Curvature of Metric Spaces

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Special Session on Metric Differential Geometry

Thursday, Friday, Saturday morning

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Joint work with Cédric Villani.

Related work was done independently by K.-T. Sturm.

Introduction

Goal : We have notions of curvature from differential geometry.

Do they make sense for metric spaces?

For example, does it make sense to say that a metric space is "positively curved"?

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Motivations :
1. Intrinsic interest
2. Understanding
3. Applications to smooth geometry

Length structure of a Riemannian manifold

Say M is a smooth n-dimensional manifold. For each $m \in M$, the tangent space $T_m M$ is an n-dimensional vector space.

A Riemannian metric g on M assigns, to each $m \in M$, an inner product on $T_m M$.

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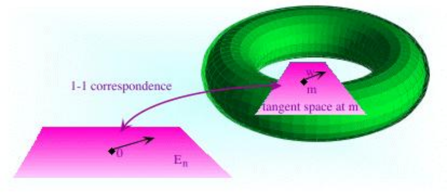
Example : M is a submanifold of $\mathbb{R}^N$

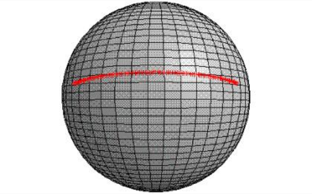


If $\mathbf{v} \in T_m M$, let $g(\mathbf{v}, \mathbf{v})$ be the square of the length of $\mathbf{v}$.

Length structure of a Riemannian manifold

The length of a smooth curve $\gamma : [0,1] \to M$ is

$$L(\gamma) = \int_0^1 \sqrt{g(\dot\gamma,\dot\gamma)}\,dt.$$



The distance between $m_0, m_1 \in M$ is the infimal length of curves joining m_0 to m_1.

Length structure of a Riemannian manifold

$$d(m_0, m_1) = \inf\{L(\gamma) \,:\, \gamma(0) = m_0, \gamma(1) = m_1\}.$$

Fact : this defines a metric on the set M.

Any length-minimizing curve from m_0 to m_1 is a geodesic.

Riemannian volume

Any Riemannian manifold M comes equipped with a smooth positive measure dvol_M.

In local coordinates, if $g = \sum_{i,j=1}^{n} g_{ij}\, dx^i\, dx^j$ then

$$\mathrm{dvol}_M \;=\; \sqrt{\det\left(g_{ij}\right)}\, dx^1 dx^2 \ldots dx^n .$$

The volume of a nice subset $A \subset M$ is

$$\mathrm{vol}(A) \;=\; \int_A \mathrm{dvol}_M \,.$$

Sectional curvature

To each $m \in M$ and each 2-plane $P \subset T_m M$ in the tangent space at m, one assigns a number $K(P)$, its sectional curvature.

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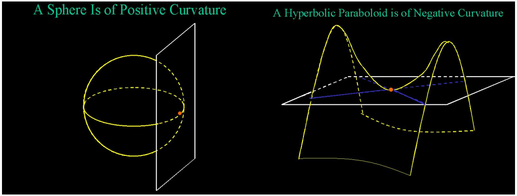
Example : If M is two-dimensional then P is all of $T_m M$ and $K(P)$ is the Gaussian curvature at m.



Ricci curvature

Ricci curvature is an averaging of sectional curvature.

Fix a unit-length vector $\mathbf{v} \in T_m M$.

Definition

$$\text{Ric}(\mathbf{v}, \mathbf{v}) = (n-1) \cdot (\text{the average sectional curvature of the 2-planes P containing } \mathbf{v}).$$

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Example : $S^2 \times S^2 \subset (\mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6)$ has *nonnegative* sectional curvatures but has *positive* Ricci curvatures.

What does Ricci curvature control?

1. Volume growth

Bishop-Gromov inequality :
If M has nonnegative Ricci curvature then balls in M grow no faster than in Euclidean space.

That is, for any $m \in M$, $r^{-n}\ \mathrm{vol}(B_r(m))$ is nonincreasing in r.

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2. The universe

Einstein says that a matter-free spacetime has vanishing Ricci curvature.

Summary

Any Riemannian manifold gets

1. Lengths of curves
2. A metric space structure
3. A measure
4. Sectional curvatures
5. Ricci curvatures

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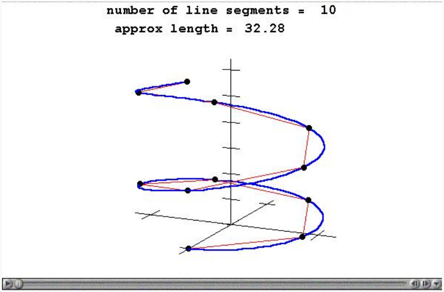
<u>Question :</u>
To what extent can we recover the sectional curvatures from just the metric space structure?

To what extent can we recover the Ricci curvatures from just the metric space structure and the measure?

Curvature of Metric Spaces

Length spaces

Say (X, d) is a compact metric space and $\gamma : [0, 1] \to X$ is a continuous map.



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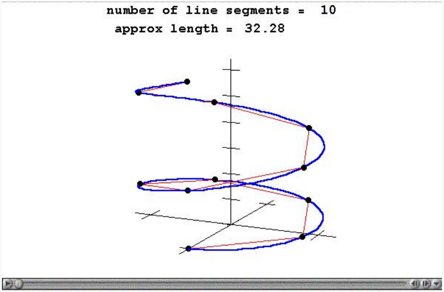

The length of γ is

$$L(\gamma) = \sup_J \sup_{0=t_0 \le t_1 \le \ldots \le t_J = 1} \sum_{j=1}^{J} d(\gamma(t_{j-1}), \gamma(t_j)).$$

Length spaces

(X, d) is a length space if the distance between two points $x_0, x_1 \in X$ equals the infimum of the lengths of curves joining them, i.e.

$$d(x_0, x_1) = \inf\{L(\gamma) \ : \ \gamma(0) = x_0, \gamma(1) = x_1\}.$$

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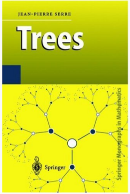
A length-minimizing curve is called a geodesic.

Length spaces

Examples of length spaces :

1. The underlying metric space of any Riemannian manifold.
2.



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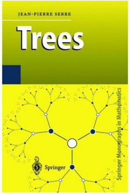
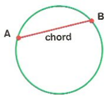


Nonexamples :

1. A finite metric space with more than one point.
2. A circle with the chordal metric.

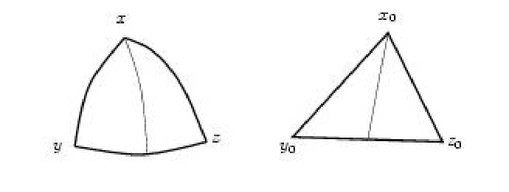


Curvature of Metric Spaces

Alexandrov curvature

Definition

A compact length space (X, d) has nonnegative Alexandrov curvature if geodesic triangles in X are at least as "fat" as corresponding triangles in $\mathbb{R}^2$.



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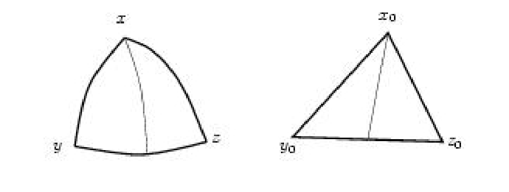
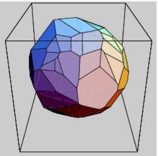


The comparison triangle in $\mathbb{R}^2$ has the same sidelengths
Fatness :
(Length of bisector from x) $\geq$ (Length of bisector from x_0)

Alexandrov curvature

Example : the boundary of a convex region in $\mathbb{R}^N$ has nonnegative Alexandrov curvature.



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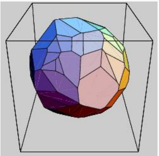

1. If (M, g) is a Riemannian manifold then its underlying metric space has nonnegative Alexandrov curvature if and only if M has nonnegative sectional curvatures.

2. If $\{(X_i, d_i)\}_{i=1}^{\infty}$ have nonnegative Alexandrov curvature and $\lim_{i\to\infty}(X_i, d_i) = (X, d)$ in the Gromov-Hausdorff topology then (X, d) has nonnegative Alexandrov curvature.

3. Applications to Riemannian geometry

Gromov-Hausdorff topology

A topology on the set of all compact metric spaces (modulo isometry).

(X_1, d_1) and (X_2, d_2) are close in the Gromov-Hausdorff topology if somebody with bad vision has trouble telling them apart.

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Example : a cylinder with a small cross-section is Gromov-Hausdorff close to a line segment.

The question

Can one extend Alexandrov's work from sectional curvature to Ricci curvature?

Motivation : Gromov's precompactness theorem

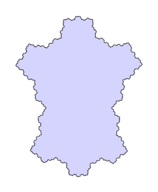
Theorem

Given $N \in \mathbb{Z}^+$ and $D > 0$,

$$\{(M, g) \; : \; \dim(M) = N, \operatorname{diam}(M) \leq D, \operatorname{Ric}_M \geq 0\}$$

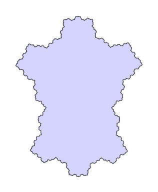
is precompact in the Gromov-Hausdorff topology on $\{$compact metric spaces$\}$/isometry.

Gromov-Hausdorff space



Each point is a compact metric space.
Each interior point is a Riemannian manifold (M, g) with $\dim(M) = N$, $\text{diam}(M) \leq D$ and $\text{Ric}_M \geq 0$.

Gromov-Hausdorff space



Each point is a compact metric space.
Each interior point is a Riemannian manifold (M, g) with $\dim(M) = N$, $\mathrm{diam}(M) \leq D$ and $\mathrm{Ric}_M \geq 0$.

The boundary points are compact metric spaces (X, d) with $\dim_H X \leq N$. They are generally not manifolds. (Example : $X = M/G$.)

In some moral sense, the boundary points are metric spaces with "nonnegative Ricci curvature".

Curvature of Metric Spaces

Dirtmoving

Given a before and an after dirtpile, what is the most efficient way to move the dirt from one place to the other?

Let's say that the cost to move a gram of dirt from x to y is $d(x, y)^2$.

Gaspard Monge

Mémoire sur la théorie des déblais et des remblais (1781)

Memoir on the theory of excavations and fillings (1781)

MÉMOIRE
SUR LA
THÉORIE DES DÉBLAIS ET DES REMBLAIS.

Par M. MONGE.

LORSQU'ON doit tranſporter des terres d'un lieu dans un autre, on a coutume de donner le nom de *Déblai* au volume des terres que l'on doit tranſporter, & le nom de *Remblai* à l'eſpace qu'elles doivent occuper après le tranſport.

Le prix du tranſport d'une molécule étant, toutes choſes d'ailleurs égales, proportionnel à ſon poids & à l'eſpace qu'on lui fait parcourir, & par conſéquent le prix du tranſport total devant être proportionnel à la ſomme des produits des molécules multipliées chacune par l'eſpace parcouru, il s'enſuit que le déblai & le remblai étant donnés de figure & de poſition, il n'eſt pas indifférent que telle molécule du déblai ſoit tranſportée dans tel ou tel autre endroit du remblai, mais qu'il y a une certaine diſtribution à faire des molécules du premier dans le ſecond, d'après laquelle la ſomme de ces produits ſera la moindre poſſible, & le prix du tranſport total ſera un *minimum.*

C'eſt la ſolution de cette queſtion que je me propoſe de donner ici. Je diviſerai ce Mémoire en deux parties, dans la première je ſuppoſerai que les déblais & les remblais ſont des aires contenues dans un même plan; dans le ſecond, je ſuppoſerai que ce ſont des volumes.

PREMIÈRE PARTIE.

Du tranſport des aires planes ſur des aires compriſes dans un même plan.

I.

QUELLE que ſoit la route que doive ſuivre une molécule

Wasserstein space

Let (X, d) be a compact metric space.

Notation

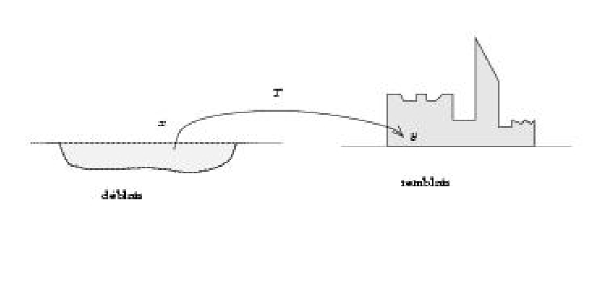
$P(X)$ is the set of Borel probability measures on X.

That is, $\mu \in P(X)$ iff μ is a nonnegative Borel measure on X with $\mu(X) = 1$.

Definition

Given $\mu_0, \mu_1 \in P(X)$, the Wasserstein distance $W_2(\mu_0, \mu_1)$ is the square root of the minimal cost to transport μ_0 to μ_1.

Wasserstein space



$$W_2(\mu_0, \mu_1)^2 = \inf \left\{ \int_{X \times X} d(x, y)^2 \, d\pi(x, y) \right\},$$

where

$$\pi \in P(X \times X), (p_0)_* \pi = \mu_0, (p_1)_* \pi = \mu_1.$$

Wasserstein space

Fact :
$(P(X), W_2)$ is a metric space, called the Wasserstein space.

The metric topology is the weak-$*$ topology, i.e. $\lim_{i\to\infty} \mu_i = \mu$ if and only if for all $f \in C(X)$, $\lim_{i\to\infty} \int_X f \, d\mu_i = \int_X f \, d\mu$.

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Proposition

If X is a length space then so is the Wasserstein space $P(X)$.

Hence we can talk about its (minimizing) geodesics $\{\mu_t\}_{t\in[0,1]}$, called Wasserstein geodesics.

Optimal transport

What is the optimal transport scheme between $\mu_0, \mu_1 \in P(X)$?

$X = \mathbb{R}^n$, μ_0 and μ_1 absolutely continuous :
Rachev-Rüschendorf, Brenier (1990)

X a Riemannian manifold, μ_0 and μ_1 absolutely continuous :
McCann (2001)

Empirical fact : The Ricci curvature of the Riemannian manifold affects the optimal transport in a quantitative way.
Otto-Villani (2000),
Cordero-Erausquin-McCann-Schmückenschläger (2001)

Metric-measure spaces

Definition

A metric-measure space is a metric space (X, d) equipped with a given probability measure $\nu \in P(X)$.

A smooth metric-measure space is a Riemannian manifold (M, g) with a smooth probability measure $d\nu = e^{-\Psi}\ \mathrm{dvol}_M$.

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Idea : Use optimal transport on X to *define* what it means for (X, ν) to have "nonnegative Ricci curvature".

$(X, d) \longrightarrow (P(X), W_2)$

To one compact length space we have assigned another.

Use the properties of the Wasserstein space $(P(X), W_2)$ to say something about the geometry of (X, d).

Measured Gromov-Hausdorff limits

An easy consequence of Gromov precompactness :

$$\left\{\left(M, g, \frac{\mathrm{dvol}_M}{\mathrm{vol}(M)}\right) \ : \ \dim(M) = N, \mathrm{diam}(M) \leq D, \mathrm{Ric}_M \geq 0\right\}$$

is precompact in the measured Gromov-Hausdorff topology on $\{$compact metric-measure spaces$\}$/isometry.

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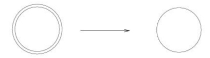
What can we say about the limit points? (Work of Cheeger-Colding)

What are the *smooth* limit points?

Measured Gromov-Hausdorff (MGH) topology



Definition

$\lim_{i \to \infty}(X_i, d_i, \nu_i) = (X, d, \nu)$ if there are Borel maps $f_i : X_i \to X$ and a sequence $\epsilon_i \to 0$ such that

1. (Almost isometry) For all $x_i, x_i' \in X_i$,

$$|d_X(f_i(x_i), f_i(x_i')) - d_{X_i}(x_i, x_i')| \le \epsilon_i.$$

2. (Almost surjective) For all $x \in X$ and all i, there is some $x_i \in X_i$ such that

$$d_X(f_i(x_i), x) \le \epsilon_i.$$

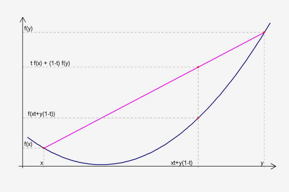
3. $\lim_{i \to \infty}(f_i)_* \nu_i = \nu$ in the weak-$*$ topology.

Notation

X a compact Hausdorff space.

$P(X) =$ Borel probability measures on X, with weak-$*$ topology.

$U : [0, \infty) \to \mathbb{R}$ a continuous convex function with $U(0) = 0$.



Fix a background measure $\nu \in P(X)$.

Entropy

The "negative entropy" of μ with respect to ν is

$$U_\nu(\mu) = \int_X U(\rho(x))\, d\nu(x) + U'(\infty)\, \mu_s(X).$$

Here

$$\mu = \rho\, \nu + \mu_s$$

is the Lebesgue decomposition of μ with respect to ν and

$$U'(\infty) = \lim_{r\to\infty} \frac{U(r)}{r}.$$

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$U_\nu(\mu)$ measures the nonuniformity of μ w.r.t. ν. It is minimized when $\mu = \nu$.

We get a function $U_\nu : P(X) \to \mathbb{R} \cup \infty$.

Effective dimension

$N \in [1, \infty]$ a new parameter (possibly infinite).

It turns out that there's not a single notion of "nonnegative Ricci curvature", but rather a 1-parameter family. That is, for each N, there's a notion of a space having "nonnegative N-Ricci curvature".

Here N is an effective dimension of the space, and must be inputted.

Displacement convexity classes

Definition

(McCann) If $N < \infty$ then DC_N is the set of such convex functions U so that the function

$$\lambda \to \lambda^N U(\lambda^{-N})$$

is convex on $(0, \infty)$.

Definition

DC_∞ is the set of such convex functions U so that the function

$$\lambda \to e^\lambda U(e^{-\lambda})$$

is convex on $(-\infty, \infty)$.

Displacement convexity classes

Example

$$U_N(r) = \begin{cases} Nr(1 - r^{-1/N}) & \text{if } 1 < N < \infty, \\ r \log r & \text{if } N = \infty. \end{cases}$$

If $U = U_\infty$ then the corresponding functional is

$$U_\nu(\mu) = \begin{cases} \int_X \rho \log \rho \, d\nu & \text{if } \mu \text{ is a.c. w.r.t. } \nu, \\ \infty & \text{otherwise,} \end{cases}$$

where $\mu = \rho \nu$.

Curvature of Metric Spaces

Convexity on Wasserstein space

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We want to ask whether the negative entropy function U_ν is a convex function on $P(X)$.

That is, given $\mu_0, \mu_1 \in P(X)$, whether U_ν restricts to a convex function along a Wasserstein geodesic $\{\mu_t\}_{t\in[0,1]}$ from μ_0 to μ_1.

Nonnegative N-Ricci curvature

Definition

Given $N \in [1, \infty]$, we say that a compact measured length space (X, d, ν) has nonnegative N-Ricci curvature if :

For all $\mu_0, \mu_1 \in P(X)$ with $\mathrm{supp}(\mu_0) \subset \mathrm{supp}(\nu)$ and $\mathrm{supp}(\mu_1) \subset \mathrm{supp}(\nu)$, there is *some* Wasserstein geodesic $\{\mu_t\}_{t \in [0,1]}$ from μ_0 to μ_1 so that for all $U \in DC_N$ and all $t \in [0, 1]$,

$$U_\nu(\mu_t) \le t\, U_\nu(\mu_1) + (1 - t)\, U_\nu(\mu_0).$$

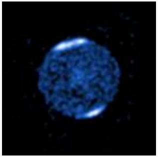
Nonnegative N-Ricci curvature

Note : We only require convexity along *some* geodesic from μ_0 to μ_1, not all geodesics.

But the same geodesic has to work for all $U \in DC_N$.

What does this have to do with curvature?

Look at optimal transport on the 2-sphere.
$\nu =$ normalized Riemannian density.
Take μ_0, μ_1 two disjoint congruent blobs. $U_\nu(\mu_0) = U_\nu(\mu_1)$.



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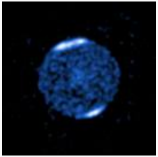
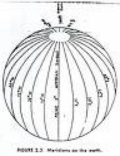
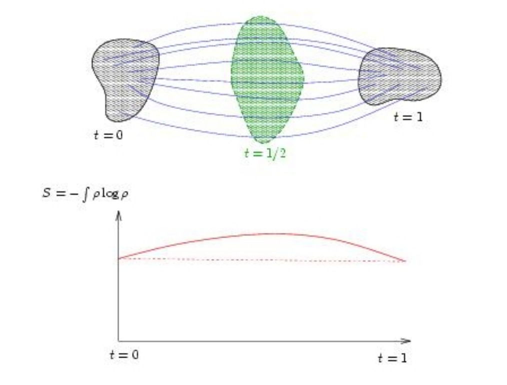


Optimal transport from μ_0 to μ_1 goes along longitudes.
Positive curvature gives **focusing** of geodesics.



Take a snapshot at time t.



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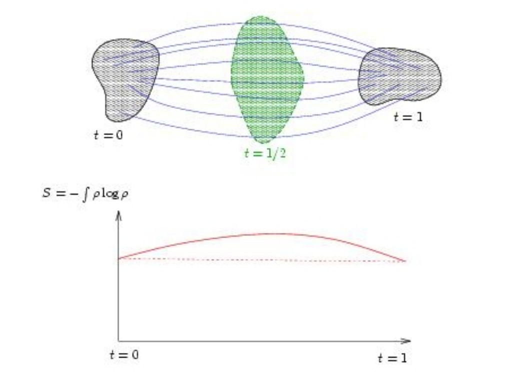

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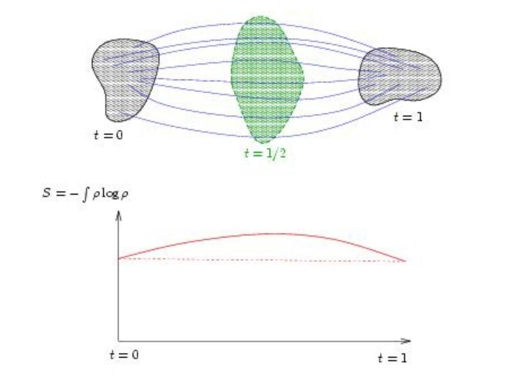

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The *more* uniform the measure, the *higher* its entropy.

So the entropy is a *concave* function of t, i.e. the negative entropy is a *convex* function of t.

Main result

Theorem

Let $\{(X_i, d_i, \nu_i)\}_{i=1}^{\infty}$ be a sequence of compact measured length spaces with

$$\lim_{i\to\infty}(X_i, d_i, \nu_i) = (X, d, \nu)$$

in the measured Gromov-Hausdorff topology.

For any $N \in [1, \infty]$, if each (X_i, d_i, ν_i) has nonnegative N-Ricci curvature then (X, d, ν) has nonnegative N-Ricci curvature.

What does all this have to do with Ricci curvature?

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Let (M, g) be a compact connected n-dimensional Riemannian manifold.
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Say $\Psi \in C^\infty(M)$ has

$$\int_M e^{-\Psi} \, \mathrm{dvol}_M = 1.$$

Put $\nu = e^{-\Psi} \, \mathrm{dvol}_M$.

Any smooth positive probability measure on M can be written in this way.

What does all this have to do with Ricci curvature?

For $N \in [1, \infty]$, define the N-Ricci tensor Ric_N of (M^n, g, ν) by

$$\begin{cases} \mathrm{Ric} + \mathrm{Hess}(\Psi) & \text{if } N = \infty, \\ \mathrm{Ric} + \mathrm{Hess}(\Psi) - \frac{1}{N-n} d\Psi \otimes d\Psi & \text{if } n < N < \infty, \\ \mathrm{Ric} + \mathrm{Hess}(\Psi) - \infty (d\Psi \otimes d\Psi) & \text{if } N = n, \\ -\infty & \text{if } N < n, \end{cases}$$

where by convention $\infty \cdot 0 = 0$.

Ric_N is a symmetric covariant 2-tensor field on M that depends on g and Ψ.

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Ric_N is a symmetric covariant 2-tensor field on M that depends on g and Ψ.

(If $N = n$ then Ric_N is $-\infty$ except where $d\Psi = 0$. There, $\mathrm{Ric}_N = \mathrm{Ric}$.)

$\mathrm{Ric}_\infty =$ Bakry-Emery tensor = right-hand side of Perelman's modified Ricci flow equation.

Abstract Ricci recovers classical Ricci

Recall that $\nu = e^{-\Psi}\,\mathrm{dvol}_M$.

Theorem

For $N \in [1, \infty]$, the measured length space (M, g, ν) has nonnegative N-Ricci curvature if and only if $\mathrm{Ric}_N \geq 0$.

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Classical case : Ψ constant, so $\nu = \frac{\mathrm{dvol}}{\mathrm{vol}(M)}$.

Then (M^n, g, ν) has <u>abstract</u> nonnegative N-Ricci curvature if and only if it has <u>classical</u> nonnegative Ricci curvature, as soon as $N \geq n$.

Smooth limit spaces

Had Gromov precompactness theorem. What are the limit spaces (X, d, ν)? Suppose that the limit space is a *smooth* measured length space, i.e.

$$(X, d, \nu) = (B, g_B, e^{-\Psi}\, \mathrm{dvol}_B)$$

for some n-dimensional smooth Riemannian manifold (B, g_B) and some $\Psi \in C^\infty(B)$.

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Note : the dimension can drop on taking limits.

The converse is true if $N \geq n + 2$.

Bishop-Gromov-type inequality

Theorem

If (X, d, ν) *has nonnegative* N*-Ricci curvature and* $x \in \mathrm{supp}(\nu)$ *then* $r^{-N}\,\nu(B_r(x))$ *is nonincreasing in* r.

Lichnerowicz inequality

If (M, g) is a compact Riemannian manifold, let λ_1 be the smallest positive eigenvalue of the Laplacian $-\nabla^2$.

Theorem

Lichnerowicz (1964)
If $\dim(M) = n$ *and* M *has Ricci curvatures bounded below by* $K > 0$ *then*

$$\lambda_1 \geq \frac{n}{n-1} K .$$

Sharp global Poincaré inequality

Theorem

If (X, d, ν) has N-Ricci curvature bounded below by $K > 0$ and f is a Lipschitz function on X with $\int_X f \, d\nu = 0$ then

$$\int_X f^2 \, d\nu \le \frac{N-1}{N} \frac{1}{K} \int_X |\nabla f|^2 \, d\nu.$$

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Here

$$|\nabla f|(x) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{d(y, x)}.$$

Open questions

1. Take any result that you know about Riemannian manifolds with nonnegative Ricci curvature.

Does it extend to measured length spaces (X, d, ν) with nonnegative N-Ricci curvature?
(Yes for Bishop-Gromov, no for splitting theorem.)

2. Take an interesting measured length space (X, d, ν). Does it have nonnegative N-Ricci curvature?

This almost always boils down to understanding the optimal transport on X.