interior.*

Proof. We use induction on n.

If $n = 1$, then since $a_{i_1} \notin acl(A)$, X is infinite and thus has an interior.

Let's assume the claim holds for n.

First we assume that $i_k < n+1$. In this case, the claim follows directly from the inductive hypothesis.

Now let's assume that $i_k = n+1$. We define $Y = \pi_{i_1,\ldots,i_{k-1},n+1}(X)$, $Z = \pi_{i_1,\ldots,i_{k-1}}(X) = \pi_{i_1,\ldots,i_{k-1}}(Y)$, and:

$$C = \{\bar{z} \in Z : \ |(\{\bar{z}\} \times M_t) \cap Y| = \infty\}$$

. Since $(a_{i_1}, \ldots, a_{i_k})$ is algebraically independent over A, $(a_{i_1}, \ldots, a_{i_{k-1}}) \in C$. So by the inductive hypothesis, C has a non-empty interior and by lemma 17, Y has an interior.

This completes the induction and the proposition. $\square$

Proposition 21. *Suppose M is an ω-saturated 1-t.t.t structure. Let $X \subset M_t^n$ be definable over A, $0 \leq k \leq n$, and $1 \leq i_1 < \cdots < i_k \leq n$ such that $\pi_{i_1,\ldots,i_k}(X)$ has an interior. Then there exists an elementary extension $M \prec N$ and a tuple $\bar{a} \in X(N)$ such that $(a_{i_1}, \ldots, a_{i_k})$ is algebraically independent over A.*

Proof. We use induction on n.

Let $n = 1$ and $X \subset M_t$ a subset definable over A. For $k = 0$ there's nothing to show.

If $k = 1$ then X is infinite so by compactness there exists an elementary extension $M \prec N$ and an element $x \in X(N)$ such that $x \notin acl(A)$. Therefore, we can take $a_{i_1} = x$.

Let's assume the claim is true for n. Let $X \subset M_t^{n+1}$ be definable over A and $0 \leq k \leq n+1$ such that $dim(X) = k$ and $\pi_{i_1,\ldots,i_k}(X)$ has an interior.

First we assume that $i_k < n+1$.

Let's define $Y = \pi_{1,\ldots n}(X)$. According to the assumption, $\pi_{i_1,\ldots,i_k}(Y)$ has an interior. So by the inductive hypothesis, there exists an elementary extension $M \prec N$ and a tuple $\bar{y} \in Y(N)$ such that $(y_{i_1}, \ldots, y_{i_k})$ is algebraically independent over A. Since $\bar{y} \in Y(N)$, there exists an element $x \in N_t$ such that $\bar{a}\hat{\ }x \in X(N)$.

Now let's assume that $i_k = n+1$.

Let's define $Y = \pi_{i_1,\ldots,i_{k-1},n+1}(X)$. According to the assumption, Y has a non-empty interior. This means that there exist basis sets $U \subset \pi_{i_1,\ldots,i_{k-1}}(X)$ and $V \subset M_t$ such that $U \times V \subset Y$. According to the inductive hypothesis, there exists an elementary extension $M \prec N$ and a tuple $\bar{u} = (u_1, \ldots, u_k) \in U(N)$ such that $(u_1, \ldots, u_k)$ is algebraically independent over A. In addition, since V is infinite, we can find an elementary extension $N \prec N'$ and an element $v \in V(N')$ such that $v \notin acl(\bar{u}/A)$. By lemma 11, $\bar{u}\hat{\ }v \in A$ is algebraically independent over A.

This finishes the induction and proves the proposition. □

Theorem 22. *Suppose M is an ω-saturated 1-t.t.t structure. Let $X \subset M_t^n$ be definable. Then $rk(X) = dim(X)$.*

Proof. Let's assume that X is definable over A.

We first prove that $rk(X) \leq dim(X)$.

Let's set $0 \leq k \leq n$ such that $rk(X) = k$. By the definition of $rk(X)$ (see remark 3), there exists an elementary extension $M \prec N$, a tuple $\bar{a} \in X(N)$, and indices $1 \leq i_1 < \cdots < i_k \leq n$ such that $(a_{i_1}, \ldots, a_{i_k})$ is algebraically independent over A.

By [D.Lowengrub(2012), 14], N is also a 1-t.t.t structure.

Therefore, by proposition 20, $\pi_{i_1,\ldots,i_k}(X(N))$ has an interior which means that $\pi_{i_1,\ldots,i_k}(X)$ has an interior as well. So by the definition of $dim(X)$, $dim(X) \geq k$.

We now prove that $rk(X) \geq dim(X)$.

Let's set $0 \leq k \leq n$ such that $dim(X) = k$. By the definition of $dim(X)$, there exist indices $1 \leq i_1 < \cdots < i_k \leq n$ such that $\pi_{i_1,\ldots,i_k}(X)$ has an interior. Therefore, by proposition 21, there exists an elementary extension $M \prec N$ and a tuple $\bar{a} \in X(N)$ such that $(a_{i_1}, \ldots, a_{i_k})$ is algebraically independent over A. So by the definition of $rk(X)$, $rk(X) \geq k$.

So together, we proved that $rk(X) = dim(X)$. □

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