Deformation Theory

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1 Introduction

In these notes we'll give an introduction to deformation theory and apply it to the special case of abelian schemes. We'll start by defining the deformation functor and show how the cohomology groups of the sheaf of derivations of a scheme can be used to both determine if deformations exist and if so, what the set of deformations looks like.

After that, we'll introduce the 800 pound gorilla of the subject, Schlessinger's criteria, which says that under fairly weak and natural assumptions the deformation functor is representable (or more precisely, pro-representable). This will allow us to produce an object encoding the set of all deformations of the given scheme.

We'll then apply this theory to the special case of abelian varieties and see that the object given to us by Schlessinger's criteria has a simple geometric interpretation.

The main source that I used to learn this material is Oort's paper [1].

2 The Deformation Functor

Given a scheme X over a ring k, we're interested in determining the ways in which the scheme can be deformed. Intuitively, a deformation of a scheme is a continuous family of schemes passing through our given scheme. One way to formalize this is to study morphisms of schemes $E \xrightarrow{\pi} Y$ such that $X = \pi^{-1}(p)$ for some k-point p of Y. In other words, the fibers of π give us a family of schemes including X which is parametrized by Y. Since we're interested in the local properties of the family around p, we focus on the case where $Y = \mathrm{Spec}(A)$ for some local artinean ring. This leads us to the formal definition of the deformation functor.

Before stating it, we first define the category of rings that we'll be working with.

Definition 1. *Let A be a local artinian ring.*

1. $\mathbf{Art}_A$ *will denote the category of local artinian A-algebras (R, m_R) such that the following diagram commutes:*

$$\begin{array}{ccc} A & \longrightarrow & R \\ \downarrow & & \downarrow \\ A/m_A & \xrightarrow{\sim} & R/m_R \end{array}$$

2. $\widehat{\mathbf{Art}_A}$ *is defined to be the category of complete local noetherian A-algebras O such that $O/{m_O}^n \in \mathbf{Art}_A$ for all n.*

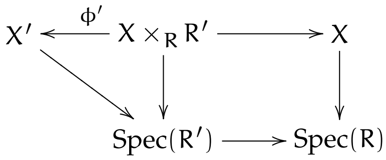
The morphisms in this category are the ones which respect the diagram as well.

Definition 2. *Let $R \to R'$ be a surjection of rings in $\mathbf{Art}_k$. Let $X' \to \mathrm{Spec}(R')$ be a smooth scheme over $\mathrm{Spec}(R')$. We define $\mathrm{Def}_k(X', R)$ to be the set of isomorphism classes of pairs*

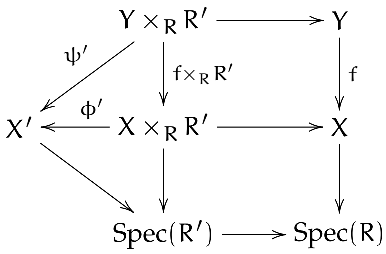
$$(X, \phi')$$

Where X *is a scheme over* $\operatorname{Spec}(R)$ *and* $X \times_R R' \xrightarrow{\phi'} X'$ *is an isomorphism over* $\operatorname{Spec}(R)$. *The functor* $\operatorname{Def}_k(X', \_) : \mathbf{Art}_k \to \mathbf{Set}$ *is called the* deformation functor.

The above definition is captured in the following diagram:



Also, in order to talk about isomorphism classes, we need to explain what a morphism of deformations is. Suppose (X, ϕ') and (Y, ψ') are two deformations of X'. A morphism f between these deformations is a morphism of schemes fitting into the following diagram:



Given a $\operatorname{Spec}(R')$-scheme X', it is natural to ask to ask when $\operatorname{Def}_k(X', R)$ is non-empty. In other words, when does there exist a $\operatorname{Spec}(R)$-scheme X that base changes back to X'? In the category of e'tale schemes, we'll see that for certain maps extensions $R \to R'$ we not only can always find such a scheme X, but there is even an equivalence of categories between schemes over $\operatorname{Spec}(R)$ and schemes over $\operatorname{Spec}(R')$.

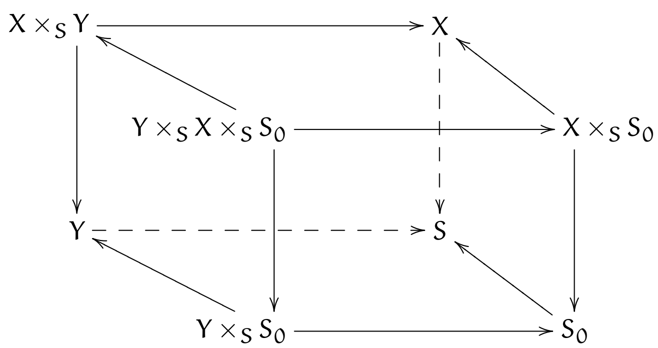
We start by studying how morphisms behave under base change.

Theorem 1. *Let* X *and* Y *be two schemes over* S. *Let* $S_0 \to S$ *be a closed subscheme with sheaf of ideals* $\mathcal{I}$ *such that* $\mathcal{I}^2 = 0$. *If* X *is etale over* S *and* $X_0 = S_0 \times_S X$ *then the natural map*

$$\operatorname{Mor}_S(Y, X) \to \operatorname{Mor}_{S_0}(Y_0, X_0)$$

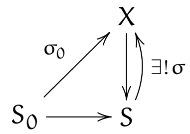
is a bijection.

Proof. First of all, consider the following fibered diagram obtained by base changing first by $Y \to S$ and then by base changing the entire fibered square by $S_0 \to S$.



By the universal property of fibered products, maps from $Y \times_S S_0$ to $X \times_S S_0$ are naturally equivalent to maps from $Y \times_S S_0$ to $X \times_S Y \times_S S_0$ which are in turn equivalent to maps from $Y \times_S S_0$ to $X \times_S Y$. Similarly, maps from Y to X are naturally equivalent to maps from Y to $X \times_S Y$.

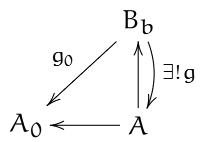
Therefore, it's enough to prove that for any map $Y \times_S S_0 \xrightarrow{\sigma_0} X \times_S Y$ there exists a unique map $Y \xrightarrow{\sigma} X \times_S Y$ such that the diagram above together with σ and σ_0 commutes. Since $X \times_S Y \to Y$ is etale as well, we can replace S with Y and prove:



In other words, for any morphism $S_0 \xrightarrow{\sigma_0} S$ there exists a unique map $S \xrightarrow{\sigma} X$ such that the diagram commutes.

Since morphisms glue, it is enough to prove the theorem locally. As X is etale over S, we can assume without loss of generality that $S = \mathrm{Spec}(A)$, $S_0 = \mathrm{Spec}(A/I)$ and that we have a rings A, $B = A[t]/(p(t))$ such that $X = \mathrm{Spec}(B_b)$ for an invertible element $b \in B$ and a monic polynomial $p \in A[t]$ such that p' is a unit in B_b. Let $A_0 = A/I$

By moving to the category of rings we have to show the following:



Assume that $g_0(t) = a + I$. In order to define g we need to find some element $a' \in A$ such that $a - a' \in I$ and $p(r') = 0$. Let $a' = a + h$ for some $h \in I$. Since $I^2 = 0$, $p(a' + h) = p(r) + hp'(a)$. Therefore, $p(a') = 0$ iff $h = -\frac{p(a)}{p'(a)}$ which proves that the map g exists and is uniquely defined by $t \mapsto a + \frac{p(a)}{p'(a)}$. ☐

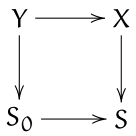
Theorem 2. *Let X and Y be schemes over S. Let $S_0 \to S$ be a closed subscheme with ideal sheaf $\mathcal{I}$ such that $\mathcal{I}^2 = 0$. Then the functor*

$$X \mapsto X_0 = X \times_S S_0$$

is an equivalence of categories between the category of e'tale schemes over S and the category of e'tale schemes over S_0.

Proof. By theorem 1, the functor is fully faithful. It remains to show that it's essentially surjective.

In other words, we need to show that for and e'tale scheme Y over S_0 there exists an e'tale scheme X over S extending Y such that the following diagram is fibered:



We'll prove this claim locally and then glue the extensions together with the uniqueness.

Locally, we can assume that $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/I)$ for a ring A and an ideal $I \subset A$ such that $I^2 = 0$. Furthermore, there exist polynomials $f_1, \ldots, f_m \in A[t_1, \ldots, t_n]$ such that for

$$g = \det\left(\partial \bar{f}_i / \partial t_j\right)$$

$\bar{g}$ is a unit in A/I. But since $I^2 = 0$, units in A/I lift to units in A so g is a unit in A. Therefore, we can define $X = \mathrm{Spec}(A[t_1, \ldots, t_n]/(f_1, \ldots, f_n))$. ☐

In particular, by theorem 2 it follows that e'tale schemes over S_0 can always be lifted to S. We can immediately deduce that a smooth scheme over S_0 can locally be lifted. As we'll see in the next corollary, we can use the equivalence of categories to prove that all of these local lifts are compatible up to isomorphism.

Corollary 1. *Let* R *be a ring and* $I \subset R$ *an ideal with* $I^2 = 0$*. Let* X' *be a smooth scheme over* $\operatorname{Spec}(R/I)$*. Then for each* $x \in X'$ *there exists an open* $x \in U' \subset X'$ *and a smooth scheme* U *over* $\operatorname{Spec}(R)$ *such that* $U \times_R R/I \cong U'$*.*

Furthermore, if $x \in U' \cap V'$ *and* V *and* U *are as above, then for every open* $W' \subset U' \cap V'$ *we have an isomorphism*

$$\begin{array}{ccc} U|_{W'} & \xrightarrow{\cong} & V|_{W'} \\ & \nwarrow \quad \nearrow & \\ & W' & \end{array}$$

Before proceeding to the proof, note that since $I^2 = 0$, U and $U \times_R R/I$ have the same topological space so $U|_{W'}$ is well defined.

Proof. Since X' is locally e'tale, the first part of the theorem follows immediately from theorem 2.

For the second part, assume that $U \times_R R/I \xrightarrow{\phi'} U'$ and $V \times_R R/I \xrightarrow{\psi'} V'$ are the isomorphisms given to us by the first part. By restricting to W' we get isomorphisms $U|_{W'} \times_R R/I \xrightarrow{\phi'} W'$ and $V|_{W'} \times_R R/I \xrightarrow{\psi'} W'$. Together this gives us an isomorphism $U|_{W'} \times_R R/I \xrightarrow{\phi'\psi'^{-1}} V|_{W'} \times_R R/I$. By theorem 2 we can extend this to an isomorphism from U to V. □

We'll now set the notation for the rest of the section.

Let $0 \to I \to R \to R' \to 0$ be a short exact sequence with R and R' in $\mathbf{Art}_k$ such that $I \cdot m_R = 0$. For any $\operatorname{Spec}(k)$-scheme Y, let Θ_Y be the sheaf of germs of k-derivations from $\mathcal{O}_Y$ to itself. Also, for each $\operatorname{Spec}(R)$-scheme Y, let $\operatorname{Aut}_R(Y, R')$ denote the $\operatorname{Spec}(R)$ automorphisms of Y which induce the identity on $Y \times_R R'$.

The following lemma will give us an extremely useful connection between $\Theta_{Z \times_R k}$ and $Aut_R(Z, R')$ where Z is a $\operatorname{Spec}(R)$-scheme. The general idea is that given an automorphism ϕ of Z which restricts to the identity on $Z \times_R R'$, measuring the amount that ϕ moves each point gives us a derivation on $\mathcal{O}_{Z \times_R k}$. Roughly speaking, the derivation that we obtain is the vector field associated to ϕ. This coincides with our more basic intuition for the relation between a function and it's associated derivative. The main technical difficulty lies in showing exactly what space the derivation should act on in order for the converse to hold. I.e, given a vector field we want the induced automorphism to be an element of $Aut_R(Z, R')$.

Lemma 1. *Suppose that* Z *is a flat finite type* $\operatorname{Spec}(R)$ *scheme. Let* $Z' = Z \times_R R'$ *and* $Z_0 = Z \times_R k$*. Then we have a canonical isomorphism:*

$$\Gamma(Z_0, \Theta_{Z_0}) \otimes I \to \operatorname{Aut}_R(Z, R')$$

where composition on the right corresponds to addition on the left.

Proof. We have the following fibered diagram:

$$\begin{array}{ccccc} Z_0 & \longrightarrow & Z' & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec}(k) & \longrightarrow & \operatorname{Spec}(R') & \longrightarrow & \operatorname{Spec}(R) \end{array}$$

We'll construct the isomorphism locally. Assume that $Z = \operatorname{Spec}(A)$. By flatness, $A \otimes_R I \cong IA$ and $A \otimes_R R/I \cong A/IA$. Therefore,

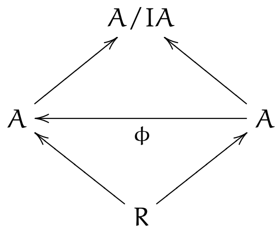
$$A/m_R A \cong A \otimes_R R/m_R \cong A \otimes_R k$$

and

$$IA \cong (A \otimes_R k) \otimes_k I$$

So it will be enough to show an equivalence between $\mathrm{Aut}_R(A, R')$ and k-derivations $A/m_R A \xrightarrow{D} IA$

Let $\phi \in \mathrm{Aut}_R(Z, R')$. This means that we have the following commutative diagram:



We define a map $A \xrightarrow{D} A$ by $D = \phi - \mathrm{id}$. We claim that D is a derivation and that $D(A) \subset IA$. Indeed, from the commutativity of the proceeding diagram we see that $D(A) \subset IA$. Furthermore,

$$\phi(ab) = \phi(a)\phi(b) = (a + D(a))(b + D(b)) =$$

$$ab + aDb + bDa + DaDb = ab + aDb + bDa$$

which implies that $D(ab) = \phi(ab) - ab = aDb + bDa$. It's clear that D is additive and that $D(k) = 0$ so D is a k derivation from A to IA.

Now, since $m_R I = 0$, this induces a map

$$A/m_R A \xrightarrow{D} IA$$

Similarly, given a derivation $A/m_R A \xrightarrow{D} IA$ we obtain a derivation $A \xrightarrow{D} IA$ which can then be used to define an automorphism

$$A \to A$$

$$a \mapsto a + Da$$

Indeed, since $D(A) \subset IA$, $\mathrm{id} + D$ induces the identity on A/IA and by flatness [4, III 4.2] this implies that $\mathrm{id} + D$ is an automorphism.

□

At first glance, the above lemma seems like an elaborate exercise in unwinding definitions. The next few glances tell a different story. One important consequence is that given an open cover of a scheme Z, it provides us with a convenient way to talk about a family of automorphisms of the open subschemes. For example, as we'll see in the proof of the following proposition, properties such as the family satisfying the cocycle condition naturally correspond to characteristics of the homology groups of Θ_{Z_0}. And as usual, the ability to translate the verification of a property to the existence of an element in a certain structure is extremely useful. For instance, in well behaved cases the structure may be trivial.

In this vein, we can use lemma 1 to give a succinct answer to the general question of when we can deform a scheme. As we saw in corollary 1, for certain extensions we can always deform our scheme on a local scale. Given a collection of local deformations, we'd like to produce a global deformation by gluing together the local ones. One way to go about this is to apply a certain automorphism to each one of the local deformations and hope that the new deformations will satisfy the cocycle condition and hence glue together. As we'll see, the existence of such a family of automorphisms is equivalent to a certain element of $H^2(X_0, \Theta_{X_0}) \otimes_k I$ being equal to 0 so by looking at this element we can tell if a global deformation exists. As in the lemma, the general idea is straight forward and the devil is in the details.

Proposition 1. *Let* X *be a smooth* $\mathrm{Spec}(R')$*-scheme. Let* $X_0 = X \times_{R'} k$*. Then:*

1. *There exists a canonical element* $\mathfrak{o} \in H^2(X_0, \Theta_{X_0}) \otimes_k I$ *such that* $\mathfrak{o} = 0$ *iff* $\mathrm{Def}_k(X', R) \neq \emptyset$

2. *If* $\mathfrak{o} = 0$ *then given any element* $(X, \phi') \in \mathrm{Def}_k(X', R)$ *we have a bijection*

$$\iota_X : H^1(X_0, \Theta_{X_0}) \otimes_k I \to \mathrm{Def}_k(X', R).$$

Proof. By corollary 1 there exists a cover $\{U'_\alpha\}$ of X' such that each map $U'_\alpha \to \mathrm{Spec}(R')$ can be lifted to a smooth affine Z_α scheme over $\mathrm{Spec}(R)$ with an automorphism $Z_\alpha \times_R R' \xrightarrow{\phi'_\alpha} U'_\alpha$.

Let $U'_{\alpha\beta} = U'_\alpha \cap U'_\beta$ and similarly for more indices. Since X' is separated, each of these open sets are affine.

On each open set $U'_{\alpha\beta}$ we have an isomorphism

$$\phi'^{-1}_\beta \phi'_\alpha : Z_\alpha \times_R R' \to Z_\beta \times_R R'$$

By theorem 2 we can extend this to an automorphism $\zeta_{\beta\alpha} : Z_\alpha|_{U'_{\alpha\beta}} \to Z_\beta|_{U'_{\alpha\beta}}$. Note that since Z_α and $Z_\alpha \times_R R'$ have the same topological space, $Z_\alpha|_{U'_{\alpha\beta}}$ is just a pullback of Z_α along an open embedding.

Similarly, we define

$$\zeta^\gamma_{\beta\alpha} = \zeta_{\beta\alpha}|_{U'_{\alpha\beta\gamma}} : Z_\alpha|_{U'_{\alpha\beta\gamma}} \to Z_\beta|_{U'_{\alpha\beta\gamma}}$$

We thus obtain an automorphism

$$c_{\alpha\beta\gamma} = \left(\zeta^\beta_{\gamma\alpha}\right)^{-1} \zeta^\alpha_{\gamma\beta}\zeta^\gamma_{\beta\alpha} \in \mathrm{Aut}_R(Z_\alpha|_{U'_{\alpha\beta\gamma}})$$

By the definition of $\zeta^\gamma_{\alpha\beta}$,

$$c_{\alpha\beta\gamma} \times_R R' = \phi'^{-1}_\alpha \phi'^{-1}_\gamma \phi'_\gamma \phi'_\beta \phi'^{-1}_\beta \phi'_\alpha = 1$$

which means that $c_{\alpha\beta\gamma} \in \mathrm{Aut}_R(Z_\alpha|_{U'_{\alpha\beta\gamma}}, R')$.

At this point, we could use lemma 1 to produce a collection of elements of $\Gamma((U_{\alpha\beta\gamma})_0, \Theta_{X_0}) \otimes I$ and thus obtain a cochain in $\Gamma(X_0, \Theta_{X_0}) \otimes I$. However, since it will be easier and more intuitive for us to work with elements of $\mathrm{Aut}_R(Z_\alpha|_{U'_{\alpha\beta\gamma}}, R')$, we'll now show that we can talk about $\Gamma((U'_{\alpha\beta\gamma})_0, \Theta_{X_0}) \otimes I$ in terms of elements in $\mathrm{Aut}_R(Z_\alpha|_{U'_{\alpha\beta\gamma}}, R')$.

We first note that by lemma 1, we have a canonical isomorphism

$$\mathrm{Aut}_R(Z_\alpha|_{U'_{\alpha\beta\gamma}}, R') \to \Gamma((Z_\alpha|_{U'_{\alpha\beta\gamma}})_0, \Theta_{Z_0}) \otimes I$$

where composition on the left corresponds to addition on the right. A happy consequence is that $\mathrm{Aut}_R(Z_\alpha|_{U'_{\alpha\beta\gamma}}, R')$ is an abelian group. Furthermore, by using the isomorphisms $(\phi_\alpha)_0 = \phi'_\alpha \times_{R'} k$ we can construct an isomorphism of differentials

$$\Gamma(U_{\alpha\beta\gamma})_0, \Theta_{X_0}) \xrightarrow{f_{\alpha\beta\gamma}} \Gamma((Z_\alpha|_{U'_{\alpha\beta\gamma}})_0, \Theta_{Z_0})$$

$$D \mapsto (\phi_\alpha)_0 D \left((\phi_\alpha)_0\right)^{-1}$$

Together, we obtain an isomorphism of abelian groups

$$\mathrm{Aut}_R(Z_\alpha|_{U'_{\alpha\beta\gamma}}, R') \to \Gamma((Z_\alpha|_{U'_{\alpha\beta\gamma}})_0, \Theta_{Z_0}) \otimes I \xrightarrow{f_{\alpha\beta\gamma}^{-1}} \Gamma(U_{\alpha\beta\gamma})_0, \Theta_{X_0}) \otimes I$$

Using this isomorphism, we can describe elements of $\Gamma(U_{\alpha\beta\gamma})_0, \Theta_{X_0}) \otimes I$ using elements of $\mathrm{Aut}_R(Z_\alpha|_{U'_{\alpha\beta\gamma}}, R')$. It will also be useful to know how elements of $\mathrm{Aut}_R(Z_\beta|_{U'_{\alpha\beta\gamma}}, R')$ correspond to elements of $\Gamma(U_{\alpha\beta\gamma})_0, \Theta_{X_0}) \otimes I$. The following commutative diagram shows us how it's done:

$$
\begin{array}{ccc}
\mathrm{Aut}_R(Z_\alpha|_{U'_{\alpha\beta\gamma}}, R') & \longrightarrow & \Gamma((Z_\alpha|_{U'_{\alpha\beta\gamma}})_0, \Theta_{Z_0}) \otimes I \\
\uparrow & & \downarrow (f_{\alpha\beta\gamma})^{-1} \\
b \mapsto (\zeta_{\beta\alpha})^{-1} b \zeta_{\beta\alpha} & & \Gamma(U_{\alpha\beta\gamma})_0, \Theta_{X_0}) \otimes I \\
 & & \uparrow (f_{\beta\alpha\gamma})^{-1} \\
\mathrm{Aut}_R(Z_\beta|_{U'_{\alpha\beta\gamma}}, R') & \longrightarrow & \Gamma((Z_\beta|_{U'_{\alpha\beta\gamma}})_0, \Theta_{Z_0}) \otimes I
\end{array}
$$

To check that this is commutative, let $b \in \mathrm{Aut}_R(Z_\beta|_{U'_{\alpha\beta\gamma}}, R')$. Going right and then left we get

$$b \mapsto \mathrm{id} - b \mapsto \phi'_\beta(\mathrm{id} - b)(\phi'_\beta)^{-1} = \phi'_\beta b(\phi'_\beta)^{-1}$$

Going up, right and then down we get

$$b \mapsto (\zeta_{\beta\alpha})^{-1} b\zeta_{\beta\alpha} \mapsto \mathrm{id} - (\zeta_{\beta\alpha})^{-1} b\zeta_{\beta\alpha} \mapsto \phi_\alpha(\phi_\alpha)^{-1}\phi_\beta b\phi_\beta(\phi_\alpha)^{-1}\phi_\alpha = \phi'_\beta b(\phi'_\beta)^{-1}$$

We'll now show that the collection $\{c_{\alpha\beta\gamma}\}$ is a cocycle and hence represents an element of $H^2(X_0, \Theta_{X_0})$ under the isomorphism given above. By the definition of the boundary,

$$(\partial\{c_{\alpha\beta\gamma}\})_{\alpha\beta\gamma\delta} = c_{\beta\gamma\delta}{c_{\alpha\gamma\delta}}^{-1} c_{\alpha\beta\delta}{c_{\alpha\beta\gamma}}^{-1} \tag{1}$$

We'll show the restriction of this element to $\mathrm{Aut}_R(Z_\alpha|_{U'_{\alpha\beta\gamma}}, R')$ is the identity. As we showed with the large commutative diagram above, the element $c_{\beta\gamma\delta} \in \mathrm{Aut}_R(Z_\beta|_{U'_{\alpha\beta\gamma}}, R')$ maps to the same element in $\Gamma(U_{\alpha\beta\gamma})_0, \Theta_{X_0}) \otimes I$ as $(\zeta_{\beta\alpha})^{-1} c_{\beta\delta\gamma}\zeta_{\beta\alpha} \in \mathrm{Aut}_R(Z_\alpha|_{U'_{\alpha\beta\gamma}}, R')$

Therefore, as elements in the abelian group $\mathrm{Aut}_R(Z_\alpha|_{U'_{\alpha\beta\gamma}}, R')$ we have

$$(\partial\{c_{\alpha\beta\gamma}\})_{\alpha\beta\gamma\delta} = (c_{\alpha\gamma\delta})^{-1} c_{\alpha\beta\delta} c_{\beta\gamma\delta}(c_{\alpha\gamma\delta})^{-1} =$$

$$((\zeta^\delta_{\gamma\alpha})^{-1}(\zeta^\alpha_{\delta\gamma})^{-1}(\zeta^\gamma_{\delta\alpha}))((\zeta^\beta_{\delta\alpha})^{-1}(\zeta^\alpha_{\delta\beta})(\zeta^\delta_{\delta\alpha}))$$

$$((\zeta_{\beta\alpha})^{-1}(\zeta^\gamma_{\delta\beta})^{-1}(\zeta^\beta_{\delta\gamma})(\zeta^\delta_{\gamma\beta})(\zeta_{\beta\alpha}))((\zeta^\gamma_{\beta\alpha})^{-1}(\zeta^\alpha_{\gamma\beta})^{-1}(\zeta^\beta_{\gamma\alpha})) = 1$$

The next thing to show is that the homology class $\{c_{\alpha\beta\delta}\}$ does not depend on the choice of isomorphisms $\zeta^\gamma_{\beta\alpha}$.

Suppose $\theta_{\beta\alpha} : Z_\alpha|_{U'_{\alpha\beta}} \to Z_\beta|_{U'_{\alpha\beta}}$ were another choice of isomorphism extending ${\phi'_\beta}^{-1}\phi'_\alpha$. Since $\theta_{\beta\alpha}$ and $\zeta_{\beta\alpha}$ agree on the pullback to Spec(R'), $\epsilon_{\beta\alpha} = \theta_{\beta\alpha}\left(\zeta_{\beta\alpha}\right)^{-1} \in \mathrm{Aut}_R(Z_\alpha|_{U'_{\alpha\beta}}, R')$. Now, if we use the new isomorphisms to generate the homology class, the elements corresponding to $c_{\alpha\beta\gamma}$ are

$$c'_{\alpha\beta\gamma} = (\theta^\beta_{\gamma\alpha})^{-1}\theta^\alpha_{\gamma\beta}\theta^\gamma_{\beta\alpha}$$

So

$$\{c'_{\alpha\beta\gamma}\} = \{c_{\alpha\beta\gamma}\}\partial(\{\epsilon_{\beta\alpha}\})$$

This shows that the holmlogy class of $\{c_{\alpha\beta\gamma}\}$ that we defined does not depend on the choice of isomorphisms $\zeta_{\alpha\beta}$. Furthermore, we see that $[\{c_{\alpha\beta\gamma}\}] = 1$ iff there exists some choice of isomorphisms $\theta_{\alpha\beta}$ as above such that the associated cocycle satisfies $\{c'_{\alpha\beta\gamma}\} = 1$. But this in turn is equivalent to

$$c'_{\alpha\beta\gamma} = (\theta^\beta_{\gamma\alpha})^{-1}\theta^\alpha_{\gamma\beta}\theta^\gamma_{\beta\alpha} = 1$$

which means that the isomorphisms $\theta_{\alpha\beta}$ satisfy the cocycle condition. This implies that we can use them to glue together the Z_α and obtain an element of $\mathrm{Def}_k(X', R)$ which concludes the first part of the proposition.

For the second part, suppose that we have an element $(X, \phi') \in \mathrm{Def}_k(X', R)$ and define $X_\alpha = X|_{U'_\alpha}$.

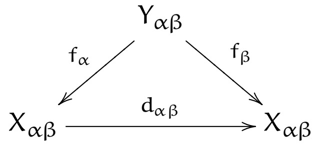
Let $\{d_{\alpha\beta}\}$ be elements of $\mathrm{Aut}_R(X|_{U'_{\alpha\beta}}, R')$ which satisfy the cocycle condition. Then these elements can be used to glue together the schemes X_α in a different way and thus obtain an isomorphic scheme X^d together with an isomorphism $X^d \times_R R' \xrightarrow{\psi'} U'$. In fact, $\psi'_\alpha = \phi'_\alpha$ since $d_{\alpha\beta}$ is trivial on $X_\alpha \times_R R'$.

We'll first show that the map $\{d_{\alpha\beta}\} \mapsto X^d$ is an injection. Assume that we have an isomorphism of deformations $X^d \xrightarrow{f} X$. Let f_α be the restriction to $X^d|_{U'_\alpha}$. Since $d_{\alpha\beta} \times_R R'$ is the identity by definition and $f_\alpha(f_\beta)^{-1} \times_R R'$ is the identity by the definition of a map of deformation, by theorem 2 we get that $d_{\alpha\beta} = f_\alpha(f_\beta)^{-1}$. This means that $\{d_{\alpha\beta}\} = \partial(\{f_\alpha\})$.

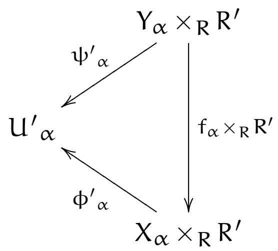
We'll now show that the map $\{d_{\alpha\beta}\} \mapsto X^d$ is surjective. Let $(Y, \psi') \in \mathrm{Def}_k(X', R)$. By theorem 2, the maps

$$f'_\alpha = (\phi'_\alpha)^{-1}\psi'_\alpha : Y_\alpha \times_R R' \to X_\alpha \times_R R'$$

extend to maps $f_\alpha : Y_\alpha \to X_\alpha$. We use these to define elements $d_{\alpha\beta} = f_\beta(f_\alpha)^{-1} \in \mathrm{Aut}_R(X|_{U'_{\alpha\beta}}, R')$. Finally, since we have



and



the maps f_α glue together to give us an isomorphism of deformations $Y \xrightarrow{f} X^d$. □

3 The Schlessinger Criterion

In proposition 1 we saw that the set of deformations of a scheme X' was in a bijection with the elements of a certain homology group. However, in order to get a better handle on the internal structure of the set of deformations and to understand them more clearly, we would like the deformation functor $\mathrm{Def}_k(X', R)$ to be representable in the following sense.

Definition 3. *A covariant functor* $F : \mathbf{Art}_A \to \mathbf{Set}$ *is called* pro-representable *if it is representable by an element of* $\widehat{\mathbf{Art}_A}$.

Schlessinger's criteria gives us necessary and conditions for such a functor to be pro-representable and in certain cases even gives us the element of $\widehat{\mathbf{Art}_A}$. Before stating Schlessinger's criteria we will need a few more definitions.

Definition 4. *Let* $\mathbf{C}$ *be a category with final object* $\emptyset$ *and fibered products. A covariant functor* $F : \mathbf{C} \to \mathbf{Set}$ *is called* left-exact *if*

1. $F(\emptyset) = \{pt\}$

2. *the canonical map* $F(X \times_Y Z) \to F(X) \times_{F(Y)} F(Z)$ *is a bijection.*

Note that if $F : \mathbf{Art}_k \to \mathbf{Set}$ is left exact then $F(k[\epsilon])$ has the structure of a vector space over k.

Definition 5. *A functor* $F : \mathbf{Art}_k \to \mathbf{Set}$ *is called* formally smooth *if for every surjection* $R \xrightarrow{\pi} R'$ *in* $\mathbf{Art}_k$, $F(R) \xrightarrow{F\pi} F(R')$ *is surjective.*

Definition 6. *A surjection* $0 \to I \to R \xrightarrow{\pi} R' \to 0$ *in* $\mathbf{Art}_k$ *is called* small *if* $Im_R = 0$.

Theorem 3. *(Schlessinger) A covariant functor* $F : \mathbf{Art}_k \to \mathbf{Set}$ *is pro-representable iff* F *is left exact and* $dim_k(F(k[\epsilon])) < \infty$.

Also, it suffices to check left exactness for products of the form:

$$\begin{array}{ccc} & & S \\ & & \downarrow \\ R' & \xrightarrow[\textit{small}]{} & R \end{array}$$

where the bottom map is a small surjection.

Finally, if F *is formally smooth and* $dim_k(F(k[\epsilon])) = n$ *then* F *is pro-representable by* $k[[t_1, \ldots, t_n]]$.

The proof of this theorem is rather hard, so instead of proving it, in the next section we'll use the criteria to prove a somewhat intuitive but deep result about abelian schemes.

4 Deformations of Abelian Schemes

In this section we'll use the material from the previous two sections in order to study deformations abelian schemes.

Intuitively, an n-dimensional abelian variety has n^2 directions in which we can deform it.

In order to make this more concrete, let's take a look at our favorite abelian variety - the n-dimensional complex torus. Since a torus is determined by a lattice in $\mathbb{C}^n$, this explains the source of the n^2 directions mentioned above.

More formally, let S be the set of matrices $s_{ij} \in M(n, \mathbb{C})$ whose imaginary part has a non zero determinant and define G to be the group of automorphisms of $\mathbb{C}^n \times S$ generated by the automorphisms δ_i and ϕ_j for $1 \leq i, j \leq n$ where

$$\delta_i(\overline{x}, s) = (\overline{x} + e_i, s)$$

$$\phi_i(\overline{x}, s) = (\overline{x} + [s]_{\downarrow j}, s)$$

and $[s]_{\downarrow j}$ is the j-th column of s. In other words, for each matrix in S we have a collection of translations of $\mathbb{C}$ corresponding to the lattice spanned (over $\mathbb{R}$) by the columns of S and the elementary basis vectors.

In addition, define $\mathcal{B} = (\mathbb{C} \times S)/G$. Then the n-dimensional complex tori are exactly the fibers of the projection $\pi : \mathcal{B} \to S$ since each fiber represents the quotient by the lattice corresponding to the columns of $s \in S$.

Now let's consider some element $s_0 \in S$ and the corresponding torus $T = \pi^{-1}(s_0)$. Given any path γ on the manifold S centered at s_0, by considering the fibers $\pi^{-1} \circ \gamma$ on each point of the path we obtain a family of tori which is a deformation of the torus T. Thus, to each tangent vector of S at s_0 we can associate a deformation of T.

Interestingly, in the paper [2], Kodira and Spencer show that *all* deformations of T can be obtained in this fashion. This shows that the tangent space of s_0 is a concrete realization of the mysterious n^2 directions mentioned above.

In a sense, the aforementioned result of Kodira and Spencer is a special case of the main theorem that we wish to prove regarding deformations of arbitrary abelian schemes.

We start by setting notation and recalling some basic results relating to abelian schemes.

Definition 7. *An* abelian scheme *over a scheme* S *is a group* S*-scheme* X *which is smooth, proper and has geometrically connected fibers. If* $S = \mathrm{Spec}(k)$ *then we will call* X *an abelian variety over* k*.*

For the remainder of this section, X_0 will be an abelian variety over k.

Definition 8. $M_{X_0} : \mathbf{Art}_k \to \mathbf{Set}$ *is the covariant functor sending a ring* R *to the isomorphism classes of pairs* (X, ϕ_0) *of the form*

$$\begin{array}{ccccc} X_0 & \xleftarrow[\sim]{\phi_0} & X \times_R k & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow \\ & & \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(R) \end{array}$$

where X *is an abelian scheme over* $\mathrm{Spec}(R)$*. We call* M_{X_0} *the* deformation functor of abelian schemes*.*

A morphism of deformations of abelian schemes is defined in the same way as it is for ordinary deformations.

We are now ready to state our main theorem for this section.

Theorem 4. *The functor* M_{X_0} *is pro-representable by* $k[[t_{11}, \dots, t_{gg}]]$ *where* $g = \mathit{dim}(X_0)$*.*

Note that this makes sense in light of our discussion above since we can think of $k[[t_{11}, \dots, t_{gg}]]$ as space of tangent vectors of dimension g^2.

Before stating the abelian scheme results that we'll use, we'll look at an example showing that non abelian group schemes do not always have deformations.

Example 1. We start by constructing a non-abelian group scheme of rank p^2 for some prime p.

Let $k = \mathbb{F}_p$ and let B be a k-algebra. We define

$$N_0(B) = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \middle| \alpha \in B,\ \beta \in B\ \alpha^p = 1\ \beta^p = 0 \right\}$$

Now, N_0 is a group $\mathrm{Spec}(k)$-scheme since $N_0 = \mathrm{Spec}(E_0)$ where $E_0 = k[\tau, \rho]$, $\tau^p = 1$ and $\rho^p = 0$ with multiplication defined by $m_0(\tau) = \tau \otimes \tau$ and $m_0(\rho) = \rho \otimes 1 + \tau \otimes \rho$.

We'll now note that $N_0(B)$ cannot be deformed to any integral domain R of characteristic 0. Suppose for contradiction that $N = \mathrm{Spec}(E)$ is a smooth group $\mathrm{Spec}(R)$ scheme where E is a free R-module of rank p^2. Define $L = \overline{\mathrm{Frac}(R)}$. It is possible (but surprisingly difficult) to show that $N \otimes_R L$ is a reduced scheme. Since L is algebraically closed, we get that $N \otimes L \to \mathrm{Spec}(L)$ is a finite $\mathrm{Spec}(L)$-scheme so $N \otimes L$ is a finite group of order p^2 which implies that it's commutative. But we can use this to show that N is commutative as well which is a contradiction.

Lemma 2. *([3, cor 6.2]) Suppose we have the following diagram*

$$\begin{array}{ccc} X & \overset{f}{\underset{g}{\rightrightarrows}} & G \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & S & \end{array}$$

where G *is a group* S*-scheme,* S *is connected,* p *is flat and* $\Gamma(X_s, \mathcal{O}_{X_s}) \cong \kappa(s)$ *for all* $s \in S$*.*

Then, if for some $s \in S$ *we know that* $f_s = g_s$ *then there is a section* $S \xrightarrow{\eta} G$ *such that* $f(x) = (\eta p)(x) \cdot g(x)$

Corollary 2. *Let* X *be an abelian scheme over* $\mathrm{Spec}(R)$*. Let* $X \xrightarrow{f} X$ *be a map of* $\mathrm{Spec}(R)$*-schemes such that* $f \times_R k = \mathit{id}$*. Then* $f = \mathit{id}$*.*

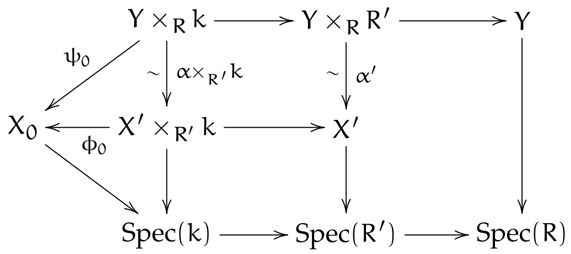
Corollary 3. *([3, cor 6.4]) Let* X *be an abelian* S*-scheme and* G *a group* S*-scheme. If* $X \xrightarrow{f} G$ *is a map of* S*-schemes and sends id to id then* f *is a group homomorphism.*

The next proposition shows us how to transfer results about Def_k to M_{X_0}.

Proposition 2. *Let* $0 \to I \to R \to R' \to 0$ *be a small surjection in* $\mathbf{Art}_k$ *and let* (X', ϕ_0) *be some deformation in* $M_{X_0}(R')$. *Then, forgetting the group structure, we have a bijection*

$$M_{X_0}(R) \supset (M_{X_0}(\pi))^{-1}(X', \phi_0) \xrightarrow{\kappa} \mathrm{Def}_k(X', R)$$

In other words, if we start with an abelian scheme X' over $\mathrm{Spec}(R')$, then we have a bijection between deformations of abelian schemes from X_0 to $\mathrm{Spec}(R)$ which restrict to X' over $\mathrm{Spec}(R')$ and deformations of ordinary schemes from X' up to $\mathrm{Spec}(R)$. The following diagram may be helpful in keeping track of what is going on. In the diagram, (Y, ψ_0) is an element of $M_{X_0}(R)$ such that $M_{X_0}(\pi)(Y, \psi_0) = (x, \phi_0)$.



Where $\alpha' : Y' \to X'$ is an isomorphism of deformations of abelian schemes.

Proof. Let (Y, ψ_0) be an element of $M_{X_0}(R)$ such that $M_{X_0}(\pi)(Y, \psi_0) = (x, \phi_0)$ as in the diagram above. Let α' be the map in the diagram as well. We define

$$\kappa(Y, \psi_0) = (Y \times_R R', \alpha') \in \mathrm{Def}_k(X', R)$$

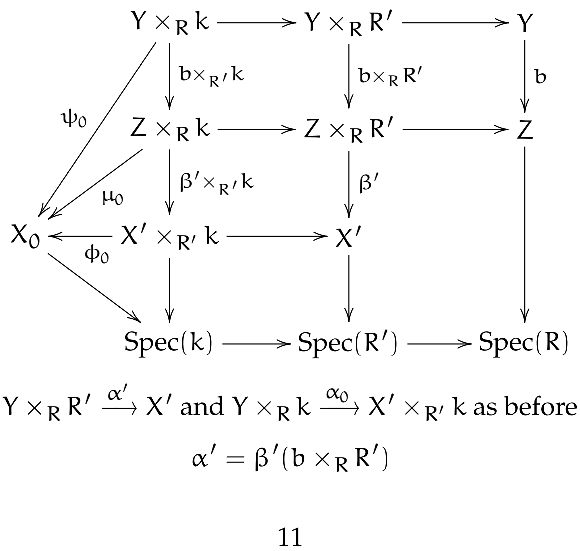
We'll first show that κ is well defined. Suppose that $Y \times_R R' \xrightarrow{\mu'} X'$ is a different map of deformations of X_0. Define a map $a = \alpha' \mu'^{-1} : X' \to X'$. Since $\alpha' \times_{R'} k$ agrees with $\mu' \times_{R'} k$, $a \times_{R'} k = \mathrm{id}$. Therefore, by lemma 2, $a = \mathrm{id}$. This shows that κ is well defined.

The next step is to show that κ is injective. Suppose that

$$(Y, \psi') = \kappa(Y, \psi_0) = \kappa(Z, \mu_0) = (Z, \mu')$$

Let α' be the map we introduced earlier in order to define $\kappa(Y, \psi_0)$ and let β' be the corresponding map for Z.

Equality as elements of $\mathrm{Def}_k(X', R)$ implies that we have some isomorphism of $\mathrm{Spec}(R)$-schemes $Y \xrightarrow{b} Z$ such that $(\mu')b = \psi'$. We want to use b to produce an isomorphism of deformations of abelian schemes. By the definition of κ we have the following commutative diagram



together with maps $Y \times_R R' \xrightarrow{\alpha'} X'$ and $Y \times_R k \xrightarrow{\alpha_0} X' \times_{R'} k$ as before and

$$\alpha' = \beta'(b \times_R R')$$

$$\alpha' \times_{R'} k = (b \times_{R'} k)(\beta' \times_{R'} k)$$

Now, let $Y \xrightarrow{f_Y} \mathrm{Spec}(R)$ be the structure map and $\mathrm{Spec}(R) \xrightarrow{\epsilon_Y} Y$ the identity morphism. In order for $b \times_R R'$ to be a group isomorphism, we fix it by defining

$$h = b - b\epsilon_Y f_Y$$

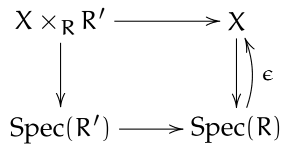
Clearly, $h\epsilon_Y$ is the identity. By corollary 3, this implies that $h : Y \to Z$ is a homomorphism of abelian Spec(R)-schemes. In addition, since $\alpha' \times_{R'} k$ and $\beta' \times_{R'} k$ are both isomorphisms of abelian varieties,

$$\alpha' \times_{R'} k = (b \times_{R'} k)(\beta' \times_{R'} k) \Rightarrow \alpha' \times_{R'} k = (h \times_{R'} k)(\beta' \times_{R'} k)$$

which means that h is indeed a morphism of deformations of abelian schemes. By applying the same procedure to b^{-1}, we get a morphism g from Z to Y such that $g \times_R k$ is the inverse of $h \times_R k$. This implies that g is the inverse of h and that h is an isomorphism of deformations of abelian schemes and that κ is injective.

The last step is to show that κ is surjective. Let (Y, ψ') be an element of $\mathrm{Def}_k(X', R)$. We need to show that $(Y, \psi') \in \kappa(M_{X_0}(R))$. For this, it will be enough to show that Y is an abelian Spec(R)-scheme. Since $Y \times_R R' \xrightarrow{\psi'} X'$ is an isomorphism by definition, the following lemma finishes the proof. □

Lemma 3. *([3, prop 6.15]) Let* X *be a smooth and proper* Spec(R)*-scheme and let* $\mathrm{Spec}(R) \xrightarrow{\epsilon} X$ *be a section.*



Assume $X \times_R R'$ *is an abelian* Spec(R')*-scheme with* $\epsilon \times_R R'$ *as the identity. Then* X *is an abelian* Spec(R)*-scheme with* ϵ *as the identity.*

The proof of this theorem in GIT is quite nice and is an interesting application of proposition 1. The main idea is to take the multiplication map on $X \times_R R'$ and show that the obstruction element in proposition 1 vanishes. This shows that the multiplication can be extended to X and functoriality of the deformation functor tells us that this is indeed multiplication.

However, since this document is already much longer than I intended, I'll refer you to GIT [3] for the details.

We are now in the position to prove theorem 4.

Proof. Our primary tool for this proof will of course be to apply Schlessinger's criteria (theorem 3) to the deformation of abelian schemes functor M_{X_0}. We'll check the conditions of the criteria one by one.

We plunge in head first with a proof of left-exactness. Let's recall exactly what it is that we have to prove. Consider the following commutative diagram in **Art**$_k$ where the bottom square is fibered, the columns are exact and π is a small surjection.

$$\begin{CD}
0 @. 0 \\
@VVV @VVV \\
J @. I \\
@VVV @VVV \\
Q @>\chi>> R \\
@V{\rho}VV @VV{\pi}V \\
T @>\mu>> R' \\
@VVV @VVV \\
0 @. 0
\end{CD} \tag{2}$$

We have to prove that the natural map

$$M_{X_0}(Q) \xrightarrow{\sim} M_{X_0}(T \times_{R'} R) \xrightarrow{\omega} M_{X_0}(T) \times_{M_{X_0}(R')} M_{X_0}(R)$$

is a bijection.

Note that since the bottom square is fibered, $\chi(J) \subset I$ and in fact $J \xrightarrow{\chi} I$ is an isomorphism. This follows from the uniqueness of the map $J \to Q$. Furthermore, we get that ρ is a small surjection as well.

Using χ we obtain a natural isomorphism

$$H^2(X_0, \Theta_{X_0}) \otimes J \xrightarrow{\mathrm{id}\otimes\chi} H^2(X_0, \Theta_{X_0}) \otimes I$$

Let $(Y, \psi_0) \in M_{X_0}(T)$ and $M_{X_0}(\mu)(Y, \psi_0) \in M_{X_0}(R')$. Furthermore, let $\mathfrak{o}(Y)$ and $\mathfrak{o}(M_{X_0}(\mu)(Y, \psi_0))$ be the deformation obstruction elements given in proposition 1.

We claim that $(\mathrm{id} \otimes \chi)(\mathfrak{o}(Y)) = \mathfrak{o}(M_{X_0}(\mu)(Y, \psi_0))$. Recall that in proposition 1 we defined the obstruction elements locally as elements in $\mathrm{Aut}_R(Z_\alpha, R')$ where U'_α was a local cover of the $\mathrm{Spec}(R')$-scheme X' and $Z_\alpha = \mathrm{Spec}(A_\alpha)$ was a lift of $X|_{U'_\alpha}$ to an affine $\mathrm{Spec}(R)$-scheme.

In our case, X' corresponds to the $\mathrm{Spec}(T)$-scheme Y. Let $\{U'_\alpha\}$ be an collection of open subschemes of Y with deformations to affine $\mathrm{Spec}(Q)$-schemes $\{Z_\alpha = \mathrm{Spec}(A_\alpha)\}$ as in proposition 1.

$$\begin{CD}
U'_\alpha @>>> \mathrm{Spec}(A_\alpha) \\
@VVV @VVV \\
\mathrm{Spec}(T) @>>> \mathrm{Spec}(Q)
\end{CD}$$

By base changing by $\mathrm{Spec}(R) \to \mathrm{Spec}(Q)$ we get

$$\begin{CD}
U'_\alpha \times_{\mathrm{Spec}(T)} \mathrm{Spec}(R') @>>> \mathrm{Spec}(A_\alpha \otimes_Q R) \\
@VVV @VVV \\
\mathrm{Spec}(R') @>>> \mathrm{Spec}(R)
\end{CD}$$

In addition, recall that locally we identify $\Gamma(X_0, \Theta_{X_0}) \otimes J$ with k-derivations $A_\alpha \otimes_Q Q/m_Q \xrightarrow{D} A_\alpha \otimes_Q J$ and the canonical map $\mathrm{Aut}_Q(Z_\alpha, T) \to \Gamma(X_0, \Theta_{X_0}) \otimes J$ has the form $\phi \mapsto D = \mathrm{id} - \phi$. Similarly, we identify $\Gamma(X_0, \Theta_{X_0}) \otimes I$ with k-derivations $(A_\alpha \otimes_Q R) \otimes_R R/m_R \xrightarrow{D} (A_\alpha \otimes_Q R) \otimes_R I$

and the canonical map $\mathrm{Aut}_R(Z_\alpha \times_{\mathrm{Spec}(Q)} \mathrm{Spec}(R), R') \to \Gamma(X_0, \Theta_{X_0}) \otimes I$ has the form $\phi \mapsto D = \mathrm{id} - \phi$.

Together, we obtain the following commutative diagram where the horizontal arrows are the ones we just described and the vertical arrows are base change

$$\begin{array}{ccc} \mathrm{Aut}_Q(Z_\alpha, T) & \longrightarrow & \Gamma(X_0, \Theta_{X_0}) \otimes J \\ \downarrow & & \downarrow \\ \mathrm{Aut}_R(Z_\alpha \times_{\mathrm{Spec}(Q)} \mathrm{Spec}(R), R') & \longrightarrow & \Gamma(X_0, \Theta_{X_0}) \otimes I \end{array}$$

Furthermore, after passing to homology, the right vertical map becomes $\mathrm{id} \otimes \chi$. This shows that indeed $(\mathrm{id} \otimes \chi)(\mathfrak{o}(Y)) = \mathfrak{o}(M_{X_0}(\mu)(Y, \psi_0))$.

We are now ready to prove that ω is a bijection. If $M_{X_0}(T) \times_{M_{X_0}(R')} M_{X_0}(R) = \emptyset$ then there is nothing to show. Otherwise, suppose we have an element

$$((Y, \psi'_0), (X, \phi_0)) \in M_{X_0}(T) \times_{M_{X_0}(R')} M_{X_0}(R)$$

Let $(X', \phi'_0) = M_{X_0}(\pi)(X, \phi_0) \in M_{X_0}(R')$ In particular, $(M_{X_0}(\pi))^{-1}((X', \phi'_0)) \neq \emptyset$ and by proposition 2, $\mathrm{Def}_k(X', R) \neq \emptyset$. By proposition 1 this means that $\mathfrak{o}(X') = 0$. Therefore,

$$(\mathrm{id} \otimes \chi)(\mathfrak{o}(Y)) = (\mathfrak{o}(M_{X_0}(\mu)(Y)) = \mathfrak{o}(X') = 0$$

Therefore, $\mathfrak{o}(Y) = 0$ so again by propositions 1 and 2 we obtain the following diagram

$$\begin{array}{ccc} H^1(X_0, \Theta_{X_0}) \otimes J & \xrightarrow[\sim]{\mathrm{id} \otimes \chi} & H^1(X_0, \Theta_{X_0}) \otimes I \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{Def}_k(Y, Q) & \longrightarrow & \mathrm{Def}_k(X', R) \\ \kappa^{-1} \downarrow \wr & & \kappa^{-1} \downarrow \wr \\ M_{X_0}(Q) & \xrightarrow{M_{X_0}(\chi)} & M_{X_0}(R) \end{array}$$

where the middle horizontal arrow is base change. By following the construction in proposition 1 we see that the upper square is commutative. Similarly, by following the construction of κ in proposition 2 we see that the lower square is commutative. This means that the map $M_{X_0}(\chi)$ defines a bijection

$$(M_{X_0}(\rho))^{-1}(Y, \psi'_0) \xrightarrow{M_{X_0}(\chi)} (M_{X_0}(\pi))^{-1}(X', \phi'_0)$$

which implies that that ω is indeed a bijection as we wanted. This concludes the proof of left exactness.

The next step is to prove that $\dim_k(M_{X_0}(k[\epsilon])) = g^2$. As we've shown above, the exact sequence

$$0 \to k\epsilon \to k[\epsilon] \to k \to 0$$

implies that

$$M_{X_0}(k[\epsilon]) \cong H^1(X_0, \Theta_{X_0}) \otimes k\epsilon \cong H^1(X_0, \Theta_{X_0})$$

Also, for any abelian variety X_0, $\Theta_{X_0} \cong \mathcal{O}_{X_0} \otimes T_{X,e}$ where $T_{X,e}$ is the tangent space at the identity. This is proved in most texts on abelian varieties but if you haven't seen this before, thinking about what happens in the case of lie groups provides good intuition. Another general fact about abelian varieties is that $H^1(X_0, \mathcal{O}_{X_0}) \cong T_{X_0{}^\nu, e}$ where $X_0{}^\nu$ is the dual abelian variety over k. Together, we get that

$$M_{X_0}(k[\epsilon]) \cong H^1(X_0, \mathcal{O}_{X_0}) \otimes T_{X_0,e} \cong T_{X_0{}^\nu,e} \otimes T_{X_0,e}$$

and in particular, $\dim_k(M_{X_0}(k[\epsilon])) = g^2$.

The last thing to prove is that M_{X_0} is formally smooth. As usual, let

$$0 \to I \to R \to R' \to 0$$

be a small surjection in $\mathbf{Art}_k$. Let (X, ϕ_0) be an element of $M_{X_0}(R')$. We have to show that $M_{X_0}(R) \neq \emptyset$. By propositions 2 and 1, it's enough to show that $\mathfrak{o}(X') = 0$. To this end, we'll use the fact that the obstruction element is invariant under automorphisms of X' to show that it must equal 0 by symmetry.

Consider the inversion map $X' \xrightarrow{i} X'$. A general fact for abelian varieties is that

$$H^2(X_0, \mathcal{O}_{X_0}) \cong H^1(X_0, \mathcal{O}_{X_0}) \wedge H^1(X_0, \mathcal{O}_{X_0})$$

Therefore,

$$H^2(X_0, \Theta_{X_0}) \otimes I \cong (T_{X_0{}^\nu,e} \otimes T_{X_0,e}) \wedge (T_{X_0{}^\nu,e} \otimes T_{X_0,e}) \otimes I \cong (T_{X_0{}^\nu,e} \wedge T_{X_0{}^\nu,e}) \otimes T_{X_0,e} \otimes I$$

By applying i, we multiply each of the first three elements by -1. So $i \otimes \mathrm{id}$ acts on $H^2(X_0, \Theta_{X_0}) \otimes I$ as -1. However, by well definedness of $\mathfrak{o}(X')$, it is invariant under automorphisms of X'. Therefore, after extending i to inversion on X', we get that $\mathfrak{o}(X') = (i \otimes \mathrm{id})(\mathfrak{o}(X')) = -\mathfrak{o}(X')$.

If $\mathrm{char}(k) \neq 2$ then $\mathfrak{o}(X') = 0$ and we're done.

For the $\mathrm{char}(k) = 2$ case we have to be slightly more sneaky.

We start be defining $P' = X' \times_{\mathrm{Spec}(R')} X'$ and $P_0 = P' \times_{\mathrm{Spec}(R')} \mathrm{Spec}(k)$. P' is an abelian $\mathrm{Spec}(R')$-scheme and $\mathfrak{o}(P') \in H^2(P_0, \Theta_{P_0}) \otimes I$. There are two natural projections from $P_0 \cong X_0 \times X_0$ to X_0 and these induce two injections

$$i_1, i_2 : H^2(X_0, \Theta_{X_0}) \to H^2(P_0, \Theta_{P_0})$$

We now claim that $\mathfrak{o}(P') = i_1(\mathfrak{o}(X')) + i_2(\mathfrak{o}(X'))$. Let $\{U'_\alpha\}$ be a covering of X' such as we used when constructing the obstruction element in the proof of proposition 1. Similarly, let $\{Z_\alpha\}$ be the lifts of the U'_α-s and let $c_{\alpha\beta\gamma} \in \mathrm{Aut}_R(Z_\alpha|_{U'_{\alpha\beta\gamma}}, R')$ be the automorphisms defined in the same proof. Recall that we identified $\mathfrak{o}(X')$ with cocycles $\{c_{\alpha\beta\gamma}\}$ and that locally, addition in $H^2(X_0, \Theta_{X_0}) \otimes I$ corresponded to composition in the $\mathrm{Aut}_R(Z_\alpha|_{U'_{\alpha\beta\gamma}}, R')$.

Now, we can form an open cover of P' by taking products of pairs of elements in the cover of X'. This gives us the open cover $\{U'_{(\alpha_1,\alpha_2)} = U'_{\alpha_1} \times_{\mathrm{Spec}(R')} U'_{\alpha_2}\}$. Furthermore, the $\mathrm{Spec}(R)$-scheme $Z_{(\alpha_1,\alpha_2)} = Z_{\alpha_1} \times_{\mathrm{Spec}(R)} Z_{\alpha_2}$ forms a lift of $U'_{(\alpha_1,\alpha_2)}$ and we can obtain the element

$$c_{(\alpha_1,\alpha_2)(\beta_1\beta_2)(\gamma_1,\gamma_2)} \in \mathrm{Aut}_R(Z_{(\alpha_1,\alpha_2)}|_{U'_{(\alpha_1,\alpha_2)(\beta_1,\beta_2)(\gamma_1,\gamma_2)}}, R')$$

by composing $c_{\alpha_1,\beta_1,\gamma_1} \times \mathrm{id}$ and $\mathrm{id} \times c_{\alpha_2,\beta_2,\gamma_2}$. This is exactly what we claimed.

We now define an additional $\mathrm{Spec}(R')$-automorphism a' of P' by $a'(x, y) = (x + y, y)$.

As we've already mentioned in the $\mathrm{char} k \neq 2$ case, in the construction of $\mathfrak{o}$ we could have first applied the automorphism a' to P' and by the well definedness of $\mathfrak{o}(P')$ we would have gotten the the same obstruction element. By looking at the induced automorphism on $H^2(P_0, \Theta_{P_0}) \otimes I$ we get that

$$i_1(\mathfrak{o}(X')) + i_2(\mathfrak{o}(X')) = i_1(\mathfrak{o}(X')) + i_2(\mathfrak{o}(X')) + i_2(\mathfrak{o}(X'))$$

The explicit way to see this is to note that if $\mathfrak{o}(X')$ is the derivation D from $\mathcal{O}_{X_0}$ to itself then $i_1(\mathfrak{o}(X')) + i_2(\mathfrak{o}(X'))$ is the derivation $f \otimes g \mapsto D(f) \otimes 1 + 1 \otimes D(g)$ from $\mathcal{O}_{X_0} \otimes \mathcal{O}_{X_0}$ to itself. and after composing with $(a')^\flat$ we get $f \otimes g \mapsto D(f) \otimes 1 + 1 \otimes D(g) + 1 \otimes D(g)$.

Since we are in characteristic 2, we get that $\mathfrak{o}(P') = i_1(\mathfrak{o}(X'))$. Similarly, we get that $\mathfrak{o}(P') = i_2(\mathfrak{o}(X'))$. By adding these together we get that $0 = 2\mathfrak{o}(P') = \mathfrak{o}(P')$ as we wanted. □

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